On the classification of contact metric (k, μ)–spaces via tangent hyperquadric bundles*

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Abstract

We classify locally the contact metric (k, μ)–spaces whose Boeckx invariant is $\leqslant -1$ as tangent hyperquadric bundles of Lorentzian space forms.

Mathematics Subject Classification (2000): 53C35, 53C25.
Keywords and phrases: contact metric (k, μ)-space, Boeckx invariant, symmetric CR manifold, tangent hyperquadric bundle.

1 Introduction

Contact (k, μ)–spaces constitute a relevant class of objects studied in contact metric geometry; these spaces were introduced by Blair, Koufogiorgos and Papantoniou in [2] as a generalization of Sasakian manifolds. Indeed, a *contact metric* (k, μ) *manifold* is a contact metric manifold $(M, \varphi, \xi, \eta, g)$ such that

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \quad (1)$$

where $X, Y \in \mathfrak{X}(M)$, k, μ are real numbers and $h := \frac{1}{2}\mathcal{L}_\xi\varphi$. Here $\mathcal{L}_\xi\varphi$ denotes the Lie derivative of φ in the direction of ξ. Recall that Sasakian manifolds are characterized by the above equation with $k = 1$ and $h = 0$.

Looking at contact metric manifolds as strongly pseudo-convex (almost) CR manifolds, Dileo and Lotta showed that the (k, μ)–condition is equivalent to the local CR-symmetry with respect to the Webster metric g (see [9] and section 2.2). In this context, another characterization was given by Boeckx and Cho and in terms of the parallelism of the Tanaka–Webster curvature and torsion [5]. Another link with CR geometry is provided by a recent result of Cho and Inoguchi stating that every orientable contact metric real hypersurface of a non-flat complex space form is a (k, μ) space [8].

Boeckx gave a crucial contribution to the problem of classifying these manifolds; after showing that every non-Sasakian contact (k, μ)-space is locally homogeneous and strongly locally φ–symmetric [3], in [4] he defined a scalar invariant

*Preprint version. Published source: Math. Nachr. **291** (2018), no. 11-12, 1851–1858.

I_M which completely determines a contact (k,μ)–space M locally up to equivalence and up to a D-homotetic deformation of its contact metric structure.

A standard example is the tangent sphere bundle T_1M of a Riemannian manifold M with constant sectional curvature $c \neq 1$. Being an hypersurface of TM, which is equipped with a natural strictly almost Kähler structure (J,G), where G is the Sasaki metric, T_1M inherits a standard contact metric structure (for details, see for instance [1]). In particular, the Webster metric g of T_1M is a scalar multiple of G. The corresponding Boeckx invariant is given by:

$$I_{T_1M} = \frac{1+c}{|1-c|}.$$

Hence, as c varies in $\mathbb{R} \setminus \{1\}$, I_{T_1M} assumes all the real values strictly greater than -1.

The case $I \leqslant -1$ seems to lead to models of different nature. Namely, Boeckx found examples of contact metric (k,μ) spaces, for every value of the invariant $I \leqslant -1$, namely a two parameter family of Lie groups with a left–invariant contact metric structure. However, he gave no geometric description of these examples.

The purpose of this paper is to show that, actually, one can construct the models with $I \leqslant -1$ simply by replacing a Riemannian space form (M,g) with a Lorentzian one, taking instead of T_1M the so-called tangent hyperquadric bundle:

$$T_{-1}M = \{(p,v) \in TM : \quad g_p(v,v) = -1\}.$$

Indeed, the formula for the Boeckx invariant changes as follows:

$$I_{T_{-1}M} = \frac{c-1}{|c+1|},$$

where c varies in $\mathbb{R} \setminus \{-1\}$, so that for $c \leqslant 0$, these examples cover all possible values of the Boeckx invariant in $(-\infty,-1]$.

We remark that, as in the Riemannian case, $T_{-1}M$ is again a strictly pseudoconvex CR hypersurface of (TM,J) (see also [14] for a recent study of these manifolds from the point of view of CR geometry). However, in this case the Webster metric g is no longer a scalar multiple of the (semi-Riemannian) Sasaki metric of TM.

2 Preliminaries

2.1 Contact metric (k,μ) manifolds

In this section we recall some basic results concerning the class of contact metric manifolds under consideration. As a general reference on contact metric geometry, we refer the reader to Blair's book [1].

Theorem ([2]). *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (k, μ) manifold. Then necessarily $k \leqslant 1$. Moreover, if $k = 1$ then $h = 0$ and $(M, \varphi, \xi, \eta, g)$ is Sasakian. If $k < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions $\mathcal{D}(0)$, $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ corresponding to the eigenspaces of h, where $\lambda = \sqrt{1-k}$.*

Non-Sasakian contact metric (k, μ) manifolds was completely classified by Boeckx in [4]. We have that $k < 1$ and the real number

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-k}},$$

is an invariant for the (k, μ) structure; moreover:

Theorem ([4]). *Let $(M_i, \varphi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, be two non-Sasakian (k_i, μ_i)-spaces of the same dimension. Then $I_{M_1} = I_{M_2}$ if and only if, up to a D-homothetic transformation, the two spaces are locally isometric as contact metric spaces. In particular, if both spaces are simply connected and complete, they are globally isometric up to a D-homothetic transformation.*

Finally we recall how the Boeckx invariant I_M of a non-Sasakian (k, μ) manifold is linked with the behavior of the Pang invariants of the Legendre foliations determined by $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$. The Pang invariant of a Legendre foliation $\mathcal{F}$ on a contact manifold (M, η) is the symmetric tensor:

$$\varPi_{\mathcal{F}}(X, Y) := -(\mathcal{L}_X \mathcal{L}_Y \eta)(\xi) = 2\mathrm{d}\eta([\xi, X], Y), \tag{2}$$

where X, Y are vectors fields tangent to $\mathcal{F}$ (cf. [13]). The Legendre foliation $\mathcal{F}$ is called positive, negative or flat according to the circumstance that the bilinear form $\varPi_{\mathcal{F}}$ is positive definite, negative definite or vanishes identically, respectively.

In our case the explicit expressions of $\varPi_{\mathcal{D}(\lambda)}$ and $\varPi_{\mathcal{D}(-\lambda)}$ are (see [1, p. 127] or [7]):

$$\varPi_{\mathcal{D}(\lambda)} = \frac{(\lambda+1)^2 - k - \mu\lambda}{\lambda} g_\eta|_{\mathcal{D}(\lambda)\times\mathcal{D}(\lambda)}, \tag{3}$$

$$\varPi_{\mathcal{D}(-\lambda)} = \frac{-(\lambda-1)^2 + k - \mu\lambda}{\lambda} g_\eta|_{\mathcal{D}(-\lambda)\times\mathcal{D}(-\lambda)}. \tag{4}$$

Using the previous equations one gets (see [6]):

Theorem. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian contact metric (k, μ) manifold. Then one of the following must hold:*

(a) both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are positive definite;

(b) $\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is negative defined;

(c) both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are negative definite;

(d) $\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is flat;

(e) $\mathcal{D}(\lambda)$ is flat and $\mathcal{D}(-\lambda)$ is negative defined.

Furthermore, M belongs to the class (a), (b), (c), (d), (e) if and only if $I_M > 1$, $-1 < I_M < -1$, $I_M < -1$, $I_M = 1$, $I_M = -1$, respectively.

2.2 Locally symmetric pseudo-Hermitian manifolds

Let M^{n+k} be a smooth manifold. A *partial complex structure* of CR-codimension k is a pair (HM, J) where HM is a smooth real subbundle of real dimension $2n$ of the tangent bundle TM, and J is a smooth bundle isomorphism $J : HM \to HM$ such that $J^2 = -I$.

An *almost CR structure* on M is a partial complex structure (HM, J) on M satisfying:

$$[X, Y] - [JX, JY] \in \mathcal{H}\ ,$$

for every $X, Y \in \mathcal{H}$, where $\mathcal{H}$ denotes the module of all the smooth sections of HM. If moreover the following equation

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0, \tag{5}$$

holds for every $X, Y \in \mathcal{H}$, then (HM, J) is said to be a *CR structure* on M, and (M, HM, J) is a *CR manifold*.

If (HM, J) is an almost CR structure of CR-codimension 1 on a orientable manifold M, one can also represent HM as

$$HM = \ker(\eta),$$

where η is a globally defined nowhere vanishing one form. If, for some choice of η, the corresponding *Levi form*

$$L_\eta(X, Y) := -\mathrm{d}\eta(X, JY), \quad X, Y \in \mathcal{H}$$

is positive definite, then the almost CR structure (HM, J) is a called *strongly pseudo-convex*, and we shall refer to (M, HM, J, η) as a *strongly pseudo-convex almost CR manifold*. When the underlying almost CR structure is also integrable, M is usually termed a *pseudo-Hermitian manifold*. See for example [11] for more information on this subject.

It is well known that any strongly pseudo-convex structure almost CR structure (HM, J, η) on a manifold M canonically determines a contact metric structure $(\varphi, \xi, \eta, g_\eta)$. Indeed, η is a contact form, so that there exists a unique nowhere vanishing globally defined vector field ξ transverse to HM (the Reeb vector field), such that

$$\eta(\xi) = 1, \quad \mathrm{d}\eta(\xi, X) = 0$$

for every $X \in \mathfrak{X}(M)$. The Levi form L_η and the bundle isomorphism J can be canonically extended respectively to a Riemannian metric g_η, called the *Webster metric*, and to a $(1,1)$-tensor field φ:

$$g_\eta(X, Y) := L_\eta(X, Y), \quad g_\eta(X, \xi) = 0, \quad g_\eta(\xi, \xi) = 1,$$

$$\varphi X := JX, \quad \varphi\xi := 0,$$

where $X, Y \in \mathcal{H}$. One can check that $(\varphi, \xi, \eta, g_\eta)$ is a contact metric structure on M in the sense of [1].

Conversely, if (φ, ξ, η, g) is a contact metric structure on M, then setting

$$HM := \mathrm{Im}(\varphi), \quad J := \varphi|_{HM},$$

one gets a strongly pseudo-convex almost CR structure, whose Webster metric g_η coincides with g.

Let (M, HM, J, η) be a strongly pseudo-convex almost CR manifold. A CR *symmetry* at a point $p \in M$ is a CR diffeomorphism

$$\sigma : M \to M,$$

which is also an isometry with respect to the Webster metric g_η, and such that

$$(\mathrm{d}\sigma)_p|_{H_pM} = -Id_{H_pM}.$$

If M admits a CR symmetry at p for every $p \in M$, then M will be called a *symmetric pseudo-Hermitian manifold*. Since the symmetry at p in uniquely determined, it makes sense also to define *locally symmetric* pseudo-Hermitian manifolds in a natural manner. Observe that, since the local CR symmetries are CR maps, for these manifolds the integrability condition (5) is automatically satisfied (see [9]).

Finally, we recall that it was showed in [9, Theorem 3.2] that a non Sasakian contact metric manifold satisfies the (k, μ) condition (1) if and only if it is a locally symmetric pseudo-Hermitian manifold.

2.3 Tangent bundles and tangent hyperquadric bundles

Here we recall some notions and properties concerning the tangent bundle of a manifold. The definition and some properties of the tangent hyperquadric bundle of a Lorentzian manifold will be also recalled.

Let M be a smooth manifold. The *vertical lift* X^V of a vector field X on M, is the vector field on TM defined by

$$X^V\omega = \omega(X) \circ \pi,$$

where ω is any 1-form on M and $\pi : TM \to M$ is the canonical projection. Furthermore, if D is an affine connection on M, the *horizontal lift* of X with respect to D, is defined by

$$X^H\omega = D_X\omega,$$

where ω is any 1-form on M. The local expression of X^H with respect to the local coordinates system (q^i, v^i) on TM associated to a local system of coordinates (x^i) on M is:

$$X^H = X^i \frac{\partial}{\partial q^i} - X^i v^j \Gamma^k_{ij} \frac{\partial}{\partial v^k}. \tag{6}$$

We denote by H_t and V_t the span of the horizontal and vertical lifts at $t \in TM$ respectively. We have that:

$$T_t(TM) = H_t \oplus V_t.$$

The *canonical vertical vector field* $\mathcal{N}$ on TM and the *geodesic flow* ζ on TM are defined by:

$$\mathcal{N}_t = u_t^V, \quad \zeta_t = u_t^H, \quad t = (p, u) \in TM.$$

The tangent bundle of an affine manifold (M, D) admits a canonical almost complex structure $\tilde{J} : TTM \to TTM$ such that:

$$\tilde{J} X^H = X^V, \quad \tilde{J} X^V = -X^H, \quad X \in \mathfrak{X}(M).$$

Observe that:

$$\tilde{J}\mathcal{N} = -\zeta, \quad \tilde{J}\zeta = \mathcal{N}.$$

For the Lie brackets between horizontal and vertical lifts of the vector fields X, Y on M, the following formulas hold (see [1]):

$$[X^V, Y^V] = 0, \quad [X^H, Y^V] = (D_X Y)^V,$$
$$[X^H, Y^H]_t = [X, Y]_t^H - (R(X, Y)u)_t^H, \tag{7}$$

where R denotes the curvature tensor of D on M.

In all that follows, we consider a Lorentzian manifold (M, g). The *Sasaki metric* $\tilde{G}$ on TM is defined according to:

$$\tilde{G}_t(X^H, Y^H) = g_p(X_p, Y_p), \quad \tilde{G}_t(X^V, Y^V) = g_p(X_p, Y_p), \quad \tilde{G}_t(X^H, Y^V) = 0,$$

where $X, Y \in \mathfrak{X}(M)$, $t = (p, u) \in TM$, and X^H, Y^H are the horizontal lifts of X, Y with respect to the Levi-Civita connection of g. Observe that the Sasaki metric $\tilde{G}$ has index 2 (see [10] for more details). It is known that the 1-form on TM:

$$\beta_t(\tilde{X}_t) := \tilde{G}_t(\tilde{X}_t, u_t^H) = g_p(\pi_\star \tilde{X}, u), \quad \tilde{X} \in \mathfrak{X}(TM),\ t = (p, u) \in TM,$$

satisfies

$$2\mathrm{d}\beta(\tilde{X}, \tilde{Y}) = \tilde{G}(\tilde{X}, \tilde{J}\tilde{Y}), \tag{8}$$

for every $\tilde{X}, \tilde{Y} \in \mathfrak{X}(TM)$ (see for instance [1, p. 171] or [10]), so that $(TM, \tilde{J}, \tilde{G})$ is an indefinite almost Kähler manifold.

Now we consider the tangent hyperquadric bundle

$$T_{-1}M := \{(p, u) \in TM \mid g_p(u, u) = -1\},$$

which is an orientable hypersurface of TM, being $\mathcal{N}$ a unit normal vector field to $T_{-1}M$. We have that:

$$\tilde{G}_t(\mathcal{N}_t, \mathcal{N}_t) = -1, \quad H_t \subset T_t(T_{-1}M),$$

$$T_t(T_{-1}M) = \{X_t^H + Y_t^V \mid X, Y \in T_pM,\ g_p(Y, u) = 0\},$$

for every $t = (p, u) \in T_{-1}M$. Being a hypersurface of $(TM, \tilde{J})$, $T_{-1}M$ inherits a canonical partial complex structure $(H(T_{-1}M), J)$, where

$$H(T_{-1}M) := \{X \in T(T_{-1}M) \mid \tilde{J}X \in T(T_{-1}M)\},$$

and

$$J : H(T_{-1}M) \to H(T_{-1}M),$$

is the restriction of the almost complex structure $\tilde{J}$. Observe that for every $t = (p, u) \in T_{-1}M$:

$$\begin{aligned} H_t(T_{-1}M) =& \{X_t^H + Y_t^V \mid X, Y \in T_pM,\ g_p(X, u) = 0,\ g_p(Y, u) = 0\} \\ =& \{X_t^O + Y_t^T \mid X, Y \in T_pM\} \end{aligned}$$

where, for every $X \in \mathfrak{X}(M)$, we introduce the following vector fields tangent to $T_{-1}M$:

$$\begin{aligned} X^O &:= X^H + \tilde{G}(X^V, \mathcal{N})\zeta, \\ X^T &:= X^V + \tilde{G}(X^V, \mathcal{N})\mathcal{N}. \end{aligned}$$

Finally, we consider the 1-form $\eta := \frac{1}{2}\beta$ on $T_{-1}M$, whose kernel is $H(T_{-1}M)$. Equation (8) implies that the Levi form L_η is positive definite and the Reeb vector field ξ of η is

$$\xi_t = -2\zeta_t, \quad t \in T_{-1}M. \tag{9}$$

Hence $(H(T_{-1}M), J, \eta)$ is a strongly pseudo-convex almost CR structure on $T_{-1}M$, that we shall call the *standard pseudo-convex structure*. The associated contact metric structure, also named *standard contact metric structure*, is determined according to:

$$\begin{aligned} &\varphi(X^O) = X^T, \quad \varphi(X^T) = -X^O, \quad \varphi(\xi) = 0, \\ &g_\eta(\tilde{X}, \tilde{Y}) = \frac{1}{4}\tilde{G}(\tilde{X}, \tilde{Y}), \quad g_\eta(\tilde{X}, \xi) = 0, \quad g_\eta(\xi, \xi) = 1, \end{aligned}$$

where $X \in \mathfrak{X}(M)$ and $\tilde{X}, \tilde{Y}$ are any smooth sections of $H(T_{-1}M)$.

3 Contact metric (k, μ) structures on tangent hyperquadric bundles

In this section we prove our main results.

Theorem 1. *Let (M, g) be a Lorentzian manifold. Then $T_{-1}M$ is a locally symmetric pseudo-Hermitian manifold if and only if (M, g) has constant sectional curvature.*

Proof. Suppose first that (M, g) has constant sectional curvature. Let $t = (p, u)$ any point on $T_{-1}M$. We have that the linear mapping

$$L : X \in T_pM \mapsto -X - 2g_p(u, X)u \in T_pM,$$

is an orthogonal transformation that preserves the Riemannian curvature tensor. Thus, there exists an isometry

$$f : U \to U,$$

where U is an open neighborhood of p, such that $\mathrm{d}_p f = L$ (cf. [12, Chapter 8]). Since f is an isometry, we see that the induced diffeomorphism

$$F = \mathrm{d}f : TU \to TU$$

satisfies:

$$(\mathrm{d}F)_s(X^H_s) = (\mathrm{d}f_x(X))^H_{F(s)}, \quad (\mathrm{d}F)_s(X^V_s) = (\mathrm{d}f_x(X))^V_{F(s)}, \tag{10}$$

for every $X \in \mathfrak{X}(M)$ and $s = (x, v) \in T_{-1}M \cap TU$, hence F is a local isometry with respect to the Sasaki metric $\tilde{G}$ on TM, preserving the almost complex structure $\tilde{J}$. It follows that F restricts to a local CR diffeomorphism of $T_{-1}M$. Moreover, (10) and (9) imply that $(\mathrm{d}F)_s(\xi_s) = \xi_{F(s)}$, yielding that F is also a local isometry with respect to the Webster metric. Moreover, being $\mathrm{d}_p f = L$, we have:

$$(\mathrm{d}F)_t|_{H_t(T_{-1}M)} = -Id,$$

and thus F is a local CR symmetry at t.

Viceversa, if $T_{-1}M$ is a locally symmetric pseudo-Hermitian manifold, then in particular $(H(T_{-1}M), J)$ is a CR structure and hence, by [14, Theorem 1], (M, g) has constant sectional curvature. □

Now we determine the Boeckx invariant of $T_{-1}M$, where M is a Lorentzian space form.

Theorem 2. *Let (M^{n+1}, g) be a Lorentzian manifold with constant sectional curvature c. Then, $T_{-1}M$ endowed with the standard contact metric structure is Sasakian if and only if $c = -1$. If $c \neq -1$ then $T_{-1}M$ is a non-Sasakian contact metric (k, μ)-space, whose Boeckx invariant is:*

$$I = \frac{c - 1}{|c + 1|}.$$

Proof. Theorem 1 ensures that the standard contact metric structure $(\varphi, \xi, \eta, g_\eta)$ of $T_{-1}M$ is a contact metric (k, μ) structure (eventally a Sasakian one). In the following we compute the spectrum of the symmetric operator h. Let $t = (p, u) \in T_{-1}M$ and $X \in T_pM$ such that $g_p(X, u) = 0$. Then:

$$\begin{aligned} 2h(X^T) &= [\xi, \varphi X^T] - \varphi[\xi, X^T] \\ &= -[\xi, X^O] - \tilde{J}[\xi, X^V + G(X^V, \mathcal{N})\mathcal{N}], \end{aligned} \tag{11}$$

where we are denoting again with X any extension of the vector X. Let (x^i) be a local coordinate system on M and (q^i, v^i) the corresponding local coordinate system on TM. Since locally

$$\xi = -2v^i(\frac{\partial}{\partial x^i})^H,$$

then using equation (7), by a standard computation (cf. also [9] in the Riemannian case), we obtain:

$$[\xi, X^V]_t = 2(X_t^H - (\nabla_u X)_t^V), \quad [\xi, X^H]_t = -2((\nabla_u X)_t^H - cX_t^V), \tag{12}$$

and hence (11) becomes:

$$\begin{aligned} 2h(X_t^T) &= 2((\nabla_u X)_t^H - cX_t^V) - 2(X_t^V + (\nabla_u X)_t^H) - \xi_t(G(X^V, \mathcal{N}))\tilde{J}\mathcal{N}_t \\ &= -2(c+1)X_t^V - \frac{1}{4}\xi_t(G(X^V, \mathcal{N}))\xi_t. \end{aligned}$$

It follows that $\xi_t(G(X^V, \mathcal{N})) = 0$, thus

$$\begin{aligned} h(X_t^V) &= -(c+1)X_t^V, \\ h(X_t^H) &= h(-\varphi X_t^V) = \varphi h X_t^V = (c+1)X_t^H, \end{aligned} \tag{13}$$

and the spectrum of the operator h is $\{0, c+1, -(c+1)\}$. It follows that $T_{-1}M$ is Sasakian iff $c = -1$.

Suppose $c \neq -1$. Let $t = (p, u)$ and $X \in T_pM$ such that $g_p(X, u) = 0$. Then using the definition (2) of the Pang invariant and equation (12), we get, being X^O a global section of $\mathcal{D}(c+1)$:

$$\begin{aligned} \mathit{\Pi}_{\mathcal{D}(c+1)}(X_t^O, X_t^O) &= 2\mathrm{d}\eta([\xi, X^O]_t, X_t^O) \\ &= 2g_\eta([\xi, X^H + \tilde{G}(X^V, \mathcal{N})\zeta]_t, X_t^T) \\ &= 2g_\eta([\xi, X^H]_t + \xi(\tilde{G}(X^V, \mathcal{N}))\zeta_t, X_t^T) \\ &= 2g_\eta([\xi, X^H]_t, X_t^T) \\ &= 4cg_\eta(X_t^V, X_t^T) \\ &= 4cg_\eta(X_t^T, X_t^T). \end{aligned} \tag{14}$$

In particular if $c > -1$, by equation (3), we obtain

$$\mathit{\Pi}_{\mathcal{D}(c+1)}(X^O, X^O) = (2c + 4 - \mu)g_\eta(X^O, X^O). \tag{15}$$

Thus comparing (14) and (15) we have that $\mu = 4 - 2c$ and hence

$$I_M = \frac{c-1}{c+1}.$$

Finally suppose $c < -1$. Then by (4):

$$\mathit{\Pi}_{\mathcal{D}(c+1)}(X^O, X^O) = (2c + 4 - \mu)g_\eta(X^O, X^O). \tag{16}$$

Comparing equations (16) and (14) we obtain that $\mu = 4 - 2c$ and

$$I_M = -\frac{c-1}{c+1}.$$

□

Corollary 1. *Every non Sasakian contact metric (k, μ) space with Boeckx invariant $I \leqslant -1$ is locally equivalent, up to a $\mathcal{D}$-homothetic deformation, to the tangent hyperquadric bundle $T_{-1}M$ of a Lorentzian manifold M with constant sectional curvature $c \leqslant 0$, $c \neq -1$, endowed with its standard contact metric structure.*

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