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Some reduction theorems for even-dimensional slant submanifolds of a $C_5 \oplus C_{12}$-manifold

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Abstract We establish conditions that allow us to reduce the codimension of a slant submanifold of an almost contact metric manifold which falls in the Chinea-Gonzales class $C_5 \oplus C_{12}$. The Reeb vector field of the ambient space is assumed to be orthogonal to the considered submanifolds.

Keywords almost contact metric manifold · slant immersion · reduction of codimension · space with constant curvature

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1 Introduction

The theory of slant submanifolds, introduced by B. Y. Chen in 1990 in the context of almost Hermitian Geometry, has been widely developed considering immersions into a Riemannian manifold endowed with an additional structure [2], [3], [11], [13], [14]. In particular, in 1996 A. Lotta introduced the concept of a slant submanifold of an almost contact metric (a.c.m.) manifold. Considering a non anti-invariant slant submanifold N, $\dim N = r$, of an a.c.m. manifold M with Reeb vector field ξ, then r is even (resp. r is odd) if and only if ξ is normal (resp. ξ is tangent) to N. The investigation of even-dimensional slant submanifolds is meaningful only if the structure of the ambient space is not a contact structure [12].

In [6] we established general properties of even-dimensional slant immersions into a manifold which falls in the Chinea-Gonzalez class $C_5 \oplus C_{12}$. This class consists of the a.c.m. manifolds that are locally realized as double-twisted products $]-\epsilon, \epsilon[\times_{(\lambda_1,\lambda_2)}\widehat{M}$, where $\lambda_1, \lambda_2\colon]-\epsilon, \epsilon[\times\widehat{M} \to \mathbb{R}$ are smooth positive functions and $(\widehat{M}, \widehat{J}, \widehat{g})$ is a Kähler manifold [10].

In [4] Chen stated a reduction of codimension theorem considering slant submanifolds of a complex space-form. The interrelation between Kähler and $C_5 \oplus C_{12}$-manifolds motivates the subject of this paper. In fact, we are going to prove three reduction of codimension theorems for even-dimensional slant submanifolds of a $C_5 \oplus C_{12}$-manifold. We apply the theorems stated by Erbacher for submanifolds of a Riemannian manifold with constant sectional curvature [9].

We point out that a local description of $C_5 \oplus C_{12}$-manifolds with constant sectional curvature has been recently obtained [8]. In particular, the hyperbolic space $\mathbb{H}^{2m+1}(-\alpha^2)$, endowed with the α-Kenmotsu structure given in [5], is the local model of non-flat $C_5 \oplus C_{12}$-manifolds with constant curvature. Note that the reduction of codimension for odd-dimensional slant submanifolds of $\mathbb{H}^{2m+1}(-\alpha^2)$ has also been discussed [7].

2 Preliminaries

Given an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$, let $f\colon (N, g') \to (M, \varphi, \xi, \eta, g)$ be an isometric immersion. For any $x \in N$, $X \in T_xN$, we adopt the identifications $x \equiv f(x)$ and $X \equiv (f_*)_x X$, $(f_*)_x$ being the tangential map. For any $X \in \Gamma(TN)$, $V \in \Gamma(T^\perp N)$, one puts

$$\varphi X = PX + FX, \quad \varphi V = tV + nV,$$

PX (resp. tV) and FX (resp. nV) denoting the tangential and normal components of φX (resp. φV), respectively. This allows us to define smooth maps

$$P\colon TN \to TN,\ F\colon TN \to T^\perp N,\ t\colon T^\perp N \to TN,\ n\colon T^\perp N \to T^\perp N,$$

inducing linear maps on each fibre. Moreover, denoting by ∇' the Levi-Civita connection of (N, g') and $\nabla^\perp$ the normal connection, the mixed covariant derivative $\overline{\nabla} F$ is defined by

$$(\overline{\nabla}_X F)Y = \nabla^\perp_X FY - F(\nabla'_X Y),$$

for any $X, Y \in \Gamma(TN)$.

The immersion f is said to be a slant immersion if for any $x \in N$, $X \in T_xN$, such that X, ξ are linearly independent, the angle $\theta(X) \in [0, \frac{\pi}{2}]$ between φX and T_xN is a constant, say θ. In this case, θ is named the slant angle of N in M and (N, f) is called a θ-slant submanifold of M. In particular, if $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$), (N, f) is an invariant (resp. anti-invariant) submanifold. If $\theta \neq 0, \frac{\pi}{2}$, (N, f) is called a proper slant submanifold [12].

We recall that, denoting by ∇ the Levi-Civita connection of $(M, \varphi, \xi, \eta, g)$, M belongs to the Chinea-Gonzalez class $C_5 \oplus C_{12}$ if and only if, for any $X, Y \in \Gamma(TM)$, the covariant derivative $\nabla\varphi$ satisfies the following condition

$$\begin{aligned}(\nabla_X \varphi)Y =& \alpha\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\} \\ & - \eta(X)\{\eta(Y)\varphi(\nabla_\xi \xi) + g(\nabla_\xi \xi, \varphi Y)\xi\},\end{aligned} \tag{2.1}$$

where $\dim M = 2m+1$, $\alpha = -\frac{\delta\eta}{2m}$. Note that the function α determines the C_5-component of $\nabla\varphi$, as well as $\nabla_\xi \xi$ defines its C_{12}-component. In particular, a C_5-manifold is also named an α-Kenmotsu manifold. Moreover, if $\nabla\varphi = 0$, then M is said to be a cosymplectic manifold [5].

Assume that (N, f) is a proper θ-slant submanifold of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ such that ξ is normal to N and denote by μ the vector subbundle of $T^\perp N$ whose fibre at any $x \in N$ is the orthogonal complement to $< \xi_x > \oplus F(T_x N)$ in $T_x^\perp N$, namely $T_x M = T_x N \oplus < \xi_x > \oplus F(T_x N) \oplus \mu_x$. We denote by h the second fundamental form of N in M and by A_V the Weingarten operator, for any $V \in \Gamma(T^\perp N)$. In [[6], Propositions 4.1, 5.4] it is proved that, if $\overline{\nabla} F = 0$, then for any $X, Y \in \Gamma(TN)$, $V \in \Gamma(\mu)$, one has

$$A_\xi X = -(\alpha \circ f)X, \quad \nabla_X^\perp \xi = 0, \quad A_V = 0, \tag{2.2}$$

$$(\sin^2\theta)\{h(X,Y) + (\alpha \circ f) g(X,Y)\xi\} = -Fth(X,Y). \tag{2.3}$$

Furthermore, $F(TN)$ is a parallel subbundle of $T^\perp N$, namely for any $V \in \Gamma(F(TN))$, $X \in \Gamma(TN)$, $\nabla_X^\perp V$ is a section of $F(TN)$.
Considering a point x of N, the first normal space at x is the linear space $N_x^1 = \mathrm{span}\{h(X,Y) | X, Y \in T_x N\}$.

Now, we recall the reduction theorems of codimension [1], [9].

Theorem 2.1 *Let (N, f) be a connected n-dimensional submanifold of a space-form $M^{n+p}(c)$. If there exists a parallel subbundle E of $T^\perp N$, rank$E = l$, such that for any $x \in N$ the first normal space N_x^1 is contained in E_x, then there exists a totally geodesic submanifold $\widehat{N}^{n+l}$ of M such that $f(N) \subset \widehat{N}^{n+l}$.*

The local version of Theorem 2.1 is obtained weakening the hypothesis that the ambient space is a space-form.

Theorem 2.2 *Let (N, f) be a connected n-dimensional submanifold of a Riemannian manifold M^{n+p} with constant sectional curvature c. Assume the existence of a parallel subbundle E of $T^\perp N$, rank$E = l$, such that for any $x \in N$, N_x^1 is a linear subspace of E_x. Then, for any $x \in N$ there exists a neighborhood U of x such that $f(U)$ is contained in an $(n+l)$-dimensional totally geodesic submanifold of M^{n+p}.*

Finally, we recall that (N, f) is said to be an austere submanifold if for any normal vector V the set of eigenvalues of the Weingarten operator A_V is invariant under multiplication by -1.

3 Reduction of the codimension

Let $(M, \varphi, \xi, \eta, g)$ be a connected $C_5 \oplus C_{12}$-manifold, $\dim M = 2m+1 \geq 5$. In [[8], Theorem 4.1] it is proved that, if M has constant sectional curvature

c, then either M is a C_5-manifold and $c = -\alpha^2 < 0$, or M is flat and falls in the class C_{12}.

As reminded in Section 1, the hyperbolic space $\mathbb{H}^{2m+1}(-\alpha^2)$, $\alpha > 0$, is the local model of space-forms carrying a non-cosymplectic α-Kenmotsu structure. On the other hand, it is well-known that the local model of flat cosymplectic space-forms is the Euclidean space $\mathbb{R}^{2m+1} = \mathbb{R} \times \mathbb{R}^{2m}$ endowed with the cosymplectic structure naturally associated to the canonical Kähler structure on $\mathbb{R}^{2m}$ [[5], section 6].

A flat C_{12}-manifold of dimension $2m+1$ is locally described as a twisted product ${}_\lambda J \times U$, $J \subset \mathbb{R}$ being an open interval such that $0 \in J$ and $U = (\mathbb{R}^*_+)^{2m}$. The manifold ${}_\lambda J \times U$ is endowed with the a.c.m. structure $(\varphi, \xi, \eta, g_\lambda)$, acting as

$$\varphi\Big(a\frac{\partial}{\partial t}, X\Big) = (0, J_0 X), \quad \eta\Big(a\frac{\partial}{\partial t}, X\Big) = a\lambda,$$
$$\xi = \frac{1}{\lambda}\Big(\frac{\partial}{\partial t}, 0\Big), \quad g_\lambda = \lambda^2 \pi_1^*(dt \otimes dt) + \pi_2^*(g_0),$$

for any $a \in \mathfrak{F}(J \times U), X \in \Gamma(TU)$, where $\pi_1 \colon J \times U \to J$, $\pi_2 \colon J \times U \to U$ are the canonical projections, (J_0, g_0) is the canonical Kähler structure on $\mathbb{R}^{2m}$ and $\lambda(t, x^1, \ldots, x^{2m}) = \sum_{i=1}^{2m} B_i(t) x^i + E(t)$ is a smooth positive function [[8], Section 7]. We recall that the action of J_0 is determined by $J_0(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^{m+i}}$, $J_0(\frac{\partial}{\partial x^{m+i}}) = -\frac{\partial}{\partial x^i}$, $i = 1, \ldots, m$.

Now, we are able to state the next theorems dealing with the reduction of the codimension for proper slant submanifolds.

Theorem 3.1 *Given $\alpha \in \mathbb{R}$, $\alpha > 0$, let (N, f) be a connected, proper θ-slant submanifold of the α-Kenmotsu manifold $\mathbb{H}^{2m+1}(-\alpha^2)$, ($m \geq 3$), such that* $\dim N = 2n$. *If $\overline{\nabla} F = 0$, then $f(N)$ is contained in a $(4n+1)$-dimensional totally geodesic submanifold of $\mathbb{H}^{2m+1}(-\alpha^2)$.*

Proof. Since $\overline{\nabla} F = 0$, $F(TN)$ is a parallel subbundle of $T^\perp N$ and it is easy to verify that it has rank $2n$. Applying (2.2), we get that $<\xi> \oplus F(TN)$ is a parallel subbundle of $T^\perp N$. Moreover, by (2.2), (2.3), for any $X, Y \in \Gamma(TN)$ one has

$$h(X, Y) = g(h(X, Y), \xi)\xi - \frac{1}{\sin^2\theta} F t h(X, Y).$$

It follows that, for any $x \in N$, the first normal space N^1_x is a subspace of $<\xi_x> \oplus F(T_x N)$. Applying Theorem 2.1, we obtain the statement. $\square$

In a similar way, also applying [[6], Corollary 5.4], we can prove the following result.

Theorem 3.2 *Let (N, f) be a connected, proper θ-slant submanifold of the flat cosymplectic manifold $\mathbb{R}^{2m+1}$ ($m \geq 3$) with* $\dim N = 2n$. *If $\overline{\nabla} F = 0$, then (N, f) is an austere submanifold of $\mathbb{R}^{4n+1}$.*

In the case that the ambient space is a flat non-cosymplectic C_{12}-manifold, we can provide the next local version of the reduction theorem.

Theorem 3.3 *Let (N, f) be a connected, proper θ-slant submanifold of a flat non-cosymplectic C_{12}-manifold M such that* $\dim N = 2n$ *and* $\overline{\nabla}F = 0$*. Then, for any $x \in N$ there exists an open neighborhood U of x such that $(U, f_{|_U})$ is an austere submanifold of a $4n$-dimensional totally geodesic submanifold of M.*

Proof. Since M belongs to the Chinea-Gonzalez class C_{12}, the function α occurring in (2.3) vanishes, so we have

$$h(X, Y) = -\frac{1}{\sin^2\theta} Fth(X, Y), \quad X, Y \in \Gamma(TN).$$

This implies that, for any $x \in N$, N_x^1 is a linear subspace of $F(T_xN)$. Moreover, $F(TN)$ is parallel in $T^\perp N$ and has rank $2n$.

Let x be a point of N. By Theorem 2.2, there exists an open neighborhood U of x such that $(U, f_{|_U})$ is contained in a totally geodesic submanifold $\widehat{N}$ of M, with $\dim \widehat{N} = 4n$. Moreover, since M belongs to the class C_{12} and $\overline{\nabla}F = 0$, (N, f) is an austere submanifold of M. Hence, $(U, f_{|_U})$ is also an austere submanifold of $\widehat{N}$.

□

Finally, we remark that the previous theorems only deal with the Riemannian aspects of the considered slant submanifolds. In general, a totally geodesic submanifold $(\overline{N}, k)$ of an a.c.m. manifold M does not inherit from M an almost Hermitian or an a.c.m. structure. This makes meaningless to ask whether a slant submanifold of M which is embedded in $(\overline{N}, k)$ is a slant submanifold of $\overline{N}$. This is clarified by the following explicit example.

Example 3.1 For any $\theta \in]0, \frac{\pi}{2}[$ the map $\widehat{f}^\theta \colon \mathbb{R}^2 \to \mathbb{R}^6$ acting as

$$\widehat{f}^\theta(x^1, x^2) = (x^1 \sin\theta, x^2, 0, 0, x^1 \cos\theta, 0)$$

is an isometric totally geodesic embedding with respect to the Euclidean metrics g_0' on $\mathbb{R}^2$, g_0 on $\mathbb{R}^6$.

Let J_0 be the canonical almost complex structure on $\mathbb{R}^6$. Putting $X_i = \widehat{f}^\theta_*(\frac{\partial}{\partial x^i})$, $i = 1, 2$, the tangential components of J_0X_1, J_0X_2 are $\widehat{P}X_1 = -\cos\theta X_2$, $\widehat{P}X_2 = \cos\theta X_1$, respectively. It follows that $\widehat{P} = -\cos\theta J_0$, hence $(\mathbb{R}^2, \widehat{f}^\theta)$ is a proper slant submanifold of the Kähler manifold $(\mathbb{R}^6, J_0, g_0)$ with slant angle θ. This implies that the map $f^\theta \colon \mathbb{R}^2 \to \mathbb{R}^7$ acting as $f^\theta(x^1, x^2) = (0, \widehat{f}^\theta(x^1, x^2))$ is an isometric embedding and $(\mathbb{R}^2, f^\theta)$ is a totally geodesic proper slant submanifold of $\mathbb{R}^7$. On $\mathbb{R}^7$ one considers the cosymplectic structure naturally associated to (J_0, g_0) [6].

Since $(\mathbb{R}^2, f^\theta)$ is totally geodesic, we have $\overline{\nabla}F = 0$. By Theorem 3.2, $(\mathbb{R}^2, f^\theta)$ can be realized as a submanifold of the Euclidean space $\mathbb{R}^5$. More precisely, we define the maps $\widehat{j} \colon \mathbb{R}^4 \to \mathbb{R}^6$, $\widehat{k}^\theta \colon \mathbb{R}^2 \to \mathbb{R}^4$ such that

$$\widehat{j}(u^1,u^2,u^3,u^4)=(u^1,u^2,0,0,u^3,u^4),\quad (u^1,u^2,u^3,u^4)\in\mathbb{R}^4,$$
$$\widehat{k}^\theta(x^1,x^2)=(x^1\sin\theta,x^2,x^1\cos\theta,0),\quad (x^1,x^2)\in\mathbb{R}^2.$$

Then we get $\widehat{f}^\theta=\widehat{j}\circ\widehat{k}^\theta$ and both the maps $\widehat{j}$, $\widehat{k}^\theta$ are isometric embeddings with respect to the Euclidean metrics. Moreover, $(\mathbb{R}^2,\widehat{k}^\theta)$ is a totally geodesic submanifold of $\mathbb{R}^4$.

Anyway, taking into account that $(\mathbb{R}^4,\widehat{j})$ is neither a holomorphic nor a proper slant submanifold of $(\mathbb{R}^6,J_0,g_0)$, the Kähler structure (J_0,g_0) on $\mathbb{R}^6$ does not induce a "canonical" Hermitian structure on $(\mathbb{R}^4,\widehat{j})$. It follows that the map $k^\theta\colon\mathbb{R}^2\to\mathbb{R}^5$, acting as $k^\theta(x^1,x^2)=(0,\widehat{k}^\theta(x^1,x^2))$ realizes an isometric immersion into a totally geodesic submanifold of $\mathbb{R}^7$, but the ambient space $\mathbb{R}^5$ does not inherit from $\mathbb{R}^7$ a "canonical" a.c.m. structure.

On the other hand, $(\mathbb{R}^2,k^\theta)$ is an anti-invariant submanifold of $\mathbb{R}^5$, which is endowed with the cosymplectic structure naturally associated to the canonical Kähler structure $(\widehat{J}_0,\widehat{g}_0)$ on $\mathbb{R}^4$. In fact, putting $\widehat{X_1}=\widehat{k}^\theta_*(\frac{\partial}{\partial x^1})$, $\widehat{X_2}=\widehat{k}^\theta_*(\frac{\partial}{\partial x^2})$, one has $\widehat{J}_0\widehat{X_1}=\sin\theta\frac{\partial}{\partial u^3}-\cos\theta\frac{\partial}{\partial u^1}$, $\widehat{J}_0\widehat{X_2}=\frac{\partial}{\partial u^4}$. It follows that $\widehat{J}_0\widehat{X}_i$, $i=1,2$, are sections of the normal bundle $T^\perp\mathbb{R}^2$. Hence $(\mathbb{R}^2,\widehat{k}^\theta)$ is a totally real submanifold of $(\mathbb{R}^4,\widehat{J}_0,\widehat{g}_0)$ and then $(\mathbb{R}^2,k^\theta)$ is anti-invariant.

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