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SOLITON SOLUTIONS FOR QUASILINEAR MODIFIED SCHRÖDINGER EQUATIONS IN APPLIED SCIENCES

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To Rosella Mininni, a beloved friend

Abstract. In this paper, we prove the existence of nontrivial weak bounded solutions of the quasilinear modified Schrödinger problem

$$\begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u) & \text{in } \mathbb{R}^3, \\ u > 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ are "good" functions and $g : \mathbb{R} \to \mathbb{R}$ is such that $g^2(u) = 1 + \frac{[(l(u^2))']^2}{2}$ for a given $l \in \mathcal{C}^2(\mathbb{R})$.

By means of variational methods and an approximation argument, here we obtain an existence result for the superfluid film equation in Plasma Physics and for the equation which models the self–channelling of a high–power ultrashort laser, which derive from our model problem by taking $l(s) = s$, respectively $l(s) = \sqrt{1+s}$, in the previous definition of $g^2(u)$.

1. Introduction and motivations

In this paper, we deal with the existence of positive solutions, often named soliton solutions, for the quasilinear modified Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where potential $V : \mathbb{R}^3 \to \mathbb{R}$ is a suitable measurable function and $g : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ are given maps.

The noteworthiness of problem (1.1) is based on its connection to the study of solitary wave solutions for the quasilinear Schrödinger equation

$$i\partial_t z = -\Delta z - \Delta l(|z|^2)l'(|z|^2)z + W(x)z - k(x,|z|)z, \quad \text{with } x \in \mathbb{R}^3,\ t \geq 0, \tag{1.2}$$

where the solution we look for $z(x,t)$ is complex in $\mathbb{R}^3 \times \mathbb{R}_+$ while $W : \mathbb{R}^3 \to \mathbb{R}$, $k : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ and $l : \mathbb{R}_+ \to \mathbb{R}$ are real functions.

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In fact, by using the ansatz $z(x,t) = \mathrm{e}^{-iEt}u(x)$ in (1.2) with $E \in \mathbb{R}$, we have that $u : \mathbb{R}^3 \to \mathbb{R}$ is an unknown real function which solves the stationary Schrödinger problem

$$\begin{cases} -\Delta u - \Delta(l(u^2))l'(u^2)u + V(x)u = f(x,u) & \text{in } \mathbb{R}^3 \\ u > 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.3}$$

with the new potential $V(x) = W(x) - E$ and $f(x,u) = k(x,u)u$.

The relation between the equation asserted in (1.1) and that one in (1.3) is exhibited through the particular choice of function $g(u)$. In fact, if $l(s)$ is a C^2 map on $\mathbb{R}_+$, having

$$g^2(u) = 1 + \frac{[(l(u^2))']^2}{2} = 1 + 2u^2[l'(u^2)]^2, \tag{1.4}$$

we obtain

$$g(u)g'(u) = 2u\, l'(u^2)[l'(u^2) + 2u^2 l''(u^2)] \tag{1.5}$$

and

$$\mathrm{div}(g^2(u)\nabla u) = 2g(u)g'(u)|\nabla u|^2 + g^2(u)\Delta u,$$

then, via direct computations, we infer that

$$\begin{aligned} \Delta u + \Delta(l(u^2))l'(u^2)u &= \Delta u + g(u)g'(u)|\nabla u|^2 + 2u^2[l'(u^2)]^2\Delta u \\ &= \mathrm{div}(g^2(u)\nabla u) - g(u)g'(u)|\nabla u|^2 \end{aligned}$$

so that the equation in (1.3) boils down exactly to equation (1.1).

Thus, the importance of problem (1.1) rests upon the wide interest in Schrödinger equation (1.2) which turns up in several fields such as, for example, Plasma Physics and Fluid Mechanics (see [15]), Mechanics (see [14]), Condensed Matter Theory (see [19]). More precisely, it has been derived as model equation of various physical phenomena according to the special form given to the nonlinear term $l(s)$.

Throughout this paper, we will focus on the equations which are originated by choosing $l(s) = s$ and $l(s) = \sqrt{1+s}$.

In particular, taking $l(s) = s$ in (1.3), definition (1.4) gives

$$g^2(u) = 1 + 2u^2 \tag{1.6}$$

and (1.1) reduces to the superfluid film equation in Plasma Physics so that we are interested in solving the model problem

$$\begin{cases} -\Delta u - u\,\Delta(u^2) + V(x)u = f(x,u) & \text{in } \mathbb{R}^3 \\ u > 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.7}$$

with a nonlinear function $f(x,u)$ at the place of $k(x,u)u$ (see [15]).

On the other hand, if $l(s) = \sqrt{1+s}$ from (1.4) we have that

$$g^2(u) = 1 + \frac{u^2}{2(1+u^2)} \tag{1.8}$$

so problem (1.1) turns into

$$\begin{cases} -\Delta u - \frac{u}{2\sqrt{1+u^2}}\,\Delta(\sqrt{1+u^2}) + V(x)u = f(x,u) & \text{in } \mathbb{R}^3 \\ u > 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.9}$$

which describess the self–channeling of a high–power ultrashort laser in matter (see [16]).

As far as we know, the first existence results for equation (1.7), via a variational approach, are gained through a constrained minimization argument (see [18, 20]). Later on, a further general existence result was derived in [17] by using a suitable change of variable which reduces the quasilinear equation (1.7) to a semilinear one so that a Orlicz space framework can be used and the existence of positive solutions follows from the Mountain Pass Theorem. The same approach was employed in [11, 13], but a most common setting involving the usual Sobolev space $H^1(\mathbb{R}^N)$ was exploited. More precisely, since the action functional associated to (1.7) is not well defined in $H^1(\mathbb{R}^N)$, taking $h(t)$ odd extension of the solution of the ordinary differential equation

$$h'(t) = \frac{1}{\sqrt{1+2h^2(t)}} \qquad \text{if } \ t \geq 0,$$

the existence of a spherically symmetric solution is established by means of the change of variable $v = h^{-1}(u)$ and of classical results provided by Berestycki and Lions in [3].

On the other hand, very few results are known about equation (1.9) (see, e.g., [12]) but, more recently, Y. Schen and Y. Wang in [21] choose to introduce a unified method for studing both (1.7) and (1.9) looking for standing wave solutions for (1.1) so that they have the existence of positive solutions for the model problems, too. As they point out, variational methods cannot apply directly so, by taking advance of the special form of (1.1), in the natural associated functional they make the change of variable

$$v = G(u) = \int_0^u g(t)dt$$

and then investigate the existence of weak solutions of the corresponding problem by means of the Mountain Pass Theorem. Clearly, with such an approach the hypotheses on the nonlinear term $f(x,u)$ are affected by $g(u)$ (for more details, see Remark 1.3).

In this paper, we deal with (1.1), too, but we use a completely different strategy. In fact, we consider (1.1) as a model problem of the quasilinear modified Schrödinger equation

$$-\operatorname{div}(A(x,u)|\nabla u|^{p-2}\nabla u)+\frac{1}{p}A_u(x,u)|\nabla u|^p+V(x)|u|^{p-2}u = f(x,u) \ \text{ in } \mathbb{R}^N \quad (1.10)$$

with $p > 1$ and $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that $A_u(x,u) = \frac{\partial A}{\partial u}(x,u)$, which has been studied in [7, 10].

Here, it has to be $N = 3$, $p = 2$, $A(x,u) = g^2(u)$ as in (1.4) and, as in [2, 10], potential $V : \mathbb{R}^3 \to \mathbb{R}$ satisfies the following conditions:

(V_1) $V(x)$ is a measurable function such that $\operatorname*{ess\,inf}_{x\in\mathbb{R}^3} V(x) > 0$;

(V_2) $\displaystyle\lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{V(y)}\, dy = 0$;

(V_3) for any $\varrho > 0$, a constant $C_\varrho > 0$ exists such that $\operatorname*{ess\,sup}_{|x|\leq\varrho} V(x) \leq C_\varrho$;

where $B_1(x)$ is the unit ball in $\mathbb{R}^3$ with center x.

Moreover, we suppose that the nonlinear term $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is such that:

(f_0^+) $f(x,u)$ is a Carathéodory function with $f(x,0) = 0$ for a.e. $x \in \mathbb{R}^3$;

(f_1^+) $a_1, a_2 > 0$ and $q \geq 2$ exist such that

$$|f(x,u)| \leq a_1 u + a_2 u^{q-1} \qquad \text{a.e. in } \mathbb{R}^3, \ \text{for all } u \geq 0;$$

(f_2^+) $\displaystyle\lim_{u\to 0^+} \frac{f(x,u)}{u} = \bar{\alpha} < \frac{1}{\tau_2^2}$ uniformly a.e. with respect to $x \in \mathbb{R}^3$;

(f_3^+) $\mu > 2$ exists such that

$$0 < \mu F(x,u) \le f(x,u)u \quad \text{a.e. in } \mathbb{R}^3, \text{ for all } u > 0;$$

where $\tau_2 > 0$ is a constant coming from a suitable embedding theorem (i.e., so that (3.4) holds), and

$$F : (x,u) \in \mathbb{R}^3 \times \mathbb{R} \ \mapsto \ \int_0^u f(x,t)dt \in \mathbb{R}.$$

In Section 4 we will state existence results for problem (1.1) (see Theorems 4.1 and 4.4), which allow us to state the following theorems for the model problems (1.7) and (1.9).

Theorem 1.1. *Assume that potential $V(x)$ satisfies assumptions (V_1)–(V_3), while the nonlinear term $f(x,u)$ is such that conditions (f_0^+)–(f_3^+) are verified.*
Then, if

$$q < 6 \qquad \text{and} \qquad \mu > 4 \tag{1.11}$$

with q as in (f_1^+) and μ as in (f_3^+), problem (1.7) admits at least one weak bounded solution.

Theorem 1.2. *Assume that potential $V(x)$ satisfies assumptions (V_1)–(V_3), while the nonlinear term $f(x,u)$ is such that conditions (f_0^+)–(f_3^+) are verified.*
Then, if (1.11) holds, problem (1.9) admits at least one weak bounded solution.

Remark 1.3. In [21] potential $V(x)$ is required to be continuous on $\mathbb{R}^3$, with $V(x) \ge V_0 > 0$ for all $x \in \mathbb{R}^3$ and so that

$$\lim_{|x|\to+\infty} V(x) = V(\infty) \quad \text{and} \quad V(x) \le V(\infty) \text{ for all } x \in \mathbb{R}^3.$$

Hence, (V_1) and (V_3) are essentially verified, but hypothesis (V_2) cannot hold.
On the other hand, here the hypotheses on $f(x,u)$, namely (f_0^+)–(f_3^+), are not affected by $g(u)$ or, more precisely, any particular form given to $l(u)$ in (1.4), hence also condition (1.11) does not change. Such independence is a great improvement when dealing with general settings but it is too rigid when working with both the special problems (1.7) and (1.9). In fact, as pointed out in [21], differently from Theorem 1.1, respectively Theorem 1.2, if (1.6) holds then it can be $q < 12$ (see [21, Remark 1.4]) while (1.8) has a solution even when the weaker estimate $\mu > \sqrt{6}$ holds (see [21, Corollary 1.5]).

Our paper is organized as follows: in Section 2 we outline the main abstract tools required for our variational approach, in particular the weak Cerami–Palais–Smale condition (see Definition 2.1), then in Section 3 we introduce the "right" variational setting and recall the existence result for the more general problem (1.10). Then, in Section 4 we state our main existence results and, at last, in Section 5 we give an hint of a direct proof of Theorem 4.1 in order to point out the different approach with respect to the previous results which allow us to have conditions on the nonlinear term $f(x,u)$ which are independent of $g(u)$.

2. Abstract framework

Throughout this section, we assume that:

- $(X, \|\cdot\|_X)$ is a Banach space with dual $(X', \|\cdot\|_{X'})$;

- $(W, \|\cdot\|_W)$ is a Banach space such that $X \hookrightarrow W$ continuously, i.e. $X \subset W$ and a constant $\sigma_0 > 0$ exists such that
$$\|\xi\|_W \leq \sigma_0 \|\xi\|_X \qquad \text{for all } \xi \in X;$$
- $J : \mathcal{D} \subset W \to \mathbb{R}$ and $J \in C^1(X, \mathbb{R})$ with $X \subset \mathcal{D}$.

For simplicity, taking $\beta \in \mathbb{R}$, we say that a sequence $(\xi_n)_n \subset X$ is a *Cerami–Palais–Smale sequence at level* β, briefly $(CPS)_\beta$*–sequence*, if
$$\lim_{n\to+\infty} J(\xi_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(\xi_n)\|_{X'}(1 + \|\xi_n\|_X) = 0.$$
Moreover, β is a *Cerami–Palais–Smale level*, briefly (CPS)*–level*, if there exists a $(CPS)_\beta$–sequence.

As $(CPS)_\beta$–sequences may exist which are unbounded in $\|\cdot\|_X$ but converge with respect to $\|\cdot\|_W$, we have to weaken the classical Cerami–Palais–Smale condition in a suitable way according to the ideas already developed in previous papers (see, e.g., [5]).

Definition 2.1. The functional J satisfies the *weak Cerami–Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ *condition*, if for every $(CPS)_\beta$–sequence $(\xi_n)_n$, a point $\xi \in X$ exists, such that

(i) $\lim_{n\to+\infty} \|\xi_n - \xi\|_W = 0$ (up to subsequences),

(ii) $J(\xi) = \beta$, $dJ(\xi) = 0$.

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

Since Definition 2.1 allows one to prove a Deformation Lemma (see [5]), the following generalization of the Ambrosetti–Rabinowitz Mountain Pass Theorem can be stated (for the proof, see [5, Theorem 1.7] with remarks in [8, Theorem 2.2] and compare it with [1, Theorem 2.1]).

Theorem 2.2 (Mountain Pass Theorem)**.** *Let $J \in C^1(X, \mathbb{R})$ be a functional such that $J(0) = 0$ and the $(wCPS)$ condition holds in $\mathbb{R}$. Moreover, assume that two constants r, $\varrho > 0$ and a point $e \in X$ exist such that*
$$u \in X,\ \|u\|_W = r \quad \Longrightarrow \quad J(u) \geq \varrho,$$
$$\|e\|_W > r \qquad \text{and} \qquad J(e) < \varrho.$$
Then, J has a critical point $u^* \in X$ such that
$$J(u^*) = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} J(\gamma(s)) \geq \varrho$$
with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\}$.

3. Variational setting and the quasilinear problem

From now on, let $N = 3$ and, taking any open subset $\Omega \subset \mathbb{R}^3$, we denote by:

- $\mathbb{N} = \{1, 2, \dots\}$ the set of the strictly positive integers;
- $B_R(x) = \{y \in \mathbb{R}^3 : |y - x| < R\}$ the open ball with center in $x \in \mathbb{R}^3$ and radius $R > 0$;
- $|D|$ the usual 3–dimensional Lebesgue measure of a measurable set D in $\mathbb{R}^3$;
- $(L^r(\Omega), |\cdot|_{\Omega,r})$ the classical Lebesgue space with norm $|u|_{\Omega,r} = \left(\int_\Omega |u|^r dx\right)^{1/r}$ if $1 \leq r < +\infty$;

- $(L^\infty(\Omega), |\cdot|_{\Omega,\infty})$ the space of Lebesgue–measurable essentially bounded functions endowed with norm $|u|_{\Omega,\infty} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$;
- $(H^1_0(\Omega), \|\cdot\|_\Omega)$ is the classical Sobolev space with $\|u\|_\Omega = (|\nabla u|^2_{\Omega,2} + |u|^2_{\Omega,2})^{\frac{1}{2}}$.

Moreover, if potential $V : \mathbb{R}^3 \to \mathbb{R}$ satisfies condition (V_1), we denote by

- $(L^r_V(\Omega), |\cdot|_{\Omega,V,r})$, $1 \le r < +\infty$, the weighted Lebesgue space with
$$L^r_V(\Omega) = \left\{ u \in L^r(\Omega) : \int_\Omega V(x)|u|^r dx < +\infty \right\}$$
endowed with the norm
$$|u|_{\Omega,V,r} = \left(\int_\Omega V(x)|u|^r dx \right)^{\frac{1}{r}};$$
- $(H^1_{0,V}(\Omega), \|\cdot\|_{\Omega,V})$ the weighted Sobolev space
$$H^1_{0,V}(\Omega) = \left\{ u \in H^1_0(\Omega) : \int_\Omega V(x)u^2 dx < +\infty \right\}$$
endowed with the norm
$$\|u\|_{\Omega,V} = (|\nabla u|^2_{\Omega,2} + |u|^2_{\Omega,V,2})^{\frac{1}{2}}.$$

For simplicity, we put $B_R = B_R(0)$ for the open ball with center in the origin and radius $R > 0$ and, if $\Omega = \mathbb{R}^3$, we avoid to write the set in the norms, i.e.,

- $|\cdot|_r = |\cdot|_{\mathbb{R}^3,r}$ is the norm in $L^r(\mathbb{R}^3)$, for any $1 \le r \le +\infty$;
- $|\cdot|_{V,r} = |\cdot|_{\mathbb{R}^3,V,r}$ is the norm in $L^r_V(\mathbb{R}^3)$, for any $1 \le r < +\infty$;
- $\|\cdot\| = \|\cdot\|_{\mathbb{R}^3}$ is the norm in $H^1(\mathbb{R}^3) = H^1_0(\mathbb{R}^3)$;
- $\|\cdot\|_V = \|\cdot\|_{\mathbb{R}^3,V}$ is the norm in $H^1_V(\mathbb{R}^3) = H^1_{0,V}(\mathbb{R}^3)$.

Remark 3.1. If weight $V(x)$ satisfies assumption (V_1), then we have the following continuous embeddings:
$$L^r_V(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3) \quad \text{for any } 1 \le r < +\infty, \tag{3.1}$$
and then
$$H^1_V(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3).$$

From Remark 3.1 and classical Sobolev Embedding Theorems, we deduce the following result (for the compact embeddings, see [2, Theorem 3.1]).

Theorem 3.2. *If weight $V : \mathbb{R}^3 \to \mathbb{R}$ satisfies assumption (V_1), then the following continuous embedding hold:*
$$H^1_V(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3) \quad \textit{for any } 2 \le r \le 6. \tag{3.2}$$
Furthermore, if assumption (V_2) also occurs, we have the compact embedding
$$H^1_V(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^r(\mathbb{R}^3) \quad \textit{for any } 2 \le r < 6. \tag{3.3}$$

For any $r \ge 2$ so that (3.2) holds, Theorem 3.2 implies that a constant $\tau_r > 0$ exists such that
$$|u|_r \le \tau_r \|u\|_V \quad \text{for all } u \in H^1_V(\mathbb{R}^3). \tag{3.4}$$

On the other hand, taking Ω open bounded domain in $\mathbb{R}^3$, the classical embedding theorem implies that a constant $\sigma_* > 0$ exists, independent of Ω, such that
$$|v|_{\Omega,2^*} \le \sigma_* \|v\|_\Omega \quad \text{for all } v \in H^1_0(\Omega).$$

From now on, we assume that potential $V(x)$ satisfies condition (V_1) and we set

$$X := H^1_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \quad \text{and} \quad \|u\|_X = \|u\|_V + |u|_\infty \quad \text{for any } u \in X. \tag{3.5}$$

Remark 3.3. From (3.1) and [10, Lemma 3.3] it follows that the continuous embedding $X \hookrightarrow L^r(\mathbb{R}^3)$ holds for every $r \geq 2$.

Now, we recall the existence result for solutions of problem (1.10) as stated in [10]. To this aim, for the coefficient $A : (x,u) \in \mathbb{R}^3 \times \mathbb{R} \mapsto A(x,u) \in \mathbb{R}$ we consider the following conditions:

(h_0) $A(x,u)$ is a $\mathcal{C}^1$–Carathéodory function with $A_u(x,u) = \frac{\partial}{\partial u}A(x,u)$, i.e., a function measurable in x for all $u \in \mathbb{R}$ and $\mathcal{C}^1$ in u for a.e. $x \in \mathbb{R}^3$;

(h_1) for any $\rho > 0$ we have that

$$\sup_{|u|\leq\rho} |A(\cdot,u)| \in L^\infty(\mathbb{R}^3), \qquad \sup_{|u|\leq\rho} |A_u(\cdot,u)| \in L^\infty(\mathbb{R}^3);$$

(h_2) a constant $\alpha_0 > 0$ exists such that

$$A(x,u) \geq \alpha_0 \quad \text{a.e. in } \mathbb{R}^3, \text{ for all } u \in \mathbb{R};$$

(h_3) some constants $\mu > 2$ and $\alpha_1 > 0$ exist so that

$$(\mu - 2)A(x,u) - A_u(x,u)u \geq \alpha_1 A(x,u) \quad \text{a.e. in } \mathbb{R}^3, \text{ for all } u \in \mathbb{R};$$

(h_4) a constant $\alpha_2 > 0$ exists such that

$$2A(x,u) + A_u(x,u)u \geq \alpha_2 A(x,u) \quad \text{a.e. in } \mathbb{R}^3, \text{ for all } u \in \mathbb{R}.$$

On the other hand, we suppose that the nonlinear term $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ may satisfy the following hypotheses:

(f_0) $f(x,u)$ is a Carathéodory function;

(f_1) $a_1, a_2 > 0$ and $q \geq 2$ exist such that

$$|f(x,u)| \leq a_1|u| + a_2|u|^{q-1} \quad \text{a.e. in } \mathbb{R}^3, \text{ for all } u \in \mathbb{R};$$

(f_2) $\displaystyle\limsup_{u\to 0} \frac{f(x,u)}{u} = \bar{\alpha} < \frac{1}{\tau_2^2}$ uniformly a.e. with respect to $x \in \mathbb{R}^3$;

(f_3) a constant $\mu > 2$ exists such that

$$0 < \mu F(x,u) \leq f(x,u)u \quad \text{a.e. in } \mathbb{R}^3, \text{ for all } u \neq 0;$$

where $\tau_2 > 0$ is so that (3.4) holds with $r = 2$.

Remark 3.4. From hypotheses $(f_0), (f_1)$, we infer that

$$F : (x,u) \in \mathbb{R}^3 \times \mathbb{R} \mapsto \int_0^u f(x,t)dt \in \mathbb{R}$$

is a well defined $\mathcal{C}^1$–Carathéodory function. Moreover, assumption (f_1) establishes also that

$$F(x,0) = f(x,0) = 0 \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{3.6}$$

Thus, from [10, Theorem 4.4] with $p = 2$ and $N = 3$, hence $p^* = 6$, the following result can be stated.

Theorem 3.5. *Under assumptions (V_1)–(V_3), (h_0)–(h_4) and (f_0)–(f_3), so that the same μ satisfies both (h_3) and (f_3), and the growth exponent q in (f_1) is such that*

$$q < 6,$$

then problem (1.10) admits at least one weak nontrivial bounded solution.

Remark 3.6. If q is as in (f_1) and μ is as in (f_3), then in the hypotheses of Theorem 3.5 it has to be $\mu \leq q$, hence we have that

$$2 < \mu \leq q < 6.$$

We note that the existence of solutions of equation (1.10) is obtained by means of a variational approach. More precisely, the hypotheses (V_1), (h_0)–(h_1) and (f_0)–(f_1) allow us to state a good variational principle so that we have to look for critical points of the functional

$$\mathcal{J}_*(u) = \frac{1}{2}\int_{\mathbb{R}^3} A(x,u)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \tag{3.7}$$

which is well defined and of class $\mathcal{C}^1$ in the Banach space X defined in (3.5) (see [10, Proposition 3.10]).

Unluckily, due to the intersection norm in (3.5) which involves both the $H^1_V(\mathbb{R}^3)$–norm and the $L^\infty(\mathbb{R}^3)$ one, functional $\mathcal{J}_*$ cannot satisfy the classical Palais–Smale condition (or its variants) in X even if we replace the whole Euclidean space with one bounded subset (see [6, Example 4.3]), so we have to consider the weak Cerami–Palais–Smale condition and consider X equipped with two different norms, namely $\|\cdot\|_X$ and $\|\cdot\|_V$.

The idea of the proof of Theorem 3.5 relies also on an approximation argument as a sequence $(u_k)_k$ is found so that for each $k \in \mathbb{N}$ function u_k is a weak bounded solution of equation (1.10) but in the open ball B_k with Dirichlet boundary condition $u_k = 0$ on ∂B_k. Then, a nontrivial critical point for $\mathcal{J}_*$ in X is constructed as a suitable limit of sequence $(u_k)_k$.

4. Existence results

Now, we are ready to state the main existence result for problem (1.1) which is considered a particular case of equation (1.10).

Theorem 4.1. *Assume that potential $V(x)$ satisfies conditions (V_1)–(V_3) and that the nonlinear term $f(x,u)$ is so that hypotheses (f_0)–(f_3) hold. Moreover, let us consider a function $l : \mathbb{R}_+ \to \mathbb{R}$ such that:*

- $(\mathcal{H}_0)$ *$l(s)$ is of class $\mathcal{C}^2$ on $\mathbb{R}_+$;*
- $(\mathcal{H}_1)$ *$l'(s)l''(s) \leq 0$ for all $s \geq 0$;*
- $(\mathcal{H}_2)$ *a constant $\alpha_* > 0$ exists such that*

$$[l'(s)]^2 + sl'(s)l''(s) \geq \alpha_*[l'(s)]^2 \quad \text{for all } s \geq 0.$$

If q in (f_1) and μ in (f_3) are such that (1.11) is verified then, taking $g(u)$ as in (1.4), problem (1.1) admits at least one weak nontrivial bounded solution.

Proof. Taking $A(x,u) = g^2(u)$, from (1.4) we have that

$$A(x,u) \;=\; 1 + 2u^2[l'(u^2)]^2, \quad A_u(x,u) \;=\; 4u\left([l'(u^2)]^2 + 2u^2 l'(u^2)l''(u^2)\right), \tag{4.1}$$

then $A(x,u) \equiv A(u) \geq 1$ for all $u \in \mathbb{R}$ and assumption $(\mathcal{H}_0)$ implies that $A(x,u)$ satisfies conditions (h_0)–(h_2).

On the other hand, in our setting (h_3) reduces to

$$\mu - 2 + 2(\mu-4)u^2[l'(u^2)]^2 - 4u^4 l'(u^2)l''(u^2) \;\geq\; \alpha_1 + 2\alpha_1 u^2[l'(u^2)]^2 \text{ for all } u \in \mathbb{R},$$

and a suitable $\alpha_1 > 0$ exists if $(\mathcal{H}_1)$ holds just taking any $\mu > 4$, in particular taking μ as in (f_3) so that (1.11) is satisfied.

At last, direct computations and (4.1) allow us to prove that (h_4) follows from $(\mathcal{H}_2)$.

Hence, the existence result for problem (1.1) is a corollary of Theorem 3.5. □

Remark 4.2. As pointed out at the end of Section 3, the weak solutions of (1.10) we look for are critical points of $\mathcal{J}_*$ defined in X as in (3.7). Thus, in the setting of equation (1.1), if hypotheses (V_1), $(\mathcal{H}_0)$ and (f_0)–(f_1) hold, such a functional reduces to

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \tag{4.2}$$

which is of class $\mathcal{C}^1$ in X with Fréchet differential in u along the direction v given by

$$\begin{aligned}\langle d\mathcal{J}(u), v\rangle = &\int_{\mathbb{R}^3} g^2(u)\nabla u\cdot\nabla v dx + \int_{\mathbb{R}^3} g(u)g'(u)v|\nabla u|^2 dx\\ &+ \int_{\mathbb{R}^3} V(x)uv dx - \int_{\mathbb{R}^3} f(x,u)v dx\end{aligned} \tag{4.3}$$

for any u, $v \in X$ (see [10, Proposition 3.10]).

In Section 1 we have justified the importance of equation (1.1) by means of its connections to the quasilinear Schrödinger equation (1.2), in particular pointing out the stationary Schrödinger problem (1.3). Then, in order to consider its possible applications, we have to look for positive solutions of (1.1), i.e., we have to solve problem

$$\begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u) & \text{in } \mathbb{R}^3,\\ u > 0 & \text{in } \mathbb{R}^3.\end{cases} \tag{4.4}$$

To this aim, we need the following Harnack type inequality for weak solutions of p–Laplacian type equations that we adapt to our setting (see [22, Theorem 1.1]).

Lemma 4.3. *Taking $p > 1$ and Ω domain in $\mathbb{R}^N$, let $\bar{u} \in W_0^{1,p}(\Omega)$ be a weak solution of the equation*

$$-\mathrm{div}(a(x,u,\nabla u)) \;=\; h(x,u,\nabla u) \qquad \text{in a cube } K(3r) \subset \Omega.$$

Assume that $M > 0$ exists such that $0 \le \bar{u}(x) < M$ for a.e. $x \in K(3r)$. If some constants $b_i \ge 0$, eventually depending on M, exist such that

$$\begin{aligned} |a(x,t,\xi)| &\le b_0\,|\xi|^{p-1} + b_1|t|^{p-1},\\ a(x,t,\xi)\cdot\xi &\ge |\xi|^p - b_2|t|^p,\\ |h(x,t,\xi)| &\le b_3|\xi|^p + b_4|\xi|^{p-1} + b_5|t|^{p-1}\end{aligned} \tag{4.5}$$

for a.e. $x \in \Omega$ and all $(t,\xi) \in]-M, M[\times\mathbb{R}^3$, then

$$\operatorname*{ess\,sup}_{x\in K(r)} \bar{u}(x) \;\le\; C \operatorname*{ess\,inf}_{x\in K(r)} \bar{u}(x),$$

where C depends only on N, p, M, r and the constants which appear in the hypotheses.

So, we can state and prove the existence result for positive solutions of (1.1).

Theorem 4.4. *Assume that potential $V(x)$ satisfies conditions (V_1)–(V_3) and a given function $l : \mathbb{R}_+ \to \mathbb{R}$ is such that $(\mathcal{H}_0)$–$(\mathcal{H}_2)$ hold. Then, if the nonlinear term $f(x,u)$ is so that hypotheses (f_0^+)–(f_3^+) are verified with q and μ satisfying estimate (1.11), taking $g(u)$ as in (1.4), equation (1.1) admits at least one weak bounded strictly positive solution, i.e., problem (4.4) has a solution.*

Proof. In the hypotheses (f_0^+)–(f_3^+), we can consider the function $f_+ : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ defined as

$$f_+(x,u) = \begin{cases} f(x,u) & \text{for a.e. } x \in \mathbb{R}^3, \text{ all } u \geq 0, \\ 0 & \text{for a.e. } x \in \mathbb{R}^3, \text{ all } u < 0, \end{cases}$$

and its related primitive

$$F_+(x,u) = \int_0^u f_+(x,t)dt = \begin{cases} F(x,u) & \text{for a.e. } x \in \mathbb{R}^3, \text{ all } u \geq 0, \\ 0 & \text{for a.e. } x \in \mathbb{R}^3, \text{ all } u < 0. \end{cases}$$

Hence, direct computations allow one to check that $f_+(x,u)$ and $F_+(x,u)$ satisfy assumptions (f_0)–(f_2) while the "partial condition" in (f_3^+) is enough for replacing (f_3) still obtaining the existence of a critical point of the $\mathcal{C}^1$ functional

$$\mathcal{J}_+(u) = \frac{1}{2}\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F_+(x,u)dx$$

in the Banach space X defined in (3.5) (see Remark 4.2). We note that, for any $u, v \in X$, its Fréchet differential is given by

$$\begin{aligned} \langle d\mathcal{J}_+(u), v\rangle = &\int_{\mathbb{R}^3} g^2(u)\nabla u \cdot \nabla v dx + \int_{\mathbb{R}^3} g(u)g'(u)v\, |\nabla u|^2 dx \\ &+ \int_{\mathbb{R}^3} V(x)uv dx - \int_{\mathbb{R}^3} f_+(x,u)v dx. \end{aligned} \tag{4.6}$$

Now, let $\bar{u} \in X$ be a critical point of $\mathcal{J}_+$ in X. We claim that it is $\bar{u} \geq 0$ a.e. in $\mathbb{R}^3$; hence, $\mathcal{J}(\bar{u}) = \mathcal{J}_+(\bar{u})$ and $\bar{u}$ is a critical point for the functional (4.2), too. The proof is similar to that one of [9, Proposition 4.5], but, for completeness, we give here the details.

Firstly, we note that, from $(\mathcal{H}_0)$ and (1.5) some constants $M > 0$ and $k_M > 0$ exist such that

$$|\bar{u}|_\infty \leq M, \qquad \max_{|t|\leq M} |g(t)g'(t)| \leq k_M, \tag{4.7}$$

which imply that

$$|g(\bar{u}(x))g'(\bar{u}(x))| \leq k_M \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{4.8}$$

Then, let us consider the real map

$$\psi(t) = te^{\eta t^2}, \qquad \text{with } \eta > \left(\frac{k_M}{2}\right)^2 > 0,$$

so, by definition, we have that $\psi(t)$ is a smooth odd function such that

$$t\psi(t) \geq t^2 \geq 0 \quad \text{and} \quad \psi'(t) - k_M|\psi(t)| > \frac{1}{2} \quad \text{for all } t \in \mathbb{R}. \tag{4.9}$$

Now, if $\bar{u}_- = \max\{-\bar{u}, 0\}$, we have that

$$\bar{u} = -\bar{u}_- \text{ in } \Omega_- = \{x \in \mathbb{R}^3 : \bar{u}(x) \leq 0\} \quad \text{and} \quad \bar{u}_- = 0 \text{ a.e. in } \mathbb{R}^3 \setminus \Omega_-. \tag{4.10}$$

Moreover, it is $\psi(-\bar{u}_-) \in X$. Hence, from $d\mathcal{J}_+(\bar{u}) = 0$, taking $v = \psi(-\bar{u}_-)$ in (4.6) we have that

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3} \psi'(-\bar{u}_-)g^2(\bar{u})\nabla\bar{u} \cdot \nabla(-\bar{u}_-)dx + \int_{\mathbb{R}^3} g(\bar{u})g'(\bar{u})\psi(-\bar{u}_-)|\nabla\bar{u}|^2 dx \\ &+ \int_{\mathbb{R}^3} V(x)\bar{u}\psi(-\bar{u}_-)dx - \int_{\mathbb{R}^3} f_+(x,\bar{u})\psi(-\bar{u}_-)dx, \end{aligned}$$

where from (4.10) it follows that

$$\int_{\mathbb{R}^3} f_+(x,\bar{u})\psi(-\bar{u}_-)dx = 0.$$

Thus, (1.4), (4.8)–(4.10), (V_1) and direct computations imply that

$$\begin{aligned}
0 \;&=\; \int_{\Omega_-} \psi'(-\bar{u}_-)g^2(-\bar{u}_-)|\nabla(-\bar{u}_-)|^2 dx \\
&\quad + \int_{\Omega_-} g(-\bar{u}_-)g'(-\bar{u}_-)\psi(-\bar{u}_-)|\nabla(-\bar{u}_-)|^2 dx + \int_{\Omega_-} V(x)(-\bar{u}_-)\psi(-\bar{u}_-)dx \\
&\geq \int_{\Omega_-} \big(\psi'(\bar{u}) - k_M|\psi(\bar{u})|\big)|\nabla(\bar{u})|^2 dx + \int_{\Omega_-} V(x)(-\bar{u}_-)\psi(-\bar{u}_-)dx \\
&\geq \frac{1}{2}\int_{\Omega_-} |\nabla(\bar{u})|^2 dx + \int_{\Omega_-} V(x)|\bar{u}|^2 dx \;\geq\; \frac{1}{2}\|\bar{u}_-\|_V^2
\end{aligned}$$

which provides that $\bar{u}_- = 0$ a.e. in $\mathbb{R}^3$.
Thus, we have proved that $\bar{u} \in X$ is a nontrivial weak solution of (1.1) with $\bar{u} \geq 0$ a.e. in $\mathbb{R}^3$ and, in order to prove $\bar{u} > 0$ a.e. in $\mathbb{R}^3$, we have to apply Lemma 4.3 with $N = 3$, $p = 2$, $\Omega = \mathbb{R}^3$ and

$$a(x,t,\xi) = g^2(t)\xi, \qquad h(x,t,\xi) = -g(t)g'(t)|\xi|^2 - V(x)t + f_+(x,t)$$

as $f_+(x,\bar{u}) = f(x,\bar{u})$ a.e. in $\mathbb{R}^3$.
To this aim, taking any cube $K(3r)$ in $\mathbb{R}^3$ with center in the origin, from $(\mathcal{H}_0)$, (V_1), (V_3), (f_1^+), (1.4), (4.7) and direct computations we have that conditions (4.5) hold with

$$b_0 = \max_{|t|\leq M} g^2(t),\ b_3 = k_M,\ b_5 = \operatorname*{ess\,sup}_{x\in K(3r)} V(x) + a_1 + a_2 M^{q-2},\ b_1 = b_2 = b_4 = 0.$$

Then, arguing by contradiction, assume that $\bar{u} = 0$ in a set with positive 3–dimensional Lebesgue measure. Thus, $\bar{r} > 0$ exists so that for all $r \geq \bar{r}$ it results

$$\operatorname*{ess\,inf}_{x\in K(r)} \bar{u}(x) = 0$$

which implies $\bar{u} = 0$ a.e. in $K(r)$ from Lemma 4.3. Hence, $\bar{u} = 0$ a.e. in $\mathbb{R}^3$ for the arbitrariness of the cube $K(r)$ and by a standard covering argument, in contradiction with $\bar{u}$ nontrivial.
So, $\bar{u}$ is a weak bounded strictly positive solution of (1.1). □

Proof of Theorems 1.1 and 1.2. From (1.4) we have that problem (4.4) reduces to (1.7) if $l(s) = s$, respectively to (1.9) if $l(s) = \sqrt{1+s}$. Hence, since direct computations allow us to prove that both the functions $l(s) = s$ and $l(s) = \sqrt{1+s}$ satisfy conditions $(\mathcal{H}_0)$–$(\mathcal{H}_2)$, then Theorem 1.1, respectively Theorem 1.2, follows from Theorem 4.4. □

5. Rehashing problem through bounded domains

In Section 4 we have proved Theorem 4.1 as a corollary of Theorem 3.5. Now, our aim is to outline the main arguments of a direct proof of Theorem 4.1 in order to point out both the variational approach and the approximating arguments we use.

Thus, throughout this section let us assume that the hypotheses of Theorem 4.1 are satisfied and, as noted in Remark 4.2, from assumptions (V_1), $(\mathcal{H}_0)$ and (f_0)–(f_1) it follows that the weak (bounded) solutions of equation (1.1) are the critical points of the $\mathcal{C}^1$ functional $\mathcal{J} : X \to \mathbb{R}$ defined in (4.2), on the Banach space X as in (3.5), with Fréchet differential given by (4.3).

On the other hand, for each $k \in \mathbb{N}$ let us consider the open ball B_k with center 0 and radius k and define

$$X_k = X_{B_k} = H^1_{0,V}(B_k) \cap L^\infty(B_k)$$

endowed with the norm

$$\|u\|_{X_k} = \|u\|_{B_k,V} + |u|_{B_k,\infty} \quad \text{for any } u \in X_k$$

and with dual space X_k'. Actually, since any function $u \in X_k$ can be trivially extended to a function $\tilde{u} \in X$ just assuming $\tilde{u}(x) = 0$ for all $x \in \mathbb{R}^3 \setminus B_k$, then

$$\|\tilde{u}\| = \|u\|_{B_k}, \quad \|\tilde{u}\|_V = \|u\|_{B_k,V}, \quad |\tilde{u}|_\infty = |u|_{B_k,\infty}, \quad \|\tilde{u}\|_X = \|u\|_{X_k},$$

where the norms $|\nabla u|_{B_k,2}$, $\|u\|_{B_k}$ and $\|u\|_{B_k,V}$ are equivalent since B_k is bounded and (V_3) holds.

So, if we consider the restriction

$$\mathcal{J}_k : u \in X_k \mapsto \mathcal{J}_k(u) = \mathcal{J}|_{X_k}(u) \in \mathbb{R},$$

we obtain that

$$\mathcal{J}_k(u) = \frac{1}{2}\int_{B_k} g^2(u)|\nabla u|^2 dx - \int_{B_k} \tilde{F}(x,u)dx, \quad u \in X_k,$$

where we take

$$\tilde{F}(x,u) = F(x,u) - \frac{1}{2}V(x)|u|^2 \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{5.1}$$

We note that, in hypotheses (V_1), (V_3) and (f_0)–(f_3) the map

$$\tilde{f}(x,u) = f(x,u) - V(x)u \quad \text{for a.e. } x \in \mathbb{R}^3, \ \text{ all } u \in \mathbb{R},$$

is a Carathéodory function such that

$$|\tilde{f}(x,u)| \le (a_1 + |V|_{B_k,\infty})|u| + a_2|u|^{q-1} \quad \text{a.e. in } B_k, \text{ for all } u \in \mathbb{R},$$

$$\limsup_{u\to 0} \frac{\tilde{f}(x,u)}{u} \le \bar{\alpha} < \frac{1}{\tau_2^2} \text{ uniformly a.e. with respect to } x \in B_k,$$

and

$$0 < \mu\tilde{F}(x,u) \le \tilde{f}(x,u)u \quad \text{a.e. in } B_k, \text{ for all } u \ne 0.$$

Thus, functional $\mathcal{J}_k$ is $\mathcal{C}^1$ in X_k with Fréchet differential given by

$$\langle d\mathcal{J}_k(u), v\rangle = \int_{B_k} g^2(u)\nabla u \cdot \nabla v dx + \int_{B_k} g(u)g'(u)v|\nabla u|^2 dx - \int_{B_k} \tilde{f}(x,u)v dx$$

for any u, $v \in X_k$. Moreover, from (1.4), (4.1) and properties $(\mathcal{H}_0)$–$(\mathcal{H}_1)$ it satisfies the $(wCPS)$ condition in $\mathbb{R}$ (see [4, Proposition 3.4]).

Now, as in [10, Section 4], we claim that in our setting the geometrical assumptions required by Theorem 2.2 are satisfied but so to obtain some constants independent of the particular radius k of the bounded domain B_k.

In fact, ϱ, $\alpha^* > 0$ exist so that

$$u \in X, \ \|u\|_V = \varrho \quad \Longrightarrow \quad \mathcal{J}(u) \ge \alpha^* \tag{5.2}$$

(see [10, Proposition 4.8]) Furthermore, fixing $\bar{u} \in X$ such that

$$\operatorname{supp} \bar{u} \subset B_1 \quad \text{and} \quad |\Omega_1^{\bar{u}}| > 0, \quad \text{with} \quad \Omega_1^{\bar{u}} = \{x \in \mathbb{R}^3 : \ |\bar{u}(x)| > 1\},$$

it follows that

$$\mathcal{J}(s\bar{u}) \to -\infty \quad \text{ as } s \to +\infty$$

(see [10, Proposition 4.9]). Thus, $\bar{s} > 0$ exists so that

$$\|u^*\|_V > \varrho \quad \text{ and } \quad \mathcal{J}(u^*) < \alpha^*$$

with $u^* = \bar{s}\bar{u}$, $\operatorname{supp} u^* \subset B_1$ and ϱ, α^* as in (5.2).

Clearly, for all $k \geq 1$ we have that $\operatorname{supp} u^* \subset B_k$, so $u^* \in X_k$, and if we consider the segment joining 0 to u^*, namely

$$\gamma^* : s \in [0,1] \mapsto su^* \in X,$$

we obtain that $\operatorname{supp} \gamma^*(s) \subset B_k$ for all $s \in [0,1]$. Hence, $\gamma^*([0,1]) \subset X_k$ and then $\gamma^* \in \Gamma_k$, with

$$\Gamma_k = \{\gamma \in C([0,1], X_{B_k}) : \ \gamma(0) = 0, \ \gamma(1) = u^*\},$$

and, for the continuity of $\mathcal{J} \circ \gamma^* : s \in [0,1] \mapsto \mathcal{J}(su^*) \in \mathbb{R}$, $\alpha^{**} \in \mathbb{R}$ exists, independent of k, such that

$$\alpha^{**} = \max_{s \in [0,1]} \mathcal{J}(su^*). \tag{5.3}$$

Moreover, from (3.6) and (5.1) we obtain $\mathcal{J}_k(0) = 0$. Thus, Theorem 2.2 applies to $\mathcal{J}_k$ in X_k, and, for the arbitrariness of $k \in \mathbb{N}$, a sequence $(u_k)_k \subset X$ exists such that for each $k \in \mathbb{N}$ it results:

(*i*) $u_k \in X_k$ with $u_k = 0$ in $\mathbb{R}^3 \setminus B_k$,
(*ii*) $\alpha^* \leq \mathcal{J}(u_k) \leq \alpha^{**}$,
(*iii*) $\langle d\mathcal{J}(u_k), v\rangle = 0$ for all $v \in X_k$,

with α^* as in (5.2) and α^{**} as in (5.3), both independent of k.

Then, as in [10, Propositions 6.3 and 6.4], a positive constant M_0 exists such that

$$\|u_k\|_X \leq M_0 \quad \text{ for all } k \in \mathbb{N}.$$

Hence, a function $u_\infty \in H^1_V(\mathbb{R}^3)$ exists such that, up to subsequences, also from (3.3) we have that

$$\begin{aligned} u_k &\rightharpoonup u_\infty \quad \text{ in } H^1_V(\mathbb{R}^3), \\ u_k &\to u_\infty \quad \text{ strongly in } L^2(\mathbb{R}^3), \\ u_k &\to u_\infty \quad \text{ a.e. in } \mathbb{R}^3. \end{aligned}$$

Actually, $u_\infty \in L^\infty(\mathbb{R}^N)$, too (see [10, Proposition 6.5]); hence, by definition (3.5), we infer that $u_\infty \in X$.

Finally, as in [10, Propositions 6.8 and 6.9], we are able to prove that

$$u_k \to u_\infty \quad \text{ strongly in } H^1_V(B_R) \text{ for all } R \geq 1$$

and also

$$\langle d\mathcal{J}(u_\infty), \varphi\rangle = 0 \quad \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^3)$$

with $C_c^\infty(\mathbb{R}^3) = \{\varphi \in C^\infty(\mathbb{R}^3) : \ \operatorname{supp} \varphi \subset\subset \mathbb{R}^3\}$.
Hence, $d\mathcal{J}(u_\infty) = 0$ in X and, by using again the compact embedding (3.3) and arguing by contradiction as in the proof of [10, Theorem 4.4], we are able to prove that u_∞ is a nontrivial weak solution of equation (1.1).

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