

Unique ergodicity and weakly monotone Fock space

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Abstract. We show that the C^* -dynamical system given by the so-called weakly monotone C^* -algebra \mathcal{WM} , and the shift automorphism on it, is not uniquely ergodic. In fact, the fixed-point subalgebra with respect to such an action is trivial, whereas there are plenty of shift-invariant states. The last assertion is proved using a Hamel basis for the linear structure of a dense $*$ -algebra of \mathcal{WM} here exhibited.

Keywords: Non-commutative Dynamical Systems, Unique Ergodicity, Weakly Monotone Fock Spaces

1 Introduction

A dynamical system (Ω, T) , where Ω is a compact Hausdorff space and T a homeomorphism on Ω , is said to be uniquely ergodic if there exists a unique T -invariant Borel probability measure on Ω . By the Riesz-Markov Theorem, any such a measure is uniquely determined by a positive unit preserving linear functional (usually called state) on $C(\Omega)$, the complex-valued algebra of continuous functions on Ω . In addition, the map $\alpha : f \in C(\Omega) \mapsto f \circ T \in C(\Omega)$ uniquely defines a $*$ -automorphism on $C(\Omega)$ (the $*$ -map being the complex conjugation). This allows to rephrase the aforementioned unique ergodicity by the existence of a unique α -invariant state on $C(\Omega)$.

Passing to a more general setting given by a non-commutative unital C^* -algebra \mathcal{A} , and an abstract $*$ -automorphism α on \mathcal{A} , one naturally arrives at a non-commutative version of the ergodic property presented above. Furthermore, this condition can be generalised again by requiring the existence of a unique α -invariant conditional expectation from \mathcal{A} to the so-called fixed-point subalgebra. The latter is nothing else than the subalgebra collecting all the α -invariant elements of \mathcal{A} . When it just consists of multiples of the unit, this notion reduces to the usual unique ergodicity.

Here, we stress that some ergodic properties with respect to the action of a particular α , namely the shift automorphism, were studied in [9, 6], for some prominent examples coming from Quantum Probability, like q -deformed (where $|q| < 1$), monotone, and Boolean cases. One can also consider the action of the

permutation group, or the monoid of strictly increasing maps on \mathbb{Z} , obtaining de Finetti or Ryll-Nardzewski Theorems (see [1–5] and references cited therein).

As for other examples coming from Quantum Probability, one can deal with weakly monotone quantisation. Weakly monotone Fock spaces were first introduced by Wysoczanski in [12], where we refer the reader for a general presentation. In the last years the investigation of the vacuum distribution for sums of position operators in such a space has been realised in [7], where it was also shown that the basic operators of annihilation and creation are monotone independent in the vacuum. For a similar study in the monotone case, see [8].

The above considerations naturally lead to the investigation of ergodic states with respect to the action of the shift for the concrete C^* -algebra generated by weakly monotone annihilation operators. This is the main content of these notes. Indeed, after recalling some useful tools from the theory of C^* -dynamical systems based on a group in Section 2, in Section 3 we present the definitions of weakly monotone Fock space, and the basic operators on it. Furthermore, in analogy to what happens in the monotone case [4], we also exhibit a Hamel basis for the dense unital $*$ -algebra generated by annihilators. Without surprise, almost all the elements of the basis are Wick-ordered words, *i.e.* they are a sequence of creators on the left and annihilators on the right. In Section 4 we find a decomposition of the weakly monotone C^* -algebra: it is indeed the sum of an irreducible C^* -subalgebra added to the multiples of the identity. Taking into account that the shift automorphism acts on the C^* -algebra generated by the weakly monotone annihilators in a natural way, the decomposition allows us to find, in Proposition 4, that the fixed-point subalgebra with respect to the shift is trivial. Namely, it contains essentially the identity. Thus, one naturally wonders if the related C^* -dynamical system is uniquely ergodic. This is false: In Proposition 5 we show that the weak*-compact subset of shift-invariant states is indeed the convex combination of the vacuum state and the so-called state at infinity.

2 Preliminaries

This section is mainly devoted to recall some definitions on discrete C^* -dynamical systems.

First, for a C^* -algebra \mathcal{A} , and a positive functional φ on \mathcal{A} , we denote by $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ the GNS representation.

A C^* -dynamical system based on a group G is given by the triplet (\mathcal{A}, G, α) , where \mathcal{A} is a C^* -algebra which will be ever supposed unital, with unit $1_{\mathcal{A}}$, and α is a group homomorphism

$$\alpha : g \in G \mapsto \alpha_g \in \text{Aut}(\mathcal{A}),$$

$\text{Aut}(\mathcal{A})$ being the group of $*$ -automorphisms on \mathcal{A} . As α_g is multiplicative for any $g \in G$, the closed space

$$\mathcal{A}_G := \{a \in \mathcal{A} \mid \alpha_g(a) = a, g \in G\}$$

is indeed a $*$ -subalgebra, called the fixed-point subalgebra.

A state $\varphi \in \mathcal{S}(\mathcal{A})$ is called G -invariant if $\varphi \circ \alpha_g = \varphi$ for any $g \in G$. Denote by $\mathcal{S}_G(\mathcal{A})$ the set of G -invariant states, which is convex and weak*-compact. Its extremal points are said to be ergodic.

A discrete C^* -dynamical system is given by the pair (\mathcal{A}, α) , where α is a single automorphism on \mathcal{A} . In this case, if $\mathcal{S}_\alpha(\mathcal{A}) = \{\omega\}$, then the C^* -dynamical system (\mathcal{A}, α) is called *uniquely ergodic*. This condition is equivalent to the following convergence in norm

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \alpha^k(a) = \omega(a)1_{\mathcal{A}}, \quad a \in \mathcal{A}$$

or to the following weak limit of averages

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(\alpha^k(a)) = \omega(a)f(1_{\mathcal{A}}), \quad a \in \mathcal{A}, f \in \mathcal{A}^*.$$

In this case, one naturally finds that the fixed-point subalgebra is trivial, namely $\mathcal{A}_\alpha = \mathbb{C}1_{\mathcal{A}}$. When the structure of \mathcal{A}_α is richer, one naturally wonders if, for any state in \mathcal{A}_α there exists a unique α -invariant extension to \mathcal{A} . This condition, which reduces to the usual notion when $\mathcal{A}_\alpha = \mathbb{C}1_{\mathcal{A}}$, by Theorem 2.1 in [10] is seen to be equivalent to the existence of a unique invariant conditional expectation¹ $E_\alpha : \mathcal{A} \rightarrow \mathcal{A}_\alpha$ given by

$$E_\alpha(a) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \alpha^k(a), \quad a \in \mathcal{A}.$$

In such a case the C^* -dynamical system (\mathcal{A}, α) is called *uniquely ergodic with respect to its fixed-point subalgebra*.

In these notes we deal with invariant states on a suitable C^* -algebra with respect to the group generated by the one-step shift $\tau(i) := i + 1$ on \mathbb{Z} , which is canonically isomorphic to \mathbb{Z} itself.

3 Weakly monotone Fock space

For any $k \geq 1$, let $I_k := \{(i_1, i_2, \dots, i_k) \mid i_1 \leq i_2 \leq \dots \leq i_k, i_j \in \mathbb{Z}\}$. When $k = 0$, we take $I_0 := \{\emptyset\}$, where \emptyset is the empty sequence. If $\mathcal{H}_k := \ell^2(I_k)$, where

¹ By a theorem of Tomiyama [11], any conditional expectation from a C^* -algebra \mathcal{A} to its C^* -subalgebra \mathcal{B} is a norm 1 projection.

\mathcal{H}_0 is identified with the scalar field \mathbb{C} , the weakly monotone Fock space is the Hilbert space $\mathcal{F}_{wm} := \bigoplus_{k=0}^{\infty} \mathcal{H}_k$. The sequence $\Omega := (1, 0, 0, \dots)$ in \mathcal{F}_{wm} is the Fock vacuum vector.

If $\beta = (i_1, i_2, \dots, i_k) \in I_k$, we denote by $e_{(\beta)}$ the generic element of the canonical basis of \mathcal{F}_{wm} , namely $e_{(\beta)} := e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}$.

The weakly monotone annihilation and creation operators are respectively firstly defined, for $i \in \mathbb{Z}$ and β as above with $k \neq 0$, by

$$a_i \Omega = 0, \quad a_i e_{(\beta)} = \delta_{i, i_1} e_{i_2} \otimes \dots \otimes e_{i_k},$$

where $\delta_{i,j}$ is the Kronecker symbol, and

$$a_i^\dagger \Omega = e_i, \quad a_i^\dagger e_{(\beta)} = \alpha_{i, i_1} e_i \otimes e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k},$$

where $\alpha_{j,k} = 1$ if $j \leq k$, and 0 otherwise.

They can be extended by linearity and continuity to the whole \mathcal{F}_{wm} , are adjoint to one another, and of norm one. Furthermore, they satisfy the following relations

$$a_i^\dagger a_j^\dagger = a_j a_i = 0 \quad \text{if } i > j, \quad (1)$$

$$a_i a_j^\dagger = 0 \quad \text{if } i \neq j. \quad (2)$$

$$a_i a_i^\dagger = I - \sum_{k < i} a_k^\dagger a_k, \quad (3)$$

where I is the identity operator on \mathcal{F}_{wm} , and (3) is understood in the strong operator topology, see [6], Proposition 3.2.

As it will be useful in the sequel, we report here Lemma 2.1 from [7].

Lemma 1. *For any $k, j \geq 1$, one has*

$$a_k a_j a_j^\dagger = \alpha_{j,k} a_k, \quad a_j a_j^\dagger a_k^\dagger = \alpha_{j,k} a_k^\dagger. \quad (4)$$

Moreover, for $j \leq k$

$$a_j a_j^\dagger a_k = a_k, \quad a_k^\dagger a_j a_j^\dagger = a_k^\dagger, \quad (5)$$

where $\alpha_{j,k} = 1$ if $k \geq j$, and 0 otherwise.

Suppose now that $m, n \geq 1$, and $i \in \mathbb{Z}$. Then, (2) and (3) give

$$a_i^m a_i^{\dagger n} = \begin{cases} a_i^{\dagger n-m} & \text{if } n > m \\ a_i^{m-n} & \text{if } n < m \\ a_i a_i^\dagger & \text{if } n = m \end{cases} \quad (6)$$

In addition,

$$a_i^\dagger a_i = a_i a_i^\dagger - a_{i+1} a_{i+1}^\dagger. \quad (7)$$

We denote by \mathcal{WM} and \mathcal{WM}_o the concrete unital C^* -algebra, together with its dense unital $*$ -algebra generated by the annihilators $\{a_i \mid i \in \mathbb{Z}\}$ in the weakly

monotone Fock space. They are both contained in $\mathcal{B}(\mathcal{F}_{wm})$.

In analogy to the monotone case [6, 4], we say that $X \in \mathcal{WM}_o$ has a $w\lambda$ -form if

$$X = a_{i_1}^{\dagger m_1} \cdots a_{i_h}^{\dagger m_h} a_{j_1}^{n_1} \cdots a_{j_k}^{n_k},$$

for all integers $h, k \geq 0$, $m_1, \dots, m_h, n_1, \dots, n_k \in \mathbb{N}$, and finally $i_1 < \cdots < i_h$, $j_1 > \cdots > j_k$. The length of X is $h + k$. When $h = k = 0$, then $X = I$. It is called the empty word, and its length is zero.

Note that every $w\lambda$ -form is reduced. Indeed, if $X_1 := a_{i_1}^{\dagger m_1} \cdots a_{i_h}^{\dagger m_h} a_{j_1}^{n_1} \cdots a_{j_k}^{n_k}$ and $X_2 := a_{p_1}^{\dagger l_1} \cdots a_{p_r}^{\dagger l_r} a_{t_1}^{s_1} \cdots a_{t_v}^{s_v}$ are equal, where $h, k, r, v \geq 0$, $i_1 < \cdots < i_h, j_1 > \cdots > j_k, p_1 < \cdots < p_r, t_1 > \cdots > t_v, m_1, \dots, m_h, n_1, \dots, n_k, l_1, \dots, l_r, s_1, \dots, s_v \in \mathbb{N}$ the identity

$$X_1(e_{j_k}^{n_k} \otimes \cdots \otimes e_{j_1}^{n_1}) = X_2(e_{j_k}^{n_k} \otimes \cdots \otimes e_{j_1}^{n_1})$$

gives that $k = v, j_u = t_u, n_u = s_u$ for each $u = 1, \dots, k$, and moreover $h = r, i_u = p_u$, and finally $m_u = l_u$ for each $u = 1, \dots, h$.

Define the index-set Λ such that $\{X_\lambda\}_{\lambda \in \Lambda}$ gives the set of all the $w\lambda$ -forms. As any word of type $a_i^\dagger a_i$ is identified with $i \in \mathbb{Z}$, in a natural way one has $\mathbb{Z} \subset \Lambda$, and we denote $\Gamma := \Lambda \setminus \mathbb{Z}$.

Since by (3) for each $i \in \mathbb{Z}$, $a_i a_i^\dagger$ is the orthogonal projection onto

$$\mathcal{F}_{wm}^{\leq i} := \overline{\text{span}\{e_{l_1} \otimes \cdots \otimes e_{l_m} \mid i \leq l_1, l_1 \leq \cdots \leq l_m, m \geq 1\}},$$

the word $a_i a_i^\dagger$ is reduced. The $w\lambda$ -forms in Γ , as well as these projections, are prominent in the linear structure of \mathcal{WM}_o , as the following proposition shows.

Proposition 1. *The family $L := \{X_\lambda\}_{\lambda \in \Gamma} \cup \{a_i a_i^\dagger\}_{i \in \mathbb{Z}}$ is a basis for \mathcal{WM}_o .*

Proof. First, we show that any non-null operator in \mathcal{WM}_o is a finite linear combination of words in $\{X_\lambda\}_{\lambda \in \Gamma} \cup \{a_i a_i^\dagger\}_{i \in \mathbb{Z}}$. To this goal, take $X := a_{i_1}^{\dagger h_1} \cdots a_{i_n}^{\dagger h_n}$, $n \in \mathbb{N}$, $i_1, \dots, i_n \in \mathbb{Z}$, $h_1, \dots, h_n \in \mathbb{N}$, $\dagger \in \{1, \dagger\}$.

When the length of X is at most 2, the identities (1), (6) and (7) give that X can be reduced to a sum of elements in L .

Denote now $X = X_1 a_{i_n}^{\dagger h_n}$, X_1 having length at least 2, and suppose by induction that X_1 belongs to the family above. If $X_1 = a_i a_i^\dagger$, $i \in \mathbb{Z}$, and $\dagger = \dagger$, then $X = 0$ when $i > i_n$, or by (4) $X = a_{i_n}^{\dagger h_n}$ if $i \leq i_n$. If $\dagger = 1$, then by (1) and (3), $X = a_{i_n}^{h_n} - \sum_{i_n \leq k < i} a_k^\dagger a_k a_{i_n}^{h_n}$ when $i > i_n$, or by (5) $X = a_{i_n}^{h_n}$ when $i \leq i_n$.

Suppose now that X_1 is a $w\lambda$ -form, namely $X_1 = a_{i_1}^{\dagger m_1} \cdots a_{i_h}^{\dagger m_h} a_{j_1}^{n_1} \cdots a_{j_k}^{n_k}$. If $k > 0$, then X is null when $j_k < i_n$. When $j_k \geq i_n$, (4), (5) and (6) give that X is a $w\lambda$ -form in the case it does not vanish. Finally, for $k = 0$, one sees that X is a $w\lambda$ -form, except the vanishing case given by $\dagger = \dagger$, and $i_n < i_h$.

Thus, we have shown that the word in L , which are reduced, generate the whole \mathcal{WM}_o . Therefore, to complete the proof, it remains to prove that the set $\{X_\lambda\}_{\lambda \in \Gamma} \cup \{a_i a_i^\dagger\}_{i \in \mathbb{Z}}$ is linearly independent in \mathcal{WM}_o . This is essentially obtained by rephrasing the arguments presented in the proof of [4], Lemma 3.2. \square

4 Shift-invariant states on weakly monotone C^* -algebra

In this section we present the convex set of shift-invariant states on the \mathcal{WM} , and show that there is no unique ergodicity for the dynamical system based on the shift automorphism.

Denote $\mathcal{W}_o := \text{span}\{X \in \mathbb{L} \mid l(X) > 0\}$, and take $\mathcal{W} := \overline{\mathcal{W}_o}$. We first show that $I \notin \mathcal{W}$.

Proposition 2. *The identity $I \in \mathcal{B}(\mathcal{F}_{wm})$ does not belong to \mathcal{W} .*

Proof. Let X be an element of \mathcal{W}_o . Thus, $X = \sum_{j \in F} \beta_j X_j$, where F is a finite set, any word X_j belongs to \mathbb{L} , and $l(X_j) > 0$. After writing $X_j = Y_j a_{l_j}^{\natural}$, $\natural \in \{1, \dagger\}$, we take an arbitrary unit vector $e_n \in \mathcal{F}_{wm}$ such that $n < \min l_j$. This gives that, for any $\alpha \in \mathbb{C}$

$$\|X + \alpha I\| \geq \|(X + \alpha I)e_n\| = |\alpha|.$$

The thesis then follows by a standard approximation argument. \square

Proposition 3. *The C^* -algebra \mathcal{W} is irreducible.*

Proof. First note that $\sum_{k \in \mathbb{Z}} a_k^\dagger a_k$ vanishes when applied to Ω , and leaves invariant any vector $\xi := e_{(\beta)}$, $\beta \in I_k$ for some $k > 0$. Since the annihilators $(a_k)_{k \in \mathbb{Z}}$ have orthogonal ranges, it follows that $\sum_{k \in \mathbb{Z}} a_k^\dagger a_k$ converges to $I - P_\Omega$ in the strong topology, P_Ω being the projection onto the space generated by the vacuum vector Ω . Thus, if $\omega := \langle \cdot, \Omega \rangle$, P_Ω belongs to $\mathcal{W}'' := \pi_\omega(\mathcal{W})''$, and then $B\Omega \in \mathbb{C}\Omega$ for any $B \in \mathcal{W}'$. Since Ω is separating for \mathcal{W}' , the thesis follows. \square

As a consequence of the previous proposition, one has the decomposition

$$\mathcal{WM} = \mathcal{W} \oplus \mathbb{C}I. \quad (8)$$

In what follows we investigate ergodic properties of a discrete dynamical system. To this goal, consider the unitary U_τ on \mathcal{F}_{wm} given by

$$U_\tau e_{(\beta)} := e_{i_1+1} \otimes e_{i_2+1} \cdots \otimes e_{i_n+1}, \quad U_\tau \Omega = \Omega$$

where $\beta = (i_1, \dots, i_n) \in I_n$ for some $n > 0$. The map α on \mathcal{WM} unitarily implemented by U_τ , namely the automorphism α defined by $\alpha(X) := U_\tau X U_\tau^*$, $X \in \mathcal{WM}$, gives rise to the discrete C^* -dynamical system (\mathcal{WM}, α) , and from now on we refer to α as the shift automorphism.

Proposition 4. *The fixed-point subalgebra of \mathcal{WM} for the action of the shift automorphism is trivial, namely $\mathcal{WM}_\alpha = \mathbb{C}I$.*

Proof. As $\alpha(\mathcal{W}) \subseteq \mathcal{W}$, we fix $X \in \mathcal{W}$ and show that $\alpha(X) \neq X$. Indeed, for any arbitrary non-null $X_o \in \mathcal{W}_o$, there exists a non decreasing sequence $\beta = (i_1, i_2, \dots, i_j) \in I_j$ for some $j > 0$, for which $X_o e_{(\beta)} \neq 0$. As $\alpha^k(X_o)$ vanishes on $e_{(\beta)}$ for sufficiently large $k \in \mathbb{N}$, it follows that $\alpha(X_o) \neq X_o$.

Take now a sequence $(X_n)_{n \geq 1}$ in \mathcal{W}_o converging to X in the uniform topology. The argument above shows that, for any n , there exists $\varepsilon_n > 0$ such that $\|\alpha(X_n) - X_n\| = \varepsilon_n$. Now denote $\varepsilon := \inf_n \varepsilon_n$, and suppose that $\alpha(X) = X$. As α is contractive, for m large enough the inequalities $\|\alpha(X_m - X)\| < \frac{\varepsilon}{2}$, and $\|X_m - X\| < \frac{\varepsilon}{2}$ hold. Hence one finds the following contradiction

$$\varepsilon \leq \|\alpha(X_m) - X_m\| \leq \|\alpha(X_m - X)\| + \|\alpha(X) - X\| + \|X - X_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

By Proposition 3, the C^* -algebra \mathcal{W} does not have an identity, and by (8) it follows that the weak*-compact set of all states $\mathcal{S}(\mathcal{WM})$ on the weakly monotone C^* -algebra is the one-point compactification of positive functionals on \mathcal{W} with norm not greater than one. The shift-invariant convex subset of $\mathcal{S}(\mathcal{WM})$ contains at least two states. One is the vacuum state ω , which is as usual given by

$$\omega(X) := \langle X\Omega, \Omega \rangle, \quad X \in \mathcal{WM}.$$

The other one is the so-called "state at infinity" ω_∞ , namely

$$\omega_\infty(A + cI) := c \quad A \in \mathcal{W}, \quad c \in \mathbb{C}.$$

Gluing these considerations with Proposition 4, one figures out that (\mathcal{WM}, α) is not uniquely ergodic. The next result shows that the two states above are exactly the only ergodic shift-invariant.

Proposition 5. *The weak*-compact convex of shift-invariant states $\mathcal{S}_{ZZ}(\mathcal{WM})$ of \mathcal{WM} is*

$$\mathcal{S}_{ZZ}(\mathcal{WM}) = \{\gamma\omega + (1 - \gamma)\omega_\infty \mid \gamma \in [0, 1]\}.$$

Proof. We first take an arbitrary element X in the dense subalgebra $\mathcal{W}_o \oplus \mathbb{C}I$. By Proposition 1, one has

$$X = cI + Y + \sum_{j \in F} \beta_j a_j a_j^\dagger.$$

where $c \in \mathbb{C}$, Y is a finite linear combination of $w\lambda$ -forms in \mathbb{L} with non-null length, F is a finite set, and finally β_j is a complex scalar. One easily finds that

$$\omega(X) = c + \sum_{j \in F} \beta_j, \quad \omega_\infty(X) = c. \quad (9)$$

We further observe that for any family of integers i_1, \dots, i_n , and vectors $(\eta_j)_{j=1}^n$ in \mathcal{H}_k , by (2) it follows that

$$\left\| \sum_{j=1}^n a_{i_j}^\dagger \eta_j \right\|^2 = \sum_{j,l=1}^n \langle a_{i_l} a_{i_j}^\dagger \eta_j, \eta_l \rangle = \sum_{j=1}^n \|a_{i_j}^\dagger \eta_j\|^2 \leq n \max_{j=1, \dots, n} \|\eta_j\|^2, \quad (10)$$

as the creators have unit norm.

Fix now the operator $a_{i_1}^\dagger a_{i_2}^\natural \cdots a_{i_m}^\natural$, where $i_1, \dots, i_m \in \mathbb{Z}$, $m \in \mathbb{N}$ and $\natural \in \{1, \dagger\}$. If l is the number of annihilators in $a_{i_1}^\dagger a_{i_2}^\natural \cdots a_{i_m}^\natural$, choose a vector $\eta \in \mathcal{H}_k$, where $\|\eta\| \leq 1$ and $k \geq l$. As $\langle e_{i+j}, e_{i+l} \rangle = \delta_{j,l}$, one has

$$\left\| \sum_{h=1}^n \alpha^{k_h} (a_{i_1}^\dagger a_{i_2}^\natural \cdots a_{i_m}^\natural) \eta \right\| = \left\| \sum_{h=1}^n a_{i_1+k_h}^\dagger a_{i_2+k_h}^\natural \cdots a_{i_m+k_h}^\natural \eta \right\| = \left\| \sum_{h=1}^n a_{i_1+k_h}^\dagger \eta_h \right\|,$$

where $\eta_h := a_{i_2+k_h}^\natural \cdots a_{i_m+k_h}^\natural \eta$. As $\|\eta_h\| \leq 1$ for any h , by (10) the last term above is less than or equal to \sqrt{n} . Likewise, by taking the adjoint, one achieves that $\left\| \sum_{h=1}^n \alpha^{k_h} (a_{i_1}^\natural \cdots a_{i_{m-1}}^\natural a_{i_m}) \eta \right\| \leq \sqrt{n}$. Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(Y) = 0$$

in norm. If φ is a shift-invariant state on \mathcal{WM} , then

$$\varphi(Y) = \varphi\left(\frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(Y)\right) = \lim_{n \rightarrow \infty} \varphi\left(\frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(Y)\right) = \varphi\left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(Y)\right) = 0.$$

In addition, by invariance, $\varphi(a_i a_i^\dagger)$ is constant for any i , and we denote by γ this common value. Hence, by (9)

$$\varphi(X) = c + \gamma \sum_{j \in F} \beta_j = \gamma \omega(X) + (1 - \gamma) \omega_\infty(X).$$

The thesis follows by a standard density argument. \square

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