

A generalized Cox-Ingersoll-Ross Equation with growing initial conditions

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Abstract

In this paper we solve the problem of the existence and strong continuity of the semigroup associated with the initial value problem for a generalized Cox-Ingersoll-Ross equation for the price of a zero-coupon bond (see [8]), on spaces of continuous functions on $[0, \infty)$ which can grow at infinity. We focus on the Banach spaces

$$Y_s = \left\{ f \in C[0, \infty) : \frac{f(x)}{1+x^s} \in C_0[0, \infty) \right\}, \quad s \geq 1,$$

which contain the nonzero constants very common as initial data in the Cauchy problems coming from financial models. In addition, a Feynman-Kac type formula is given.

Key words. Cox-Ingersoll-Ross equation, (C_0) semigroups, Feynman-Kac type formula, large initial condition.

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1 Introduction

Of concern is the initial value problem studied in [8],

$$\begin{cases} \frac{\partial u}{\partial t} = \nu^2 x \frac{\partial^2 u}{\partial x^2} + (\gamma + \beta x) \frac{\partial u}{\partial x} - r x u \\ u(0, x) = f(x), \end{cases} \quad (1)$$

for $x \geq 0$, $t \geq 0$. The constant coefficients satisfy $\nu > 0$, $\gamma > 0$, $\beta \in \mathbb{R}$ and $r > 0$. This problem generalizes the so-called CIR problem introduced by Cox, Ingersoll and Ross in 1985 ([3]) to price a discount zero-coupon bond, that is a contract promising to pay a certain “face” amount, conventionally taken equal to 1 (currency unit), at a fixed

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maturity date T . In the CIR framework, the variable x in (1) denotes the current value of the interest rate assumed to be stochastic, the potential term is $-xu$ ($r = 1$), the initial condition is $f(x) = 1$, which corresponds to the face value 1 of the bond at the maturity T , and $\nu = \sigma/\sqrt{2}$ with $\sigma > 0$ being the volatility of the stochastic interest rate (for more details the reader can refer to [8]).

This problem can be compared to the Black-Scholes equation to price European options without dividends ([1])

$$\begin{cases} \frac{\partial v}{\partial t} = \nu^2 x^2 \frac{\partial^2 v}{\partial x^2} + \mu x \frac{\partial v}{\partial x} - \mu v, \\ v(0, x) = g(x) \end{cases} \quad (2)$$

for $x \geq 0$, $t \geq 0$, with ν as before and associated constant interest rate $\mu > 0$. Here the most important initial condition is

$$g(x) = (x - K)_+ = \max\{x - K, 0\} \quad \text{or} \quad g(x) = (K - x)_+ = \max\{K - x, 0\},$$

which indicates the payoff of a call (resp. put) European option at a fixed maturity date T , where the strike price K is a given positive constant. Observe that the semigroup governing (2) is easy to construct and study. Indeed, it can factor into the product of the semigroups generated by the commuting operators $H_1 v = \nu^2 x^2 v''$, $H_2 v = \mu x v'$ and $H_3 u = -\mu v$, while the semigroup governing (1) does not factor into the product of the four semigroups generated by the operators $A_1 u = \nu^2 x u''$, $A_2 u = \gamma u'$, $A_3 u = \beta x u'$ and $V(x)u = -rxu$. Each pair A_i and A_j with i and j different fails to commute. The noncommuting makes (1) a much harder problem than (2).

As a consequence of the results obtained in [8], we know that the semigroup \mathcal{T}_{CIR} governing (1) is a (C_0) semigroup on $C_0[0, \infty)$, the space of continuous functions on $[0, \infty)$ which vanish at ∞ equipped with the sup norm $\|\cdot\|_\infty$. In fall 2017, we learned that the (C_0) property of \mathcal{T}_{CIR} on $C_0[0, \infty)$ had been established in [4], but there it is expressed in a very different context using affine processes and Feller semigroups. Our paper [8] contained a completely different proof of the strong continuity and additional results, such as a characterization of the domain of the generator and a new type of Feynman-Kac formula. This is a significant step towards understanding how to represent the solution of the Cauchy problem (1).

It is worth noting that \mathcal{T}_{CIR} fails to be of class (C_0) on the space $C[0, \infty]$ of all continuous complex valued functions on $[0, \infty)$ having finite limit at ∞ . Thus, the previous results cannot be applied to the special initial function $f_0(x) \equiv 1$. To remedy this, one must find a space Y such that (1) is governed by a (C_0) semigroup on Y and $f_0 \in Y$. The present paper is devoted to solving this problem. We also replace the potential term $V(x)u = -rxu$ in (1) by a more general nonpositive potential term. In Mathematical Finance to make some more assumptions on the derivative's price dynamics when other issues in the financial markets are considered (e.g. jumps that actually happen because of government fiscal and monetary decisions, changes in investors' expectations, etc.). We use polynomials, which generalize positive polynomials on $(0, \infty)$.

The spaces that we need are

$$Y_s := C_0 \left([0, \infty), \frac{1}{1+x^s} \right)$$

for $s > 0$. Here, for a weight function $0 < w \in C[0, \infty)$, we define

$$C_0([0, \infty), w) = \{h \in C[0, \infty) : hw \in C_0[0, \infty)\},$$

and for $h \in C_0([0, \infty), w)$, the norm $\|h\|_w$ is defined by

$$\|h\|_w = \sup_{x \geq 0} |h(x)| w(x) < \infty.$$

The space Y_s corresponds to $w(x) = \frac{1}{1+x^s}$. Thus, the *weighted space* Y_s has its norm depending on $s > 0$. This will be denoted by $\|\cdot\|_s$. If $s = 0$, then $C_0[0, \infty)$ is equipped with the norm

$$\|f\|_0 = \frac{1}{2} \|f\|_\infty$$

for $f \in C_0[0, \infty)$. In a similar way, if $C_0(0, \infty)$ denotes the space of all continuous complex valued functions on $(0, \infty)$ that vanish at both 0 and ∞ , then one can define

$$X_s := C_0 \left((0, \infty), \frac{1}{1+x^s} \right)$$

for $s > 0$. Notice that Y_s is a strictly larger space than X_s since functions $h \in Y_s$ need not satisfy the condition that $h(x)/(1+x^s)$ vanishes at the origin.

We remark that the CIR equation is often treated with potential term in (1) missing. Then the generator becomes $\nu^2 x \frac{d^2}{dx^2} + (\gamma + \beta x) \frac{d}{dx}$, and generators like this have a long history, starting with Feller around 1950. But the treatments of these operators on $C[0, \infty]$, the space of all continuous complex valued functions on $[0, \infty)$ having finite limit at ∞ are often incomplete.

2 Preliminaries

Denote by $\mathcal{M}(0, \infty)$ the set of all finite complex Borel measures on $(0, \infty)$. As observed in [6, Section 2], by the Riesz Representation Theorem, the dual space of $(C_0[0, \infty), \|\cdot\|_0)$ can be identified by $\mathcal{M}(0, \infty)$ equipped with the norm

$$\|\psi\| = 2 (\text{Total Variation of } \psi).$$

Similarly we can define the dual space Y_s^* of Y_s . Using the pairing, for $u \in Y_s$, $\psi \in Y_s^*$,

$$\begin{aligned} \langle u, \psi \rangle &= \int_0^\infty \frac{u(x)}{1+x^s} (1+x^s) \psi(dx) \\ &= \int_0^\infty u(x) \psi(dx), \end{aligned}$$

the dual space of Y_s can be identified by

$$Y_s^* = \{\varphi \in \mathcal{M}_{loc}(0, \infty) : (1 + x^s) \varphi(dx) \in \mathcal{M}(0, \infty)\} \quad (3)$$

for all $s > 0$ (see [6, Lemmas 2.1–2.2]), with the norm of $\varphi \in Y_s^*$ being the total variation of the finite complex measure $(1 + x^s) \varphi(dx)$.

Observe that the initial value problem (1) can be written as

$$u_t = B_\nu u + M_r u, \quad u(0, x) = f(x), \quad t, x \geq 0,$$

with $r > 0$ and $f \in C[0, \infty]$ (resp. $f \in C_0[0, \infty)$), where M_r is the multiplicative operator, $M_r u(x) = -rxu$, and the operator B_ν is given by

$$B_\nu u := \nu^2 x u'' + (\gamma + \beta x) u', \quad (4)$$

with $\nu > 0$, $\gamma > 0$, $\beta \in \mathbb{R}$. The operators B_ν and M_r act on $C[0, \infty]$ (resp. $C_0[0, \infty)$). For any $\nu > 0$, in [8] we considered the operator defined formally by

$$G_\nu = \nu \sqrt{x} \frac{d}{dx}; \quad (5)$$

its square is given by

$$G_\nu^2 = \nu^2 x \frac{d^2}{dx^2} + \frac{\nu^2}{2} \frac{d}{dx}.$$

Thus the operator B_ν defined by (4) can be represented as a perturbation of G_ν^2 , namely

$$\begin{aligned} B_\nu &= G_\nu^2 + \left(\gamma - \frac{\nu^2}{2}\right) \frac{d}{dx} + \beta x \frac{d}{dx} \\ &= G_\nu^2 + P_1 + P_2, \end{aligned} \quad (6)$$

where

$$P_1 = \alpha \frac{d}{dx} \quad (\text{with } \alpha = \gamma - \frac{\nu^2}{2}), \quad P_2 = \beta x \frac{d}{dx}. \quad (7)$$

According to the results in [8] the operators G_ν^2 , B_ν , $\overline{P_1 + P_2 + M_r}$ with $\alpha > 0$, $\beta \neq 0$, $r > 0$ are infinitesimal generators of (C_0) semigroups on $C_0[0, \infty)$, while the operator $A_r := \overline{B_\nu + M_r}$, $r > 0$, generates a once-integrated semigroup on $C[0, \infty]$ which is (C_0) on the space $C_0[0, \infty)$. Furthermore, we showed that the semigroups generated by the operators G_ν^2 and $\overline{P_1 + P_2 + M_r}$ have explicit representations on $C_0[0, \infty)$.

In this paper our aim is to study the operators G_ν^2 , $\overline{P_1 + P_2 + M_r}$, $\overline{A_r} := \overline{B_\nu + M_r}$, and their associated semigroups on the weighted Banach space Y_s , $s > 0$, in order to solve the CIR problem ($r = 1$) mentioned in the Introduction

$$u_t = A_1 u, \quad u(0, x) \equiv 1, \quad t > 0, x \geq 0.$$

Observe that the constant function 1 is in Y_s for all $s > 0$. Further, we will show that some generation results work even when the multiplicative operator M_r is replaced by a more general nonpositive multiplicative operator of type $M_{-P} = -P(x)I$ with $P(x)$ posynomial (see Definition 1 in the next section). We only need to consider real functions and real Banach spaces. So, from now on, all our function spaces consist of only real functions.

3 Generation results

Recall that if X is a Banach space equipped with norm $\|\cdot\|_X$, for any $x \in X, x \neq 0$, we can define the (nonempty) subset $\mathcal{I}(x)$ of the dual of X, X^* , by

$$\mathcal{I}(x) = \{\varphi \in X^* : \|\varphi\|_{X^*} = 1, \langle x, \varphi \rangle = \|x\|_X\}.$$

Here $\langle \cdot, \cdot \rangle$ is the duality between X and X^* . A linear operator \hat{A} on X is called *quasidissipative* if

for any $x \in \mathcal{D}(\hat{A}) \subset X$ there exists $\varphi \in \mathcal{I}(x)$ such that

$$\operatorname{Re} \langle \hat{A}x, \varphi \rangle \leq \omega \|x\|_X \quad (8)$$

for some constant $\omega \in \mathbb{R}$ depending only on \hat{A} . Also, $\hat{A} - \omega I$ is called *dissipative*.

Consider the operator G_ν defined in (5) and acting on $C[0, \infty]$ with domain

$$\mathcal{D}(G_\nu) = \{u \in C[0, \infty] \cap C^1(0, \infty) : G_\nu u \in C[0, \infty]\}, \quad (9)$$

and the operator $G_{\nu,s}$ defined as G_ν and acting on Y_s (resp. on $C_0[0, \infty)$ when $s = 0$) with domain

$$\mathcal{D}(G_{\nu,s}) = \{u \in Y_s \cap C^1(0, \infty) : G_{\nu,s} u \in Y_s\}. \quad (10)$$

As in [11, Corollary 4.5], let us consider the family of operators $\mathcal{T}_\nu := (T_\nu(t))_{t \in \mathbb{R}}$ defined by

$$T_\nu(t) f(x) := f \left(\left[\sqrt{x} + \frac{\nu t}{2} \right]^2 \right) \quad (11)$$

for $t \in \mathbb{R}, x \geq 0, f \in C[0, \infty]$.

Lemma 1. *Fix $\nu > 0$. The family \mathcal{T}_ν defined in (11) is a (C_0) group of isometries on $C[0, \infty]$ having as generator G_ν with domain $\mathcal{D}(G_\nu)$ given by (9).*

Proof. For any $t \in \mathbb{R}$ and $x \geq 0$ let us define

$$\xi_t(x) := \left[\sqrt{x} + \frac{\nu t}{2} \right]^2$$

and observe that the Cauchy problem

$$u_t = \nu \sqrt{x} u_x, \quad u(0, x) = f(x), \quad t \in \mathbb{R}, \quad x \geq 0$$

is solved by $u(t, x) = f(\xi_t(x))$. In addition, it can be easily seen that $(T_\nu(t))_{t \in \mathbb{R}}$ is a group of bounded linear continuous operators on $C[0, \infty]$ which consists of isometries. Now, let us prove that the group \mathcal{T}_ν is strongly continuous. Preliminarily, we remark that

$$\lim_{x \rightarrow +\infty} \xi_t(x) = +\infty \quad \text{uniformly for } t \geq q \text{ for each } q \in \mathbb{R}$$

and

$$\lim_{t \rightarrow 0} \xi_t(x) = x \quad (12)$$

uniformly on compact sets of $[0, \infty)$. Let us fix $f \in C[0, \infty]$. We have to prove that

$$\lim_{t \rightarrow 0} \|T_\nu(t)f - f\|_\infty = \lim_{t \rightarrow 0} \|f(\xi_t) - f\|_\infty = 0. \quad (13)$$

Observe that $f(x)$ converges as x approaches to $+\infty$ and denote by $L \in \mathbb{R}$ its limit. It follows that

$$\lim_{x \rightarrow \infty} f(x) = L = \lim_{x \rightarrow \infty} f(\xi_t(x)).$$

Hence, fixing any $\varepsilon > 0$, there exists $M > 0$ such that

$$\sup_{x > M} |f(\xi_t(x)) - L| < \frac{\varepsilon}{2}$$

uniformly for $t \in \mathbb{R}$ and

$$\sup_{x > M} |f(x) - L| < \frac{\varepsilon}{2}.$$

Therefore,

$$\sup_{x > M} |f(\xi_t(x)) - f(x)| < \varepsilon \quad (14)$$

uniformly for $t \in \mathbb{R}$. On the other hand, from the uniform continuity of f and (12), we deduce that there exists $\delta > 0$ such that, for any $t \in \mathbb{R}$,

$$0 < |t| < \delta \text{ implies } \sup_{0 \leq x \leq M} |f(\xi_t(x)) - f(x)| < \varepsilon. \quad (15)$$

From (14) and (15) we can deduce that, for any $t \in \mathbb{R}$,

$$0 < |t| < \delta \text{ implies } \sup_{x \in \mathbb{R}} |f(\xi_t(x)) - f(x)| = \|T_\nu(t)f - f\|_\infty < \varepsilon,$$

and the assertion (13) follows. Let us consider the resolvent equation

$$\lambda u - G_\nu u = h \quad (16)$$

for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $h \in C[0, \infty]$. The general solution is parametrized by $u(0) \in \mathbb{R}$, but only one solution is bounded and in $C[0, \infty]$. Suppose $\lambda > 0$ and $h \in C[0, \infty]$ satisfies

$$h(x) = a \in \mathbb{R}$$

for any $x > b$. As a, b vary in \mathbb{R} the functions h describe a dense subspace of $C[0, \infty]$. Observe that (16) implies

$$\frac{\lambda u(x)}{\nu \sqrt{x}} - u'(x) = \frac{h(x)}{\nu \sqrt{x}}, \quad x > 0,$$

and $u(0) = c$ is to be determined. Thus,

$$\frac{d}{dx} \left(e^{-\frac{2\lambda\sqrt{x}}{\nu}} u(x) \right) = e^{-\frac{2\lambda\sqrt{x}}{\nu}} \left(u'(x) - \frac{\lambda u(x)}{\nu \sqrt{x}} \right) = -e^{-\frac{2\lambda\sqrt{x}}{\nu}} \left(\frac{h(x)}{\nu \sqrt{x}} \right) \quad (17)$$

for any $x > 0$. Integrating (17) gives

$$u(x) = e^{\frac{2\lambda\sqrt{x}}{\nu}} u(0) - e^{\frac{2\lambda\sqrt{x}}{\nu}} \int_0^x e^{-\frac{2\lambda\sqrt{y}}{\nu}} \left(\frac{h(y)}{\nu\sqrt{y}} \right) dy.$$

For $x > b$ we obtain that

$$u(x) = e^{\frac{2\lambda\sqrt{x}}{\nu}} \left(u(0) - \int_0^b e^{-\frac{2\lambda\sqrt{y}}{\nu}} \left(\frac{h(y)}{\nu\sqrt{y}} \right) dy - \int_b^x e^{-\frac{2\lambda\sqrt{y}}{\nu}} \left(\frac{a}{\nu\sqrt{y}} \right) dy \right).$$

Since

$$\int_b^x e^{-\frac{2\lambda\sqrt{y}}{\nu}} \left(\frac{a}{\nu\sqrt{y}} \right) dy = -\frac{a}{\lambda} \left(e^{-\frac{2\lambda\sqrt{x}}{\nu}} - e^{-\frac{2\lambda\sqrt{b}}{\nu}} \right) := C(a, b, \nu, \lambda, x),$$

this yields that

$$\lim_{x \rightarrow \infty} C(a, b, \nu, \lambda, x) = \frac{a}{\lambda} e^{-\frac{2\lambda\sqrt{b}}{\nu}} := C(a, b, \nu, \lambda).$$

Because $e^{\frac{2\lambda\sqrt{x}}{\nu}} \rightarrow \infty$ as $x \rightarrow \infty$, the only choice of $u(0)$ that makes $u \in C[0, \infty]$ is

$$u(0) = \int_0^b e^{-\frac{2\lambda\sqrt{y}}{\nu}} \left(\frac{h(y)}{\nu\sqrt{y}} \right) dy + C(a, b, \nu, \lambda). \quad (18)$$

A similar calculation works for $\lambda < 0$. Thus, $\overline{G}_\nu = G_\nu$ generates a (C_0) group on $C[0, \infty]$ and for $\lambda u - G_\nu u = h$ with $h(x) = a$ for $x > b$, $u(0)$ is given by (18). \square

Now, let us focus on the family of operators \mathcal{T}_ν acting on the space Y_s .

Lemma 2. Fix $\nu > 0$. The family $\mathcal{T}_\nu = (T_\nu(t))_{t \in \mathbb{R}}$, defined in (11) is a (C_0) quasicontractive group on Y_s , for any $s \geq 1/2$.

Proof. Fix $\nu > 0$. We will prove that, for any $s \geq 1/2$:

- i) The operators $\pm G_{\nu,s}$ are quasidissipative on Y_s .
- ii) The operators $\pm \overline{G}_{\nu,s}$ satisfy the range condition on Y_s .
- iii) The group \mathcal{T}_ν is strongly continuous on the space Y_s and there exists $\omega \in \mathbb{R}_+$ such that

$$\|T_\nu(t)f\|_s \leq e^{\omega|t|} \|f\|_s,$$

for any $t \in \mathbb{R}$, $f \in Y_s$.

Proof of i). **Step 1.** For $\nu > 0$ and $s \geq \frac{1}{2}$ we consider the operator $G_{\nu,s}$, see (??), (10). In order to prove the assertion, let $0 \neq f \in \mathcal{D}(G_{\nu,s})$ and $x_0 \geq 0$ be such that for $w(x) := \frac{1}{1+x^s}$, $s > 0$, the real function $|f(x)|w(x)$ is maximized at x_0 , i.e.,

$$|f(x_0)|w(x_0) = \|f\|_s.$$

Without any loss of generality, we may assume f is real and $f(x_0) > 0$. We choose notation in the usual way and write $\langle f, \delta_{x_0} \rangle = f(x_0)$, where δ_{x_0} is the Dirac measure. From (3) it follows that

$$\varphi = \delta_{x_0} w(x_0) \in \mathcal{I}(f).$$

Assume now $x_0 > 0$ and define $g(x) := f(x) w(x)$ for any $x > 0$. Then

$$g'(x_0) = 0 \text{ if and only if } f'(x_0) = -\frac{f(x_0)}{w(x_0)} w'(x_0) = \|f\|_s s x_0^{s-1},$$

and so we deduce that

$$\langle G_{\nu,s} f, \varphi \rangle = \nu \sqrt{x_0} f'(x_0) w(x_0) = \|f\|_s \nu s \frac{x_0^{s-1/2}}{1+x_0^s} \leq \nu s \|f\|_s,$$

because $\frac{x_0^{s-1/2}}{1+x_0^s} \in (0, 1]$ for all $s \geq 1/2$ and $x_0 > 0$. Indeed, if $0 < x_0 < 1$, then the assertion is trivial. If $x_0 \geq 1$, then taking into account that $1+x_0^s \geq x_0^s$, we have that

$$\frac{x_0^{s-1/2}}{1+x_0^s} \leq \frac{x_0^{s-1/2}}{x_0^s} = \frac{1}{\sqrt{x_0}} \leq 1.$$

Then condition (8) is satisfied with $\omega = \omega_s := \nu s$ and $s \geq 1/2$. In case $x_0 = 0$, then f has a positive maximum at 0. Thus,

$$\frac{d^+}{dx} f(x)|_{x=0} \leq 0$$

whence

$$\langle G_{\nu,s} f, \varphi \rangle \leq 0.$$

Step 2. We consider the operator $-G_{\nu,s}$. From the above arguments it follows that

$$\langle -G_{\nu,s} f, \varphi \rangle = -\|f\|_s \nu s \frac{x_0^{s-1/2}}{1+x_0^s} \leq 0$$

for all $x_0 > 0$ and $s \geq 1/2$. If in the above inequality we let $x_0 \rightarrow 0$, then

$$\langle -G_{\nu,s} f, \varphi \rangle \leq 0.$$

Then the operator $-G_{\nu,s}$ is dissipative, and hence quasi-dissipative on Y_s for all $s \geq 1/2$. Thus the proof of assertion *i*) is complete.

Proof of *ii*). Fix $s > 0$. Let $h \in Y_s$ and observe that $\frac{h}{1+x^s} \in C_0[0, \infty)$. Since G_ν generates \mathcal{T}_ν on $C[0, \infty]$ and \mathcal{T}_ν leaves invariant $C_0[0, \infty)$ (as in [8, Proposition 3]) we deduce that for λ sufficiently large, we can find $v \in C_0[0, \infty) \cap C^1(0, \infty)$ with $G_\nu v \in C_0[0, \infty)$ such that

$$\lambda v - G_\nu v = \frac{h}{1+x^s}$$

and the assertion follows. Similar arguments work for $-G_{\nu,s}$.

Thus for any $s > 0$,

$$D_s := \{\tilde{h} : \tilde{h}(x) = \frac{h(x)}{1+x^s}, h \in Y_s\}$$

is in the range of $\lambda - G_{\nu,s}$ for large real λ , and D_s is dense in Y_s . It follows that both $\pm \overline{G_{\nu,s}}$ are m -quasidissipative. This works for all $s \geq 1/2$. And we may choose $\omega = \nu s$ in all cases.

Part *iii*) now follows from the Hille-Yosida Theorem. \square

Lemma 2, [8, Lemma 1], and Romanov's theorem (see [10]) imply the following result.

Theorem 1. *For any $\nu > 0$, the closure of the operator $G_{\nu,s}^2 = \nu^2 x \frac{d^2}{dx^2} + \frac{\nu^2}{2} \frac{d}{dx}$ with domain*

$$\mathcal{D}(G_{\nu,s}^2) = \{u \in \mathcal{D}(G_{\nu,s}) : G_{\nu,s}u \in \mathcal{D}(G_{\nu,s})\},$$

generates a positive (C_0) quasicontractive semigroup $\mathcal{W}_\nu = (\mathcal{W}_\nu(t))_{t \geq 0}$ on Y_s , $s \geq 1/2$, having the explicit representation

$$\mathcal{W}_\nu(t)f(x) = e^{tG_\nu^2} f(x) = \int_{-\infty}^{+\infty} p(t,y) f\left(\left[\sqrt{x} + \frac{\nu y}{2}\right]^2\right) dy, \quad t > 0, x \geq 0,$$

for $f \in Y_s$, where p is the probability density function of the normal distribution with mean zero and variance $2t$,

$$p(t,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}.$$

As our next step, according to (6), we consider the operator Q_s acting on Y_s such that $Q_s u(x) := (\alpha + \beta x) u'(x)$ for $x \geq 0$. Here $\alpha > 0$ and $\beta \neq 0$.

Theorem 2. *Assume $\nu > 0$, $\alpha > 0$ and $P_{i,s}$, $i = 1, 2$ defined as P_i in (7), acting on Y_s . Then the closure of the operator*

$$\begin{aligned} B_{\nu,s} &= G_{\nu,s}^2 + P_{1,s} + P_{2,s} \\ &= G_{\nu,s}^2 + Q_s \end{aligned}$$

with domain $\mathcal{D}(B_{\nu,s}) = \mathcal{D}(G_{\nu,s}^2)$ generates a positive quasicontractive (C_0) semigroup on Y_s , for all $s \geq 1$.

To prove this theorem we need the following lemmas.

Lemma 3. *The closure of the operator $P_{1,s}$ defined as $P_{1,s}u(x) := \alpha u'(x)$, for $x \geq 0$ and $\alpha > 0$, with domain*

$$\mathcal{D}(P_{1,s}) = \{u \in Y_s \cap C^1(0, \infty) : u' \in Y_s\},$$

generates a positive quasicontractive (C_0) semigroup on Y_s for all $s \geq 1$, given by

$$e^{tP_{1,s}} f(x) = f(x + \alpha t), \quad t, x \geq 0.$$

Proof. Let $s \geq 1$. We first prove that for any $u \in \mathcal{D}(P_{1,s})$,

there exists $\varphi \in \mathcal{I}(u)$ such that $\langle P_{1,s}u, \varphi \rangle \leq \tilde{\omega}_s \|u\|_s$ (*quasidissipativity*)

for some constant $\tilde{\omega}_s \in \mathbb{R}$ depending only on P_1 and s .

By using the same arguments as in the proof of Lemma 1 *i*), consider $0 \neq f \in \mathcal{D}(P_{1,s})$, with $x_0 \geq 0$ a maximizing value yielding $\|f\|_s$, and $\varphi = \delta_{x_0} w(x_0) \in \mathcal{I}(f)$, where $w(x) = \frac{1}{1+x^s}$ and $f(x_0) > 0$. We deduce that, if $x_0 > 0$,

$$\langle P_{1,s}f, \varphi \rangle = \alpha f'(x_0) w(x_0) = \|f\|_s \alpha s \frac{x_0^{s-1}}{1+x_0^s} \leq \alpha s \|f\|_s,$$

because $\frac{x_0^{s-1}}{1+x_0^s} \in (0, 1]$ for all $s \geq 1$.

Suppose $|f(x)|w(x)$ is maximized at $x_0 = 0$. Without loss of generality we suppose $f(0)$ is positive. Then

$$\langle P_{1,s}f, \varphi \rangle = \alpha f'(0) \leq 0 \quad \text{since } \alpha > 0 \text{ and } f'(0) \leq 0.$$

But $(f w)'(0) = f'(0)w(0) + f(0)w'(0) = f'(0) + f(0)w'(0) \leq 0$, because

$$w'(x) = -\frac{s x^{s-1}}{(1+x^s)^2},$$

$w'(0) = -1$ if $s = 1$ and $w'(0) = 0$ if $s > 1$. It follows that

$$\langle P_{1,s}f, \varphi \rangle \leq \alpha \|f\|_s$$

if $s \geq 1$ in case $x_0 = 0$. Then the quasidissipativity is satisfied with $\tilde{\omega}_s := \alpha s$ and $s \geq 1$.

Concerning the range condition, we observe that $P_{1,0}$ generates the (C_0) contraction semigroup $(T_{P_{1,0}}(t)f)(x) = f(x + \alpha t)$, for $\alpha > 0$, $x \geq 0$, $t \geq 0$, on the space $C_0[0, \infty)$. Thus, by the Hille-Yosida theorem, the range of $\lambda I - P_{1,0}$ contains $C_0[0, \infty)$ for each $\lambda > 0$. This yields that on the weighted Y_s space, for large enough $\lambda > 0$, the range of $\lambda I - P_{1,s}$ contains $C_0[0, \infty)$ which is dense in Y_s . Hence the closure of $P_{1,s}$ on Y_s is essentially m -quasidissipative. \square

Lemma 4. *For any $s \geq 1$, $\beta \neq 0$, the closure of the operator $P_{2,s}$ defined as $P_{2,s}u(x) := \beta x u'(x)$, $x \geq 0$, with domain*

$$\mathcal{D}(P_{2,s}) = \{u \in Y_s \cap C^1(0, \infty) : xu' \in Y_s\}$$

generates a positive quasicontractive (C_0) semigroup on Y_s given by

$$e^{tP_{2,s}}f(x) = f(x e^{\beta t}), \quad t, x \geq 0.$$

Proof. As was the case for Lemma 3, it is enough to verify the quasidissipativity and the range condition. Observe that for any $u \in \mathcal{D}(P_{2,s})$, there exists $\varphi \in \mathcal{I}(u)$ such that $\langle P_{2,s}u, \varphi \rangle \leq \omega_s^* \|u\|_s$ for some constant $\omega_s^* \in \mathbb{R}$ depending only on $P_{2,s}$ and s .

Hence, by using the same arguments as in the proof of Lemma 3, quasidissipativity holds for $\omega_s^* := |\beta| s$ and $s \geq 1$.

The range condition follows by using analogous arguments as in the proof of Lemma 3. Indeed, for any real $\beta \neq 0$, the realization of $P_{2,0}$ on $C_0[0, \infty)$ generates a (C_0) group of isometries given by $T_{P_{2,0}}f(x) = f(xe^{\beta t})$ for $x \geq 0, t \in \mathbb{R}$. By the Hille-Yosida theorem, $P_{2,0}$ and $-P_{2,0}$ are both m -dissipative, thus the equation $u - \beta xu' = h$ has a unique solution in the space $C_0[0, \infty)$ for any initial condition $u(0) = a$ and any $h \in Y_0$. In particular, for any large enough $\lambda > 0$, $C_0[0, \infty)$ is in the range of $\lambda I - P_{2,0}$ and $C_0[0, \infty)$ is dense in all our weighted Y_s spaces. Hence the closure of $P_{2,s}$ is m -quasidissipative. \square

Let us define the domain of the operator $Q_s = P_{1,s} + P_{2,s}$ on the space Y_s to be

$$\mathcal{D}(Q_s) = \mathcal{D}(P_{1,s}) \cap \mathcal{D}(P_{2,s}) = \{u \in Y_s \cap C^1(0, \infty) : u', xu' \in Y_s\},$$

for any $s > 0$.

Lemma 5. *For any $s \geq 1, \alpha > 0, \beta \in \mathbb{R}$, the closure of the operator Q_s generates a (C_0) quasicontractive semigroup $\mathcal{V}^{\alpha\beta} = (\mathcal{V}^{\alpha\beta}(t))_{t \geq 0}$ on Y_s given by*

$$\mathcal{V}^{\alpha\beta}(t)f(x) = e^{tQ_s}f(x) = f\left(e^{t\beta}x + \frac{\alpha}{\beta}(e^{t\beta} - 1)\right), \quad t \geq 0, x \geq 0, f \in Y_s. \quad (19)$$

Proof. Here assume $\beta \neq 0$. If $\beta = 0$, we can replace $\frac{e^{t\beta} - 1}{\beta}$ by t , its limit as $\beta \rightarrow 0$.

In [8, Lemma 3] we observed that the solution of the Cauchy problem

$$\begin{cases} v_t = (\alpha + \beta x)v_x = Q_0v, & t, x \geq 0 \\ v(0, x) = f(x), & x \geq 0 \end{cases}$$

where Q_0 acts on $C_0[0, \infty)$, is given by

$$v(t, x) = f\left(e^{t\beta}x + \frac{\alpha}{\beta}(e^{t\beta} - 1)\right).$$

Both $P_{1,s}$ and $P_{2,s}$ are quasidissipative on Y_s for $s \geq 1$. It follows that $Q_s = P_{1,s} + P_{2,s}$ has quasidissipative closure on Y_s for $s \geq 1$. The constant ω_s for Q_s is the sum of $\tilde{\omega}_s$ and ω_s^* corresponding to $P_{1,s}$ and $P_{2,s}$ respectively.

From [8, Lemma 3] it follows that, for any large enough $\lambda > 0$, the range of $\lambda I - Q_0$ is dense in $C_0[0, \infty)$. This implies the density of the range of $\lambda I - Q_s$ in Y_s for all $s > 0$. Thus the closure \overline{Q}_s is m -quasidissipative for $s \geq 1$ and we are done. \square

Proof of Theorem 2. The quasidissipativity of $B_{\nu,s}$ is an immediate consequence of Theorem 1 and Lemma 5. The range condition follows from the range condition proved in [8] for $C_0[0, \infty)$ since the range of $\lambda I - B_{\nu,s}$ contains $C_0[0, \infty)$ which is dense in Y_s for all $s > 0$. \square

Using a terminology due to Dick Duffin (see [5]), we introduce the definition of a *posynomial*.

Definition 1. We call a function $f : [0, \infty) \rightarrow \mathbb{R}$ a *posynomial* if for any $x \geq 0$,

$$f(x) = \sum_{j=1}^N r_j x^{k_j} \geq 0,$$

where $N \geq 1$, each $r_j \geq 0$ and $0 \leq k_1 < k_2 < \dots < k_N$.

Let us fix a weight function $0 < w \in C[0, \infty)$ and denote by $C_{0,w}$ the Banach space $C_0([0, \infty), w)$ already introduced in Section 1. Then the following result holds. Its easy proof is omitted.

Lemma 6. Let P be a posynomial and let M_{-P} be the multiplication operator

$$M_{-P}f(x) = -P(x)f(x), \quad x \geq 0$$

with domain

$$D(M_{-P}) = \{f \in C_{0,w} : M_{-P}f \in C_{0,w}\}.$$

Then M_{-P} generates a (C_0) contraction semigroup on $C_{0,w}$ given by

$$e^{tM_{-P}}f(x) = e^{-tP(x)}f(x), \quad x \geq 0$$

for any $f \in C_{0,w}$.

Remark 1. The previous Lemma works in the particular case of the posynomial $P(x) = s + r x^k$, $x \geq 0$, with $s \geq 0$, $r > 0$ and $0 < k < \infty$.

The next theorem generalizes the result proved in [8, Theorem 3] in the case of the more general multiplication operator M_{-P} .

Theorem 3. Let us assume $\gamma > 0$ and consider the operator $B_{\nu,0} = G_{\nu,0}^2 + Q_0$ with domain

$$D_0(B_{\nu,0}) = \{u \in D(G_{\nu,0}^2) : G_{\nu,0}(0) = 0 = G_{\nu,0}^2(0)\}.$$

Then the operator $A_{P,0} := B_{\nu,0} + M_{-P}$ with domain

$$D(A_{P,0}) = \{u \in D_0(B_{\nu,0}) : M_{-P}u \in Y_0\} \tag{20}$$

is essentially m -dissipative (and densely defined) on $C_0[0, \infty)$.

Proof: Let $h \in C_0[0, \infty)$, $\lambda > 0$. We want solve the equation

$$\lambda u - A_{P,0}u = h. \tag{21}$$

Without any loss of generality, it is enough to do this for $h \geq 0$. For any $m \in \mathbb{N}$, consider the sequence

$$W_m(x) = \begin{cases} -P(x), & \text{if } x \in [0, m], \\ -P(m), & \text{if } x \in [m, \infty). \end{cases}$$

Note that, by the properties of P ,

$$0 \geq W_m(x) \geq W_{m+1}(x) \geq -P(x) \quad (22)$$

for all $x \geq 0$, $m \in \mathbb{N}$, and $W_m(x) \rightarrow -P(x)$ as $m \rightarrow \infty$, for all $x \geq 0$. Thus the multiplication operator M_{W_m} is a bounded perturbation of $B_{\nu,0}$ in $C_0[0, \infty)$. It follows that $A_{m,0} := B_{\nu,0} + M_{W_m}$ with domain $\mathcal{D}(A_{m,0}) = \mathcal{D}(B_{\nu,0})$ generates a positive (C_0) contraction semigroup on $C_0[0, \infty)$. Therefore for any $m \in \mathbb{N}$ there exists a unique solution $u_m \in \mathcal{D}(B_{\nu,0})$ of

$$\lambda u - A_{m,0}u = h. \quad (23)$$

Also, (22) implies that $0 \leq u_{m+1}(x) \leq u_m(x)$ for all $x \geq 0$, $m \in \mathbb{N}$. Since $h \geq 0$ and $(\lambda I - A_{m,0})^{-1}$ is positive, for $\lambda > 0$, $\|u_m\| \leq \|u_1\| < \infty$, so by (22) $\sup_m \|A_{m,0}u_m\| < \infty$. Rewrite (23) as

$$\lambda u_m - (\nu^2 x u_m'' + (\gamma + \beta x) u_m' + W_m u_m) = h$$

or

$$x u_m'' + (\gamma + \beta x) u_m' + (W_m - \lambda I)u_m = -h \quad (24)$$

by replacing each $c \in \{\gamma, \beta, W_m, h, \lambda\}$ by $\tilde{c} = \frac{c}{\nu^2}$ and then erasing each tilde. Multiply (24) by $x^{\gamma-1}e^{\beta x}$ to get

$$\frac{d}{dx} \left(x^\gamma e^{\beta x} u_m'(x) \right) = x^{\gamma-1} e^{\beta x} ((\lambda - W_m)u_m(x) - h(x)). \quad (25)$$

Let $\epsilon > 0$ be given. Integrating (25) over $[\epsilon, x]$ with $x \leq 1/\epsilon$ gives

$$|x^\gamma e^{\beta x} u_m'(x) - \epsilon^\gamma e^{\beta \epsilon} u_m'(\epsilon)| \leq K_\epsilon$$

for some $K_\epsilon > 0$ and all $m \in \mathbb{N}$ and $x \in [\epsilon, 1/\epsilon]$.

Since u_m satisfies (24) on $[\epsilon, 1/\epsilon]$, the operator at the left hand side of (24) is uniformly elliptic on $[\epsilon, 1/\epsilon]$, and $\sup_m \|u_m\| < \infty$, we have that $\frac{d}{dx} (x^\gamma e^{\beta x} u_m'(x))$, $x^\gamma e^{\beta x} u_m'(x)$ and $u_m'(x)$ are all uniformly bounded on $[\epsilon, 1/\epsilon]$, for all $m \in \mathbb{N}$, as is u_m'' by subtraction (by (24)). By the Arzela-Ascoli theorem and Cantor diagonalization, there exists a subsequence $\{v_n\}$ of $\{u_m\}$ such that

$$v_n \rightarrow u \quad \text{in } C_{loc}^2(0, \infty).$$

To prove that the pointwise solution u of (21) belongs to $\mathcal{D}(A_{P,0})$, it remains to control u near $x = \infty$ and near to $x = 0$.

Note that $0 \leq u(x) \leq u_1(x) \rightarrow 0$ as $x \rightarrow \infty$, because $u_1 \in C_0[0, \infty)$. Further, $u, u', u'' \in C(0, \infty)$, thus $u \in C^2(0, \infty) \cap C(0, \infty]$ and $u(\infty) = 0$. This takes care of u near $x = \infty$.

Concerning the case of x near to 0, let us recall a result of [2]. Let $U(x, t)$ for $x, t \geq 0$ satisfy

$$U_t = \frac{1}{2}U_{xx} + \frac{c_1}{x}U_x, \quad U(x, 0) = f(x),$$

with Neumann boundary condition at the origin, $U_x(0, t) = 0 (= f'(0))$. Then $Z(x, t) := U\left(\sqrt{x}, \frac{t}{2}\right)$ satisfies

$$Z_t = xZ_{xx} + c_2Z_x, \quad Z(x, 0) = g(x) = f(\sqrt{x}),$$

where $c_2 = c_1 + \frac{1}{2} > 0$, which is valid when $c_1 > -1/2$. The boundary condition for Z resulting from the boundary condition for U (which is independent of $c_2 > 0$) is

$$\sqrt{x}Z_x(x, t) \rightarrow 0 \tag{26}$$

as $x \rightarrow 0^+$ for all $t \geq 0$. Z satisfies $Z_t = (A + B)Z$ in $C_0[0, \infty)$ if we take $c_2 = \frac{\gamma}{\nu^2} > 0$ and $\beta = 0$. But for the operator $G_{\nu,0} = \nu\sqrt{x}D = \nu\sqrt{x}\frac{d}{dx}$ with boundary condition $G_{\nu,0}u(x) \rightarrow 0$ as $x \rightarrow 0^+$, i.e. (26), we get

$$G_{\nu,0}^2u(x) = \nu^2\sqrt{x}D(\sqrt{x}u'(x)) = \nu^2(xu''(x) + \frac{1}{2}u'(x)) = \nu^2(xD^2 + \frac{1}{2}D)u(x).$$

And for $u \in \mathcal{D}(G_{\nu,0}^2)$, $G_{\nu,0}u(x)$ and $G_{\nu,0}^2u(x)$ vanish when $x \rightarrow 0^+$. In [11, Theorem 3.4] and in the results proved in [8] we showed that $G_{\nu,0}$ is the infinitesimal operator of a non (C_0) , but a once integrated positive contraction semigroup on $C_0[0, \infty)$. The Wentzell boundary condition for $G_{\nu,0}^2$ is $G_{\nu,0}u(0) = G_{\nu,0}^2u(0) = 0$, and $G_{\nu,0}^2$ is m-dissipative and densely defined on $C_0[0, \infty)$. Now we must strengthen this to strong continuity.

Recall that $\alpha = \gamma - \nu^2/2$. We now give a new direct proof that the Wentzell boundary condition for $B_{\nu,0}$ on $C_0[0, \infty)$, namely $B_{\nu,0}u(0) = 0$, is independent of $\gamma > 0$ and $\beta \in \mathbb{R}$. Thus the Wentzell boundary condition is independent of ν , γ and β , provided that $\gamma > 0$.

The next Lemma is an extension of the Kallman-Rota inequality [12].

Lemma 7. *Let L generate a strongly measurable contraction semigroup $\{e^{sL} : s \geq 0\}$ on a Banach space Z . Then for all $f \in \mathcal{D}(L^2)$,*

$$\|L f\|^2 \leq 4\|L^2 f\| \|f\|.$$

Proof: $\{e^{sL} : s \geq 0\}$ is strongly continuous for $t > 0$, but it need not be of class (C_0) . The usual Kallman-Rota inequality assumes that $\{e^{sL} : s \geq 0\}$ is of class (C_0) , but the conclusion holds in the more general case when L need not be densely defined.

Let $f \in \mathcal{D}(L^2)$. Then by Taylor's formula,

$$e^{tL}f = f + tLf + \int_0^t (t-s)e^{sL}L^2f ds,$$

whence

$$Lf = \frac{e^{tL}f - f}{t} + \frac{1}{t} \int_0^t (t-s)e^{sL}L^2f ds,$$

and

$$\|Lf\| \leq \frac{2}{t}\|f\| + \frac{1}{t} \int_0^t (t-s)\|L^2f\| ds = \frac{2}{t}\|f\| + \frac{t}{2}\|L^2f\|.$$

Minimizing over $t > 0$ gives, if $L^2 f \neq 0$, $t = 2 \|f\|^{1/2} \|L^2 f\|^{-1/2}$, and so

$$\|Lf\| \leq 2 \|L^2 f\|^{1/2} \|f\|^{1/2} \leq \delta \|L^2 f\| + \frac{1}{\delta} \|f\| \quad (27)$$

for every $\delta > 0$. This is also valid if $L^2 f = 0$, which implies $Lf = 0$. Then Lemma now follows. \square

First we take $\beta = 0$. Thus L in Lemma 7 is a dissipative Kato perturbation of L^2 (and so is $-L$ if L is the infinitesimal operator of a group of isometries). Therefore, $L^2 + kL$ is m -dissipative on $\mathcal{D}(L^2)$ for all $k \geq 0$ (and for all $k \in \mathbb{R}$ in the isometric group case).

Thus the Wentzell boundary condition for $G_{\nu,0}^2 + kG_{\nu,0}$ is independent of k and has the form $G_{\nu,0} u(0) = G_{\nu,0}^2 u(0) = 0$ for $u \in \mathcal{D}(\widehat{B}_{\nu,0})$, where $\widehat{B}_{\nu,0} = \nu^2 x D^2 + \gamma D$, and this holds for all $\gamma > 0$ since it holds for all relevant k (see (27)).

We take now $\beta \neq 0$. Note that for $x > 0$ small, $\widehat{B}_{\nu,0} u(x) = B_{\nu,0} u(x) - \sqrt{x}(\beta \sqrt{x} u'(x))$. Since $\widehat{B}_{\nu,0} u(x) \rightarrow 0$ and $\sqrt{x}(\beta \sqrt{x} u'(x)) \rightarrow 0$ as $x \rightarrow 0^+$, it follows that $B_{\nu,0} u(0) = 0$. Thus, $u, A_{P,0} u \in C_0[0, \infty)$, $B_{\nu,0} u(0) = 0$ and so $u \in \mathcal{D}(A_{P,0})$.

This now completes our proof of strong continuity of the our problem on $C_0[0, \infty)$, including the characterization of $\mathcal{D}(B_{\nu,0})$ which is independent of γ and β , provided $\gamma > 0$. Theorem 3 is now fully proved.

Hence the claim follows. \square

Remark 2. At $x = \infty$, the multiplicative operator M_{-P} is unbounded, so it preserves the boundary condition $u(\infty) = 0$ whenever $u \in D(A_{P,0})$, but $D(A_{P,0})$ will be smaller than $D(B_{\nu,0})$ because $u(x) \rightarrow 0$ as $x \rightarrow \infty$ does not imply $P(x)u(x) \rightarrow 0$ as $x \rightarrow \infty$.

Theorem 4. *Assume that $\nu > 0, \gamma > 0, \beta \in \mathbb{R}$, P is a posynomial. Then the closure of the operator $A_{P,s}$ with domain*

$$D(A_{P,s}) = \{u \in D(G_{\nu,s}^2) : G_{\nu,s} u(0) = 0 = G_{\nu,s}^2 u(0), M_{-P} u \in Y_s\}$$

generates a positive (C_0) quasicontractive semigroup on Y_s for all $s \geq 1$.

Proof. The quasidissipative assertion follows from the previous results. The range condition is a consequence of the range condition proved in the previous theorem for $C_0[0, \infty)$ since the range of $\lambda I - B_{\nu,s}$ contains $C_0[0, \infty)$ which is dense in Y_s for all $s > 0$. \square

4 A Feynman-Kac type formula

As already mentioned in the Introduction, in [8] we proved a new type of Feynman-Kac formula for the generalized CIR problem (1) on $C_0[0, \infty)$. Consider the posynomial $P_r(x) = rx$, $x \geq 0$, with $r > 0$. Denote Thus the problem (1) can be written as

$$\frac{du}{dt} = C_1 u + C_2 u, \quad u(0) = f$$

where

$$C_1 = G_{\nu,0}^2, \quad C_2 = Q_0 + M_{-P_r},$$

and M_{-P_r} reduces to the multiplication operator M_r defined in Section 2. The Trotter product formula implies that the semigroup generated by $C = \overline{C_1 + C_2}$ is given by

$$e^{tC} f = \lim_{n \rightarrow \infty} \left(e^{\frac{t}{n} C_1} e^{\frac{t}{n} C_2} \right)^n f. \quad (28)$$

Using our explicit formulas for e^{tC_1} and e^{tC_2} , we found a very complicated explicit formula for $(e^{\frac{t}{n} C_1} e^{\frac{t}{n} C_2})^n f$ for $n = 2^k$, for any positive integer k . Using this formula in (28) gives our Feynman-Kac formula. In the Feynman case of the Schrödinger equation, $z_n = (e^{\frac{t}{n} C_1} e^{\frac{t}{n} C_2})^n f$ was regarded as a Riemann sum for the Feynman path integral. The latter integral does not exist in the usual measure theoretic context, but it is very useful nonetheless. For the heat equation with a potential, Kac showed directly that z_n converges to a Wiener integral solution of the heat equation. Our z_n looks nothing like an approximation to an integral. But still, z_n converges to the desired solution. We showed this in [8] for $C_0[0, \infty)$ and here the analogous proof establishes it for $Y_s, s > 1$. This is our Theorem 5.

Before stating Theorem 5, we note the following result.

Lemma 8. *For any $s > 1, \alpha > 0, \beta \in \mathbb{R}, r > 0$, the closure of the operator $Q_s + M_{-P_r}$ with domain*

$$\mathcal{D}(Q_s + M_{-P_r}) = \{u \in \mathcal{D}(Q_s) : M_{-P_r} u \in Y_s\}$$

generates a (C_0) semigroup $(\mathcal{U}^{\alpha\beta}(t))_{t \geq 0}$ on Y_s satisfying

$$\mathcal{U}^{\alpha\beta}(t) f(x) = \exp\left(-\frac{r}{\beta} \left[(e^{t\beta} - 1) \left(\frac{\alpha}{\beta} + x \right) - \alpha t \right]\right) f\left(e^{t\beta} x + \frac{\alpha}{\beta} (e^{t\beta} - 1)\right) \quad (29)$$

for $t \geq 0, x \geq 0$.

Proof. As in Lemma 5 assume $\beta \neq 0$. If $\beta = 0$, we can replace $\frac{e^{t\beta}-1}{\beta}$ by t , its limit as $\beta \rightarrow 0$. The assertion follows from [8, Theorem 4] where we showed that the family $(\mathcal{U}^{\alpha\beta}(t))_{t \geq 0}$ of linear operators satisfying (29) is a (C_0) semigroup on $C_0[0, \infty)$ obtained by the approximating formula

$$\mathcal{U}^{\alpha\beta}(t) f = \lim_{m \rightarrow \infty} \left(e^{\frac{t}{m} Q_0} e^{\frac{t}{m} M_{-P_r}} \right),$$

for all $t \geq 0$ and $f \in C_0[0, \infty)$. □

Theorem 5. *Assume that $\nu > 0, \gamma > 0, \beta \in \mathbb{R}, r > 0, P_r(x) = rx$. Then the semigroup $(\mathcal{T}_{A_{P,s}}(t))_{t \geq 0}$ on Y_s with $s > 1$ is given by*

$$\mathcal{T}_{A_{P,s}}(t) f(x) = \lim_{n \rightarrow \infty} \left(W_\nu\left(\frac{t}{n}\right) \mathcal{U}^{\alpha\beta}\left(\frac{t}{n}\right) \right)^n f \quad (30)$$

$$= \lim_{n \rightarrow \infty} \underbrace{\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty}}_{n \text{ times}} L(t, n, \nu, \{y_j\}_{1 \leq j \leq n}, x, f) \cdot \prod_{j=1}^n p\left(\frac{t}{n}, y_j\right) dy_1 \dots dy_n. \quad (31)$$

with $n = 2^k$ for $k \in \mathbb{N}$, and for L given by

$$\begin{aligned} L(t, n, \nu, \{y_j\}_{1 \leq j \leq n}, x, f) &= e^{Z(\frac{t}{n}, S(y_1, x)) + Z(\frac{t}{n}, S(y_2, R(\frac{t}{n}, S(y_n(1, x))))} \\ &\cdot e^{Z(\frac{t}{n}, S(y_3, R(\frac{t}{n}, S(y_2, R(\frac{t}{n}, S(y_1, x)))))) + \dots + Z(\frac{t}{n}, S(y_n, R(\frac{t}{n}, S(y_{n-1}, R(\frac{t}{n}, \dots, R(\frac{t}{n}, S(y_1, x))))))} \\ &\cdot f(R(\frac{t}{n}, S(y_n, R(\frac{t}{n}, S(y_{n-1}, R(\frac{t}{n}, \dots, R(\frac{t}{n}, S(y_1, x)))))))). \end{aligned}$$

where

$$\begin{aligned} R(t, S(y, x)) &= e^{t\beta} x + \frac{\alpha}{\beta} (e^{t\beta} - 1) + e^{t\beta} \nu y (\sqrt{x} + \frac{\nu y}{4}), \\ Z(t, S(y, x)) &= -\frac{r}{\beta} \left[\frac{\alpha}{\beta} (e^{t\beta} - 1) + (e^{t\beta} - 1)x - \alpha t \right] - r \left[\frac{(e^{t\beta} - 1)}{\beta} \nu y (\sqrt{x} + \frac{\nu y}{4}) \right]. \end{aligned}$$

and $t, x \geq 0$. The convergence is uniform for $x \in [0, \infty)$ and for t in bounded intervals of $[0, \infty)$.

Proof. See the proof of [8, Theorem 5]. □

Remark 3. Theorem 5 implies that the unique solution of the CIR problem (1) in the special case $r = 1$ and $f(x) = 1$, as mentioned in the Introduction, is given by

$$u(t, x) = \lim_{n \rightarrow \infty} \underbrace{\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty}}_{n \text{ times}} \tilde{L}(t, n, \nu, \{y_j\}_{1 \leq j \leq n}, x) \cdot \prod_{j=1}^n p(\frac{t}{n}, y_j) dy_1 \dots dy_n, \quad (32)$$

where

$$\begin{aligned} \tilde{L}(t, n, \nu, \{y_j\}_{1 \leq j \leq n}, x) &= e^{Z(\frac{t}{n}, S(y_1, x)) + Z(\frac{t}{n}, S(y_2, R(\frac{t}{n}, S(y_n(1, x))))} \\ &\cdot e^{Z(\frac{t}{n}, S(y_3, R(\frac{t}{n}, S(y_2, R(\frac{t}{n}, S(y_1, x)))))) + \dots + Z(\frac{t}{n}, S(y_n, R(\frac{t}{n}, S(y_{n-1}, R(\frac{t}{n}, \dots, R(\frac{t}{n}, S(y_1, x))))))}. \end{aligned}$$

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