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Contact metric manifolds with large automorphism group and (κ, μ)-spaces

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Abstract: We discuss the classification of simply connected, complete (κ, μ)-spaces from the point of view of homogeneous spaces. In particular, we exhibit new models of (κ, μ)-spaces having Boeckx invariant -1. Finally, we prove that the number $\frac{(n+1)(n+2)}{2}$ is the maximum dimension of the automorphism group of a contact metric manifold of dimension $2n + 1$, $n \geq 2$, whose symmetric operator h has rank at least 3 at some point; if this dimension is attained, and the dimension of the manifold is not 7, it must be a (κ, μ)-space. The same conclusion holds also in dimension 7 provided the almost CR structure of the contact metric manifold under consideration is integrable.

Keywords: contact metric manifold, (κ, μ)-space

MSC: Primary 53C25, 53D10; Secondary 53C30

1 Introduction

Among the contact metric spaces $(M, \varphi, \xi, \eta, g)$, the so-called (κ, μ)-spaces form a special and significant class with remarkable geometric properties; they were originally defined by Blair, Koufogiorgos and Papantoniou in 1995, by the following curvature condition:

$$R(X, Y)\xi = (\kappa Id + \mu h)(\eta(Y)X - \eta(X)Y), \tag{1}$$

where $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ and κ, μ are real numbers; see [2]. For the notation and basic facts of contact metric geometry we refer the reader to Blair's book [1]. Of course the above condition is satisfied by the Sasakian manifolds (corresponding to $h = 0$ and $k = 1$); in all that follows we shall deal exclusively with non Sasakian (κ, μ)-spaces.

The main motivation for studying these manifolds was a previous result of Blair, stating that the unique simply connected, complete contact metric manifold of dimension $2n + 1$, $n \geq 2$, satisfying $R(X, Y)\xi = 0$ is the Riemannian product $\mathbb{S}^n \times \mathbb{R}^{n+1}$, where the metric on the sphere is chosen of curvature 4; this is the unique contact metric non Sasakian orientable hypersurface of $\mathbb{C}^{n+1}$ (see [17]).

In the paper [2], it was established that every (κ, μ)-space is a CR manifold, and that its curvature tensor is completely determined by (1). Blair, Koufogiorgos and Papantoniou also exhibited models of type $T_1N(c)$, i.e. tangent sphere bundles over Riemannian manifolds $N(c)$ with constant curvature $c \in \mathbb{R}$, $c \neq 1$. Moreover, they provided a complete classification in the 3-dimensional case.

Later other characterizations of the (κ, μ)-spaces and related geometric results appeared in the literature. See for instance [5], [6], [12], [10], [11].

Concerning the classification problem in higher dimension, in [3] and [4] Boeckx proved that every (κ, μ)-space is locally homogeneous, and showed that, up to equivalence and D-homothetic deformations, the fam-

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ily of simply connected, complete (κ,μ)-spaces of fixed dimension is parametrized by $\mathbb{R}$. Indeed, he determined a number I_M which is invariant under D-homothetic deformations, namely

$$I_M := \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}},$$

which completely determines locally a (κ,μ)-manifold M, up to equivalence:

Theorem 1 (Boeckx). *Let $(M_i,\varphi_i,\xi_i,\eta_i,g_i)$, $i=1,2$, be two (κ,μ)-spaces of the same dimension. Then $I_{M_1} = I_{M_2}$ if and only if, up to a D-homothetic deformation of the contact metric structure, the two spaces are locally equivalent as contact metric spaces.*

The tangent sphere bundles $T_1N(c)$, as c varies in $\mathbb{R}-\{1\}$, provide examples exhausting all possible values $I_M > -1$. The case $I_M \leqslant -1$ seemed to lead to models of different nature. Namely, Boeckx himself found examples of contact metric (κ,μ)-spaces, for every value of the invariant $\leqslant -1$, namely a two parameter family of abstractly constructed Lie groups with a left–invariant contact metric structure. However, he gave no geometric description of these examples. A geometrical construction of models with $I_M \leq -1$ has been provided more recently by E. Loiudice and the present author in [15] and [16]. Namely, in [15] they are obtained as tangent hyperquadric bundles

$$T_{-1}N(c) = \{(p,v) \in TN(c): \quad g_p(v,v) = -1\}.$$

over a Lorentzian space form $N(c)$ with curvature $c \neq -1$, endowed with a suitable contact metric structure, which is a pseudohermitian structure associated to its natural CR structure, inherited from the standard almost complex structure of the tangent bundle $TN(c)$ (concerning this CR structure, see also [18]). Moreover, in [16] all the simply connected (κ,μ)-spaces M with $I_M \neq \pm 1$ of dimension at least five have been classified in a uniform way as homogeneous contact metric manifolds. Below we give a classification table, including also the cases $I_M = \pm 1$. Consider the following Stiefel manifolds:

$V_0(2,n+2)$:= set of orthonormal frames $\{u,v\}$ of the Euclidean space $\mathbb{R}^{n+2}_0$:

$$u\cdot u = 1,\ v\cdot v = 1,\ u\cdot v = 0;$$

$V_1(2,n+2)$:= set of frames $\{u,v\}$ of the Minkowski space $\mathbb{R}^{n+2}_1$ such that:

$$u\cdot u = -1,\ v\cdot v = 1,\ u\cdot v = 0;$$

$V_2(2,n+2)$:= set of frames $\{u,v\}$ of the index 2 space $\mathbb{R}^{n+2}_2$ such that:

$$u\cdot u = -1,\ v\cdot v = -1,\ u\cdot v = 0.$$

Theorem 2. *Up to equivalence and D-homothetic deformations, the simply connected, complete, (κ,μ)-spaces of dimension at least five are:*

Boeckx invariant	*Homogeneous model M*, $\dim(M) = 2n+1 \geq 5$	*Alternative interpretation*
$I_M > 1$	$V_0(2,n+2) = SO(n+2)/SO(n)$	$T_1(\mathbb{S}^{n+1})$
$-1 < I_M < 1$	$V_1(2,n+2) = SO(1,n+1)/SO(n)$	$T_1(\mathbb{H}^{n+1})$
$I_M < -1$	$V_2(2,n+2) = SO(2,n)/SO(n)$	$T_{-1}(\mathbb{H}^{n+1}_1)$
$I_M = 1$	$\mathbb{S}^n \times \mathbb{R}^{n+1} = SO(n+1)\times\mathbb{R}^{n+1}/SO(n)$	$T_1(\mathbb{R}^{n+1})$
$I_M = -1$	$\mathbb{H}^n \times \mathbb{R}^{n+1}_1 = SO(1,n)\times\mathbb{R}^{n+1}/SO(n)$	$T_{-1}(\mathbb{R}^{n+1}_1)$

Each of the Stiefel manifolds listed in the table carries a one parameter family of inequivalent invariant contact metrics structures satisfying the (κ, μ) *condition, whose Boeckx invariants cover the range indicated in the first column.*

For $I_M \neq \pm 1$, this classification was achieved in [16] by using the fact the (κ, μ)-spaces can be characterized as (locally) CR-symmetric spaces in the sense of Kaup-Zaitsev [13]; namely, a non K-contact, contact metric space M is a (κ, μ)-space if and only if each point p admits a local isometric CR automorphism $\sigma_p : M \to M$ such that:

$$\sigma_p(p) = p, \quad (d\sigma)_p(X) = -X, \quad \forall X \in D_p,$$

where D is the contact distribution (cf. [12]). This yields that the simply connected, complete (κ, μ)-spaces admit a transitive Lie group G of automorphisms whose Lie algebra $\mathfrak{g}$ has a canonical symmetric decomposition (see Theorem 4 in the next section). These facts lead to the conclusion that the base space M/ξ of the canonical fibration of M according to Boothby-Wang [8] is an affine symmetric space, which turns out to be a suitable Grassmannian. For details concerning the proof see [16].

Here we shall concentrate on the case $I_M = -1$ (not included in [16]), giving an explicit description of the $SO(1, n) \ltimes \mathbb{R}^{n+1}$-invariant contact metric structure on the manifold $\mathbb{H}^n \times \mathbb{R}^{n+1}_1$ appearing in the above table (see Theorem 5 in the next section).

Looking at the same classification table, one can observe that the spaces under discussion have a rich amount of symmetry, as expected from their above description as CR-symmetric spaces. Namely, for each of the models M in the list, letting $\dim(M) = 2n + 1$, we see that:

$$\dim(\mathrm{Aut}(M)) \geq \frac{(n+1)(n+2)}{2}.$$

In this connection, in this paper we prove the following result, yielding that the above inequality is in fact an equality and showing that the (κ, μ)-spaces are the only manifolds having largest automorphism group, within the class of contact metric manifolds of prescribed dimension, whose operator h has rank at least 3 at some point:

Theorem 3. *Let* $(M, \varphi, \xi, \eta, g)$ *be a connected contact metric manifold of dimension* $2n + 1$, $n \geq 2$. *Assume that at some point* $p \in M$ *the operator* h_p *has rank* > 2. *Then:*

a) $\dim(\mathrm{Aut}(M)) \leq \frac{(n+1)(n+2)}{2}$.

b) If $\dim(\mathrm{Aut}(M)) = \frac{(n+1)(n+2)}{2}$, *then* M *is a* (κ, μ)*-space, provided* $n \neq 3$ *or* $n = 3$ *and the almost CR structure of M is integrable.*

We remark that for the entire class of contact metric manifolds of dimension $2n + 1$, Tanno proved that the maximum dimension of $\mathrm{Aut}(M)$ is $(n + 1)^2$, which is attained only by three types of homogeneous Sasakian manifolds with constant φ-sectional curvature [20].

2 A Lie-theoretic characterization of (κ, μ)-spaces

Given a contact metric manifold $(M, \varphi, \xi, \eta, g)$, we shall denote by $\mathrm{Aut}(M)$ be the group of all automorphisms of M, i.e. the group consisting of all isometries $f : M \to M$ with respect to g, preserving the contact form η. It follows that such an f also preserves ξ and φ. Clearly, $\mathrm{Aut}(M)$ is a closed Lie subgroup of the isometry group $I(M, g)$. The contact distribution $\ker(\eta)$ of M will be denoted by D.

Assume now that $(M, \varphi, \xi, \eta, g)$ is a homogeneous contact metric manifold, so that $M = G/H$ where G is a Lie subgroup of $\mathrm{Aut}(M)$ acting transitively on M.

Fix a reductive decomposition of $\mathfrak{g} := \mathrm{Lie}(G)$ (it is known that such a decomposition always exists, being G a group of isometries, cf. e.g. [19]):

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{h} = \mathrm{Lie}(H), \ \mathfrak{m} \simeq T_o M, \ o = H. \tag{2}$$

Set:

$$\zeta := \xi_o, \qquad \mathfrak{b} := D_o,$$

so that

$$\mathfrak{m} = \mathfrak{b} \oplus \mathbb{R}\zeta.$$

We have another decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}, \quad \bar{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{R}\zeta. \tag{3}$$

Keeping this notation, the following characterization of the (κ,μ)-spaces has been obtained in [16]:

Theorem 4. *Let $(M, \varphi, \xi, \eta, g)$ be a simply connected, complete, contact metric manifold. Assume M is not K-contact. Then the following conditions are equivalent:*

1) M is a (κ,μ)-space.

2) M admits a transitive, effective Lie group of automorphisms G whose Lie algebra $\mathfrak{g}$ is a symmetric Lie algebra with respect to the decomposition $\mathfrak{g} = \bar{\mathfrak{h}} \oplus \mathfrak{b}$.

Here, the condition concerning the Lie algebra $\mathfrak{g}$ stated in 2) means that the following usual Cartan relations hold:

$$[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subset \bar{\mathfrak{h}}, \ [\bar{\mathfrak{h}}, \mathfrak{b}] \subset \mathfrak{b}, \ [\mathfrak{b}, \mathfrak{b}] \subset \bar{\mathfrak{h}}.$$

The proof of 1) $\Rightarrow$ 2) is based on the existence of CR-symmetries ([12]), while 2) $\Rightarrow$ 1) relies on the characterization of (κ,μ)-spaces by η–parallelism of h, due to Boeckx and Cho [5].

As an application, we shall discuss the simply connected, complete (κ,μ)-spaces with Boeckx invariant -1.

Actually, among the (κ,μ)-spaces, those having Boeckx invariant $I_M = \pm 1$ are particular: they can be characterized by using the so-called *Pang invariants* $\Pi^{\pm}$ of the Legendrian foliations $D^+ = D(\lambda)$ and $D^- = D(-\lambda)$, see [9]. Here D^+ and D^- are the (integrable) eigendistributions of the symmetric operator h relative to its non null eigenvalues; recall that for every (κ,μ)-space, the spectrum of h is $\{\lambda, -\lambda, 0\}$, where $\lambda = \sqrt{1-k}$ (cf. e.g. [1]).

By definition, the Pang invariant of a Legendrian foliation $\mathfrak{F}$ on a contact manifold (M,η) is the symmetric tensor field:

$$\Pi^{\mathfrak{F}}(X, Y) = 2d\eta([\xi, X], Y),$$

where X, Y are sections of $\mathfrak{F}$. In the case of a (κ,μ)-space, one has [9]:

$$I_M = 1 \iff \Pi^- = 0, \ \Pi^+ \text{ is positive definite}$$

$$I_M = -1 \iff \Pi^+ = 0, \ \Pi^- \text{ is negative definite.}$$

Recall that the model having Boeckx invariant $I_M = 1$ is the Riemannian product $\mathbb{S}^n \times \mathbb{R}^{n+1}$ (where $\mathbb{S}^n$ has curvature 4), i.e, the tangent sphere bundle of the Euclidean space $\mathbb{R}^{n+1}$, which is in a natural way a homogeneous space under the action of the subgroup $SO(n+1) \ltimes \mathbb{R}^{n+1}$ of the group of isometries of the Euclidean space $\mathbb{R}^{n+1}$.

To construct a model with $I_M = -1$, it is natural to think it should be the "dual" $M = \mathbb{H}^n \times \mathbb{R}^{n+1}$, where $\mathbb{H}^n$ is an hyperbolic space. Of course, this cannot be a Riemannian product, due to the fact that, by a well-know result of Boeckx and Cho [7], $\mathbb{S}^n \times \mathbb{R}^{n+1}$ is the unique non Sasakian, simply connected Riemannian symmetric contact metric manifold of dimension $2n + 1$.

We shall think to our product manifold M as:

$$M = \mathbb{H}^n \times \mathbb{R}^{n+1}_1$$

where $\mathbb{R}^{n+1}_1$ is the Minkowski space, i.e. $M = T_{-1}(\mathbb{R}^{n+1}_1)$, and we shall construct a homogeneous contact metric structure by using the natural transitive action of the Poincarè group $E(1, n)$; actually we consider motions of the form:

$$x \mapsto Ax + v, \qquad A \in SO(1, n), \quad v \in \mathbb{R}^{n+1}.$$

Then $M = G/H$, where $G = SO(1, n) \ltimes \mathbb{R}^{n+1}$ consists of the matrices:

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}, \quad A \in SO(1, n),\ v \in \mathbb{R}^{n+1},$$

while H consists of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B \in SO(n),$$

so that $H \cong SO(n)$.

Accordingly, the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ can be described as follows:

$$\text{Generic element of } \mathfrak{g}\text{:} \begin{pmatrix} 0 & {}^t w & \alpha \\ w & A & u \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{matrix} A \in \mathfrak{so}(n) \\ w, u \in \mathbb{R}^n \\ \alpha \in \mathbb{R}. \end{matrix}$$

$$\text{Generic element of } \mathfrak{h} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ A \in \mathfrak{so}(n).$$

We have a natural reductive decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} := \left\{ \begin{pmatrix} 0 & {}^t w & \alpha \\ w & 0 & u \\ 0 & 0 & 0 \end{pmatrix} : w, u \in \mathbb{R}^n,\ \alpha \in \mathbb{R} \right\}.$$

We shall denote a matrix belonging to the vector space $\mathfrak{m}$ by using the notation $(w\ u\ \alpha)$. A G-invariant contact metric structure can be introduced in a natural way by using the splitting:

$$\mathfrak{m} = \mathfrak{b} \oplus \mathbb{R}\zeta, \quad \zeta := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{b} := \left\{ \begin{pmatrix} 0 & {}^t w & 0 \\ w & 0 & u \\ 0 & 0 & 0 \end{pmatrix} : w, u \in \mathbb{R}^n \right\}.$$

We consider the G-invariant 1-form η and the G-invariant Riemannian metric g on M, determined at the base point $o = H$ by setting:

$$\eta_o|_{\mathfrak{b}} = 0,\ \eta_o(\zeta) = 1,\ g_o((w, u, \alpha), (\bar{w}, \bar{u}, \bar{\alpha})) = \frac{1}{2}(< w, \bar{w} > + < u, \bar{u} >) + \alpha\bar{\alpha},$$

where $< , >$ denotes the standard inner product on $\mathbb{R}^n$. Then a routine check yields that η is a contact form whose Reeb vector field ξ satisfies $\xi_o = \zeta$, and g is an associated metric.

Moreover, easy matrix computations show:

- $\mathfrak{g} = (\mathfrak{h} \oplus \mathbb{R}\zeta) \oplus \mathfrak{b}$ is a symmetric decomposition of the Lie algebra $\mathfrak{g}$.

- The spectrum of the operator h is $\{\frac{1}{2}, -\frac{1}{2}, 0\}$, and the eigendistributions D^+ and D^- of h relative to the non null eigenvalues are the G-invariant distributions whose determinations at o are given by the following subspaces of $\mathfrak{b}$:

$$\mathfrak{b}_+ = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} : u \in \mathbb{R}^n \right\}, \quad \mathfrak{b}_- = \left\{ \begin{pmatrix} 0 & {}^t w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : w \in \mathbb{R}^n \right\}.$$

The first fact yields that M is a (κ,μ)-space according to Theorem 4. In order to verify that $I_M = -1$, it is convenient to compute the Pang invariants of D^+ and D^-.

Lemma 1. *For a G-invariant Legendre foliation $\mathfrak{F}$ on a homogeneous redutive contact metric manifold $M = G/H$, with a given reductive decomposition (2) of the Lie algebra $\mathfrak{g} = Lie(G)$, the Pang invariant $\Pi^{\mathfrak{F}}$ is given in terms of Lie algebra bracket as follows:*

$$\Pi^{\mathfrak{F}}_o(X, Y) = 2 < [\zeta, X]_{\mathfrak{m}}, \varphi Y >, \quad X, Y \in \mathfrak{e},$$

where $\mathfrak{e} \subset \mathfrak{m}$ is the determination of $\mathfrak{F}$ at the base point $o = H$, $\zeta = \xi_o$ and $<,>= g_o$.

Proof. We shall make use of the canonical G-invariant linear connection $\tilde{\nabla}$ on M relative to the fixed reductive decomposition of $\mathfrak{g}$. Recall that all the G-invariant tensor fields on M are $\tilde{\nabla}$-parallel (cf. [14, Prop. 2.7, Ch. X, p. 193]). In particular $\tilde{\nabla}\xi = 0$, whence, for every X, Y smooth sections of $\mathfrak{F}$:

$$\Pi^{\mathfrak{F}}(X, Y) = 2d\eta([\xi, X], Y) = 2d\eta(\tilde{\nabla}_\xi X, Y) - 2d\eta(\tilde{T}(\xi, X), Y)$$

where $\tilde{T}$ denotes the torsion of $\tilde{\nabla}$. Moreover, since $\mathfrak{F}$ is G-invariant, $\tilde{\nabla}_\xi X$ is also tangent to $\mathfrak{F}$, so that $d\eta(\tilde{\nabla}_\xi X, Y) = 0$; hence

$$\Pi^{\mathfrak{F}}(X, Y) = -2d\eta(\tilde{T}(\xi, X), Y), \quad \forall X, Y \in \Gamma\mathfrak{F}.$$

Evaluating at o, this gives

$$\Pi^{\mathfrak{F}}_o(X, Y) = 2d\eta([\zeta, X]_{\mathfrak{m}}, Y), \quad X, Y \in \mathfrak{e}$$

taking into account the expression of $\tilde{T}$ given in [14, Prop. 2.3, Ch. X, p. 192]. □

In order to apply this lemma in our case, we observe that:

$$ad_\zeta(w\ u\ 0) = (0\ -w\ 0), \quad \varphi(w\ u\ 0) = (-u\ w\ 0),$$

from which we get

$$\Pi^+ = 0, \ \Pi^- = -2g|_{D^- \times D^-}$$

yielding $I_M = -1$.

In conclusion, we have proved

Theorem 5. *Up to equivalence and D-homothetic deformations, the simply connected, complete (κ,μ)-space of dimension $2n + 1 \geq 5$ and with Boeckx invariant -1 is the homogeneous space $\mathbb{H}^n \times \mathbb{R}^{n+1}_1 = SO(1, n) \ltimes \mathbb{R}^{n+1}/SO(n)$, endowed with an invariant contact metric structure.*

3 Proof of Theorem 3

We start the proof with the following algebraic remark:

Lemma 2. *Let* $(V, <,>)$ *be an Euclidean vector space of dimension* $2n$, $n \geq 2$, *endowed with an orthogonal complex structure* $J : V \to V$. *Let* $h : V \to V$ *be a symmetric endomorphism anticommuting with* J. *Then the group* $U(h, J) \subset O(V)$ *consisting of the orthogonal transformations commuting with both* J *and* h *is isomorphic to*

$$O(k_1) \times O(k_2) \times \cdots \times O(k_m) \times U(s)$$

where the integers k_i *are the multiplicities of the distinct positive eigenvalues* $\alpha_1, \ldots, \alpha_m$ *of* h, *and* $2s$ *is the dimension of* $Ker(h)$.

Proof. Since h anti-commutes with J, the spectrum of h must be of the form

$$\{\alpha_1, \ldots, \alpha_m, -\alpha_1, \ldots, -\alpha_m\}, \qquad \alpha_i > 0,\ \alpha_i \neq \alpha_j$$

provided h is non singular, or of the form

$$\{\alpha_1, \ldots, \alpha_m, -\alpha_1, \ldots, -\alpha_m, 0\}$$

if $s \geq 1$. Moreover, we have $J(Ker(h)) = Ker(h)$ and the eigenspaces $V(\alpha_i)$ and $V(-\alpha_i)$ of h satisfy:

$$J(V(\alpha_i)) = V(-\alpha_i).$$

Hence V admits an orthonormal basis of the form

$$\{v_1^i, \ldots, v_{k_i}^i, Jv_1^i, \ldots, Jv_{k_i}^i\}$$

where $i = 1, \ldots, m$, or of the form

$$\{v_1^i, \ldots, v_{k_i}^i, Jv_1^i, \ldots, Jv_{k_i}^i, u_1, \ldots, u_s, Ju_1, \ldots, Ju_s\}$$

where $h(v_j^i) = \alpha_i v_j^i$ and $h(u_t) = 0$. With respect to such a basis, every automorphism $L \in U(h, J)$ is represented by a block diagonal matrix A of the form

$$A = \mathrm{diag}(A_1, \ldots, A_m, A_1, \ldots A_m) \tag{4}$$

or of the form

$$A = \mathrm{diag}(A_1, \ldots, A_m, A_1, \ldots A_m, B),$$

where $A_i \in O(k_i)$ and B belongs to the subgroup of $O(2s)$ consisting of the orthogonal matrices commuting with the matrix

$$\begin{pmatrix} 0 & -I_s \\ I_s & 0 \end{pmatrix}$$

corresponding to the standard complex structure of $\mathbb{R}^{2s}$, which is naturally isomorphic to $U(s)$. This proves the assertion. □

Now we can start proving Theorem 3. It is known that since $\mathrm{Aut}(M)$ is a closed subgroup of the isometry group $I(M, g)$, the $\mathrm{Aut}(M)$-orbit S of p is a closed submanifold of M, diffeomorphic to $\mathrm{Aut}(M)/\mathrm{Aut}_p(M)$, where $\mathrm{Aut}_p(M)$ denotes the isotropy subgroup of $\mathrm{Aut}(M)$ relative to the point p.

Consider the linear isotropy representation $\rho : \mathrm{Aut}_p(M) \to O(T_pM, g_p)$. Observe that every automorphism of the contact metric structure must also preserve the tensor field h. Hence, denoting again by D the contact distribution and by $\bar{\rho} : \mathrm{Aut}_p(M) \to O(D_p, g_p)$ the representation induced by ρ, i.e.

$$\bar{\rho}(F) := (dF)_p|_{D_p},$$

and keeping the notation of Lemma 2, we have that $\mathrm{Im}(\bar{\rho})$ is contained in the group $U(h_p, J_p) \subset O(D_p, g_p)$ determined by the symmetric operator $h_p : D_p \to D_p$ and by the complex structure $J_p : D_p \to D_p$ given by the restriction of φ_p. According to the assumption on the rank of h_p and the same lemma, this group has dimension at most $\frac{n(n-1)}{2} = \dim O(n)$. This yields a).

To prove b), assume that $\dim \mathrm{Aut}(M) = \frac{(n+1)(n+2)}{2} = 2n + 1 + \frac{n(n-1)}{2}$; then $\mathrm{Aut}_p(M)$ has dimension $\frac{n(n-1)}{2}$ and $\dim(S) = 2n + 1$; this yields $S = M$ since S is closed. Hence in this case $\mathrm{Aut}(M)$ is transitive on M, i.e. M is a homogeneous contact metric manifold. Moreover, being $\dim U(h_p, J_p) \geq \dim O(n)$, using again Lemma 2, we see that $h_p : D_p \to D_p$ must be non singular and must admit exactly one positive eigenvalue $\alpha = \alpha_1$. By homogeneity, this actually holds at every point of M. Observe also that actually $\dim U(h_p, J_p) = \dim O(n)$, and since $\mathrm{Aut}_p(M)$ is compact, $\mathrm{Im}(\bar{\rho})$ contains the connected component of the identity of $U(h_p, J_p)$.

Denote by D^+ and D^- the eigendistributions of h relative to the constant eigenvalues α and $-\alpha$. In order to complete the proof, we shall verify that M is a η-parallel contact metric manifold in the sense of [5]. Denote by Θ the $(0, 3)$ tensor field on M defined by

$$\Theta(X, Y, Z) := g((\nabla_X h)Y, Z),$$

where ∇ is the Levi-Civita connection. We need to show that $\Theta(X, Y, Z) = 0$ whenever the vector fields X, Y, Z belong to D.

First of all, we remark that this automatically holds if Y and Z both belong to D^+ or to D^-. Hence, since Θ is symmetric in Y and Z, it suffices to prove that:

$$\Theta(X, Y, JZ) = \Theta(JX, Y, JZ) = 0 \qquad X, Y, Z \in \Gamma D^+$$

and, by homogeneity, since every $F \in \mathrm{Aut}(M)$ clearly preserves Θ, this in turn reduces to:

$$\Theta_p(X, Y, JZ) = \Theta_p(JX, Y, JZ) = 0 \qquad X, Y, Z \in \mathcal{B}, \tag{5}$$

where $\mathcal{B} = \{X_1, \ldots, X_n\}$ is a fixed orthonormal basis of D_p^+.

For the sake of convenience, let us put, for every $X, Y, Z \in D_p^+$:

$$\Psi(X, Y, Z) := \Theta_p(X, Y, JZ), \quad \Psi'(X, Y, Z) := \Theta_p(JX, Y, JZ),$$

so that Ψ and Ψ' are $(0, 3)$ tensors on the vector space D_p^+.

Now, assuming $n \neq 3$, using the matrix description (4) of the automorphisms of D_p belonging to the group $U(h_p, J_p)$, with $m = 1$, we readily see that given X, Y, Z in $\mathcal{B}$, there always exists a linear transformation $L : D_p \to D_p$ belonging to the connected component of the identity $U^0(h_p, J_p) \cong SO(D_p^+) \cong SO(n)$ of $U(h_p, J_p)$, and such that:

$$L(X) = -X, \; L(Y) = -Y, \; L(Z) = -Z.$$

Since $U^0(h_p, J_p)$ is contained in $\mathrm{Im}(\bar{\rho})$, there exists an automorphism $F \in \mathrm{Aut}_p(M)$ such that $(dF)_p|_{D_p} = L$. Hence, taking again into account the $\mathrm{Aut}(M)$-invariance of Θ and J, we get:

$$\Psi(X, Y, Z) = \Psi(LX, LY, LZ) = -\Psi(X, Y, Z)$$

and similarly for Ψ'. This proves (5) and thus M is a (k, μ)-space.

Concerning the case $n = 3$, assume henceforth that the almost CR structure (D, J), $J = \varphi|_D$, is integrable. Then, according to [1, Theorem 6.7], we have that φ is η-parallel:

$$g((\nabla_X \varphi)Y, Z) = 0$$

for every X, Y, Z sections of D. Using this, the symmetry of $\nabla_X h$ and of $\nabla_{JX} h$ and the fact that h anticommutes with φ, it is immediate to verify that Ψ and Ψ' are symmetric in Y and Z. On the other hand, the same argument as above yields that both Ψ and Ψ' are preserved by every orthogonal transformation in $SO(D_p^+)$, hence they must be 3-forms on D_p^+. Thus we arrive again at $\Psi = \Psi' = 0$, proving our assertion.

References

[1] D. E. Blair: Riemannian geometry of contact and symplectic manifolds, Volume 203 of *Progress in Mathematics*. Birkhäuser, Boston Inc., Boston, MA, second edition (2010).

[2] D. E. Blair, T. Koufogiorgos, B. J. Papantoniou: *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91** (1995), 189–214.

[3] E. Boeckx: *A class of locally φ-symmetric contact metric spaces*, Arch. Math. (Basel) **72** (1999), 466–472.

[4] E. Boeckx: *A full classification of contact metric (k, μ)-spaces*, Illinois J. Math. **44** (2000), 212–219.

[5] E. Boeckx, J. T. Cho: *η-parallel contact metric spaces*, Differential Geom. Appl. **22** (2005), 275–285.

[6] E. Boeckx, J. T. Cho: *Pseudo-Hermitian symmetries*, Israel J. Math. **166** (2008), 125–145.

[7] E. Boeckx, J. T. Cho: *Locally symmetric contact metric manifolds*, Monatsh. Math. **148** (2006), no. 4, 269–281.

[8] W. M. Boothby, H. C. Wang: *On contact manifolds*, Ann. of Math. (2) **68** (1958), 721–734.

[9] B. Cappelletti-Montano: *The foliated structure of contact metric (κ, μ)-spaces*, Illinois J. Math. **53** (2009), no. 4, 1157–1172.

[10] B. Cappelletti-Montano, A. Carriazo, V. Martin-Molina: *Sasaki-Einstein and paraSasaki-Einstein metrics from (k, μ)-structures*, J. Geom. Phys. **73** (2013), 20–36.

[11] J.T. Cho, J. Inoguchi: *Contact metric hypersurfaces in complex space forms*, Differential geometry of submanifolds and its related topics, 87–97, World Sci. Publ., Hackensack, NJ, 2014.

[12] G. Dileo, A. Lotta: *A classification of spherical symmetric CR manifolds*, Bull. Aust. Math. Soc. **80** (2009), 251–274.

[13] W. Kaup, D. Zaitsev: *On symmetric Cauchy-Riemann manifolds*, Adv. Math. **149** (2000), 145–181.

[14] S. Kobayashi, K Nomizu: Foundations of differential geometry, Vol. II, Interscience Publishers, 1969.

[15] E. Loiudice, A. Lotta: *On the classification of contact metric (κ, μ)-spaces via tangent hyperquadric bundles*, Math. Nachr. **291** (2018), no. 11-12, 1851–1858.

[16] E. Loiudice, A. Lotta: *Canonical fibrations of contact metric (κ, μ)-spaces*, preprint (2017), arXiv:1704.01310, to appear on Pacific J. Math.

[17] Z. Olszak: *Contact metric hypersurfaces in complex spaces*, Demonstratio Math. **16** (1983), 95–102.

[18] D. Perrone: *On the standard nondegenerate almost CR structure of tangent hyperquadric bundles*, Geom. Dedicata **185** (2016), 15–33.

[19] F. Tricerri, L. Vanhecke: Homogeneous structures on Riemannian manifolds, London Mathematical Society Lecture Note Series, 83. Cambridge University Press, Cambridge, 1983.

[20] S. Tanno: *The automorphism groups of almost contact Riemannian manifolds*, Tohoku Math. J. **21** (1969), 21–38.