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ORIGINAL PAPER

Asymptotic profile for a two-terms time fractional diffusion problem

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Abstract

We consider the Cauchy-type problem associated to the time fractional partial differential equation:

$$\begin{cases} \partial_t u + \partial_t^\beta u - \Delta u = g(t,x), \quad t>0,\ x\in\mathbb{R}^n \\ u(0,x) = u_0(x), \end{cases}$$

with $\beta \in (0,1)$, where the fractional derivative ∂_t^β is in Caputo sense. We provide a sufficient condition on the right-hand term $g(t,x)$ to obtain a solution in $\mathcal{C}_b([0,\infty), H^s)$. We exploit a dissipative-smoothing effect which allows to describe the asymptotic profile of the solution in low space dimension.

Keywords Multi-terms fractional ordinary and partial differential equations · Asymptotic profile · Critical exponent · Global existence · Small data

Mathematics Subject Classification 35R11 (primary) · 35A01

1 Introduction

In the present paper we consider the Cauchy-type problem for a fractional (in time) partial differential equation

$$\begin{cases} \partial_t u + \partial_t^\beta u - \Delta u = g(t,x) \quad t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

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where $\partial_t^\beta u$ denotes the (forward) Caputo fractional derivative of order $\beta \in (0,1)$, with starting time 0, with respect to the time variable (see, for instance, [22]). Namely,

$$\partial_t^\beta u(t,x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial_t u(s,x)}{(t-s)^\beta}\, ds,$$

for any $t > 0$ and $x \in \mathbb{R}^n$; here, $\Gamma(\cdot)$ denotes the gamma function. In Theorem 1, we show that solutions to (1.1) are bounded in H^s, with respect to $t \in [0,\infty)$, namely, are in $C_b([0,\infty), H^s)$, if $g \in L_t^\infty H^{s-2+\varepsilon}$ and $\langle t\rangle^\beta g \in L_t^\infty H^{s-4}$, where $\langle t\rangle = \sqrt{1+t^2}$.

In Theorem 2, in low space dimension $n = 1, 2, 3$, assuming initial data in $L^1 \cap H^s$ with $s \in [0, 2-n/2)$ we prove that the asymptotic profile of the solution to (1.1) is independent of g, provided that suitable decay assumptions on $g(t,\cdot)$ are satisfied.

As a corollary of this latter result, we investigate a class of nonlinear perturbations of the problem, for which global-in-time small data solutions exist and we show that their asymptotic profile is independent on the nonlinear perturbation.

One crucial property which allow us to get the previous results is a smoothing effect. In particular, the H^s norm of $u(t,\cdot)$ at any time $t > 0$ can be controlled by $C(t)\,\|u_0\|_{H^{s-2}}$, if $g \equiv 0$. However, $C(t) \to \infty$ as $t \to 0$ (see (3.10) and (3.12)). This effect is analogous to the smoothing effect of the heat equation and related parabolic equations, but it only allows to gain a finite amount of regularity. The smoothing effect also appears with respect to the inhomogeneous term $g(t,\cdot)$, with a different singular power (see (3.9) and (3.13)), since the Duhamel's principle does not hold in classical sense, for Cauchy-type problems with Caputo fractional derivatives, as (1.1) (see later, Lemma 1).

The counterpart of this limited smoothing effect is a limited dissipative effect which appears at long time: higher order derivatives of the solution vanish as $t \to \infty$ with a faster speed, but not faster than $t^{-\beta}$. As for the smoothing effect, this limitation is due to the structure of the fundamental solution of the equation. In particular, to show the optimality of the decay estimates, at least in low space dimension, we describe the asymptotic profile of the solution, under suitable decay assumption on $g(t,x)$. Under the moment condition $M \neq 0$, where

$$M = \int_{\mathbb{R}^n} u_0(x)\, dx\,, \tag{1.2}$$

this profile is described by $M\, K_0^\dagger(t,x)$, where $K_0^\dagger$ is the fundamental solution to the homogeneous Cauchy-type problem for the sub-diffusive fractional equation

$$\begin{cases} \partial_t^\beta v - \Delta v = 0 & t > 0,\ x \in \mathbb{R}^n, \\ v(0,x) = v_0(x). \end{cases} \tag{1.3}$$

Explicitly, $K_0^\dagger(t,x) = t^{-\frac{n\beta}{2}}\, K_0^\dagger(1, t^{-\frac{\beta}{2}}x)$ is given by:

$$K_0^\dagger(1,\cdot) = \frac{\sin(\beta\pi)}{\beta\pi}\, \mathscr{F}^{-1}\left(\int_0^\infty e^{-\tau^{\frac{1}{\beta}}} \frac{|\xi|^2}{|\xi|^4 + \tau^2 + 2\tau|\xi|^2\cos(\beta\pi)}\, d\tau \right); \tag{1.4}$$

in particular, $K_0^\dagger(1,\cdot)$ belongs to $H^{2-\frac{n}{2}}$ (see for instance [10]). Here, and in the following, $\mathscr{F}$ denotes the Fourier transform operator acting on the space variable x, and $\hat{f}(t,\xi) = (\mathscr{F}(t,\cdot))(\xi)$. By representation (1.4), the limited amount of smoothing effect is motivated by the fact that $|\hat{K}_0^\dagger(1,\xi)| \approx |\xi|^2\langle\xi\rangle^{-4}$.

The dissipative-smoothing effect also appears in other evolution equations, for instance, in the case of strongly damped waves [41] (see also [8]), and of more general damped evolution equations [15]. However, those cases are more related to the heat equation and other diffusive equations, since the smoothing effect is not limited by $\langle\xi\rangle^{-2}$. The study of the H^s well-posedness for multi-point value problems for partial differential equations of fractional order similar to (1.1) is already faced, for instance, in [18].

The main difficulty in dealing with the equation in (1.1) is its lack of homogeneity. Theorems 1 and 2 are based on the representation formula provided by Lemma 1 for the solution to the Cauchy-type problem

$$\begin{cases} y'(t) + \partial_t^\beta y(t) + \lambda y(t) = g(t) & t > 0, \\ y(0) = c_0, \end{cases} \tag{1.5}$$

with $\lambda > 0$ and $c_0 \in \mathbb{R}$.

1.1 Background

We refer to [22] or [33] for a deep study about the theory of fractional derivatives. It is well known that differential equations with fractional derivatives turned out to be suitable to describe in a very good way various physical phenomena in areas like rheology, biology, engineering, mathematical physics, etc. (see for instance [16, 25, 26, 28, 33] and the reference given therein). Open problem in this field is finding some easy and effective methods for solving such equations. Such problem becomes even more difficult when multiple fractional in time derivatives are involved in the equation. In the literature some authors considered the two-term time fractional diffusion-wave equation of the type

$$b_1\partial_t^{\delta_1} w + b_2\partial_t^{\delta_2} w - c^2\Delta w = F(t,x,w), \tag{1.6}$$

for $b_1, b_2 \in \mathbb{R}$, $\delta_1, \delta_2 > 0$ and $F \equiv 0$ or F nonlinear; then, they investigate the existence of solution to the Cauchy-type problem associated to (1.6) in suitable spaces, under given assumptions on the exponents δ_1 and δ_2 and on the function F. A deep review can be found for instance in [43]; here, the authors find the upper viscosity solutions to (1.6) for $b_1 + b_2 = 1$, $c = 1$ and $\delta_1, \delta_2 \in (0, 2)$, considering a non-linear lipschitz term F, in the $L^p(\mathbb{R}^n)$ framework, for $1 \le p \le \infty$. Equation (1.6) with $\delta_1 = 2\delta_2$ is known as the time-fractional telegraph equation; it is studied for instance in [39] where the authors obtain the Fourier transform of the solutions for any $\delta_2 \in (0, 1]$ expressed in terms of Mittag-Leffler functions and they give a representation of their inverse, in terms of stable densities; the special case $\delta_2 = 1/2$ can be interpreted as a heat equation subject to a damping effect, represented by the

1/2-order time-derivative; in this case they show that the fundamental solution is the distribution of a telegraph process with Brownian time. In [42] the author investigates the existence and uniqueness of local (in time) solutions to the nonlinear n-term time-fractional differential equation with constant coefficients in the Banach space $C([0, T])$. Some results about the well-posedness and regularity of solutions to (1.6) in bounded domains are presented for instance in [4, 9, 45].

Having in mind to apply the Fourier transform to the linear equation associated to (1.6), it is understandable as the problem of finding a suitable representation of solution is strictly related to solving fractional ordinary differential equations in the form

$$\partial_t^{\delta_1} w + \partial_t^{\delta_2} w + \lambda w = 0, \tag{1.7}$$

for $\lambda \in \mathbb{R}$. In [24] the authors develop the operational calculus of Mikusiński's type for the Caputo fractional differential equation, in order to obtain exact solutions of the initial value problem associated to (1.7) through Mittlag-Leffler type functions. The special cases $\delta_1 = 1, \delta_2 \in (0, 1)$ and, respectively, $\delta_1 = 2, \delta_2 \in (1, 2)$ have been deeply investigated in [17] taking $\lambda = 1$ and are referred as the *composite fractional relaxation equation* and, respectively, the *composite fractional oscillation equation*; here, by applying the technique of Laplace transforms they derive the analytical solutions to such equations.

In particular, the fractional differential equation in (1.7) with $\delta_1 = 1$, $\delta_2 = 1/2$ corresponds to the Basset problem: it represents a classical problem in fluid dynamics where the unsteady motion of a particle accelerates in a viscous fluid due to the force of gravity. The situation of a sphere subjected to gravity was first considered independently by Boussinesq [6] and by Basset [2], who introduced a special hydrodynamic force, which is nowadays referred to as Basset force. The whole was summarized by Basset himself in a later paper [3], and, in more recent times, by Hughes and Gilliand [19]. Nowadays the dynamics of impurities in unsteady flows is investigated as shown by several publications, which aim to provide more general expressions for the hydrodynamic forces, including the Basset force, in order to fit experimental data and numerical simulations, see e.g. [5, 23, 29–32, 37, 38]. For a complete history of the Basset problem one can refer to [7].

In [14] the Cauchy-type problem

$$\begin{cases} \partial_t^{\beta} w - \Delta w = g(t, x) & t > 0,\ x \in \mathbb{R}^n, \\ w(0, x) = w_0(x), \\ w_t(0, x) = w_1(x), \end{cases}$$

is investigated in the case $\beta \in (1, 2)$. Under suitable assumptions on the nonhomogeneous term $g(t, x)$, the authors investigate some $L^p - L^q$ decay estimates for the solution w; then, they apply such estimates to study the corresponding semilinear problem.

Notation

In this paper, $L_t^\infty X$ denotes $L^\infty([0,\infty), X)$, i.e., the space of essentially bounded functions from $[0,\infty)$ to X. Moreover, $\mathcal{C}_b([0,\infty), X)$ denotes the space of continuous bounded functions from $[0,\infty)$ to X. In the both cases, $\|g\|_{L_t^\infty X}$ denotes the norm $\sup_{t\geq 0}\|g(t,\cdot)\|_X$. For any $s\in\mathbb{R}$,

$$H^s = \left\{ f\in\mathcal{S}' : \langle\xi\rangle^s\hat{f}\in L^2 \right\},$$

is the (fractional) Sobolev space equipped with norm $\|f\|_{H^s} = \|\langle\xi\rangle^s\hat{f}\|_{L^2}$, where the symbol $\langle\xi\rangle$ denotes the quantity $\sqrt{1+|\xi|^2}$. For $f\in H^s$ with $s\geq 0$, we also define $\|f\|_{\dot{H}^s} = \||\xi|^s\hat{f}\|_{L^2}$. It is clear that $H^0=L^2$ and $\|f\|_{\dot{H}^0}=\|f\|_{L^2}$.

For $q\in(1,\infty)$ and $s\in\mathbb{R}$ we also define the Bessel potential space [1]

$$H^{s,q} = \left\{ f\in\mathcal{S}' : \langle\xi\rangle^s\hat{f}\in L^q \right\},$$

equipped with norm $\|f\|_{H^{s,q}} = \|\mathfrak{F}^{-1}(\langle\xi\rangle^s\hat{f})\|_{L^q}$. We recall that $\|f\|_{H^{s,q}}\leq\|f\|_{L^q}$ for any $s<0$.

In this paper, $f\lesssim g$ means that $f\leq Cg$ for some constant $C>0$, and $f\approx g$ means that $f\lesssim g\lesssim f$.

1.2 Results

We first present a sufficient condition on g such that the solution to (1.1) remains bounded in H^s.

Theorem 1 *Let $n\geq 1$ and $s\in\mathbb{R}$. Assume that $u_0\in H^s$ and that $g\in L_t^\infty H^{s-2+\varepsilon}$ for some $\varepsilon>0$. Then the solution u to (1.1) is in $\mathcal{C}([0,\infty), H^s)$ and*

$$\|u(t,\cdot)\|_{H^s}\leq C\,\|u_0\|_{H^s}+C_\varepsilon\|g\|_{L_t^\infty H^{s-2+\varepsilon}}+C\int_0^t(t-\tau)^{-(1-\beta)}\|g(\tau,\cdot)\|_{H^{s-4}}\,d\tau, \tag{1.8}$$

for any $t\geq 0$, where $C>0$ and $C_\varepsilon>0$ are independent of t. In particular, u is in $\mathcal{C}_b([0,\infty), H^s)$ and

$$\|u\|_{L_t^\infty H^s}\leq C\left(\|u_0\|_{H^s}+A\right)+C_\varepsilon\|g\|_{L_t^\infty H^{s-2+\varepsilon}},$$

if $A=\sup_{t\geq 0}\langle t\rangle^\beta\|g(t,\cdot)\|_{H^{s-4}}$ is finite.

When $n\leq 3$ and $s\in[0,2-n/2)$, the embedding $L^1\hookrightarrow H^{s-2}$ holds, hence the smoothing effect is sufficient to describe the asymptotic profile in H^s of the solution to (1.1), in the form $M\,K_0^\dagger(t,x)$, where M is as in (1.2), provided that we assume the moment condition $M\neq 0$ and that we make suitable decay assumptions on $g(t,\cdot)$ as $t\to\infty$.

Theorem 2 *Let $n = 1, 2, 3$, and $s \in [0, 2 - n/2)$. Assume that $u_0 \in L^1 \cap H^s$, and that $g \in L^\infty_t H^{s-2+\varepsilon}$ for some $\varepsilon > 0$, also satisfies*

$$B_1 := \sup_{t\geq 0} Q_1(t) < \infty, \tag{1.9}$$

where

$$Q_1(t) := \langle t\rangle^{\frac{n\beta}{4}} \left(\langle t\rangle^\beta \|g(t,\cdot)\|_{H^{s-4}} + \langle t\rangle^{\frac{s\beta}{2}} \|g(t,\cdot)\|_{H^{s-2+\varepsilon}}\right).$$

If

$$\frac{n}{4}\beta + \beta \geq 1, \tag{1.10}$$

we also assume that $g \in L^\infty_t H^{s+a-4,q}$ for some $q \in (1, 2)$, where

$$a = n\left(\frac{1}{q} - \frac{1}{2}\right), \qquad \frac{n}{2}\beta\left(1 - \frac{1}{q}\right) + \beta < 1, \tag{1.11}$$

and that

$$B_2 := \sup_{t\geq 0} Q_2(t) < \infty, \tag{1.12}$$

where

$$Q_2(t) := \langle t\rangle^{\frac{n\beta}{2}\left(1-\frac{1}{q}\right)+\beta} \|g(t,\cdot)\|_{H^{s+a-4,q}}.$$

Then, u is in $C_b([0,\infty), H^s)$ and there exists $C > 0$ independent of t, such that ($B_2 = 0$ in the following, if (1.10) does not hold)

$$\|u(t,\cdot)\|_{\dot H^s} \leq C(1+t)^{-\frac{n\beta}{4}-\frac{s\beta}{2}}(\|u_0\|_{H^s\cap L^1} + B_1 + B_2). \tag{1.13}$$

In particular, if $\limsup_{t\to\infty} Q_1(t) = 0$ and, in addition, $\limsup_{t\to\infty} Q_2(t) = 0$ when (1.10) holds, then the solution u also satisfies

$$\|u(t,\cdot) - MK_0^\dagger(t,\cdot)\|_{\dot H^s} = o(t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}), \quad t\to\infty. \tag{1.14}$$

When $M \neq 0$, we may say that the asymptotic profile of $u(t,\cdot)$ in H^s as $t\to\infty$, is $MK_0^\dagger(t,\cdot)$.

Remark 1 We stress that $s - 4 < s - 2 + \varepsilon < 0$ for sufficiently small $\varepsilon > 0$, and $s + a - 4 < 0$, in Theorem 2, so that B_1 and B_2 in (1.9) and (1.12) are finite if there exists $B > 0$ such that

$$\langle t\rangle^\beta \left(\langle t\rangle^{\frac{n\beta}{4}} \|g(t,\cdot)\|_{L^2} + \langle t\rangle^{\frac{n\beta}{2}\left(1-\frac{1}{q}\right)} \|g(t,\cdot)\|_{L^q}\right) \leq B; \tag{1.15}$$

in particular, conditions (1.9) and (1.12) are satisfied for $B_1 = 2B$ and $B_2 = B$. Similarly, the conditions $\lim_{t\to\infty} Q_i(t) = 0$ hold for $i = 1, 2$ if

$$\lim_{t\to\infty} t^\beta \left(t^{\frac{n\beta}{4}} \|g(t,\cdot)\|_{L^2} + t^{\frac{n\beta}{2}\left(1-\frac{1}{q}\right)} \|g(t,\cdot)\|_{L^q}\right) = 0. \tag{1.16}$$

We may apply Theorem 2 to study the semilinear problem

$$\begin{cases} \partial_t u + \partial_t^\beta u - \Delta u = f(u), & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{1.17}$$

where $f(u) = |u|^p$ for some $p \geq 2$, or, more in general,

$$|f(u) - f(v)| \leq C\,|u - v|(|u|^{p-1} + |v|^{p-1}). \tag{1.18}$$

Then, as a consequence of Theorem 2 we have the following result:

Corollary 1 *Let $n = 1, 2$ and assume that $p \geq 1 + 2/n$ in (1.18). Fix s such that*

$$\frac{n}{2}\left(1 - \frac{1}{p}\right) \leq s < 2 - \frac{n}{2}.$$

Then there exists $\varepsilon > 0$ such that for any initial data

$$u_0 \in L^1 \cap H^s, \quad \text{with } \|u_0\|_{L^1} + \|u_0\|_{H^s} \leq \varepsilon, \tag{1.19}$$

there is a uniquely determined solution $u \in \mathcal{C}_b([0,\infty), H^s)$ to (1.17). Moreover, if $p > 1 + 2/n$ then

$$\|u(t,\cdot) - MK_0^\dagger(t,\cdot)\|_{\dot{H}^\kappa} = o(t^{-\frac{n\beta}{4} - \frac{\kappa\beta}{2}}), \quad \text{for } \kappa = 0, s, \ \text{as } t \to \infty. \tag{1.20}$$

Thus, when $M \neq 0$ and $p > 1 + 2/n$, Corollary 1 means that the nonlinearity does not influence the asymptotic profile of the solution to (1.17). The critical exponent $1 + 2/n$ is sharp [10]. We notice that in the critical case $p = 1 + 2/n$, Corollary 1 guarantees the existence of a global small data solution, but the asymptotic profile of the solution to (1.17) depends on the nonlinearity, in general.

Theorems 1 and 2 are based on the following representation formula for the solution to (1.5).

Lemma 1 *Assume that $y = y(t)$ solves the Cauchy problem (1.5). Then*

$$y(t) = c_0 K_0(t) + \int_0^t g(t-\tau) K_1(\tau)\, d\tau,$$

where K_0 and K_1 have the following integral representations:

$$K_0(t) = \frac{\sin(\beta\pi)}{\pi} \int_0^\infty e^{-xt} \frac{\lambda\, x^{-1+\beta}}{(\lambda - x)^2 + x^{2\beta} + 2x^\beta \cos(\beta\pi)(\lambda - x)}\, dx, \tag{1.21}$$

$$K_1(t) = \frac{\sin(\beta\pi)}{\pi} \int_0^\infty e^{-xt} \frac{x^\beta}{(\lambda - x)^2 + x^{2\beta} + 2x^\beta \cos(\beta\pi)(\lambda - x)}\, dx, \tag{1.22}$$

for any $t \geq 0$*. Moreover, we may write* $K_0(t) = 1 - \lambda \int_0^t K_1(r)dr$*, that is,* $\partial_t K_0(t) = -\lambda K_1(t)$*.*

The fact that K_0 and K_1 are different in Lemma 1 means that the Duhamel's principle does not hold in classical sense.

Remark 2 Applying the change of variable $xt = \tau^{\frac{1}{\beta}}$ we obtain the following representations for $K_0(t)$ and $K_1(t)$:

$$K_0(t) = \frac{\sin(\beta\pi)}{\beta\pi} \int_0^\infty \frac{e^{-\tau^{\frac{1}{\beta}}} \lambda t^{2-\beta}}{(t\lambda - \tau^{\frac{1}{\beta}})^2 + \tau^2 t^{2(1-\beta)} + 2\tau t^{1-\beta} \cos(\beta\pi)(t\lambda - \tau^{\frac{1}{\beta}})}\, d\tau,$$

$$K_1(t) = \frac{\sin(\beta\pi)}{\beta\pi} \int_0^\infty \frac{e^{-\tau^{\frac{1}{\beta}}} \tau^{\frac{1}{\beta}} t^{1-\beta}}{(t\lambda - \tau^{\frac{1}{\beta}})^2 + \tau^2 t^{2(1-\beta)} + 2\tau t^{1-\beta} \cos(\beta\pi)(t\lambda - \tau^{\frac{1}{\beta}})}\, d\tau.$$

1.3 Comparison with the damped wave equation

By Theorem 2 we deduce that the solution u to the homogeneous Cauchy-type problem associated to (1.1), asymptotically behaves as the solution v to (1.3). There are several analogies with the diffusion phenomenon studied for the damped wave equation [27, 34, 36]:

$$\begin{cases} u_{tt} - \Delta u + u_t = 0 & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

The asymptotic profile of the solution is described by $u \sim MG(t, x)$, where $G = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the fundamental solution to the heat equation and

$$M = \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\, dx,$$

under the assumption of nonzero moment condition $M \neq 0$. This diffusion phenomenon allowed to prove the global existence of small data solutions to the semilinear problem with power nonlinearity $f(u)$, in the supercritical case $p > 1 + 2/n$ (see [44]), as for the semilinear heat equation. A nonlinearity, in general, influences the asymptotic profile of the solution, see [21]. In Corollary 1, we showed that this is not the case for our equation in the supercritical case. This latter phenomenon is a consequence of the special structure of the Cauchy-type problem for fractional equations, and of the fact that the Duhamel's principle does not hold in classical sense, see Lemma 1.

Diffusion phenomena hold, more in general, for evolution equations

$$\begin{cases} u_{tt} + (-\Delta)^{\sigma} u + (-\Delta)^{\theta} u_t = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.23}$$

when the damping is *effective* damping according to the classification introduced by the authors in [12], that is, $2\theta < \sigma$. Here, $(-\Delta)^{\alpha}$ denotes the fractional Laplace operator of order $\alpha > 0$ defined on $\mathcal{S}$ as $(-\Delta)^{\alpha} f = \mathscr{F}^{-1}(|\xi|^{2\alpha} \hat{f})$. If $\theta = 0$ the solution to (1.23) behaves asymptotically like the solution to the corresponding diffusive equation, with initial data $u_0 + u_1$, namely by $e^{-t(-\Delta)^{\sigma}}(u_0 + u_1)$; for $\theta > 0$ a double *diffusion phenomenon* holds, that is, two different diffusive equations compete to describe the asymptotic profile of the solution to (1.23) (see [11]). On the other hand, when $2\theta > \sigma$, the asymptotic profile to (1.23) is completely different; in particular, the wave structure appears and oscillations come into play (see [20]).

Inspired by the results just described, the main goal of the present paper is to show how the fractional in time derivative $\partial_t^{\beta} u$ in (1.1) influences the asymptotic profile of the solution with respect to the undamped heat equation: the presence of the fractional order member deeply influences the structure of the fundamental solution of equation (1.1); as a consequence, a dissipative-smoothing effect appears and the asymptotic profile of the solution to (1.1) is described by $MK_0^{\dagger}(t,\cdot)$, independently on the non-homogeneous term, under suitable decay assumptions on $g(t,\cdot)$ (see Theorem 2).

2 Proof of Lemma 1

In order to prove Lemma 1 we will use the Laplace transform method. Given a function $\varphi = \varphi(t)$ of a real variable $t \in \mathbb{R}_+ = [0,\infty)$, $\mathcal{L}(\varphi)$ denotes its Laplace transform defined by

$$(\mathcal{L}\varphi)(s) := \int_0^{\infty} e^{-st} \varphi(t)\, dt \quad (s \in \mathbb{C}).$$

Under suitable assumptions, the inverse Laplace transform of a given function $F = F(s)$, holomorphic in some half-plane $\{\Re s > \lambda\}$, is given for any $t \in \mathbb{R}_+$ by the formula

$$f(t) = \mathcal{L}^{-1}(F(s))(t) = \lim_{b\to\infty} \frac{1}{2\pi i} \int_{a-ib}^{a+ib} e^{st} F(s)\, ds, \tag{2.1}$$

where $a > \lambda$.

The Laplace transform has many properties which are useful for studying dynamical systems. In particular, we mention that for any $\alpha \in (0,1]$ the following transform rule holds

$$\mathcal{L}\big(\partial_t^{\alpha} \varphi\big)(s) = s^{\alpha} \mathcal{L}(\varphi)(s) - s^{\alpha-1} \varphi(0), \tag{2.2}$$

for suitable good functions φ (see, for instance, [22]); such formula will allows us to transform the fractional differential equation in (1.5) in a functional equation.

Let us apply the Laplace transform to the fractional differential equation in (1.5). Applying the identity (2.2) we get the functional equation

$$s\mathcal{L}(y)(s) - c_0 + s^\beta \mathcal{L}(y)(s) - s^{\beta-1}c_0 + \lambda\mathcal{L}(y)(s) = \mathcal{L}(g)(s),$$

that is

$$\mathcal{L}(y)(s) = c_0 \frac{1+s^{\beta-1}}{s+s^\beta+\lambda} + \frac{\mathcal{L}(g)(s)}{s+s^\beta+\lambda}. \tag{2.3}$$

Here and hereafter, for any $s \in \mathbb{C}$, with $s = re^{i\theta}$, and $\alpha \in (0, 1)$ we are denoting by s^α its root of order α on the principal branch, i.e., $s^\alpha := r^\alpha e^{i\alpha\theta}$ with $\theta \in (-\pi, \pi)$.

Thus, using the convolution theorem $\mathcal{L}(f * h) = \mathcal{L}(f)\mathcal{L}(h)$ we find

$$y(t) = c_0 K_0(t) + \int_0^t g(t-\tau)K_1(\tau)\,d\tau,$$

where for any $t \geq 0$ we set

$$K_0(t) := \mathcal{L}^{-1}\left(\frac{1+s^{\beta-1}}{s+s^\beta+\lambda}\right)(t), \qquad K_1(t) := \mathcal{L}^{-1}\left(\frac{1}{s+s^\beta+\lambda}\right)(t). \tag{2.4}$$

We notice that

$$K_0(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) - \lambda\,\mathcal{L}^{-1}\left(\frac{1}{s}\,\frac{1}{s+s^\beta+\lambda}\right)(t) = 1 - \lambda\int_0^t K_1(r)\,dr,$$

thanks to the properties of the Laplace transform.

The remaining part of this section is devoted to the proof of Lemma 1 starting from the identities in (2.4). In order to get this aim we will use two different approaches: the first approach is based on the direct evaluation of the inverse Laplace transforms in (2.4); on the other hand, in the second approach we will express $K_0(t)$ and $K_1(t)$ as a combination of Mittag-Leffler functions.

2.1 Laplace Transfrom method

In order to get the desired integral representations in (1.21)-(1.22) we will use the integral formula for the inverse Laplace transform given in (2.1).

Let us define the function $\omega : \mathbb{C} \to \mathbb{C}$ such that $\omega(s) := s + s^\beta + \lambda$. We remark that ω has no zeros in $\mathbb{C}$: suppose that $s_0 = r_0\cos\theta_0 + ir\sin\theta_0$ is a zero of ω; then the couple (r_0, θ_0) satisfies the system

$$\begin{cases} r\cos\theta + r^\beta\cos(\beta\theta) + \lambda = 0, \\ r\sin\theta + r^\beta\sin(\beta\theta) = 0; \end{cases} \tag{2.5}$$

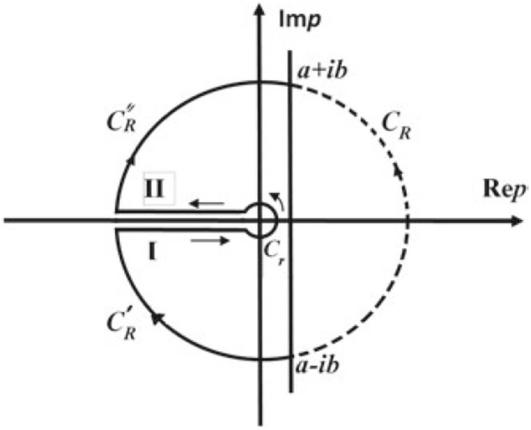

Fig. 1 Hankel path (from [35])

but this system has no solutions $(r_0, \theta_0) \in \mathbb{R}_+ \times [-\pi, \pi)$; in fact, the second equation admits as solutions only the couples $(r, 0)$, and such couple never solves the first equation.

By definition (2.1), in order to give an integral representation of $K_1(t)$ we evaluate the limit

$$\lim_{b\to\infty} \frac{1}{2\pi i} \int_{a-ib}^{a+ib} F(t,s)\, ds,$$

where $a > 0$ and $F(t,s) := e^{st}/\omega(s)$ for any $t \geq 0$. In order to calculate such integral, we consider the region delimited by the so called *Hankel path*, defined by the segment $(a - ib, a + ib)$, arcs C_R' and C_R'', segments I and II, and the circle C_r as represented in Fig. 1.

The path is contained in $\mathbb{C} \setminus \mathbb{R}_-$, where the function $F(t, s)$ is holomorphic; thus, we can apply the Cauchy theorem to get

$$\frac{1}{2\pi i} \int_{a-ib}^{a+ib} F(t,s)\, ds = \frac{1}{2\pi i}\Big(\int_I F(t,s)\, ds + \int_{II} F(t,s)\, ds + \int_{C_r} F(t,s)\, ds + \int_{C_R'} F(t,s)\, ds + \int_{C_R''} F(t,s)\, ds \Big).$$

As a consequence of the Jordan lemma we immediately obtain that

$$\lim_{R\to\infty} \int_{C_R'} F(t,s)\, ds = \lim_{R\to\infty} \int_{C_R''} F(t,s)\, ds = 0;$$

moreover, it is easy to check that also the integral over C_r tends to 0 as $r \to 0$. Therefore, we conclude

$$\begin{aligned}K_1(t) &= \frac{1}{2\pi i}\int_0^\infty e^{-xt}\Big(\frac{1}{-x+x^\beta(\cos(\beta\pi)-i\sin(\beta\pi))+\lambda}\\&\quad-\frac{1}{-x+x^\beta(\cos(\beta\pi)+i\sin(\beta\pi))+\lambda}\Big)\,dx\\&= \frac{1}{\pi}\int_0^\infty e^{-xt}\frac{x^\beta\sin(\beta\pi)}{(\lambda-x)^2+x^{2\beta}+2x^\beta\cos(\beta\pi)(\lambda-x)}\,dx.\end{aligned}$$

We use the same approach to get the desired integral representation of K_0, defined as

$$K_0(t) = \lim_{b\to\infty}\frac{1}{2\pi i}\int_{a-ib}^{a+ib} G(t,s)\,ds,$$

where $G(t,s) := e^{st}(1+s^{\beta-1})/\omega(s)$. Also in this case, the path is contained in $\mathbb{C}\setminus\mathbb{R}_-$, where the function $G(t,s)$ is holomorphic. Thus, using the same notation as for K_1, we get

$$\begin{aligned}K_0(t) = \frac{1}{2\pi i}\Big(\int_I G(t,s)\,ds + \int_{II} G(t,s)\,ds\\+\int_{C_r} G(t,s)\,ds + \int_{C'_R} G(t,s)\,ds + \int_{C''_R} G(t,s)\,ds\Big).\end{aligned}$$

As a consequence of the Jordan lemma we can conclude that the integrals over C'_R and C''_R tend to 0 as $R\to\infty$; moreover, it is easy to check that also the integral over C_r goes to 0 as $r\to 0$. Thus, we get

$$\begin{aligned}K_0(t) &= \frac{1}{2\pi i}\int_0^\infty e^{-xt}\Big(\frac{1+x^{\beta-1}(\cos((\beta-1)\pi)-i\sin((\beta-1)\pi))}{-x+x^\beta(\cos(\beta\pi)-i\sin(\beta\pi))+\lambda}\\&\quad-\frac{1+x^{\beta-1}(\cos((\beta-1)\pi)+i\sin((\beta-1)\pi))}{-x+x^\beta(\cos(\beta\pi)+i\sin(\beta\pi))+\lambda}\Big)\,dx\\&= \frac{1}{\pi}\int_0^\infty e^{-xt}\frac{\lambda x^{\beta-1}\sin(\beta\pi)}{(\lambda-x)^2+x^{2\beta}+2x^\beta\cos(\beta\pi)(\lambda-x)}\,dx.\end{aligned}$$

This complete the proof of Lemma 1.

2.2 Mittag-Leffler functions

In this section we show how the simplest case $\beta = 1/2$ can be treated with an alternative approach. In order to obtain the desired representations (1.21) and (1.22) we follow the idea given in [17] to provide the solutions K_0 and K_1 in terms of Mittag-Leffler functions. By (2.3) we know that the Laplace tranform of the solution $y = y(t)$ to (1.5) has the representation

$$y(t) = c_0K_0(t) + \int_0^t g(t-\tau)K_1(\tau)\,d\tau,$$

where for any $t \geq 0$, the values of $K_0(t)$ and $K_1(t)$ are defined in (2.4).

Let $s_\pm$ denote the two roots of the second degree polynomial $s^2 + s + \lambda$; then, it holds

$$\omega(s) = (s^{\frac{1}{2}} - s_+)(s^{\frac{1}{2}} - s_-), \qquad s_\pm = \frac{-1 \pm \sqrt{1-4\lambda}}{2}.$$

Here, $s_\pm$ satisfies the useful relations

$$s_+ s_- = \lambda, \qquad s_+ - s_- = \sqrt{1-4\lambda}.$$

Thus, we can write

$$\frac{1 + s^{-\frac{1}{2}}}{\omega(s)} = \frac{A_-}{s^{\frac{1}{2}}(s^{\frac{1}{2}} - s_+)} + \frac{A_+}{s^{\frac{1}{2}}(s^{\frac{1}{2}} - s_-)} \tag{2.6}$$

and

$$\frac{1}{\omega(s)} = \frac{A_+}{s^{\frac{1}{2}}(s^{\frac{1}{2}} - s_+)} + \frac{A_-}{s^{\frac{1}{2}}(s^{\frac{1}{2}} - s_-)}, \tag{2.7}$$

where $A_\pm := \pm s_\pm/(s_+ - s_-)$. As a consequence it is possible to write $K_0(t)$ and $K_1(t)$ as a linear combination of Mittag Leffler functions; such functions are defined by the following series representation,

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \ z \in \mathbb{C}.$$

In particular, it is possible to prove that for any $\mu \in \mathbb{R}$,

$$\mathcal{L}^{-1}\left(\frac{1}{s^{\frac{1}{2}}(s^{\frac{1}{2}} - \mu)}\right) = E_{\frac{1}{2}}(\mu\sqrt{t}); \tag{2.8}$$

in fact, by formula (1.9.13) in [22] we know:

$$\mathcal{L}\big((E_{\frac{1}{2}}(\mu\sqrt{t})\big)(s) = \frac{1}{s^{\frac{1}{2}}(s^{\frac{1}{2}} - \mu)};$$

thus, (2.8) follows since $E_{\frac{1}{2}}(\mu\sqrt{\cdot})$ is continuous with respect to $t \in [0, \infty)$.

As a consequence of (2.8), by (2.6) and (2.7) we obtain

$$K_0(t) = A_- E_{\frac{1}{2}}(s_+\sqrt{t}) + A_+ E_{\frac{1}{2}}(s_-\sqrt{t}), \tag{2.9}$$

and

$$K_1(t) = A_+ E_{\frac{1}{2}}(s_+\sqrt{t}) + A_- E_{\frac{1}{2}}(s_-\sqrt{t}). \tag{2.10}$$

In order to get the desired integral representations in (1.21)-(1.22) we will use the following useful lemma that is a particular case of Theorem 1 in [40].

Lemma 2 *The following representation holds*

$$E_{\frac{1}{2}}(z) = -\frac{2}{\pi}\int_0^\infty e^{-\tau^2}\frac{z}{\tau^2+z^2}\,d\tau,$$

for any $z \in \mathbb{C}$ such that $|\arg(z)| \in (\pi/2, \pi]$.

We remark that $\Re(s_\pm) < 0$; thus, $|\arg(s_\pm)| \in (\pi/2, \pi]$. Therefore, the proof of Lemma 1 follows as a consequence of (2.9) and (2.10), applying Lemma 2.

Remark 3 In the general case $\beta \in (0, 1)$ one can express the kernels K_0 and K_1 in terms of multivariate Mittag-Leffler functions (see [24]):

$$E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n) := \sum_{k=0}^{\infty}\sum_{\substack{\ell_1+\cdots+\ell_n=k\\ \ell_1\geq 0,\ldots,\ell_n\geq 0}}\frac{k!}{\ell_1!\times\cdots\times\ell_n!}\frac{\Pi_{i=1}^n z_i^{\ell_i}}{\Gamma(b+\sum_{i=1}^n a_i\ell_i)}.$$

Indeed, one can prove

$$K_0(t) = 1 - \lambda t E_{(1-\beta,1),2}(-t^{1-\beta}, -\lambda t),$$

and

$$K_1(t) = E_{(1-\beta,1),1}(-t^{1-\beta}, -\lambda t).$$

3 Decay estimates

Applying the Fourier transform with respect to the space variable in (1.1) we get the following Cauchy problem for a parameter dependent fractional differential equation:

$$\begin{cases}\partial_t\hat{u} + \partial_t^\beta\hat{u} + |\xi|^2\hat{u} = \hat{g}(t,\xi)\\ \hat{u}(0,\xi) = \hat{u}_0(\xi).\end{cases} \tag{3.1}$$

By Lemma 1, the solution is

$$\hat{u}(t,\xi) = \hat{K}_0(t,\xi)\hat{u}_0(\xi) + \int_0^t \hat{K}_1(t-\tau,\xi)\hat{g}(\tau,\xi)\,d\tau,$$

where

$$\hat{K}_0(t,\xi) = \frac{\sin(\beta\pi)}{\beta\pi}\int_0^\infty \frac{e^{-\tau^{\frac{1}{\beta}}}|\xi|^2 t^{2-\beta}}{(t|\xi|^2-\tau^{\frac{1}{\beta}})^2+t^{2(1-\beta)}\tau^2+2\tau t^{1-\beta}\cos(\beta\pi)(|\xi|^2 t-\tau^{\frac{1}{\beta}})}\,d\tau, \tag{3.2}$$

$$\hat{K}_1(t,\xi) = \frac{\sin(\beta\pi)}{\beta\pi}\int_0^\infty \frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}} t^{1-\beta}}{(t|\xi|^2-\tau^{\frac{1}{\beta}})^2+t^{2(1-\beta)}\tau^2+2\tau t^{1-\beta}\cos(\beta\pi)(|\xi|^2 t-\tau^{\frac{1}{\beta}})}\,d\tau, \tag{3.3}$$

As a consequence, we obtain the representation

$$u(t,x) = K_0(t,\cdot) *_{(x)} u_0 + \int_0^t g(\tau,\cdot) *_{(x)} K_1(t-\tau,\cdot)\,d\tau. \tag{3.4}$$

By the change of variable $\xi \mapsto t^{\frac{\beta}{2}}\xi$, for any $s \geq 0$ and $1 \leq q \leq \infty$, we obtain

$$\|\hat{K}_j(t,\cdot)|\xi|^s\|_{L^q} = t^{-\frac{n\beta}{2q}-\frac{s\beta}{2}}\|\hat{R}_j(t,\cdot)|\xi|^s\|_{L^q}, \tag{3.5}$$

where

$$\hat{R}_0(t,\cdot) = \frac{\sin(\beta\pi)}{\beta\pi}\,t^{2(1-\beta)}\int_0^\infty e^{-\tau^{\frac{1}{\beta}}}\,\frac{|\xi|^2}{\varphi(t,\tau,\xi)}\,d\tau,$$
$$\hat{R}_1(t,\cdot) = \frac{\sin(\beta\pi)}{\beta\pi}\,t^{1-\beta}\int_0^\infty e^{-\tau^{\frac{1}{\beta}}}\,\frac{\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau,$$
$$\varphi(t,\tau,\xi) = (t^{1-\beta}|\xi|^2-\tau^{\frac{1}{\beta}})^2+t^{2(1-\beta)}\tau^2+2\tau t^{1-\beta}\cos(\beta\pi)(t^{1-\beta}|\xi|^2-\tau^{\frac{1}{\beta}}),$$

Noticing that

$$\varphi(t,\tau,\xi) \geq (1-|\cos(\beta\pi)|)\big((t^{1-\beta}|\xi|^2-\tau^{\frac{1}{\beta}})^2+t^{2(1-\beta)}\tau^2\big), \tag{3.6}$$

it is useful to divide the half-line $\mathbb{R}_+$ in two regions, depending on t and ξ:

$$I_{t,\xi} = \left[\frac{\tau_0}{2},\,2\tau_0\right], \quad \tau_0 = \big(|\xi|^2 t^{1-\beta}\big)^\beta, \qquad J_{t,\xi} = \mathbb{R}_+ \setminus I_{t,\xi}. \tag{3.7}$$

Therefore, we can estimate

$$\varphi(t,\tau,\xi) \geq \begin{cases} c_1\,t^{2(1-\beta)}\tau^2 & \text{if } \tau \in I_{t,\xi}, \\ c_2\left(t^{2(1-\beta)}(|\xi|^4+\tau^2)+\tau^{\frac{2}{\beta}}\right) & \text{if } \tau \in J_{t,\xi}. \end{cases} \tag{3.8}$$

Thanks to (3.8), we prepare the pointwise estimates for $\hat{R}_0(t,\xi)$ and $\hat{R}_1(t,\xi)$.

Lemma 3 *The following estimates hold:*

$$|\hat{R}_1(t,\xi)| \lesssim \begin{cases} t^{-(1-\beta)}\langle\xi\rangle^{-4} & \textit{if } t \geq 1, \\ \langle t^{\frac{1-\beta}{2}}\xi\rangle^{-4} & \textit{if } t \leq 1, \end{cases} \tag{3.9}$$

$$|\hat{R}_0(t,\xi)| \lesssim \begin{cases} \langle\xi\rangle^{-2} & \textit{if } t \geq 1, \\ \langle t^{\frac{1-\beta}{2}}\xi\rangle^{-2} & \textit{if } t \leq 1. \end{cases} \tag{3.10}$$

Proof In order to get the desired estimates, we split the integral in the two regions $I_{t,\xi}$ and $J_{t,\xi}$ defined in (3.7).

We first consider $\hat{R}_1(t,\xi)$. By using estimate (3.8) in $I_{t,\xi}$, we get

$$\begin{aligned} t^{1-\beta}\int_{\tau_0/2}^{2\tau_0}\frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau &\lesssim t^{-(1-\beta)}\,e^{-(\tau_0/2)^{\frac{1}{\beta}}}\int_{\tau_0/2}^{2\tau_0}\tau^{\frac{1}{\beta}-2}\,d\tau \\ &\approx t^{-(1-\beta)}\,\tau_0^{\frac{1}{\beta}-1}\,e^{-(\tau_0/2)^{\frac{1}{\beta}}}. \end{aligned}$$

In particular, since $\beta < 1$ this latter term may be estimated by the quantity $t^{-(1-\beta)}\langle t^{1-\beta}|\xi|^2\rangle^{-M}$, for any $M \geq 0$. However, for short times, the estimate above is singular at $t = 0$, so we proceed in a different way:

$$\begin{aligned} t^{1-\beta}\int_{\tau_0/2}^{2\tau_0}\frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau &\lesssim t^{1-\beta}\tau_0 e^{-(\tau_0/2)^{\frac{1}{\beta}}}\int_{\tau_0/2}^{2\tau_0}\frac{\tau^{\frac{1}{\beta}-1}}{(t^{1-\beta}|\xi|^2-\tau^{\frac{1}{\beta}})^2+t^{2(1-\beta)}\tau_0^2}\,d\tau \\ &\lesssim t^{1-\beta}\,\tau_0\,e^{-(\tau_0/2)^{\frac{1}{\beta}}}\int_{-\infty}^{+\infty}\frac{1}{s^2+t^{2(1-\beta)}\tau_0^2}\,ds \leq C\,e^{-(\tau_0/2)^{\frac{1}{\beta}}}, \end{aligned}$$

where we first used the change of variable $s = \tau^{\frac{1}{\beta}} - t^{1-\beta}|\xi|^2$ and then the change of variable $r = s/(t^{1-\beta}\tau_0)$. Summarizing, we proved

$$t^{1-\beta}\int_{\tau_0/2}^{2\tau_0}\frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau \lesssim \begin{cases} t^{-(1-\beta)}\,\langle t^{1-\beta}|\xi|^2\rangle^{-M} & \text{if } t \geq 1, \\ \langle t^{1-\beta}|\xi|^2\rangle^{-M} & \text{if } t \leq 1, \end{cases}$$

for any $M \geq 0$. We now consider the integral over $J_{t,\xi}$. If we use (3.8) to estimate $\varphi(t,\tau,\xi) \gtrsim t^{2(1-\beta)}|\xi|^4$, then we find

$$t^{1-\beta}\int_{J_{t,\xi}}\frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau \lesssim t^{-(1-\beta)}|\xi|^{-4}\int_0^\infty e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}\,d\tau \lesssim t^{-(1-\beta)}|\xi|^{-4},$$

whereas, if we use (3.8) to estimate $\varphi(\tau,\xi)\geq t^{2(1-\beta)}\tau^2$ as we did in $I_{t,\xi}$, we find

$$t^{1-\beta}\int_{J_{t,\xi}}\frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau\lesssim t^{-(1-\beta)}\int_0^\infty e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}-2}\,d\tau\lesssim t^{-(1-\beta)}.$$

Therefore, we obtain

$$t^{1-\beta}\int_{J_{t,\xi}}\frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau\lesssim t^{-(1-\beta)}\langle\xi\rangle^{-4}.$$

Comparing with the estimate of the integral over $I_{t,\xi}$, we conclude the proof of (3.9) for $t\geq 1$, using $\langle t^{\frac{1-\beta}{2}}\xi\rangle^{-4}\leq\langle\xi\rangle^{-4}$.

However, for short times, the estimate above is singular at $t=0$, then we estimate

$$\varphi(t,\tau,\xi)\gtrsim t^{2(1-\beta)}\tau^2+\tau^{\frac{2}{\beta}},$$

so that

$$\begin{aligned}t^{1-\beta}\int_{J_{t,\xi}}\frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau&\lesssim t^{1-\beta}\int_{J_{t,\xi}}\frac{\tau^{\frac{1}{\beta}-2}}{t^{2(1-\beta)}+\tau^{2\left(\frac{1}{\beta}-1\right)}}\,d\tau\\&\lesssim t^{1-\beta}\int_0^\infty\frac{1}{t^{2(1-\beta)}+s^2}\,ds\leq C,\end{aligned}$$

where we used first the change of variable $s=\tau^{\frac{1}{\beta}-1}$ and then the change of variable $r=s/t^{1-\beta}$. Summarizing, at short time, we may estimate

$$t^{1-\beta}\int_{J_{t,\xi}}\frac{e^{-\tau^{\frac{1}{\beta}}}\tau^{\frac{1}{\beta}}}{\varphi(t,\tau,\xi)}\,d\tau\lesssim\langle t^{\frac{1-\beta}{4}}\xi\rangle^{-4}.$$

Comparing with the estimate of the integral over $I_{t,\xi}$, we conclude the proof of (3.9) for $t\leq 1$, using $\langle t^{\frac{1-\beta}{4}}\xi\rangle^{-4}\leq\langle t^{\frac{1-\beta}{2}}\xi\rangle^{-4}$.

We now prove (3.10). If we use (3.8) to estimate $\varphi(t,\tau,\xi)\gtrsim t^{2(1-\beta)}|\xi|^4$ when we integrate over $J_{t,\xi}$, then we find

$$\int_{J_{t,\xi}}e^{-\tau^{\frac{1}{\beta}}}\frac{t^{2(1-\beta)}|\xi|^2}{\varphi(t,\tau,\xi)}\,d\tau\lesssim|\xi|^{-2}\int_0^\infty e^{-\tau^{\frac{1}{\beta}}}\,d\tau\lesssim|\xi|^{-2}.$$

On the other hand, if we use (3.8) to estimate $\varphi(t,\tau,\xi)\gtrsim t^{2(1-\beta)}(|\xi|^4+\tau^2)$, we obtain

$$\int_{J_{t,\xi}}e^{-\tau^{\frac{1}{\beta}}}\frac{t^{2(1-\beta)}|\xi|^2}{\varphi(t,\tau,\xi)}\,d\tau\lesssim|\xi|^2\int_0^\infty\frac{1}{|\xi|^4+\tau^2}\,d\tau=\int_0^\infty\frac{1}{1+\rho^2}\,d\rho=\frac{\pi}{2}.$$

Therefore,

$$\int_{J_{t,\xi}} e^{-\tau^{\frac{1}{\beta}}} \frac{t^{2(1-\beta)}|\xi|^2}{\varphi(t,\tau,\xi)}\, d\tau \lesssim \langle\xi\rangle^{-2}.$$

Now we consider the integral over $I_{t,\xi}$. In this case, we get

$$\begin{aligned} t^{2(1-\beta)}|\xi|^2 & \int_{I_{t,\xi}} e^{-\tau^{\frac{1}{\beta}}} \frac{1}{\varphi(t,\tau,\xi)}\, d\tau \\ & \lesssim t^{2(1-\beta)}|\xi|^2\, \tau_0^{1-\frac{1}{\beta}}\, e^{-(\tau_0/2)^{\frac{1}{\beta}}} \int_{\tau_0/2}^{2\tau_0} \frac{\tau^{\frac{1}{\beta}-1}}{(t^{1-\beta}|\xi|^2 - \tau^{\frac{1}{\beta}})^2 + t^{2(1-\beta)}\tau_0^2}\, d\tau \\ & \lesssim t^{2(1-\beta)}|\xi|^2\, \tau_0^{1-\frac{1}{\beta}}\, e^{-(\tau_0/2)^{\frac{1}{\beta}}} \int_{-\infty}^{+\infty} \frac{1}{s^2 + t^{2(1-\beta)}\tau_0^2}\, ds \approx e^{-(\tau_0/2)^{\frac{1}{\beta}}}, \end{aligned}$$

where we used first the change of variable $s = \tau^{\frac{1}{\beta}} - t^{1-\beta}|\xi|^2$ and then the change of variable $r = s/(t^{1-\beta}\tau_0)$. Therefore,

$$t^{2(1-\beta)}|\xi|^2 \int_{I_{t,\xi}} e^{-\tau^{\frac{1}{\beta}}} \frac{1}{\varphi(t,\tau,\xi)}\, d\tau \lesssim \langle t^{1-\beta}|\xi|^2\rangle^{-M},$$

for any $M \geq 0$. Comparing with the estimate over $J_{t,\xi}$, we conclude the proof of (3.10). □

As a straightforward consequence of (3.5) and of Lemma 3, we have the following estimates.

Lemma 4 *For any* $s \in [0, 4]$, *we have*

$$\|\hat{K}_1(t,\cdot)|\xi|^s\|_{L^\infty} \leq \begin{cases} Ct^{-(1-\beta)-\frac{s\beta}{2}} & \text{if } t \geq 1, \\ Ct^{-\frac{s}{2}} & \text{if } t \leq 1; \end{cases} \tag{3.11}$$

furthermore, it holds

$$\|\hat{K}_0(t,\cdot)\|_{L^\infty} \leq C, \tag{3.12}$$

for any $t \geq 0$. *Moreover, for any* $t \geq 1$, *and* $q \in [1, \infty)$,

$$\|\hat{K}_1(t,\cdot)|\xi|^s\|_{L^q} \leq Ct^{-(1-\beta)-\frac{n\beta}{2q}-\frac{s\beta}{2}}, \tag{3.13}$$

provided that $s + n/q < 4$, *and*

$$\|\hat{K}_0(t,\cdot)|\xi|^s\|_{L^q} \leq Ct^{-\frac{n\beta}{2q}-\frac{s\beta}{2}}, \tag{3.14}$$

provided that $s + n/q < 2$. *Here* C *is positive constant which does not depend on* t.

We stress that since (3.11) holds for any $s \in [0, 4]$, we may also write it in the form

$$\|\hat{K}_1(t,\cdot)|\xi|^s\langle\xi\rangle^{4-s}\|_{L^\infty} \leq C t^{-(1-\beta)-\frac{s\beta}{2}} \qquad \text{if } t \geq 1, \tag{3.15}$$

$$\|\hat{K}_1(t,\cdot)\langle\xi\rangle^s\|_{L^\infty} \leq C t^{-\frac{s}{2}} \qquad \text{if } t \leq 1, \tag{3.16}$$

for any $s \in [0, 4]$.

Proof For any $\xi \in \mathbb{R}^n$ and $s \in [0, 4]$, we use (3.9) to obtain

$$|\hat{R}_1(t,\xi)|\xi|^s| \lesssim \begin{cases} t^{-(1-\beta)} & \text{if } t \geq 1, \\ t^{-(1-\beta)\frac{s}{2}} & \text{if } t \leq 1. \end{cases}$$

On the other hand, if $t \geq 1$, we get

$$\left(\int_{\mathbb{R}^n} |\hat{R}_1(t,\xi)|\xi|^s|^q \, d\xi\right)^{\frac{1}{q}} \lesssim t^{-(1-\beta)} \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{-4q} \, |\xi|^{sq} \, d\xi\right)^{\frac{1}{q}} \approx t^{-(1-\beta)},$$

provided that $s + n/q < 4$. Estimate (3.12) immediately follows from (3.10). Moreover, if $t \geq 1$, we get

$$\left(\int_{\mathbb{R}^n} |\hat{R}_0(t,\xi)|\xi|^s|^q \, d\xi\right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{-2q} \, |\xi|^{sq} \, d\xi\right)^{\frac{1}{q}} \leq C,$$

provided that $s + n/q < 2$. □

4 Proof of the main results

Proof of Theorem 1 Using (3.4), we may estimate

$$\|u(t,\cdot)\|_{H^s} = \|\langle\xi\rangle^s \, \hat{u}(t,\cdot)\|_{L^2} \leq \|\hat{K}_0(t,\cdot)\|_{L^\infty} \, \|\langle\xi\rangle^s \, \hat{u}_0\|_{L^2} + \int_0^t \|\langle\xi\rangle^s \, \hat{K}_1(t-\tau,\cdot) \, \hat{g}(\tau,\cdot)\|_{L^2} \, d\tau;$$

in order to estimate the last term, we split the integral in the two domains $[0, (t-1)_+]$ and $[(t-1)_+, t]$. Then, we apply estimate (3.15) in the first interval and (3.16) in the second one:

$$\begin{aligned} &\int_0^t \|\langle\xi\rangle^s \, \hat{K}_1(t-\tau,\cdot) \, \hat{g}(\tau,\cdot)\|_{L^2} \, d\tau \\ &\quad \leq \int_0^{(t-1)_+} \|\langle\xi\rangle^4 \, \hat{K}_1(t-\tau,\cdot)\|_{L^\infty} \, \|\langle\xi\rangle^{s-4} \, \hat{g}(\tau,\cdot)\|_{L^2} \, d\tau \\ &\qquad + \int_{(t-1)_+}^t \|\langle\xi\rangle^{2-\varepsilon} \, \hat{K}_1(t-\tau,\cdot)\|_{L^\infty} \, \|\langle\xi\rangle^{s+\varepsilon-2} \hat{g}(\tau,\cdot)\|_{L^2} \, d\tau \end{aligned}$$

$$\leq C \int_0^{(t-1)_+} (t-\tau)^{-(1-\beta)} \|g(\tau,\cdot)\|_{H^{s-4}} \, d\tau \\ + C \|g\|_{L^\infty_t H^{s+\varepsilon-2}} \int_{(t-1)_+}^t (t-\tau)^{-1+\frac{\varepsilon}{2}} \, d\tau.$$

This proves (1.8). The second part of the statement follows noticing that

$$\int_0^t (t-\tau)^{-(1-\beta)} \langle\tau\rangle^{-\beta} \, d\tau$$

is bounded with respect to t, since $\beta \in (0,1)$. □

The proof of Theorem 2 will be given throughout the following two main lemmas.

Lemma 5 *Let $n \leq 3$ and $s \in [0, 2-n/2)$. For any $u_0 \in H^s \cap L^1$ it holds*

$$\left\| K_0^\dagger(t,\cdot) * u_0 - \left(\int_{\mathbb{R}^n} u_0(x)\, dx \right) K_0^\dagger(t,\cdot) \right\|_{\dot{H}^s} = o(t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}), \tag{4.1}$$

as $t \to \infty$, where $K_0^\dagger$ is the fundamental solution to (1.3).

Proof We follow as in [13, Lemma 4]. We first remark that, as in (3.14), with $q = 2$, we have

$$\|K_0^\dagger(t,\cdot)\|_{\dot{H}^s} = t^{-\frac{n\beta}{4}-\frac{s\beta}{2}} \|K_0^\dagger(1,\cdot)\|_{\dot{H}^s} = C\, t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}, \tag{4.2}$$

for any $t > 0$ and $s \in [0, 2-n/2)$. Morover, it holds

$$\left\| K_0^\dagger(t,\cdot) * u_0 - \left(\int_{\mathbb{R}^n} u_0(x)\, dx \right) K_0^\dagger(t,\cdot) \right\|_{\dot{H}^s} = \| |\xi|^s K_0^\dagger(t,\cdot)\rho \|_{L^2},$$

where $\rho(\xi) = \hat{u}_0(\xi) - \hat{u}_0(0)$. Since we are assuming $u_0 \in L^1(\mathbb{R}^n)$, the function ρ is continuous and then, for any $\varepsilon > 0$ there exists δ sufficiently small such that $|\rho(\xi)| \leq \varepsilon$ for any $|\xi| < \delta$. Thus, on the one hand we can estimate

$$\int_{|\xi|<\delta} |\xi|^{2s} |\hat{K}_0^\dagger(t,\xi)\rho(\xi)|^2 \, d\xi \lesssim \varepsilon^2 \|K_0^\dagger(t,\cdot)\|^2_{\dot{H}^s} = C\, \varepsilon^2 t^{-\frac{n\beta}{2}-s\beta}, \tag{4.3}$$

due to (4.2). On the other hand, applying the change of variable $\eta = \xi t^{\frac{\beta}{2}}$, for t large enough we find

$$\int_{|\xi|\geq\delta} |\xi|^{2s} |\hat{K}_0^\dagger(t,\xi)\rho(\xi)|^2\,d\xi$$
$$\lesssim \int_{|\xi|\geq\delta} \Big| \int_0^\infty e^{-\tau^{\frac{1}{\beta}}} \frac{|\xi|^{2+s} t^{2\beta-1}}{t^{2\beta}|\xi|^4 + 2\tau|\xi|^2 t^\beta \cos(\beta\pi) + \tau^2}\,d\tau \Big|^2 d\xi$$
$$= t^{-\frac{n\beta}{2}-\beta s+2\beta-2} \int_{|\eta|\geq\delta t^{\frac{\beta}{2}}} \Big| \int_0^\infty e^{-\tau^{\frac{1}{\beta}}} \frac{|\eta|^{2+s}}{|\eta|^4+\tau^2}\,d\tau \Big|^2 d\eta$$
$$\lesssim t^{-\frac{n\beta}{2}-\beta s+2\beta-2} \int_{|\eta|\geq\delta t^{\frac{\beta}{2}}} |\eta|^{-4+2s}\,d\eta \lesssim \varepsilon^2 t^{-\frac{n\beta}{2}-\beta s},$$

for sufficiently large t. Indeed, due to $s < 2 - n/2$, the last integral tends to 0 as $t \to \infty$. □

Lemma 6 *Let $s \in [0, 2 - n/2)$. Then, it holds*

$$\|K_0(t,\cdot) - K_0^\dagger(t,\cdot)\|_{\dot{H}^s} \leq C t^{-\frac{n\beta}{4}-\frac{s\beta}{2}-(1-\beta)}, \tag{4.4}$$

for $t \geq 1$, where C is a positive constant which does not depend on t.

Proof We first note that it holds

$$\|K_0(t,\cdot) - K_0^\dagger(t,\cdot)\|_{\dot{H}^s} = t^{-\frac{n\beta}{4}-\frac{s\beta}{2}} \|R_0(t,\cdot) - K_0^\dagger(1,\cdot)\|_{\dot{H}^s}. \tag{4.5}$$

In particular, we have

$$\hat{R}_0(t,\xi) - \hat{K}_0^\dagger(1,\xi) = \frac{\sin(\beta\pi)}{\beta\pi} \int_0^\infty e^{-\tau^{\frac{1}{\beta}}} \psi(\tau,\xi)\,d\tau,$$

where

$$\psi(\tau,\xi) = \frac{|\xi|^2 \tau^{\frac{1}{\beta}} (2t^{1-\beta}|\xi|^2 - \tau^{\frac{1}{\beta}} + 2\tau t^{1-\beta}\cos(\beta\pi))}{\varphi(t,\tau,\xi)(|\xi|^4 + 2\tau|\xi|^2\cos(\beta\pi) + \tau^2)}.$$

By using (3.8), on the one hand, being $t^{1-\beta}|\xi|^2 \sim \tau^{\frac{1}{\beta}}$ in $I_{t,\xi}$ for any $t \geq 1$ we can estimate

$$|\psi(\tau,\xi)| \lesssim \frac{t^{-(1-\beta)}(\tau^{\frac{2}{\beta}} + \tau^{1+\frac{1}{\beta}})}{\tau^2(|\xi|^4+\tau^2)^{\frac{1}{2}}} \lesssim t^{-(1-\beta)} \langle\xi\rangle^{-2} \tau^{-p_\beta}, \qquad \text{if } \tau \in I_{t,\xi},$$

for some $p_\beta > -1$; on the other hand, for any $t \geq 1$ we have

$$|\psi(\tau,\xi)| \lesssim \frac{t^{-(1-\beta)}\tau^{\frac{1}{\beta}}(|\xi|^2 + \tau^{\frac{1}{\beta}} + \tau)}{(|\xi|^4+\tau^2)^{\frac{3}{2}}} \lesssim t^{-(1-\beta)} \langle\xi\rangle^{-2} \tau^{-q_\beta}, \qquad \text{if } \tau \in J_{t,\xi},$$

for some $q_\beta > -1$.

As a consequence, we can easily conclude

$$\|R_0(t,\cdot) - K_0^\dagger(1,\cdot)\|_{\dot H^s} \lesssim t^{-(1-\beta)},$$

since $n < 4 - 2s$. The proof of the desired result follows by identity (4.5). □

Proof of Theorem 2 The solution to (1.1) is in $\mathcal{C}_b([0,\infty), H^s)$ thanks to Theorem 1. By using (3.4), on the one hand, we may estimate

$$\begin{aligned}\|u(t,\cdot)\|_{\dot H^s} &\leq C(1+t)^{-\frac{n\beta}{4}-\frac{s\beta}{2}}(\|u_0\|_{\dot H^s} + \|u_0\|_{L^1}) \\ &\quad + \int_0^t \||\xi|^s \hat K_1(t-\tau)\hat g(\tau,\cdot)\|_{L^2}\,d\tau; \end{aligned} \tag{4.6}$$

here, we used (3.12) for $t \leq 1$ and (3.14), together with Plancherel and Riemann-Lebesgue theorem, for $t > 1$; on the other hand, we have

$$\begin{aligned}\|u(t,\cdot) - MK_0^\dagger(t,\cdot)\|_{\dot H^s} &= \||\xi|^s\,(\hat u(t,\cdot) - M\hat K_0^\dagger(t,\cdot))\|_{L^2} \\ &\leq \||\xi|^s\,(\hat K_0(t,\cdot)\hat u_0 - M\hat K_0^\dagger(t,\cdot))\|_{L^2} \\ &\quad + \int_0^t \||\xi|^s\,\hat K_1(t-\tau,\cdot)\hat g(\tau,\cdot)\|_{L^2}\,d\tau. \end{aligned} \tag{4.7}$$

Recalling that $s < 2 - n/2$, we use Plancherel and Riemann-Lebesgue theorem together with Lemmas 5 and 6 to conclude that

$$\begin{aligned}\||\xi|^s\,(\hat u(t,\cdot) - M\hat K_0^\dagger(t,\cdot))\|_{L^2} &\lesssim \|K_0(t,\cdot) - K_0^\dagger(t,\cdot)\|_{\dot H^s}\|u_0\|_{L^1} \\ &\quad + \|K_0^\dagger(t,\cdot) * u_0 - MK_0^\dagger(t,\cdot)\|_{\dot H^s} = o(t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}).\end{aligned}$$

It remains to estimate the integral term in (4.6) and (4.7): let $t \leq T$, with $T > 0$ arbitrarily large; then, as a consequence of assumption (1.12) and estimate (3.11) we obtain

$$\int_0^t \||\xi|^s\,\hat K_1(t-\tau,\cdot)\hat g(\tau,\cdot)\|_{L^2}\,d\tau \leq B_1 \int_0^t (t-\tau)^{-1+\frac{\varepsilon}{2}}d\tau \leq CB_1; \tag{4.8}$$

Let us consider now $t > T$; we separately estimate three integrals:

$$\begin{aligned} I_1 &= \int_0^{t/2} \||\xi|^s\,\hat K_1(t-\tau,\cdot)\hat g(\tau,\cdot)\|_{L^2}\,d\tau, \\ I_2 &= \int_{t/2}^{t-1} \||\xi|^s\,\hat K_1(t-\tau,\cdot)\hat g(\tau,\cdot)\|_{L^2}\,d\tau, \\ I_3 &= \int_{t-1}^{t} \||\xi|^s\,\hat K_1(t-\tau,\cdot)\hat g(\tau,\cdot)\|_{L^2}\,d\tau. \end{aligned}$$

Due to $s < 2$, using (3.15), we may estimate

$$
\begin{aligned}
I_2 &\leq \int_{t/2}^{t-1} \||\xi|^s \langle\xi\rangle^{4-s} \hat{K}_1(t-\tau,\cdot)\|_{L^\infty} \|\langle\xi\rangle^{s-4} \hat{g}(\tau,\cdot)\|_{L^2}\, d\tau \\
&\leq C \int_{t/2}^{t-1} (t-\tau)^{-(1-\beta)-\frac{s\beta}{2}} \|g(\tau,\cdot)\|_{H^{s-4}}\, d\tau \\
&\leq C B_1 \langle t\rangle^{-\frac{n}{4}\beta-\beta} \int_{t/2}^{t-1} (t-\tau)^{-(1-\beta)-\frac{s\beta}{2}}\, d\tau \leq C B_1 \langle t\rangle^{-\frac{n}{4}\beta-\frac{s\beta}{2}}. \qquad (4.9)
\end{aligned}
$$

Similarly, if we assume that

$$
\frac{n}{4}\beta + \beta < 1, \qquad (4.10)
$$

then, using (3.15), we may estimate

$$
\begin{aligned}
I_1 &\leq \int_0^{t/2} \||\xi|^s \langle\xi\rangle^{4-s} \hat{K}_1(t-\tau,\cdot)\|_{L^\infty} \|\langle\xi\rangle^{s-4} \hat{g}(\tau,\cdot)\|_{L^2}\, d\tau \\
&\leq C \int_0^{t/2} (t-\tau)^{-(1-\beta)-\frac{s\beta}{2}} \|g(\tau,\cdot)\|_{H^{s-4}}\, d\tau \\
&\leq C B_1\, t^{-(1-\beta)-\frac{s\beta}{2}} \int_0^{t/2} \langle\tau\rangle^{-\frac{n}{4}\beta-\beta}\, d\tau \leq C B_1\, t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}. \qquad (4.11)
\end{aligned}
$$

If (4.10) does not hold, that is, (1.10) holds, using (3.15) we may estimate

$$
I_1 \leq \int_0^{t/2} \||\xi|^{s+a} \langle\xi\rangle^{4-s-a} \hat{K}_1(t-\tau,\cdot)\|_{L^\infty} \||\xi|^{-a} \langle\xi\rangle^{s+a-4} \hat{g}(\tau,\cdot)\|_{L^2}\, d\tau,
$$

where a is as in (1.11). By the Hardy-Littlewoow-Sobolev theorem,

$$
\||\xi|^{-a} \hat{f}\|_{L^2} = \|I_a f\|_{L^2} \leq C\, \|f\|_{L^q},
$$

provided that $q \in (1, 2)$, where a is as in (1.11). Therefore,

$$
\||\xi|^{-a} \langle\xi\rangle^{s+a-4} \hat{g}(\tau,\cdot)\|_{L^2} \leq C\, \|g(\tau,\cdot)\|_{H^{s-4+a,q}}.
$$

We obtain

$$
\begin{aligned}
I_1 &\leq C \int_0^{t/2} (t-\tau)^{-(1-\beta)-\frac{s\beta}{2}-\frac{n}{2}\beta\left(\frac{1}{q}-\frac{1}{2}\right)} \|g(\tau,\cdot)\|_{H^{s-4+a,q}}\, d\tau \\
&\leq C B_2\, t^{-(1-\beta)-\frac{s\beta}{2}-\frac{n}{2}\beta\left(\frac{1}{q}-\frac{1}{2}\right)} \int_0^{t/2} \langle\tau\rangle^{-\frac{n}{2}\beta\left(1-\frac{1}{q}\right)-\beta}\, d\tau \leq C B_2\, t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}. \qquad (4.12)
\end{aligned}
$$

Due to $s < 2$, by using estimate (3.16), we obtain

$$\begin{aligned} I_3 &\leq \int_{t-1}^{t} \| |\xi|^s \langle\xi\rangle^{2-s-\varepsilon} \hat{K}_1(t-\tau,\cdot)\|_{L^\infty} \|\langle\xi\rangle^{s-2+\varepsilon}\hat{g}(\tau,\cdot)\|_{L^2}\, d\tau \\ &\leq C \int_{t-1}^{t} (t-\tau)^{-1+\frac{\varepsilon}{2}} \|g(\tau,\cdot)\|_{H^{s-2+\varepsilon}}\, d\tau \\ &\leq C B_1\, t^{-\frac{n\beta}{4}-\frac{s\beta}{2}} \int_{t-1}^{t} (t-\tau)^{-1+\frac{\varepsilon}{2}}\, d\tau \leq C B_1\, t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}. \end{aligned} \tag{4.13}$$

As a consequence, by (4.6) we conclude

$$\|u(t,\cdot)\|_{H^s} \leq C(1+t)^{-\frac{n\beta}{4}-\frac{s\beta}{2}}(\|u_0\|_{H^s\cap L^1} + B_1 + B_2).$$

Moreover, if $\limsup_{t\to\infty} Q_1(t) = 0$ and, in addition, $\limsup_{t\to\infty} Q_2(t) = 0$ when (1.10) holds, then we can also conclude

$$\|u(t,\cdot) - M K_0^{\dagger}(t,\cdot)\|_{\dot{H}^s} = o(t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}),$$

as a consequence of (4.7); indeed, for any $\varepsilon > 0$ we can choose $T > 0$ sufficiently large such that for any $t > T/2$

$$\|g(t,\cdot)\|_{H^{s-4}} \leq \varepsilon\langle t\rangle^{-\frac{n\beta}{4}-\beta}, \qquad \|g(t,\cdot)\|_{H^{s-2+\varepsilon}} \leq \varepsilon\langle t\rangle^{-\frac{n\beta}{4}-\frac{s\beta}{2}},$$

and

$$\|g(t,\cdot)\|_{H^{s-4+a,q}} \leq \varepsilon\langle t\rangle^{-\frac{n\beta}{2}(1-\frac{1}{q})-\beta},$$

where a and q are as in (1.11); then, estimates (4.9) and (4.13) still hold replacing B_1 and B_2 by ε. Additionally, if (4.10) holds we can estimate

$$\begin{aligned} I_1 &\leq C t^{-(1-\beta)-\frac{s\beta}{2}} \left(B_1 \int_0^{T/2} \langle\tau\rangle^{-\frac{n\beta}{4}-\beta}\, d\tau + \varepsilon \int_{T/2}^{t} \langle\tau\rangle^{-\frac{n\beta}{4}-\beta}\, d\tau \right) \\ &\leq C B_1\, t^{-(1-\beta)-\frac{s\beta}{2}} + C\varepsilon t^{-\frac{n\beta}{4}-\frac{s\beta}{2}} \leq C_1 \varepsilon t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}, \end{aligned}$$

due to condition (4.10); similarly, if (4.10) does not hold we have

$$\begin{aligned} I_1 &\leq C\, t^{-(1-\beta)-\frac{s\beta}{2}-\frac{n}{2}\beta\left(\frac{1}{q}-\frac{1}{2}\right)} \left(B_2 \int_0^{T/2} \langle\tau\rangle^{-\frac{n}{2}\beta\left(1-\frac{1}{q}\right)-\beta}\, d\tau \right. \\ &\quad \left. + \varepsilon \int_{T/2}^{t} \langle\tau\rangle^{-\frac{n}{2}\beta\left(1-\frac{1}{q}\right)-\beta}\, d\tau \right) \\ &\leq C B_2\, t^{-(1-\beta)-\frac{s\beta}{2}-\frac{n}{2}\beta\left(\frac{1}{q}-\frac{1}{2}\right)} + C\varepsilon t^{-\frac{n\beta}{4}-\frac{s\beta}{2}} \leq C_1 \varepsilon t^{-\frac{n\beta}{4}-\frac{s\beta}{2}}, \end{aligned}$$

as a consequence of condition (1.11). This completes the proof of Theorem 2. □

We now prove Corollary 1.

Proof of Corollary 1 We equip the evolution space $X(T) = \mathcal{C}_b([0,T], H^s)$ with the norm

$$\|u\|_{X(T)} = \sup_{0\le t\le T} (1+t)^{\frac{n\beta}{4}} \Big(\|u(t,\cdot)\|_{L^2} + (1+t)^{\frac{s\beta}{2}} \|u(t,\cdot)\|_{\dot{H}^s} \Big),$$

and we define the operator

$$N : u \in X(T) \to Nu(t,x) = K_0(t,\cdot) * u_0 + Fu(t,\cdot),$$

where

$$Fu(t,\cdot) = \int_0^t K_1(t-\tau,\cdot) * f(u(\tau,\cdot))\, d\tau.$$

We will prove the existence of the unique global (in time) solution to (1.17) as the fixed point of the operator N. Hence, in order to get the global (in time) existence and uniqueness of the solution in $X(T)$, we need to prove the following two crucial estimates:

$$\|Nu\|_{X(T)} \le C\, \|u_0\|_{L^1\cap \dot{H}^s} + \|u\|_{X(T)}^p, \tag{4.14}$$

$$\|Nu - Nv\|_{X(T)} \le C\, \|u-v\|_{X(T)} \Big(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \Big), \tag{4.15}$$

with $C > 0$, independent of T.

As a consequence of Banach's fixed point theorem, the conditions (4.14) and (4.15) guarantee the existence of a uniquely determined solution u to (1.17). We simultaneously gain a local and a global (in time) existence result.

Indeed, let $R > 0$ be such that $CR^{p-1} < 1/2$. Then N is a contraction on $X_R(T) = \{u \in X(T) :\ \|u\|_{X(T)} \le R\}$, thanks to (4.15). The solution to (1.17) is a fixed point for N, so if $\|K_0(t,\cdot) *_{(x)} u_0\|_{X(T)} \le R/2$, then $u \in X_R(T)$, thanks to (4.14). As a consequence, the uniqueness and existence of the solution in $X_R(T)$ follows by the Banach fixed point theorem on contractions. The condition $\|K_0(t,\cdot) *_{(x)} u_0\|_{X(T)} \le R/2$ is obtained taking initial data verifying $\|u_0\|_{L^1\cap H^s} \le \varepsilon$, with ε such that $C\varepsilon \le R/2$. Since C, R and ε do not depend on T, the solution is global (in time).

Let us prove estimates (4.14) and (4.15).

For any $u \in X(T)$, by Gagliardo-Nirenberg inequality, we obtain

$$\|u\|_{L^r} \le (1+t)^{-\frac{n}{2}\beta\left(1-\frac{1}{r}\right)} \|u\|_{X(T)}, \quad r \in [2, 2p],$$

since $H^s \hookrightarrow L^{2p}$. Therefore, for any $u, v \in X(T)$, the function

$$g(t,x) = f(u(t,x)) - f(v(t,x))$$

verifies the estimate

$$\begin{aligned}\|g(t,\cdot)\|_{L^q} &\leq C\,\|(u-v)(t,\cdot)\|_{L^{qp}}\left(\|u(t,\cdot)\|_{L^{qp}}^{p-1}+\|v(t,\cdot)\|_{L^{qp}}^{p-1}\right)\\ &\leq C'\,(1+t)^{-(p-1)\frac{n}{2}\beta-\frac{n}{2}\beta\left(1-\frac{1}{q}\right)}\,\|u-v\|_{X(T)}\left(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\right),\end{aligned}$$

for any $q\in[1,2]$. Due to $p\geq 1+2/n$, we obtain

$$\|g(t,\cdot)\|_{L^q}=\|u-v\|_{X(T)}\left(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\right)t^{-\frac{n\beta}{2}\left(1-\frac{1}{q}\right)-\beta}.$$

Thus, (1.15) holds and we can apply Theorem 2. Thus, (1.13) implies estimate (4.14) and, taking $u_0\equiv 0$ in (1.13) we obtain (4.15).

In particular, if $p>1+2/n$ we have

$$\|g(t,\cdot)\|_{L^q}=\|u-v\|_{X(T)}\left(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\right)o(t^{-\frac{n\beta}{2}\left(1-\frac{1}{q}\right)-\beta}),$$

and then, we prove (1.20) as a consequence of (1.14). □

5 Conclusions

In Theorem 1, we provided sufficient conditions for the non-homogeneous term $g(t,x)$ which allow to obtain the boundness of the solution to (1.1) in H^s; then, we describe its asymptotic profile in Theorem 2, showing that it is independent of g, provided that $g(t,\cdot)$ satisfies some additional decay assumptions. This latter effect is related to the special structure of the solution to the Cauchy-type problem for equations with Caputo fractional derivatives, in relation to the non-homogeneous term $g(t,x)$.

There exist many functions $g(t,\cdot)$ which satisfy the desired conditions. Let us test the assumptions of Theorem 1 for the special class of auto-similar g:

$$g(t,x)=t^\gamma\,h(t^{\frac{2\gamma}{n}}x),\quad\text{for some }\gamma>0\text{ and for any }t>0.$$

Let $h\in L^1\cap L^2$ and $s<\min\{4-n/2,2\}$. In particular, $g\in L^\infty_t L^2$, since

$$\|g(t,\cdot)\|_{L^2}=t^\gamma\,\|h(t^{\frac{2\gamma}{n}}\cdot)\|_{L^2}=\|h\|_{L^2}.$$

On the other hand, for any $t>0$ it holds

$$\|g(t,\cdot)\|_{L^1}=t^\gamma\,\|h(t^{\frac{2\gamma}{n}}\cdot)\|_{L^1}=t^{-\gamma}\,\|h\|_{L^1}.$$

Therefore, we can estimate

$$\|g(t,\cdot)\|_{H^{s-4}}\leq\|g(t,\cdot)\|_{H^{s-2+\varepsilon}}\leq\|g\|_{L^\infty_t L^2_x}=\|h\|_{L^2},\tag{5.1}$$

for any $t > 0$ (here ε is sufficiently small, namely $\varepsilon \le 2 - s$); on the other hand, for any $t > 1$ we may produce a decay rate $t^{-\gamma}$, by the estimate

$$\begin{aligned}\|g(t,\cdot)\|_{H^{s-4}} &\le \|\langle\xi\rangle^{s-4}\|_{L^2}\|\hat{g}(t,\cdot)\|_{L^\infty}\\ &\le \|\langle\xi\rangle^{s-4}\|_{L^2}\|g(t,\cdot)\|_{L^1} = C(n,s)\,t^{-\gamma}\|h\|_{L^1},\end{aligned}$$

where $C(n,s)$ is finite, due to the assumption $s < 4 - n/2$. Therefore, if $\gamma \ge \beta$, Theorem 1 may be applied with

$$A = \sup_{t\ge 0}\langle t\rangle^{\beta}\|g(t,\cdot)\|_{H^{s-4}} \lesssim \|h\|_{L^1\cap L^2},$$

so that we get the estimate

$$\|u\|_{L^\infty_t H^s} \le C\left(\|u_0\|_{H^s} + \|h\|_{L^1\cap L^2}\right).$$

In high space dimension $n \ge 5$, it may happen that $4 - n/2 \le s < 2$. In this latter case, let $r \in (2,\infty)$ be such that $r(4-s) > n$, and fix $m \in (1,2)$ be such that

$$\frac{1}{2} = \frac{1}{r} + \frac{1}{m'}, \quad \text{i.e.} \quad \frac{1}{r} = \frac{1}{m} - \frac{1}{2}.$$

Then we may produce a decay rate $t^{-2\gamma\left(\frac{1}{m}-\frac{1}{2}\right)}$ for any $t > 1$, by the estimate

$$\begin{aligned}\|g(t,\cdot)\|_{H^{s-4}} &\le \|\langle\xi\rangle^{s-4}\|_{L^r}\|\hat{g}(t,\cdot)\|_{L^{m'}}\\ &\lesssim \|\langle\xi\rangle^{s-4}\|_{L^r}\|g(t,\cdot)\|_{L^m} = C(n,s,m)\,t^{-2\gamma\left(\frac{1}{m}-\frac{1}{2}\right)}\|h\|_{L^m},\end{aligned}$$

where $C(n,s,m)$ is finite, due to the assumption $r(4-s) > n$. Therefore, if $2\gamma\left(\frac{1}{m}-\frac{1}{2}\right) \ge \beta$, Theorem 1 applies. Given a fixed s and n, such m exists if $2\gamma \ge r\beta$ for some $r > n/(4-s)$, that is, if $2\gamma > n\beta/(4-s)$.

One may proceed with similar reasoning to test the assumptions of Theorem 2 for a self-similar g.

One could also investigate the possibility to take $g(t,\cdot)$ in different functional spaces; for instance, following the proof of Theorem 1 one could also prove that u is in $\mathcal{C}_b([0,\infty), H^s)$ and

$$\|u\|_{L^\infty_t H^s} \le C\left(\|u_0\|_{H^s} + A\right) + C_\varepsilon\|g\|_{L^\infty_t H^{s-2+\varepsilon+n\left(\frac{1}{2}-\frac{1}{q}\right),q}},$$

assuming

$$g \in L^\infty_t H^{s-2+\varepsilon+n\left(\frac{1}{2}-\frac{1}{q}\right),q} \cap L^\infty_t H^{s-4+\varepsilon+n\left(\frac{1}{2}-\frac{1}{r}\right),r},$$

and

$$A = \sup_{t\geq 0} \langle t\rangle^{\left(\beta - \frac{n\beta}{4} + \frac{n\beta}{2r}\right)_+} \|g\|_{L_t^\infty H^{s-4+\varepsilon+n\left(\frac{1}{2}-\frac{1}{r}\right),r}} < \infty,$$

for some $q \in [1, 2n/(n-4)_+]$ and $r \geq 1$.

The validity of our main results could be discussed also for the Cauchy-type problem associated to a more general fractional differential equation in the form

$$\partial_t^\alpha u + \partial_t^\beta u - \Delta u = g(t,x), \quad t > 0, \ x \in \mathbb{R}^n,$$

with $0 < \beta < \alpha < 2$. On the one hand, if both α and β belong to $(0, 1)$, then one can apply the same approach used in the present paper to investigate the H^s-boundness of the associated Cauchy-type problem and the asymptotic profile of its solution, assuming the initial data $u_0 = u(0, \cdot)$ to be integrable; on the other hand, if $0 < \beta < 1 < \alpha < 2$ new difficulties arise; in particular, the solution to the corresponding fractional ODE

$$\partial_t^\alpha y + \partial_t^\beta y + \lambda y = \hat{g}, \quad y(0) = c_0, \ y'(0) = c_1,$$

can be represented as

$$y(t) = c_0 H_0(t) + c_1 H_1(t) + \int_0^t \hat{g}(t-\tau) H_2(\tau)\, d\tau,$$

where

$$H_0(t) := \mathcal{L}^{-1}\left(\frac{s^{\alpha-1} + s^{\beta-1}}{h(s)}\right)(t), \qquad H_1(t) := \mathcal{L}^{-1}\left(\frac{s^{\alpha-2}}{h(s)}\right)(t),$$
$$H_2(t) := \mathcal{L}^{-1}\left(\frac{1}{h(s)}\right)(t),$$

with $h(s) := s^\alpha + s^\beta + \lambda$. Here, unlike the case $\alpha, \ \beta \in (0, 1)$ one need to take account of the roots of $h(s)$ in the complex plane in the derivation of an integral representation of the kernels H_i, $i = 0, 1, 2$. The study of these and other evolution models with multi-term time-fractional derivatives will be object of future investigations.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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