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Calculus of Variations

Nodal cluster solutions for the Brezis–Nirenberg problem in dimensions $N \geq 7$

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Abstract

We show that the classical Brezis–Nirenberg problem

$$\Delta u + |u|^{\frac{4}{N-2}} u + \varepsilon u = 0, \quad \text{in} \quad \Omega, \quad u = 0, \quad \text{on} \quad \partial\Omega$$

admits nodal solutions clustering around a point on the boundary of Ω as $\varepsilon \to 0$, for smooth bounded domains $\Omega \subset \mathbb{R}^N$ in dimensions $N \geq 7$.

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1 Introduction

In this paper we find a new family of sign-changing solutions to the classical Brezis–Nirenberg problem

$$-\Delta u = |u|^{\frac{4}{N-2}} u + \varepsilon u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{BN}$$

where $\varepsilon > 0$ is a small parameter and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 7$.

In their seminal 1983 paper [6], Brezis and Nirenberg initiated the study of positive solutions to (BN) and demonstrated that for dimensions $N \geq 4$, the problem admits a solution for $\varepsilon \in (0, \lambda_{1,\Omega})$, where $\lambda_{1,\Omega}$ represents the first eigenvalue of $-\Delta$ with 0-Dirichlet boundary conditions on $\partial\Omega$. If dimension N is 3, they proved the existence of $\lambda_{*,\Omega} > 0$

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(whose definition depends on Ω) and of a positive solution to (BN) if $\lambda \in (0, \lambda_{*,\Omega})$. If $\Omega = B$ the unit ball, then $\lambda_{*,B} = \frac{\lambda_{1,B}}{4}$; for general domains see [13]. Multiplying the equation in (BN) against the eigen-function associated to $\lambda_{1,\Omega}$ and integrating by parts on Ω show that no positive solutions exist for $\varepsilon \geq \lambda_{1,\Omega}$. Additionally, Pohozaev identity [23] gives that problem (BN) has no non-trivial solutions when Ω is star-shaped and $\varepsilon = 0$. Conversely, Bahri and Coron [2] presented an existence result for a positive solution to problem (1.1) for Ω with a nontrivial topology and $\varepsilon = 0$. Subsequently, considerable attention has been devoted to understanding the possibility of multiple positive solutions to (BN) in the regime $\varepsilon \to 0$ [3, 11, 19, 20, 25] and also to understanding the limiting behavior of the positive solutions u_ε of (BN) as $\varepsilon \to 0$ [14, 26].

Concerning the existence of sign-changing solutions to (BN), this has been established for all range of $\varepsilon > 0$: it has been proven in [9] for $\varepsilon \in (0, \lambda_{1,\Omega})$ and $N \geq 6$, and in [7] for $\varepsilon \geq \lambda_{1,\Omega}$ and $N \geq 4$. Devillanova and Solimini [12] proved the existence of infinitely many sign-changing solutions to (BN) for any $\varepsilon > 0$ when $N \geq 7$. Dimension 7 seems to be a threshold case as for $4 \leqslant N \leqslant 6$ there are no radial sign-changing solutions for (BN), when Ω is a ball and $\varepsilon \in (0, \lambda_{**})$, for some $\lambda_{**} > 0$ [1].

This paper wants to give a contribution in the understanding of multiple sign-changing solutions to (BN) in the regime $\varepsilon \to 0$. It is well known that in this regime a crucial role is played by the *bubbles*, namely the positive solutions to (BN) when $\varepsilon = 0$ and $\Omega = \mathbb{R}^N$. For any $\delta > 0$ and $\xi \in \mathbb{R}^N$, the bubbles

$$\mathcal{U}_{\delta,\xi}(x) = \alpha_N \frac{\delta^{\frac{N-2}{2}}}{\left(\delta^2 + |x-\xi|^2\right)^{\frac{N-2}{2}}}, \quad \alpha_N := [N(N-2)]^{\frac{N-2}{4}} \tag{1.1}$$

are all the solutions of the problem

$$\Delta u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \tag{1.2}$$

The asymptotic analysis of low-energy sign-changing solutions to (BN) as $\varepsilon \to 0$ has been studied in [5] for $N \geq 4$: assuming their existence, such solutions u_ε have a simple positive and negative blow-up behaviour at two distinct points of Ω as $\varepsilon \to 0$, provided the rates of blow-up for the positive and the negative parts are comparable. Roughly speaking, they can be described as follows

$$u_\varepsilon(x) \sim \mathcal{U}_{\delta_{1\varepsilon},\xi_{1\varepsilon}}(x) - \mathcal{U}_{\delta_{2\varepsilon},\xi_{2\varepsilon}}(x) \quad \text{with} \quad \delta_{i\varepsilon}, \to 0, \ \xi_{i\varepsilon} \to \xi_i \in \Omega \quad i = 1, 2,$$
$$\frac{\delta_{1\varepsilon}}{\delta_{2\varepsilon}} = O(1), \quad \xi_1 \neq \xi_2, \quad \text{as} \quad \varepsilon \to 0.$$

Construction, asymptotic analysis and multiplicity of sign-changing solutions exhibiting this type of simple blow-up as $\varepsilon \to 0$ were obtained in [4, 8, 18].

However, in the case of the unit ball, the low-energy radial sign-changing solutions obtained in [9] do not have a simple blow-up if $N \geq 7$. Indeed, both their positive and negative parts blow-up in the form of a positive and a negative bubble both centered at the center of the ball as $\varepsilon \to 0$, with non comparable rates of blow-up [24]. Roughly speaking, in this case solutions look like

$$u_\varepsilon(x) \sim \mathcal{U}_{\delta_{1\varepsilon},0}(x) - \mathcal{U}_{\delta_{2\varepsilon},0}(x) \quad \text{with} \quad \delta_{i\varepsilon} \to 0, \quad i = 1, 2,$$
$$\frac{\delta_{1\varepsilon}}{\delta_{2\varepsilon}} = o(1), \quad \text{as} \quad \varepsilon \to 0.$$

This behaviour is known as *tower of bubbles* (see [10]). In [24] it is proven that sign-changing tower of bubbles for (BN) exist as $\varepsilon \to 0$ for dimensions $N \geq 7$ in a general domain. In contrast, in low dimensions $N = 4, 5, 6$, sign-changing bubble-towers cannot exist, as shown in [15].

In [30] Vaira constructed a different type of sign-changing solutions to (BN) which blow-up in the form of a concentrated bubble and blow-up occurs at a point of the boundary of Ω. Bubbling at the boundary is not always allowed [26], and some extra requirement on the domain Ω seems to be necessary. In [30] it is assumed that Ω is a smooth bounded domain with non-trivial topology such that the problem

$$-\Delta u_0 = |u_0|^{\frac{4}{N-2}} u_0 \quad \text{in } \Omega, \quad u_0 = 0 \quad \text{on } \partial\Omega, \quad u_0 > 0 \quad \text{in } \Omega \tag{1.3}$$

has a positive solution u_0 which is non-degenerate, in the sense that the following linear problem

$$-\Delta \boldsymbol{\tau} = p|u_0|^{\frac{4}{N-2}} v \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{in} \quad \partial\Omega \tag{1.4}$$

admits only the trivial solution $v = 0$. Existence of solutions to (1.3) for domain with non-trivial topology has been obtained by [2]. Besides, for generic Ω these solutions are non-degenerate [28].

Let ν be the unitary outer normal to $\partial\Omega$. Assuming that the function $\xi \in \partial\Omega \mapsto \nabla u_0(\xi) \cdot \nu(\xi)$ has a non-degenerate critical point ξ_0, Vaira proves the existence of a sign-changing solution to problem (BN) of the form

$$u_\varepsilon(x) \sim u_0(x) - \mathcal{U}_{\delta,\xi}(x), \quad \text{with} \quad \delta \sim \varepsilon^{\frac{2(N-1)}{N^2-6N+4}}$$
$$\xi - \xi_0 \sim \varepsilon^{\frac{N-2}{N^2-6N+4}} \quad \text{as} \quad \varepsilon \to 0.$$

Here $\mathcal{U}_{\delta,\xi}$ is again the bubble defined in (1.1).

The main result of this paper is to prove that a sign-changing cluster solution to (BN) around ξ_0 is possible. Clustering configurations are those where the solutions blow-up as the sum of a finite number of bubbles, of comparable heights, whose centers converge to the same point. Clustering configurations are known to exist in several problems related to semi-linear elliptic equations with critical non-linearity, but none was known for the Brezis–Nirenberg problem (BN).

To state our result, let us denote by PW the projection of a function W onto $H_0^1(\Omega)$, i.e.

$$\Delta PW = \Delta W \quad \text{in } \Omega, \qquad PW = 0 \quad \text{on } \partial\Omega.$$

Our main result is the following

Theorem 1.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 7$, such that Problem* (1.3) *has a solution u_0, which is non-degenerate in the sense that the linear problem* (1.4) *has only the trivial solution. Assume there exists a critical point $\xi_0 \in \partial\Omega$ of the function for $\xi \in \partial\Omega \to \nabla u_0(\xi) \cdot \nu(\xi)$, where ν is the unitary outer normal to $\partial\Omega$, such that the second variation $D^2_{N-1}\,(\nabla u_0(\xi) \cdot \nu(\xi))$ is positive definite.*

Let $k \in \mathbb{N}$. Then there exist $\bar\varepsilon > 0$ and a constant $C > 0$ such that, for all $\varepsilon \in (0, \bar\varepsilon)$ there exists a sign-changing solution u_ε to (BN) *given by*

$$u_\varepsilon(x) = u_0(x) - \sum_{j=1}^{k} P\mathcal{U}_{\delta_{j\varepsilon},\xi_{j\varepsilon}}(x) + \phi_\varepsilon(x)$$

where

$$\delta_{j\varepsilon} = \varepsilon^{\frac{2(N-1)}{N^2-6N+4}} d_{j\varepsilon}, \quad \xi_{j\varepsilon} \neq \xi_{i\varepsilon} \ \ \textit{for} \ \ i \neq j, \quad \xi_{j\varepsilon} = \xi_0 + \varepsilon^{\frac{N-2}{N^2-6N+4}} \hat{\xi}_{j\varepsilon} \in \Omega$$

with

$$|d_{j\varepsilon}| \leqslant C, \quad |\hat{\xi}_{j\varepsilon}| \leqslant C \quad \forall j = 1, \ldots, k,$$

and

$$\|\phi_\varepsilon\|_{H_0^1(\Omega)} \leqslant C\varepsilon^{\frac{N^3-8N+8}{2N(N^2-6N+4)}+\sigma}$$

for any $\sigma > 0$ *arbitrarily small.*

The solutions described in the theorem are rather delicate to capture, and precise expansions of the parameters $\delta_{j\epsilon}$ and the points $\xi_{j\epsilon}$ at two consecutive scales are required in the construction. This is described in details in Sect. 2.

The method we use to prove Theorem 1.1 also applies to the construction of sign-changing solutions exhibiting a cluster configuration near the boundary of Ω for the almost critical problem

$$-\Delta u = |u|^{\frac{4}{N-2}-\varepsilon}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where $\varepsilon > 0$ is a small parameter and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 7$. This observation is already present in [30] and we will not elaborate further on this point.

Clustering configurations are known in the literature for perturbation of the Yamabe problem to find metrics on Riemannian manifolds with constant scalar curvature. These have been found in high dimensions $N \geq 7$ in [21], in dimensions 4 and 5 in [29], see also [27]. We dont't know if clustering sign-changing solutions exist for the Brezis–Nirenberg problem (BN) in low dimensions 4, 5, 6, but if it does the form of the solution should though be different from the one obtained in Theorem 1.1.

Finally, we mention that several interesting results have been obtained on the existence of sign changing solutions to the Brezis–Nirenberg problem in regimes different from the one treated in this paper, namely when ε converges to some fixed $\varepsilon_* > 0$. Results in this direction are contained for instance in [16, 22].

2 The setting of the problem

We consider the Hilbert space $H_0^1(\Omega)$ equipped with the usual inner product

$$\langle u, \boldsymbol{\tau} \rangle = \int_\Omega \nabla u \cdot \nabla \boldsymbol{\tau}$$

which induces the norm

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}.$$

For $r \in [1, \infty)$ and $u \in L^r(\Omega)$ we set $|u|_{r,\Omega} = \left(\int_\Omega |u|^r\right)^{\frac{1}{r}}$.

Let $i^\star : L^{\frac{2N}{N+2}}(\Omega) \to H_0^1(\Omega)$ be the adjoint operator of the immersion $i : H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$. By definition $u = i^\star(f)$ if and only if

$$\langle u, \varphi \rangle = \int_\Omega f\varphi \text{ for all } \varphi \in H_0^1(\Omega)$$

or equivalently u weakly solves

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

The operator $i^\star : L^{\frac{2N}{N+2}}(\Omega) \to H_0^1(\Omega)$ is continuous as

$$\|i^\star(f)\|_{H_0^1(\Omega)} \leqslant S^{-1}|f|_{\frac{2N}{N+2};\Omega} \tag{2.1}$$

where S is the best constant for the Sobolev embedding.

In terms of the operator $i^\star$, problem (BN) can be formulated as

$$u = i^\star(|u|^{p-1}u + \varepsilon u). \tag{2.2}$$

We look for cluster solutions of the problem (BN) which change sign. They have the form

$$u_\varepsilon(x) = W_{\boldsymbol{\delta},\boldsymbol{\xi}}(x) + \phi_{\boldsymbol{\delta},\boldsymbol{\xi}}(x) \quad \text{where} \quad W_{\boldsymbol{\delta},\boldsymbol{\xi}}(x) = u_0(x) - \sum_{i=1}^{k} P\mathcal{U}_i(x). \tag{2.3}$$

Here k is a fixed given integer, u_0 is the positive non-degenerate solution to (1.3),

$$P\mathcal{U}_i = P\mathcal{U}_{\delta_i,\xi_i} = i^\star(\mathcal{U}_i^p), \quad \boldsymbol{\delta} = (\delta_1, \ldots, \delta_k) \in \mathbb{R}^k \quad \text{and} \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_k) \in \Omega^k.$$

In our construction the scaling parameters δ_i will be positive and small, while the points ξ_i will collapse into each other, as $\varepsilon \to 0$.

In [30] Vaira constructs a solution to problem (BN) of the form (2.3), with $k = 1$ under the assumption that there exists a non-degenerate critical point ξ_0 of the function $\xi \in \partial\Omega \mapsto \partial_\nu u_0(\xi)$, where ν is the inner unit normal on the boundary. Such solution blows-up, as $\varepsilon \to 0$, at ξ_0, in the sense that the scaling parameter δ and the point ξ in (2.3) can be described at main order as

$$\begin{aligned} &\delta \sim \varepsilon^\alpha d_0 && \text{with } \alpha = \frac{2(N-1)}{N^2-6N+4} \\ &\xi \sim \xi_0 + t_0\varepsilon^\beta \nu(\xi_0) && \text{with } \beta = \frac{N-2}{N^2-6N+4}, \end{aligned} \quad \text{as } \varepsilon \to 0. \tag{2.4}$$

One has $\delta \to 0$ and $\xi \to \xi_0$ as $\varepsilon \to 0$. The point $(d_0, t_0, \xi_0) \in \mathbb{R}^+ \times \mathbb{R} \times \partial\Omega$ is a critical point of the function

$$\Psi(d, t, \xi) = -\mathtt{C}\partial_\nu u_0(\xi) d^{\frac{N-2}{2}} t + \frac{\alpha_N}{2^{N-1}}\mathtt{C}\frac{d^{N-2}}{t^{N-2}} - \mathtt{B}d^2 \tag{2.5}$$

where $\mathtt{C}$ and $\mathtt{B}$ are the explicit positive constants

$$\mathtt{C} = \int_{\mathbb{R}^N} \mathcal{U}_{1,0}^p \quad \textit{and} \quad \mathtt{B} = \frac{1}{2}\int_{\mathbb{R}^N} \mathcal{U}_{1,0}^2. \tag{2.6}$$

A direct computation gives that (d_0, t_0, ξ_0) satisfies the system

$$\begin{cases} -2\mathtt{B}d_0 + \dfrac{\alpha_N}{2^{N-1}}\dfrac{d_0^{N-3}}{t_0^{N-2}}\mathtt{C} - \mathtt{C}\dfrac{N-2}{2}d_0^{\frac{N-4}{2}} t_0 \partial_\nu u_0(\xi_0) = 0 \\ -\dfrac{\alpha_N(N-2)}{2^{N-1}}\dfrac{d_0^{N-2}}{t_0^{N-1}}\mathtt{C} - \mathtt{C}d_0^{\frac{N-2}{2}}\partial_\nu u_0(\xi_0) = 0 \\ \nabla_{\xi_0}\partial_\nu u_0(\xi_0) = 0. \end{cases} \tag{2.7}$$

Since by Hopf's Lemma $\partial_\nu u_0(\xi_0) < 0$, the function Ψ has a critical point in the considered region.

The result in [30] indicates that a solution with the form (2.3) and $k > 1$ would possibly exhibit a cluster behaviour around the point ξ_0. This suggests the form for the scaling parameters δ_i and the points ξ_i in (2.3). Let us be more precise.

Locally around ξ_0, the boundary $\partial\Omega$ can be described as the set of points $x = (x', \vartheta(x'))$, for a certain smooth function $\vartheta : \mathbb{R}^{N-1} \to \mathbb{R}$. Without loss of generality, we assume $\partial_{x_j}\vartheta(\xi_0) = 0$ for all $j = 1, \ldots, N-1$ and we write $\xi_0 = (\xi_0', \vartheta(\xi_0'))$. For the construction of a cluster solution, we assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k) \in \Omega^k$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k) \in \mathbb{R}^k$ in (2.3) have the form

$$\begin{aligned} \delta_i &= \varepsilon^{\alpha} d_0 + \varepsilon^{\hat\alpha} d_i && \text{with } \hat\alpha = \frac{3N^2-6N+4}{N(N^2-6N+4)}; \\ \hat\xi_i &= (\xi_0' + \varepsilon^{\hat\beta}\tau_i, \vartheta(\xi_0' + \varepsilon^{\hat\beta}\tau_i)) && \text{with } \hat\beta = \frac{(N-2)^2}{N(N^2-6N+4)}; \\ \xi_i &= \hat\xi_i + (\varepsilon^{\beta} t_0 + \varepsilon^{\tilde\beta} t_i)\nu(\hat\xi_i) && \text{with } \tilde\beta := \frac{2N^2-6N+4}{N(N^2-6N+4)}. \end{aligned} \tag{2.8}$$

where d_0, t_0, ξ_0, α and β are defined in (2.4), $\hat\xi_i \in \partial\Omega$, $\tau_i \in \mathbb{R}^{N-1}$, $\tau_i, d_i \in \mathbb{R}$ are parameters to be found. For the moment, we make the following assumptions on these parameters: we assume there exist $a > 0$ and $\rho > 0$ such that τ_i, τ_i and d_i

$$|d_i|,\ |t_i| < a, \forall i = 1, \ldots, k, \quad |\tau_i - \tau_h| > \rho \quad \forall i \neq h. \tag{2.9}$$

Remark 2.1 Without loss of generality, we can choose a coordinate system such that $\nabla_{N-1}\vartheta(\xi_0') = 0$. Now a computation shows that

$$\begin{aligned} |\hat\xi_i - \hat\xi_h|^2 &= \varepsilon^{2\hat\beta}\sum_{\ell=1}^{N-1}(\tau_i - \tau_h)_\ell^2 + (\vartheta(\xi_0' + \varepsilon^{\hat\beta}\tau_i) - \vartheta(\xi_0' + \varepsilon^{\hat\beta}\tau_h))^2 \\ &= \varepsilon^{2\hat\beta}|\tau_i - \tau_h|^2 + \varepsilon^{2\hat\beta}\underbrace{\nabla_{N-1}\vartheta(\xi_0')}_{=0}\cdot(\tau_i - \tau_h) + o(\varepsilon^{2\hat\beta}). \end{aligned}$$

and then

$$\begin{aligned} |\xi_i - \xi_h|^2 &= |\hat\xi_i + (\varepsilon^{\beta} t_0 + \varepsilon^{\tilde\beta} t_i)\nu(\hat\xi_i) - \hat\xi_h - (\varepsilon^{\beta} t_0 + \varepsilon^{\tilde\beta} t_h)\nu(\hat\xi_h)|^2 \\ &= |\hat\xi_i - \hat\xi_h + o(\varepsilon^{\hat\beta})|^2 \\ &= \varepsilon^{2\hat\beta}|\tau_i - \tau_h|^2 + o\left(\varepsilon^{2\hat\beta}\right). \end{aligned} \tag{2.10}$$

It is important to observe that $\alpha < \hat\alpha$ and $\hat\beta < \beta < \tilde\beta$. Let us call

$$\theta = 1 + 2\alpha = (\alpha - \beta)(N-2) = \frac{N(N-2)}{N^2-6N+4}, \tag{2.11}$$

and

$$\hat\theta = 1 + 2\hat\alpha = (\alpha - \hat\beta)(N-2) = \frac{N^3-8N+8}{N(N^2-6N+4)}. \tag{2.12}$$

Now we are able to state our main result.

Theorem 2.1 *Assume there exists is a C^1-stable critical point ξ_0 of the function $\xi \in \partial\Omega \mapsto \partial_\nu u_0(\xi)$ such that $D^2_{N-1}\partial_\nu u_0(\xi_0)$ is positive definite, and let $k \in \mathbb{N}$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the problem (BN) has a sign-changing cluster solution which blows-up at ξ_0. More precisely, there exist constants C, a, ρ, a function $\phi_\varepsilon \in H^1_0(\Omega)$, points $\boldsymbol{\xi}_\varepsilon = (\xi_{1\varepsilon}, \dots, \xi_{k\varepsilon}) \in \Omega^k$ with $\xi_{i\varepsilon} \neq \xi_{h\varepsilon}$ if $i \neq h$, and parameters $\boldsymbol{\delta}_\varepsilon = (\delta_{1\varepsilon}, \dots, \delta_{k\varepsilon}) \in \mathbb{R}^k$ satisfying (2.8)–(2.9) such that u_ε defined in (2.3) is a solution to (BN). Moreover*

$$\|\phi_\varepsilon\|_{H^1_0(\Omega)} \leqslant C\varepsilon^{\frac{\hat{\theta}}{2}+\sigma}$$

for some $\sigma > 0$ arbitrarily small.

Theorem 1.1 is a direct consequence of Theorem 2.1.

The rest of the paper is devoted to prove Theorem 2.1. The proof is done via a reduction method.

In Sect. 3 we prove that, for given $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \Omega^k$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k) \in \mathbb{R}^k$ satisfying (2.8)–(2.9), there exists ϕ_ε solution of a projected problem. We then show that a true solution to our problem can be achieved by finding a specific set of parameters $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \Omega^k$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k) \in \mathbb{R}^k$. This is the reduced finite dimensional problem that we treat in the subsequent sections.

3 Reduction to a finite dimensional problem

The purpose of this section is to find the term ϕ_ε in (2.3) for given $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \Omega^k$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k) \in \mathbb{R}^k$ satisfying (2.8)–(2.9). The term ϕ_ε will be small in ε and satisfy a set of orthogonal conditions. To introduce these orthogonality conditions, consider the linear problem

$$-\Delta\psi = p\mathcal{U}_{1,0}^{p-1}\psi \text{ in } \mathbb{R}^N,$$

whose set of solution is spanned by the functions

$$\psi^0(x) = \frac{N-2}{2}\mathcal{U}_{1,0}(x) + \nabla\mathcal{U}_{1,0}(x)\cdot x = \alpha_N\frac{N-2}{2}\frac{1-|x|^2}{(1+|x|^2)^{N/2}}$$

and

$$\psi^j(x) = \frac{\partial\mathcal{U}_{1,0}}{\partial x_j}(x) = -\alpha_N(N-2)\frac{x_j}{(1+|x|^2)^{N/2}}.$$

Set

$$\psi^j_{\delta,\xi}(x) = \frac{1}{\delta^{\frac{N-2}{2}}}\psi^j\left(\frac{x-\xi}{\delta}\right) \text{ for all } j = 0, \dots, N$$

and define

$$P\psi^j_i = i^\star\left(p\mathcal{U}_i^{p-1}\psi^j_i\right) \text{ for all } i = 1, \dots, k \text{ and } j = 0, \dots, N$$

where $\mathcal{U}_i = \mathcal{U}_{\delta_i,\xi_i}$ and $\psi^j_i = \psi^j_{\delta_i,\xi_i}$. For any $\boldsymbol{\delta} \in \mathbb{R}^k$ and $\boldsymbol{\xi} \in \Omega^k$, consider the space

$$K_{\boldsymbol{\delta},\boldsymbol{\xi}} = \text{span}\{P\psi^j_i \text{ for all } i = 1, \dots, k \text{ and } j = 0, \dots, N\}$$

and its orthogonal

$$K^{\perp}_{\boldsymbol{\delta},\boldsymbol{\xi}} = \{\phi \in H^1_0(\Omega) : \langle \phi, P\psi_i^j \rangle = 0 \text{ for all } i = 1, \dots, k \text{ and } j = 0, \dots, N\}.$$

The reminder term introduced in (2.3) will satisfy $\phi_\varepsilon \in K^{\perp}_{\boldsymbol{\delta},\boldsymbol{\xi}}$.

Now let us recall some useful results.

Lemma 3.1 *Let $N > 6$ and $q \in \left(\frac{p+1}{2}, p+1\right]$. Let $\delta \in (0,\infty)$, $\xi \in \Omega$ and $\mathcal{U}_{\delta,\xi}$ as in (1.1). Then there exist there exists a constant $C = C(a) > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$|\mathcal{U}_{\delta,\xi}|_{2^\star,\Omega} \leqslant C, \tag{3.1}$$

$$|\mathcal{U}_{\delta,\xi}|_{\frac{2N}{N+2},\Omega} \leqslant C\delta^2, \tag{3.2}$$

$$|\psi^j_{\delta,\xi}|_{\frac{2N}{N+2},\Omega} \leqslant C\delta^2. \tag{3.3}$$

Let us denote by $G(x, y)$ the Green's function of the Laplace operator with Dirichlet boundary condition and let $H(x, y)$ be its regular part, namely

$$H(x, y) = \gamma_N \left(\frac{1}{|x-y|^{N-2}} - G(x, y) \right), \quad \forall\, x, y \in \Omega, \text{ with } \gamma_N = \frac{1}{(N-2)|\mathbb{S}^{N-1}|}.$$

We remark that in [25] it was shown that for every $\xi \in \Omega$

$$H(\xi, \xi) = \frac{1}{2^{N-2}\operatorname{dist}(\xi, \partial\Omega)^{N-2}} + o\left(\frac{1}{\operatorname{dist}(\xi, \partial\Omega)^{N-2}}\right).$$

By using the previous observations we get that

$$H(\xi_h, \xi_i) = \frac{1}{2^{N-2}} \frac{1}{\eta_i^{N-2}} + o\left(\frac{1}{\eta_i^{N-2}}\right)$$

where

$$\eta_i = \operatorname{dist}(\xi_i, \partial\Omega) = \varepsilon^{\beta} t_0 + \varepsilon^{\tilde\beta} t_i. \tag{3.4}$$

Lemma 3.2 *Let $\delta \in (0,\infty)$, $\xi \in \Omega$ and $\varphi_{\delta,\xi} = \mathcal{U}_{\delta,\xi} - P\mathcal{U}_{\delta,\xi}$. We have*

$$\varphi_{\delta,\xi}(x) = \alpha_N \delta^{\frac{N-2}{2}} H(\xi, x) + \mathcal{O}\left(\frac{\delta^{\frac{N+2}{2}}}{\operatorname{dist}(\xi, \partial\Omega)^N}\right)$$

and

$$0 \leqslant \varphi_{\delta,\xi} \leqslant \mathcal{U}_{\delta,\xi}.$$

Moreover

$$\psi^0_{\delta,\xi} - P\psi^0_{\delta,\xi} = \alpha_N \frac{N-2}{2} \delta^{\frac{N-2}{2}} H(\xi, x) + \mathcal{O}\left(\frac{\delta^{\frac{N+2}{2}}}{\operatorname{dist}(\xi, \partial\Omega)^N}\right)$$

and for all $j = 1, \dots, N$

$$\psi^j_{\delta,\xi} - P\psi^j_{\delta,\xi} = \alpha_N \delta^{\frac{N}{2}} \frac{\partial H}{\partial \xi_j}(\xi, x) + \mathcal{O}\left(\frac{\delta^{\frac{N+4}{2}}}{\operatorname{dist}(\xi, \partial\Omega)^N}\right).$$

Proof See Proposition 1 in [25]. □

In particular Lemma 3.2 implies that $P\mathcal{U}_{\delta,\xi} \geq 0$ and that there exists a positive constant C such that

$$|\varphi_{\delta,\xi}|_{\infty,\Omega} \leqslant C\frac{\delta^{\frac{N-2}{2}}}{\mathrm{dist}(\xi,\partial\Omega)^{N-2}}. \tag{3.5}$$

In [25] it is also shown that

$$|\varphi_{\delta,\xi}|_{2^\star,\Omega} \lesssim \frac{\delta^{\frac{N-2}{2}}}{\mathrm{dist}(\xi,\partial\Omega)^{\frac{N-2}{2}}}. \tag{3.6}$$

More generally we can estimate the $L^q(\Omega)$ norm of $\varphi_{\delta,\xi}$ and the $L^{\frac{2N}{N-2}}(\Omega)$ of $P\psi^j_{\delta,\xi} - \psi^j_{\delta,\xi}$.

Lemma 3.3 *Let* $N > 6$, $q \in \left(\frac{p+1}{2}, p+1\right]$ *and* $\boldsymbol{\delta}, \boldsymbol{\xi}$ *satisfy* (2.8) *and* (2.9) *as in Sect.* 2. *Then there exists a constant* $C = C(a) > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *for all* $i = 1, \ldots, k$ *and* $j = 0, \ldots, N$ *it holds*

$$|\varphi_{\delta_i,\xi_i}|_{q,\Omega} \leqslant C\varepsilon^{\frac{\theta}{2}}$$

and

$$|P\psi^j_i - \psi^j_i|_{\frac{2N}{N-2},\Omega} \leqslant C\varepsilon^{\sigma}$$

for some $\sigma > 0$ *arbitrarily small.*

Proof The proof easily follows from Lemma 2.2 in [30]. □

Now let us define the operators

$$\Pi(u) = \sum_{i=1}^{k}\sum_{j=0}^{N}\langle u, P\psi^j_i\rangle P\psi^j_i$$

and

$$\Pi^\perp(u) = u - \Pi(u).$$

Moreover the problem (2.2) is equivalent to solve the system

$$\Pi\{u_\varepsilon - i^\star(f(u_\varepsilon) + \varepsilon u_\varepsilon)\} = 0 \tag{3.7}$$

and

$$\Pi^\perp\{u_\varepsilon - i^\star(f(u_\varepsilon) + \varepsilon u_\varepsilon)\} = 0. \tag{3.8}$$

Let us also define the linear operator

$$\mathcal{L}_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi = \Pi^\perp\left\{\phi - i^\star\left[f'\left(u_0 - \sum_{h=1}^{k}\mathcal{U}_h\right)\phi\right]\right\}, \tag{3.9}$$

the error term

$$\mathcal{E}_{\boldsymbol{\delta},\boldsymbol{\xi}} = \Pi^\perp\left\{W_{\boldsymbol{\delta},\boldsymbol{\xi}} - i^\star\left[f\left(W_{\boldsymbol{\delta},\boldsymbol{\xi}}\right) + \varepsilon W_{\boldsymbol{\delta},\boldsymbol{\xi}}\right]\right\} \tag{3.10}$$

and the nonlinear terms

$$\mathcal{N}^1_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi = \Pi^{\perp}\{i^{\star}\left[f(W_{\boldsymbol{\delta},\boldsymbol{\xi}}+\phi) - f(W_{\boldsymbol{\delta},\boldsymbol{\xi}}) - f'(W_{\boldsymbol{\delta},\boldsymbol{\xi}})\phi + \varepsilon\phi\ \right]\}, \tag{3.11}$$

$$\mathcal{N}^2_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi = \Pi^{\perp}\left\{i^{\star}\left[\left(f'\left(u_0 - \sum_{h=1}^{k} P\mathcal{U}_h\right) - f'\left(u_0 - \sum_{h=1}^{k}\mathcal{U}_h\right)\right)\right]\phi\right\}, \tag{3.12}$$

where $f(u) = |u|^{p-1}u$.

Remark 3.1 We want to stress that $\Pi^{\perp}_{\boldsymbol{\delta},\boldsymbol{\xi}}$ is a continuous map, i.e. there exists a constant $C > 0$ such that for any $\varepsilon > 0$, $\boldsymbol{\delta} \in \mathbb{R}^k$, $\boldsymbol{\xi} \in \Omega^k$ it holds

$$\|\Pi^{\perp}_{\boldsymbol{\delta},\boldsymbol{\xi}} u\|_{H^1_0(\Omega)} \leqslant C\|u\|_{H^1_0(\Omega)} \quad \text{for all} \quad u \in H^1_0(\Omega).$$

Two important inequalities we will use throughout all our work are the following.

Lemma 3.4 *For all $a > 0$ and $b \in \mathbb{R}$, we have*

$$||a+b|^r - a^r| \leqslant \begin{cases} C(r)\min\{a^{r-1}b, b^r\} & \text{if } r < 1, \\ C(r)\left(a^{r-1}b + b^r\right) & \text{if } r \geq 1 \end{cases}$$

and

$$||a+b|^r(a+b) - a^{r+1} - (1+r)a^r b| \leqslant \begin{cases} C(r)\min\{|b|^{r+1}, a^{r-1}b^2\} & \text{if } 0 \leqslant r < 1, \\ C(r)\max\left(|b|^{r+1}, a^{r-1}b^2\right) & \text{if } r \geq 1. \end{cases}$$

Proof See Lemma 2.2 in [17]. □

Now we want to solve (3.8). For this aim we start proving the invertibility of $\mathcal{L}_{\boldsymbol{\delta},\boldsymbol{\xi}}$ defined in (3.9).

Proposition 3.5 *Let $N > 6$ and $\boldsymbol{\delta}, \boldsymbol{\xi}$ satisfy (2.8) and (2.9) as in Sect. 2. Then there exists $C = C(a) > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\|\mathcal{L}_{\boldsymbol{\delta},\boldsymbol{\xi}}(\phi)\|_{H^1_0(\Omega)} \geq C\|\phi\|_{H^1_0(\Omega)} \text{ for any } \phi \in K^{\perp}_{\boldsymbol{\delta},\boldsymbol{\xi}}. \tag{3.13}$$

Furthermore, the operator $\mathcal{L}_{\boldsymbol{\delta},\boldsymbol{\xi}}$ is invertible and its inverse is continuous.

Proof We will argue as in the proof of Lemma 1.7 in [19]. The main difference is that our points are converging to the same point $\xi_0 \in \partial\Omega$ while in [19] they are converging to different points in Ω. In view of a contradiction, assume that there exist

· sequences $\varepsilon_n \to 0$ as $n \to \infty$, $|d_i|, |t_i| < a$ for all $i = 1, \ldots, k$ as $n \to \infty$,
· $\phi_n \in K^{\perp}_n := K^{\perp}_{\delta_n,\xi_n}$, with $\|\phi_n\|_{H^1_0(\Omega)} = 1$;
· $\mathcal{L}_n\phi_n := \mathcal{L}_{\boldsymbol{\delta_n},\boldsymbol{\xi_n}}(\phi_n) = h_n$ with $\|h_n\|_{H^1_0(\Omega)} \to 0$.

We have that

$$\phi_n = i^*\left[f'\left(u_0 - \sum \mathcal{U}_{h_n}\right)\phi_n\right] + h_n + w_n \tag{3.14}$$

where $w_n \in K_n = K_{\delta_n,\xi_n}$, i.e.

$$w_n = \sum_{i=1}^{k}\sum_{j=0}^{N} c^j_{i_n} P\psi^j_{i_n}.$$

Observe that

$$|f'\left(\sum \mathcal{U}_h\right) - \sum f'\left(\mathcal{U}_h\right)|_{\frac{N}{2},\Omega} = \mathcal{O}\left(\varepsilon^{2(\alpha-2\beta)}\right). \tag{3.15}$$

Indeed

$$\begin{aligned}
&|f'\left(\sum \mathcal{U}_h\right) - \sum f'\left(\mathcal{U}_h\right)|_{\frac{N}{2},\Omega} \\
&\quad \leqslant C|f'\left(\sum \mathcal{U}_h\right) - \sum f'\left(\mathcal{U}_h\right)|_{\frac{N}{2}\Omega\setminus\cup B_{\eta_i}\xi_i} \\
&\qquad + C\sum_i \left(|f'\left(\sum_{h=1}^{k}\mathcal{U}_h\right) - f'(\mathcal{U}_i)|_{\frac{N}{2},B_{\eta_i}\xi_i} + \sum_{h\neq i}|f'(\mathcal{U}_h)|_{\frac{N}{2},B_{\eta_i}\xi_i}\right) \\
&\quad \leqslant C\sum_i \left(|f'\left(\sum_{h\neq i}\mathcal{U}_h\right)|_{\frac{N}{2},B_{\eta_i}\xi_i} + \sum_{h\neq i}|f'(\mathcal{U}_h)|_{\frac{N}{2},B_{\eta_i}\xi_i}\right) \\
&\qquad + C\sum|f'\left(\sum \mathcal{U}_h\right) - \sum f'\left(\mathcal{U}_h\right))|_{\frac{N}{2},\Omega\setminus\cup B_{\eta_i}\xi_i} \\
&\quad \leqslant C\sum_{i,h}\left(\left(\sum_{h\neq i}\left(\frac{\delta_h}{\eta_h^2}\right)^{\frac{N-2}{2}}\right)^{\frac{4}{N-2}} + \sum_{h\neq i}\left(\frac{\delta_h}{\eta_h^2}\right)^2\right)|B_{\eta_i}\xi_i|^{\frac{2}{N}} \\
&\qquad + C\left(\sum\left(\frac{\delta_h}{\eta_h^2}\right)^{\frac{N-2}{2}}\right)^{\frac{4}{N-2}} + \sum\left(\frac{\delta_h}{\eta_h^2}\right)^2
\end{aligned}$$

where η_i is defined in 3.4. Let us also observe that $w_n \in K_n$ is orthogonal to $\phi_n, h_n \in K_n^{\perp}$.

· Step 1. *It holds that*

$$\|w_n\|_{H_0^1(\Omega)} \to 0 \text{ as } n \to \infty.$$

Observe that

$$0 = \langle \phi_n, w_n\rangle = \sum_{i=1}^{k}\sum_{j=0}^{N} c_{i_n}^{j}\int_\Omega \phi_n f'(\mathcal{U}_{i_n})\psi_{i_n}^{j}.$$

Using (3.14) and using Lemma 3.4, we have

$$\begin{aligned}
\|w_n\|_{H_0^1(\Omega)}^2 &= \underbrace{\langle \phi_n, w_n\rangle}_{=0} - \langle i^*\left[f'\left(u_0 - \sum_{h=1}^{k}\mathcal{U}_{h_n}\right)\phi_n\right], w_n\rangle - \underbrace{\langle h_n, w_n\rangle}_{=0} \\
&= -\int_\Omega \left[f'(u_0 - \sum_{h=1}^{k}\mathcal{U}_{h_n}) - f'\left(\sum_{h=1}^{k}\mathcal{U}_{h_n}\right)\right]\phi_n w_n \\
&\quad - \int_\Omega \left[f'\left(\sum_{h=1}^{k}\mathcal{U}_{h_n}\right) - \sum_{h=1}^{k} f'(\mathcal{U}_{h_n})\right]\phi_n w_n - \sum_{h=1}^{k}\int_\Omega f'(\mathcal{U}_{h_n})\phi_n w_n \\
&= -\sum_{i=1}^{k}\sum_{j=0}^{N} c_{i_n}^{j}\int_\Omega \left[\underbrace{f'\left(u_0 - \sum_{h=1}^{k}\mathcal{U}_{h_n}\right) - f'\left(\sum_{h=1}^{k}\mathcal{U}_{h_n}\right)}_{\leqslant f'(u_0)}\right]\phi_n P\psi_{i_n}^{j}
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\Omega}\left[f'\left(\sum_{h=1}^{k}\mathcal{U}_{h_n}\right)-\sum_{h=1}^{k}f'(\mathcal{U}_{h_n})\right]\phi_n w_n\\
&-\sum_{h,i=1}^{k}\sum_{j=0}^{N}c_{i_n}^{j}\int_{\Omega}f'(\mathcal{U}_{h_n})\phi_n(P\psi_{i_n}^{j}-\psi_{i_n}^{j})+\underbrace{\sum_{h,i=1}^{k}\sum_{j=0}^{N}c_{i_n}^{j}\int_{\Omega}f'(\mathcal{U}_{h_n})\phi_n\psi_{i_n}^{j}}_{=0}\\
\leqslant\ & f'(u_0)\sum_{h,i=1}^{k}\sum_{j=1}^{N}|c_{i_n}^{j}||\phi|_{\frac{2N}{N-2},\Omega}\underbrace{|P\psi_{i_n}^{j}|_{\frac{2N}{N+2},\Omega}}_{\mathcal{O}(\delta_{i_n})^2}\\
&+|f'\left(\sum_{h,i=1}^{k}\mathcal{U}_{h_n}\right)-\sum_{h,i=1}^{k}f'(\mathcal{U}_{h_n})|_{\frac{N}{2},\Omega}|\phi_n|_{\frac{2N}{N-2},\Omega}|w_n|_{\frac{2N}{N-2},\Omega}\\
&+\sum_{h,i=1}^{k}\sum_{j=0}^{N}|c_{i_n}^{j}||f'(U_{h_n})|_{\frac{N}{2},\Omega}|\phi_n|_{\frac{2N}{N-2},\Omega}|P\psi_{i_n}^{j}-\psi_{i_n}^{j}|_{\frac{2N}{N-2},\Omega}
\end{aligned}
$$

where $|P\psi_{i_n}^{j}|_{\frac{2N}{N+2},\Omega}$, $|f'(\sum\mathcal{U}_{h_n})-\sum f'(\mathcal{U}_{h_n})|_{\frac{N}{2},\Omega}$ and $|P\psi_{i_n}^{j}-\psi_{i_n}^{j}|_{\frac{2N}{N-2},\Omega}$ go to zero as $\varepsilon_n\to 0$ by (3.3), (3.15) and Lemma 3.3 and $|f'(\mathcal{U}_{h_n})|_{\frac{N}{2},\Omega}$ is bounded as $\mathcal{U}_{1,0}\in L^{\frac{N}{2}}(\mathbb{R}^N)$ and

$$|f'(\mathcal{U}_h)|_{\frac{N}{2},\Omega}\leqslant\alpha^{p-1}|U_{1,0}|_{\frac{N}{2},\mathbb{R}^N}.$$

Then we have

$$\|w_n\|_{H_0^1(\Omega)}^2\leqslant o_n(1)\sum_{j=0}^{N}\sum_{i=1}^{k}|c_{i_n}^{j}|+o_n(1)\|w_n\|_{H_0^1(\Omega)}. \tag{3.16}$$

Moreover,

$$\|w_n\|_{H_0^1(\Omega)}^2=\sum c_{i_n}^{j}c_{r_n}^{s}\langle P\psi_{i_n}^{j},P\psi_{r_n}^{s}\rangle=\sum c_{i_n}^{j}c_{r_n}^{s}[\delta_{j,s}\delta_{i,r}+o_n(1)]. \tag{3.17}$$

Putting together (3.16) and (3.17), we have that $c_{i_n}^{j}$ are bounded and consequently $\|w_n\|_{H_0^1(\Omega)}\to 0$ as $n\to\infty$. Now as $\|\phi_n\|$ is bounded in $H_0^1(\Omega)$, it converges weakly in $\mathcal{D}^{1,2}(\Omega)$ to ϕ_∞. We want to show that $\phi_\infty=0$.

· Step 2. For all $i=1,\dots,k$ let us define

$$\tilde{\phi}_{i_n}(y)=\delta_{i_n}^{\frac{N-2}{2}}\phi_n(\delta_{i_n}y+\xi_{i_n})\text{ for all }y\in\Omega_{i_n}=\frac{\Omega-\xi_{i_n}}{\delta_{i_n}},$$

such that $\|\tilde{\phi}_{i_n}\|_{H_0^1(\Omega_{i_n})}=\|\phi_n\|_{H_0^1(\Omega)}=1$. Then $\tilde{\phi}_{i_n}$ is bounded and there exists $\tilde{\phi}_{i_\infty}$ such that $\tilde{\phi}_{i_n}\to\tilde{\phi}_{i_\infty}$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. We observe that

$$
\begin{aligned}
\tilde{\phi}_{i_n}=\ &\delta_{i_n}^{2}i_{i_n}^{\star}\left[f'\left(u_0(\delta_{i_n}y+\xi_{i_n})-\delta_{i_n}^{-\frac{N-2}{2}}\mathcal{U}_{1,0}-\sum_{h\neq i}\mathcal{U}_h(\delta_{i_n}y+\xi_{i_n})\right)\tilde{\phi}_{i_n}\right]\\
&+\tilde{h}_{i_n}(y)+\tilde{w}_{i_n}(y),
\end{aligned}
$$

where $i^{\star}_{i_n}$ is the adjoint operator of the immersion $i^{\star}_{i_n} : H^1_0(\Omega_{i_n}) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega_{i_n})$, $\tilde{h}_{i_n}(y) = h_n(\delta_{i_n} y + \xi_{i_n})$ and $\tilde{w}_{i_n}(y) = (\delta_{i_n} y + \xi_{i_n})$. Recalling that $|\xi_i - \xi_h|^2 = \mathcal{O}(\varepsilon^{2\beta})$ by (2.1), for all $\varphi \in C^\infty_c$ such that $\operatorname{supp}\varphi \subset \Omega_{i_n}$, we have

$$\begin{aligned}
\langle \tilde{\phi}_{i_n}, \varphi\rangle_{H^1_0(\Omega_{i_n})} &= \delta^2_{i_n} \int_{\Omega_{i_n}} f'\left(u_0 - \sum \mathcal{U}_h\right) \tilde{\phi}_{i_n}\varphi + \underbrace{\langle h^i_n + w^i_n, \varphi\rangle}_{=o_n(1)} \\
&\leqslant \delta^2_{i_n} \int\limits_{\Omega_{i_n}} f'\left(u_0(\delta_{i_n} y + \xi_{i_n}) - \delta_{i_n}^{-\frac{N-2}{2}} \mathcal{U}_{1,0} - \alpha_N \sum_{h\neq i} \frac{\delta_h^{\frac{N-2}{2}}}{(\delta_h^2 + |\delta_i y + \xi_{i_n} - \xi_{h_n}|^2)^{\frac{N-2}{2}}}\right) \tilde{\phi}_{i_n}\varphi \\
&\quad + o_n(1) \\
&= \int\limits_{\Omega_{i_n}} \Bigg[f'\left(\mathcal{U}_{1,0}(y) - \delta_{i_n}^{\frac{N-2}{2}} u_0(\delta_{i_n} y + \xi_{i_n}) + \alpha_N \sum_{h\neq i} \frac{\delta_{h_n}^{\frac{N-2}{2}} \delta_{i_n}^{\frac{N-2}{2}}}{(\delta_h^2 + |\delta_i y + \xi_{i_n} - \xi_{h_n}|^2)^{\frac{N-2}{2}}})\right) \\
&\quad - f'(\mathcal{U}_{1,0})\Bigg] \tilde{\phi}_{i_n}\varphi + p \int\limits_{\Omega_{i_n}} \mathcal{U}^{p-1}_{1,0} \tilde{\phi}_{i_n}\varphi + o_n(1) \\
&\leqslant \int\limits_{\Omega_{i_n}} f'\left(\delta_{i_n}^{\frac{N-2}{2}} u_0(\delta_{i_n} y + \xi_{i_n}) - \alpha_N \sum_{h\neq i} \underbrace{\frac{\delta_{h_n}^{\frac{N-2}{2}} \delta_{i_n}^{\frac{N-2}{2}}}{(\delta_h^2 + |\delta_i y + \xi_{i_n} - \xi_{h_n}|^2)^{\frac{N-2}{2}}})}_{\mathcal{O}(\varepsilon_n^{(\alpha-\hat{\beta})(N-2)})=o_n(1)}\right) \tilde{\phi}_{i_n}\varphi \\
&\quad + \int_{\Omega_{i_n}} f'(\mathcal{U}_{1,0})\tilde{\phi}_{i_n}\varphi + o_n(1) \\
&\quad \to p \int_{\mathbb{R}^N} \mathcal{U}^{p-1}_{1,0} \varphi\tilde{\phi}_{i_\infty}
\end{aligned}$$

where the first term goes to zero by the Dominated Convergence Theorem. Hence at the limit

$$-\Delta\tilde{\phi}_{i_\infty} = p\mathcal{U}^{p-1}_{1,0}\tilde{\phi}_{i_\infty} \qquad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N).$$

We want to prove that the limit function $\tilde{\phi}_{i_\infty}$ is null. It is sufficient to show that

$$\tilde{\phi}_{i_\infty} \in \ker(-\Delta - pU^{p-1})^{\perp},$$

i.e.

$$\int_{\mathbb{R}^N} \nabla\tilde{\phi}_{i_\infty} \cdot \nabla\psi^j = \int_{\mathbb{R}^N} \tilde{\phi}_{i_\infty} f'(\mathcal{U}_{1,0})\psi^j = 0 \text{ for all } j = 0, \dots, N.$$

Indeed, if $j = 0$ we have that

$$\begin{aligned}
0 = \langle \phi_n, P\psi^j_{i_n}\rangle = \int_\Omega \nabla\phi_n(x) \cdot \nabla P\psi^j_{i_n} dx &= \int_\Omega \phi_n(x) f'(U_i(x))\psi^j_i(x) dx \\
= \int_{\Omega_{i_n}} \tilde{\phi}_{i_n}(y) f'(\mathcal{U}_{1,0}(y))\psi^0(y) dy &\to \int_{\mathbb{R}^N} \tilde{\phi}_{i_\infty} f'(\mathcal{U}_{1,0})\psi^0
\end{aligned}$$

and similarly for $j = 1, \dots, N$. As $\tilde{\phi}_{i_n} \rightharpoonup \tilde{\phi}_{i_\infty}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $f'(U_{1,0})\psi^0_{1,0} \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ then $\tilde{\phi}_{i_\infty} \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

· Step 3. Now for all $\varphi \in C_c^\infty(\Omega)$ we have that

$$\begin{aligned}
\langle \phi_n, \varphi\rangle_{H_0^1(\Omega)} &= \int_\Omega f'\left(u_0 - \sum \mathcal{U}_{h_n}\right)\phi_n\varphi + \underbrace{\langle h_n + w_n, \varphi\rangle}_{=o_n(1)}\\
&= \int_\Omega f'(u_0)\,\phi_n\varphi + \int_\Omega \underbrace{\left[f'(u_0 - \sum \mathcal{U}_{h_n}) - f'(u_0)\right]}_{\leqslant f'(\sum \mathcal{U}_{h_n})}\phi_n\varphi + o_n(1)\\
&\leqslant \int_\Omega f'(u_0)\phi_n\varphi + \int_\Omega \left[f'\left(\sum \mathcal{U}_{h_n}\right) - f'\left(\mathcal{U}_{1_n}\right)\right]|\phi_n||\varphi|\\
&\quad + \int_\Omega f'\left(\mathcal{U}_{1_n}\right)|\phi_n||\varphi| + o_n(1)\\
&\leqslant \int_\Omega f'(u_0)\phi_n\varphi + \sum \int_\Omega f'\left(\mathcal{U}_{i_n}\right)|\phi_n||\varphi| + o_n(1)\\
&= \int_\Omega f'(u_0)\phi_n\varphi + \sum \delta_{i_n}^{\frac{N+2}{2}} \int_{\Omega_{i_n}} f'\left(\sum \mathcal{U}_{h_n}(\delta_{i_n}y + \xi_{i_n})\right)|\tilde{\phi}_{i_n}||\varphi| + o_n(1)\\
&= \int_\Omega f'(u_0)\phi_n\varphi + \sum \delta_{i_n}^{\frac{N-2}{2}} \int_{\Omega_{i_n}} f'\left(U_{1,0}\right)|\tilde{\phi}_{i_n}||\varphi| + o_n(1)\\
&\to p\int_\Omega u_0^{p-1}\varphi\phi_\infty
\end{aligned}$$

as $\tilde{\phi}_{i_n}$ converges to zero weakly in $\mathfrak{D}^{1,2}(\mathbb{R}^N)$ and $\mathcal{U}_{1,0} \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$. Hence ϕ_∞ weakly satisfies $-\Delta\phi_\infty = pu_0^{p-1}\phi_\infty$ in Ω and, by the non degeneracy of u_0, we have that $\phi_\infty = 0$.

· Step 4. Now we want to show that $\phi_n \to 0$ strongly in $H_0^1(\Omega)$. Indeed

$$\begin{aligned}
\|\phi_n\|_{H_0^1(\Omega)} &= \langle i^\star(f'\left(u_0 - \sum \mathcal{U}_{h_n}\right)\phi_n, \phi_n\rangle + \underbrace{\langle h_n + w_n, \phi_n\rangle}_{=0}\\
&= \int_\Omega \left(f'\left(u_0 - \sum \mathcal{U}_{h_n}\right) - f'\left(\sum \mathcal{U}_{h_n}\right)\right)\phi_n^2 + \int_\Omega f'\left(\sum \mathcal{U}_{h_n}\right)\phi_n^2\\
&\leqslant \int_\Omega f'(u_0)\,\phi_n^2 + \int_{\Omega_{i_n}} f'\left(U_{1,0} + \delta_{i_n}^{\frac{N-2}{2}} \sum_{h_n \neq i_n} \mathcal{U}_{h_n}(\delta_{i_n}y + \xi_{i_n})\right)\tilde{\phi}_{i_n}^2\\
&\leqslant \int_\Omega f'(u_0)\phi_n^2 + \int_{\Omega_{i_n}} f'(U_{1,0})\tilde{\phi}_{i_n}^2\\
&\quad + \sum_{h\neq i} \delta_{i_n}^2 \int_{\Omega_{i_n}} \alpha_N^{\frac{4}{N-2}}\left(\frac{\delta_{h_n}}{\delta_{h_n}^2 + |\delta_{i_n}y + \xi_{i_n} - \xi_{h_n}|^2}\right)^2 \tilde{\phi}_{i_n}^2
\end{aligned}$$

where the last term converges to zero by the Dominated Convergence Theorem. Moreover ϕ_n^2 and $\tilde{\phi}_{h_n}^2$ are uniformly bounded respectively in $L^{\frac{N}{N-2}}(\Omega)$ and $L^{\frac{N}{N-2}}(\mathbb{R}^N)$ and they converge to zero almost everywhere in Ω and $\mathbb{R}^N$. Hence ϕ_n^2 and $\tilde{\phi}_{h_n}^2$ converge weakly to zero in $L^{\frac{N}{N-2}}(\Omega)$ and $L^{\frac{N}{N-2}}(\mathbb{R}^N)$. As $f'(u_0) \in L^{\frac{N}{2}}(\Omega)$ and $f'\left(U_{1,0}\right) \in L^{\frac{N}{2}}(\mathbb{R}^N)$, we have that $\|\phi_n\|_{H_0^1(\Omega)} = o_n(1)$, and this contradicts the hypothesis $\|\phi_n\| = 1$ and prove that $\|\mathcal{L}_{\delta,\xi}\phi\| \geq C\,\|\phi\|$.

· Step 5. Now we have to prove that $\mathcal{L}_{\delta,\boldsymbol{\xi}}$ is invertible and its inverse is continuous. Indeed, we know that $\Pi^{\perp} \circ i^{\star} : L^{\frac{2N}{N-2}}(\Omega) \to H_0^1(\Omega)$ is a compact operator, then $\mathcal{L}_{\delta,\boldsymbol{\xi}} = Id - K$ where K is a compact operator. By (3.13) we also know that $\mathcal{L}_{\delta,\boldsymbol{\xi}}$ is injective. Thus, by the Fredholm's alternative theorem is also surjective.

□

Now we want to estimate the error term defined in (3.10).

Proposition 3.6 *Let $N > 6$ and $\boldsymbol{\delta}, \boldsymbol{\xi}$ satisfy (2.8) and (2.9) as in Sect. 2. Then there exists $C = C(a) > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the error term satisfies*

$$\|\mathcal{E}_{\boldsymbol{\delta},\boldsymbol{\xi}}\|_{H_0^1(\Omega)} \leqslant C\varepsilon^{\frac{\hat{\theta}}{2}+\sigma}$$

for some $\sigma > 0$ arbitrarily small, where $\hat{\theta}$ is given by (2.12).

Proof Observing that

$$\mathcal{E}_{\boldsymbol{\delta},\boldsymbol{\xi}} = \Pi^{\perp}\{i^{\star}\left[f(W_{\boldsymbol{\delta},\boldsymbol{\xi}}) + \varepsilon W_{\boldsymbol{\delta},\boldsymbol{\xi}}\right] - i^{\star}[f(u_0) - \sum_{h=1}^{k} f(\mathcal{U}_h)]\}$$

and by (2.1) we get

$$\begin{aligned}
\|\mathcal{E}_{\boldsymbol{\delta},\boldsymbol{\xi}}\|_{H_0^1(\Omega)} &\leqslant |f(W_{\boldsymbol{\delta},\boldsymbol{\xi}}) - f(u_0) + \sum f(P\mathcal{U}_h) + \varepsilon(W_{\boldsymbol{\delta},\boldsymbol{\xi}})|_{\frac{2N}{N+2},\Omega} \\
&\leqslant \underbrace{\left|f(W_{\boldsymbol{\delta},\boldsymbol{\xi}}) - f\left(\sum P\mathcal{U}_h\right)\right|_{\frac{2N}{N+2};\cup B_{\eta_i}(\xi_i)}}_{(I)} + \sum_h \underbrace{|f(\mathcal{U}_h)|_{\frac{2N}{N+2};\Omega\setminus\cup B_{\eta_i}(\xi_i)}}_{(II)} \\
&\quad + \underbrace{\left|\sum f(\mathcal{U}_h) - f\left(\sum P\mathcal{U}_h\right)\right|_{\frac{2N}{N+2},\cup B_{\eta_i}(\xi_i)}}_{(III)} \\
&\quad + \underbrace{|f(W_{\boldsymbol{\delta},\boldsymbol{\xi}}) - f(u_0)|_{\frac{2N}{N+2};\Omega\setminus\cup B_{\eta_i}(\xi_i)}}_{(IV)} + \underbrace{|f(u_0)|_{\frac{2N}{N+2};\cup B_{\eta_i}(\xi_i)}}_{(V)} \\
&\quad + \underbrace{\varepsilon|u_0|_{\frac{2N}{N+2}}}_{(VI)} + \underbrace{\varepsilon\sum_h |P\mathcal{U}_h|_{\frac{2N}{N+2};\Omega}}_{(VII)}.
\end{aligned}$$

Recalling that $\eta_h = \mathrm{dist}(\xi_h, \partial\Omega) = \mathcal{O}(\varepsilon^{\beta})$, we observe that

$$\begin{aligned}
(V) &\leqslant \sum |u_0^p|_{\frac{2N}{N+2},B_{\eta_i}(\xi_i)} \leqslant \sum_i \left(\int_0^{\eta_i} r^{N-1}\right)^{\frac{N+2}{2N}} \\
&= \sum \mathcal{O}\left(\eta_i^{\frac{N+2}{2}}\right) = \mathcal{O}\left(\varepsilon^{\frac{N^2-4}{N^2-6N+4}}\right) = \mathcal{O}\left(\varepsilon^{\frac{\hat{\theta}}{2}+\sigma}\right)
\end{aligned}$$

and that

$$\mathcal{U}_{\delta_h,\xi_h} \lesssim \left(\frac{\delta_h}{\eta_h^2}\right)^{\frac{N-2}{2}} = \mathcal{O}\left(\varepsilon^{(\alpha-2\beta)\frac{N-2}{2}}\right) = \mathcal{O}\left(\varepsilon^{\frac{N-2}{N^2-6N+4}}\right) \tag{3.18}$$

in $\Omega\backslash B_{\eta_h}(\xi_h)$ for all $h=1,\dots,k$. Then for all $N\ge 7$

$$\begin{aligned}
(I) &\leqslant \sum_i \left| f(u_0-\sum_h P\mathcal{U}_h) - f\left(\sum_h P\mathcal{U}_h\right) \right|_{\frac{2N}{N+2};B_{\eta_i}(\xi_i)} \\
&\leqslant \sum_i \left(\left| f'\left(\sum_h P\mathcal{U}_h\right) u_0 \right|_{\frac{2N}{N+2};B_{\eta_i}(\xi_i)} + |u_0^p|_{\frac{2N}{N+2};B_{\eta_i}(\xi_i)} \right) \\
&\leqslant C\sum_i \left(\left| f'\left(\sum_h P\mathcal{U}_h\right) - f'(\mathcal{U}_i) \right|_{\frac{2N}{N+2};B_{\eta_1}(\xi_1)} + |f'(\mathcal{U}_i)|_{\frac{2N}{N+2};B_{\eta_1}(\xi_1)} \right. \\
&\quad \left. + \left(\int_0^{\eta_i} r^{N-1}\right)^{\frac{N+2}{2N}} \right) \\
&\leqslant C\sum_i \left(|f'(\mathcal{U}_i - P\mathcal{U}_i)|_{\frac{2N}{N+2};B_{\eta_i}(\xi_i)} + \sum_{h\neq i} |f'(P\mathcal{U}_h)|_{\frac{2N}{N+2};B_{\eta_i}(\xi_i)} \right) \\
&\quad + C\sum_i \left(|f'(\mathcal{U}_i)|_{\frac{2N}{N+2};B_{\eta_i}(\xi_i)} + \eta_i^{\frac{N+2}{2}} \right) \\
&\leqslant C\sum_i \left(|\varphi_{\delta_i,\xi_i}|_{\infty,\Omega}^{\frac{4}{N-2}} \eta_i^{\frac{N+2}{2}} + \eta_i^{\frac{N+2}{2}} \sum_{h\neq i} \frac{\delta_h^2}{\eta_h^4} + \delta_i^2 \left(\int_0^{\eta_i} r^{N\frac{N-6}{N+2}-1}\right)^{\frac{N+2}{2}} + \eta_i^{\frac{N+2}{2}} \right) \\
&\leqslant C\sum_i \frac{\delta_i^2}{\eta_i^4}\eta_i^{\frac{N+2}{2}} + +\eta_i^{\frac{N+2}{2}} \sum_{h\neq i}\frac{\delta_h^2}{\eta_h^4} + \delta_i^2\eta_i^{\frac{N-6}{2}} + \eta_i^{\frac{N+2}{2}} = \mathcal{O}(\varepsilon^{\frac{\hat\theta}{2}+\sigma})
\end{aligned}$$

by (3.18) and (3.5). Now

$$\begin{aligned}
(II) &\leqslant |f(\mathcal{U}_h)|_{\frac{2N}{N+2},\Omega\backslash B_{\eta_h}(\xi_h)} = \alpha_N^p \left(\int_{\Omega\backslash B_{\eta_h}(0)} \left(\frac{\delta_h}{\delta_h^2+|y|^2} \right)^N \right)^{\frac{N+2}{2N}} \\
&\leqslant C\left(\int_{\frac{\eta_h}{\delta_h}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^N} \right)^{\frac{N+2}{2N}} \leqslant C\left(\frac{\delta_h}{\eta_h}\right)^{\frac{N+2}{2}} \\
&= \mathcal{O}\left(\varepsilon^{\frac{N+2}{2}(\alpha-\beta)}\right) = \mathcal{O}\left(\varepsilon^{\frac{N(N+2)}{N^2-6N+4}}\right) = \mathcal{O}\left(\varepsilon^{\frac{\hat\theta}{2}+\sigma}\right)
\end{aligned} \tag{3.19}$$

and analogously

$$\begin{aligned}
|\mathcal{U}_h|_{\frac{2N}{N+2},\Omega\backslash\cup B_{\eta_i}(\xi_i)} &\leqslant |\mathcal{U}_h|_{\frac{2N}{N+2},\Omega\backslash B_{\eta_h}(\xi_h)} = \alpha_N \left(\int_{\Omega\backslash B_{\eta_h}(0)} \left(\frac{\delta_h}{\delta_h^2+|y|^2} \right)^{N\frac{N-2}{N+2}} \right)^{\frac{N+2}{2N}} \\
&\leqslant C\delta_h^2 \left(\int_{\frac{\eta_h}{\delta_h}}^{+\infty} \alpha_N \frac{r^{N-1}}{(1+r^2)^{N\frac{N-2}{N+2}}} \right)^{\frac{N+2}{2N}} \leqslant C\delta_h^2 \left(\frac{\delta_h}{\eta_h}\right)^{\frac{N-6}{2}} \\
&= \mathcal{O}\left(\varepsilon^{2\alpha+\frac{N-6}{2}(\alpha-\beta)}\right) = \mathcal{O}\left(\varepsilon^{\frac{N^2+2N-8}{2(N^2-6N+4)}}\right) = \mathcal{O}\left(\varepsilon^{\frac{\hat\theta}{2}+\sigma}\right).
\end{aligned} \tag{3.20}$$

Then, by (3.19) and (3.20)

$$\begin{aligned}(IV) &\leqslant \Big| \Big(\sum P\mathcal{U}_h\Big)^p \Big|_{\frac{2N}{N+2};\Omega\setminus\cup B_{\eta_i}(\xi_i)} + |u_0^{p-1}\sum P\mathcal{U}_h|_{\frac{2N}{N+2};\Omega\setminus\cup B_{\eta_i}(\xi_i)}\\ &\leqslant C\sum |(P\mathcal{U}_h)^p|_{\frac{2N}{N+2};\Omega\setminus\cup B_{\eta_i}(\xi_i)} + C\sum |P\mathcal{U}_h|_{\frac{2N}{N+2};\Omega\setminus\cup B_{\eta_i}(\xi_i)}\\ &\leqslant C\sum |(\mathcal{U}_h)^p|_{\frac{2N}{N+2};\Omega\setminus\cup B_{\eta_i}(\xi_i)} + C\sum |\mathcal{U}_h|_{\frac{2N}{N+2};\Omega\setminus\cup B_{\eta_i}(\xi_i)} = \mathcal{O}(\varepsilon^{\frac{\hat\theta}{2}+\sigma}).\end{aligned}$$

In a similar way, by (3.5) and (3.18)

$$\begin{aligned}(III) &\leqslant \sum_i \left| \sum_h f(\mathcal{U}_h) - f\left(\sum_h P\mathcal{U}_h\right)\right|_{\frac{2N}{N+2},B_{\eta_i}(\xi_i)}\\ &\leqslant \sum_i \left(|f(\mathcal{U}_i) - f(\sum P\mathcal{U}_h)|_{\frac{2N}{N+2},B_{\eta_i}(\xi_i)} + \sum_{h\neq i} |f(\mathcal{U}_h)|_{\frac{2N}{N+2},B_{\eta_i}(\xi_i)}\right)\\ &\leqslant C\sum_i \left(|f'(\mathcal{U}_i)\left(\varphi_i + \sum_{h\neq i} P\mathcal{U}_h\right)|_{\frac{2N}{N+2},B_{\eta_i}(\xi_i)} + \sum_{h\neq i}|f(\mathcal{U}_h)|_{\frac{2N}{N+2},B_{\eta_i}(\xi_i)}\right)\\ &\leqslant C\sum_i \left(|f'(\mathcal{U}_i)|_{\frac{2N}{N+2},B_{\eta_i}(\xi_i)} \left(|\varphi_{\delta_i,\xi_i}|_{\infty,\Omega} + \sum_{h\neq i}\left(\frac{\delta_h}{\eta_h^2}\right)^{\frac{N-2}{2}}\right)\right.\\ &\qquad \left. + |B_{\eta_i}(\xi_i)|^{\frac{N+2}{2N}} \sum_{h\neq i}\left(\frac{\delta_h}{\eta_h^2}\right)^{\frac{N-2}{2}}\right)\\ &\leqslant C\sum_i \left(\delta_i^{\frac{N-2}{2}} \left(\int_0^{\frac{\eta_h}{\delta_h}} r^{\frac{N^2-7N-2}{N+2}}\right)^{\frac{N+2}{2N}}\right)\sum\left(\frac{\delta_h^{\frac{N-2}{2}}}{\eta_h^{N-2}}\right) + \eta_i^{\frac{N+2}{2}}\sum_{h\neq i}\left(\frac{\delta_h}{\eta_h^2}\right)^{\frac{N-2}{2}}\\ &\leqslant C\sum_i \left(\delta_i^{\frac{N-2}{2}} \left(\frac{\eta_i}{\delta_i}\right)^{\frac{N-6}{2}} \sum\left(\frac{\delta_h^{\frac{N-2}{2}}}{\eta_h^{N-2}}\right) + \eta_i^{\frac{N+2}{2}}\sum_{h\neq i}\left(\frac{\delta_h}{\eta_h^2}\right)^{\frac{N-2}{2}}\right)\\ &= \mathcal{O}\left(\varepsilon^{(\alpha-\beta)\frac{N+2}{2}}\right) + \mathcal{O}\left(\varepsilon^{\alpha\frac{N-2}{2}-\beta\frac{N-6}{2}}\right) = \mathcal{O}\left(\varepsilon^{\hat\theta+\sigma}\right).\end{aligned}$$

□

Now we can solve (3.8) using a fixed point argument.

Proposition 3.7 *Let $N > 6$ and $\boldsymbol{\delta}, \boldsymbol{\xi}$ satisfy (2.8) and (2.9) as in Sect. 2. Then there exists $C = C(a) > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a unique $\phi_{\boldsymbol{\delta},\boldsymbol{\xi}} \in K^{\perp}_{\boldsymbol{\delta},\boldsymbol{\xi}}$ solving*

$$\mathcal{L}_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_{\boldsymbol{\delta},\boldsymbol{\xi}} = \mathcal{E}_{\boldsymbol{\delta},\boldsymbol{\xi}} + \mathcal{N}^1_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_{\boldsymbol{\delta},\boldsymbol{\xi}} + \mathcal{N}^2_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_{\boldsymbol{\delta},\boldsymbol{\xi}} \tag{3.21}$$

and satisfying

$$\|\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}\|_{H_0^1(\Omega)} \leqslant C\varepsilon^{\frac{\hat\theta}{2}+\sigma} \tag{3.22}$$

for some $\sigma > 0$.

Proof We want to show that $\mathcal{T}_{\delta,\xi} : K^{\perp}_{\delta,\xi} \to K^{\perp}_{\delta,\xi}$ is a contraction where

$$\mathcal{T}_{\delta,\xi}\phi = \mathcal{L}^{-1}_{\delta,\xi}[\mathcal{E}_{\delta,\xi} + \mathcal{N}^1_{\delta,\xi}\phi + \mathcal{N}^2_{\delta,\xi}\phi] \text{ for all } \phi \in K^{\perp}_{\delta,\xi}$$

and the operators $\mathcal{L}_{\delta,\xi}$, $\mathcal{E}_{\delta,\xi}$, $\mathcal{N}^1_{\delta,\xi}$ and $\mathcal{N}^2_{\delta,\xi}$ are defined (3.9), (3.10), (3.11) and (3.12).

Let $r > 0$ and $\mathcal{B}_{\delta,\xi} = \{\phi \in K^{\perp}_{\delta,\xi} : \|\phi\|_{H^1_0(\Omega)} \leqslant r\varepsilon^{\frac{\hat{\theta}}{2}+\sigma}\}$. We observe that

$$\begin{aligned}
\|\mathcal{N}^1_{\delta,\xi}\phi\|_{H^1_0(\Omega)} \leqslant& C\|i^{\star}[f(W_{\delta,\xi}+\phi) - f(W_{\delta,\xi}) - f'(W_{\delta,\xi})\phi + \varepsilon\phi]\|_{H^1_0(\Omega)}\\
\leqslant& C|f(W_{\delta,\xi}+\phi) - f(W_{\delta,\xi}) - f'(W_{\delta,\xi})\phi|_{\frac{2N}{N+2},\Omega} + \varepsilon|\phi|_{\frac{2N}{N+2},\Omega}\\
\leqslant& C|f(\phi)|_{\frac{2N}{N+2},\Omega} + \varepsilon\|\phi\|_{H^1_0(\Omega)} \leqslant C\|\phi\|^p_{H^1_0(\Omega)} + C\varepsilon\|\phi\|_{H^1_0(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{N}^2_{\delta,\xi}\phi\|_{H^1_0(\Omega)} \leqslant& C\left\|i^{\star}\left[\left(f'\left(u_0 - \sum_{h=1}^{k} P\mathcal{U}_h\right) - f'\left(u_0 - \sum_{h=1}^{k}\mathcal{U}_h\right)\right)\phi\right]\right\|_{H^1_0(\Omega)}\\
\leqslant& C\left|\left(f'\left(u_0 - \sum_{h=1}^{k} P\mathcal{U}_h\right) - f'\left(u_0 - \sum_{h=1}^{k}\mathcal{U}_h\right)\right)\phi\right|_{\frac{2N}{N+2},\Omega}\\
\leqslant& C\left|f'\left(\sum \varphi_{\delta_h,\xi_h}\right)\right|_{\frac{N}{2},\Omega}|\phi|_{\frac{2N}{N-2},\Omega} \leqslant C\left(\sum|\varphi_{\delta_h,\xi_h}|_{\frac{2N}{N-2},\Omega}\right)^{\frac{4}{N-2}}\|\phi\|_{H^1_0(\Omega)}.
\end{aligned}$$

Hence by (3.6)

$$\|\mathcal{T}_{\delta,\xi}\phi\|_{H^1_0(\Omega)} \leqslant \|\mathcal{N}^1_{\delta,\xi}\phi\|_{H^1_0(\Omega)} + \|\mathcal{N}^2_{\delta,\xi}\phi\|_{H^1_0(\Omega)} \leqslant C\varepsilon^{\frac{\hat{\theta}}{2}+\sigma}$$

and so $\mathcal{T}_{\delta,\xi}$ maps $\mathcal{B}_{\delta,\xi}$ into itself. Moreover for all $\phi_1, \phi_2 \in K^{\perp}_{\delta,\xi}$ we have

$$\begin{aligned}
&\|\mathcal{N}^1_{\delta,\xi}\phi_1 - \mathcal{N}^1_{\delta,\xi}\phi_2\|_{H^1_0(\Omega)} \leqslant C|f(W_{\delta,\xi}+\phi_1) - f(W_{\delta,\xi}+\phi_2) - f'(W_{\delta,\xi})(\phi_1-\phi_2)|_{\frac{2N}{N+2},\Omega}\\
&\quad + \varepsilon|\phi_1-\phi_2|_{\frac{2N}{N+2},\Omega}\\
&\leqslant |f(\underbrace{W_{\delta,\xi}+\phi_1}_{=a+b}) - f(\underbrace{W_{\delta,\xi}+\phi_2}_{=a;\ b=\phi_1-\phi_2}) - f'(W_{\delta,\xi}+\phi_2)(\phi_1-\phi_2)|_{\frac{2N}{N+2},\Omega}\\
&\quad |f'(W_{\delta,\xi}+\phi_2)(\phi_1-\phi_2) - f'(W_{\delta,\xi})(\phi_1-\phi_2)|_{\frac{2N}{N+2},\Omega} + \varepsilon|\phi_1-\phi_2|_{\frac{2N}{N+2},\Omega}\\
&\leqslant C\left(||\phi_1-\phi_2|^p|_{\frac{2N}{N+2},\Omega} + ||\phi_2|^p|\phi_1-\phi_2|_{\frac{2N}{N+2},\Omega} + \varepsilon|\phi_1-\phi_2|_{\frac{2N}{N+2},\Omega}\right)\\
&\leqslant C\left(|(\phi_1-\phi_2)^{p-1}|_{\frac{N}{2},\Omega}|\phi_1-\phi_2|_{\frac{2N}{N-2},\Omega} + |\phi_2^{p-1}|_{\frac{N}{2},\Omega}|\phi_1-\phi_2|_{\frac{2N}{N-2},\Omega}\right)\\
&\quad + C\varepsilon\|\phi_1-\phi_2\|^{p-1}_{H^1_0(\Omega)}\\
&\leqslant C\left(\underbrace{\|\phi_1-\phi_2\|_{H^1_0(\Omega)}}_{\lesssim\|\phi_1\|_{H^1_0(\Omega)}+\|\phi_2\|_{H^1_0(\Omega)}}\|\phi_1-\phi_2\|_{H^1_0(\Omega)} + \|\phi_2\|^{p-1}_{H^1_0(\Omega)}\|\phi_1-\phi_2\|_{H^1_0(\Omega)}\right)\\
&\quad + C\varepsilon\|\phi_1-\phi_2\|_{H^1_0(\Omega)}\\
&\leqslant C\left(\|\phi_1\|^{p-1}_{H^1_0(\Omega)} + \|\phi_2\|^{p-1}_{H^1_0(\Omega)} + \varepsilon\right)\|\phi_1-\phi_2\|_{H^1_0(\Omega)}
\end{aligned}$$

and

$$
\begin{aligned}
\|\mathcal{N}^2_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_1-\mathcal{N}^2_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_2\|_{H^1_0(\Omega)} \leqslant& C|[f'(u_0-\sum P\mathcal{U}_h)-f'(u_0-\sum\mathcal{U}_h)](\phi_1-\phi_2)|_{\frac{2N}{N+2},\Omega}\\
\leqslant& C|f'(\sum\varphi_{\delta_h,\xi_h})|_{\frac{N}{2},\Omega}|\phi_1-\phi_2|_{\frac{2N}{N-2},\Omega}\\
\leqslant& C|\sum\varphi_{\delta_h,\xi_h}|^{p-1}_{\frac{2N}{N-2},\Omega}\|\phi_1-\phi_2\|_{H^1_0(\Omega)}\\
\leqslant& C\left(\sum|\varphi_{\delta_h,\xi_h}|_{\frac{2N}{N-2},\Omega}\right)^{p-1}\|\phi_1-\phi_2\|_{H^1_0(\Omega)}\\
\leqslant& C\sum|\varphi_{\delta_h,\xi_h}|^{p-1}_{\frac{2N}{N-2},\Omega}\|\phi_1-\phi_2\|_{H^1_0(\Omega)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|\mathcal{T}_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_1-\mathcal{T}_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_2\|_{H^1_0(\Omega)} \leqslant& C\|\mathcal{N}^1_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_1-\mathcal{N}^1_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_2\|_{H^1_0(\Omega)}+\|\mathcal{N}^2_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_1-\mathcal{N}^2_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_2\|_{H^1_0(\Omega)}\\
\leqslant& C\left(\|\phi_1\|_{H^1_0(\Omega)}+\|\phi_2\|_{H^1_0(\Omega)}+\varepsilon+\sum|\varphi_{\delta_h,\xi_h}|^{p-1}_{\frac{2N}{N-2},\Omega}\right)\\
&\|\phi_1-\phi_2\|_{H^1_0(\Omega)}
\end{aligned}
$$

where $|\varphi_{\delta_h,\xi_h}|^{p-1}_{\frac{2N}{N-2},\Omega}=\mathcal{O}(\varepsilon^{2(\alpha-\beta)})$. Then, for ε sufficiently small, $\mathcal{T}_{\boldsymbol{\delta},\boldsymbol{\xi}}$ is a contraction and the claim follows. □

4 The reduced energy

In this section we want to reduce the original problem to a finite dimensional one and solve (3.7). Define $J_\varepsilon : H^1_0(\Omega)\to\mathbb{R}^N$ as

$$
J_\varepsilon(u)=\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\frac{1}{p+1}\int_\Omega|u|^{p+1}\,dx-\frac{\varepsilon}{2}\int_\Omega u^2\,dx. \tag{4.1}
$$

Then the critical points of J_ε are solutions to (BN). Let us also define the reduced functional

$$
\tilde{J}_\varepsilon(\boldsymbol{t},\boldsymbol{d},\boldsymbol{\tau})=J_\varepsilon(W_{\boldsymbol{\delta},\boldsymbol{\xi}}+\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}) \tag{4.2}
$$

where $\boldsymbol{t}=(t_1,\ldots,t_k)$, $\boldsymbol{d}=(d_1,\ldots,d_k)\in\mathbb{R}^k$, $\boldsymbol{\tau}=(\tau_1,\ldots,\tau_k)\in(\mathbb{R}^{N-1})^k$ as in Sect. 2, $W_{\boldsymbol{\delta},\boldsymbol{\xi}}=u_0-\sum P\mathcal{U}_h$ and $\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}$ is as in Proposition 3.7 such that

$$
\|\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}\|_{H^1_0(\Omega)}\leqslant C\varepsilon^{\frac{\hat{\theta}}{2}+\sigma}
$$

for some $\sigma>0$ and $\hat{\theta}=\frac{N^3-8N+8}{N(N^2-6N+4)}$.

Proposition 4.1 *Let $N>6$ and $\boldsymbol{\delta}$, $\boldsymbol{\xi}$ satisfy (2.8) and (2.9) as in Sect. 2. There exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ it holds*

$$
J_\varepsilon(W_{\boldsymbol{\delta},\boldsymbol{\xi}}+\phi_{\boldsymbol{\delta},\boldsymbol{\xi}})=J_\varepsilon(W_{\boldsymbol{\delta},\boldsymbol{\xi}})+\mathcal{O}(\varepsilon^{\hat{\theta}+\sigma}),
$$

where J_ε is defined in (4.1).

Proof It's easy to see that

$$\begin{aligned}J_\varepsilon(W_{\boldsymbol{\delta},\boldsymbol{\xi}}+\phi_{\boldsymbol{\delta},\boldsymbol{\xi}})-J_\varepsilon(W_{\boldsymbol{\delta},\boldsymbol{\xi}}) &= \frac{1}{2}\|\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}\|^2_{H^1_0(\Omega)}+\int_\Omega \nabla W_{\boldsymbol{\delta},\boldsymbol{\xi}}\cdot\nabla\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}-\int_\Omega W^p_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}\\ &-\frac{1}{p+1}\int_\Omega\left[|W_{\boldsymbol{\delta},\xi}+\phi_{\boldsymbol{\delta},\xi}|^{p+1}+|W_{\boldsymbol{\delta},\xi}|^{p+1}-(p+1)W^p_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}\right]\\ &-\frac{\varepsilon}{2}\int_\Omega\phi^2_{\boldsymbol{\delta},\xi}-\varepsilon\int_\Omega W_{\boldsymbol{\delta},\xi}\phi_{\boldsymbol{\delta},\xi}.\end{aligned}$$

Using (3.22)

$$\varepsilon\int_\Omega\phi^2_{\boldsymbol{\delta},\xi}\leqslant\varepsilon|\phi|_{2,\Omega}\leqslant\varepsilon|\phi_{\boldsymbol{\delta},\xi}|^2_{\frac{2N}{N-2},\Omega}\leqslant\varepsilon\|\phi_{\boldsymbol{\delta},\xi}\|^2_{H^1_0(\Omega)}\leqslant C\varepsilon^{\hat{\theta}+\sigma}$$

and

$$\varepsilon\int_\Omega W_{\boldsymbol{\delta},\xi}\phi_{\boldsymbol{\delta},\xi}\leqslant\varepsilon|W_{\boldsymbol{\delta},\xi}|_{\frac{2N}{N+2}}|\phi_{\boldsymbol{\delta},\xi}|_{\frac{2N}{N-2},\Omega}\leqslant C\varepsilon^{1+2\alpha}\|\phi_{\boldsymbol{\delta},\xi}\|_{H^1_0(\Omega)}\leqslant C\varepsilon^{\hat{\theta}+\sigma}$$

by (3.2). Now by (3.1) and (3.22)

$$\begin{aligned}\int_\Omega\left[|W_{\boldsymbol{\delta},\boldsymbol{\xi}}+\phi_{\boldsymbol{\delta},\xi}|^{p+1}+|W_{\boldsymbol{\delta},\xi}|^{p+1}\ -(p+1)W^p_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_{\boldsymbol{\delta},\xi}\right]&\leqslant\int_\Omega\left[|\phi_{\boldsymbol{\delta},\xi}|^{p+1}+\phi^2W^{p-1}_{\boldsymbol{\delta},\boldsymbol{\xi}}\right]\\ \leqslant|\phi^{p+1}_{\boldsymbol{\delta},\boldsymbol{\xi}}|_{\frac{2N}{N+2},\Omega}+|\phi_{\boldsymbol{\delta},\xi}|_{\frac{N}{N-2},\Omega}|W^{p-1}_{\boldsymbol{\delta},\boldsymbol{\xi}}|_{\frac{N}{2},\Omega}&\\ \leqslant C\|\phi_{\boldsymbol{\delta},\xi}\|^{p+1}_{H^1_0(\Omega)}+\|\phi_{\boldsymbol{\delta},\xi}\|^2_{H^1_0(\Omega)}|W_{\boldsymbol{\delta},\xi}|^{p-1}_{\frac{2N}{N-2},\Omega}&\\ \leqslant C\left(\varepsilon^{(p+1)\frac{\hat{\theta}}{2}+\sigma}+\varepsilon^{\hat{\theta}+\sigma}\right)\leqslant C\varepsilon^{\hat{\theta}+\sigma}.&\end{aligned}$$

Moreover

$$\begin{aligned}&\int_\Omega\nabla W_{\boldsymbol{\delta},\boldsymbol{\xi}}\cdot\nabla\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}-\int_\Omega W^p_{\boldsymbol{\delta},\boldsymbol{\xi}}\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}\\ &=-\int_\Omega\left[\Delta W_{\boldsymbol{\delta},\boldsymbol{\xi}}-W^p_{\boldsymbol{\delta},\boldsymbol{\xi}}\right]\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}\\ &=\int_\Omega\left[f(u_0)-\sum f(\mathcal{U}_h)-f(u_0-\sum P\mathcal{U}_h)\right]\phi_{\boldsymbol{\delta},\boldsymbol{\xi}}\\ &\leqslant|f(u_0)-\sum f(\mathcal{U}_h)-f(u_0-\sum P\mathcal{U}_h)|_{\frac{2N}{N+2},\Omega}|\phi_{\boldsymbol{\delta},\xi}|_{\frac{2N}{N-2},\Omega}\\ &\leqslant C\varepsilon^{\hat{\theta}+\sigma}\end{aligned}$$

as in the proof of Proposition 3.6. □

Now we want to evaluate and find an expansion of

$$J_\varepsilon\left(u_0-\sum_{i=1}^k P\mathcal{U}_i\right).$$

Proposition 4.2 *Let $N>6$ and $\boldsymbol{\delta},\boldsymbol{\xi}$ satisfy (2.8) and (2.9) as in Sect. 2. Then it holds*

$$\tilde{J}_\varepsilon(\boldsymbol{d},\boldsymbol{t},\boldsymbol{\tau})\ =\varepsilon^{\hat{\theta}}\left(\mathfrak{A}\sum_i d_i^2+\mathfrak{C}\sum_i t_i^2-\alpha_N\mathrm{C}\sum_{h<i}\frac{d_0^{N-2}}{|\tau_i-\tau_h|^{N-2}}\right.$$

$$-\frac{\mathsf{C}}{2} d_0^{\frac{N-2}{2}} t_0 \sum_i \langle D^2_{N-1}\partial_\nu u_0(\xi_0)\tau_i, \tau_i\rangle \Bigg)$$
$$+\mathbb{A}_1 + \mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right).$$

C^0-uniformly with respect to $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k) \in (\mathbb{R}^{N-1})^k$, $\boldsymbol{d} = (d_1, \ldots, d_k), \boldsymbol{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$ such that $|d_i|, |t_i| < a$ and $|\tau_i - \tau_h| > \rho$ for all $i = 1, \ldots, k$ and $i \neq h$, and where

$$\mathbb{A}_1 = \mathbb{A} + \varepsilon^{\theta}\mathfrak{g}(d_0, t_0) - k\varepsilon^{\theta+\sigma}\mathfrak{f}(d_0, t_0).$$

For the definition of $\tilde{J}_\varepsilon$ we refer to (4.2).

Proof We can write $J_\varepsilon(u_0 - \sum_{i=1}^k P\mathcal{U}_i)$ as

$$J_\varepsilon(u_0 - \sum_{i=1}^k P\mathcal{U}_i) = \underbrace{\frac{1}{2}\int_\Omega |\nabla u_0|^2 - \frac{1}{2^\star}\int_\Omega |u_0|^{2^\star}}_{:=J_0(u_0)} - \varepsilon \underbrace{\frac{1}{2}\int_\Omega u_0^2}_{:=\tilde{J}_0(u_0)}$$
$$+ \underbrace{\sum_{i=1}^k \left(\frac{1}{2}\int_\Omega |\nabla P\mathcal{U}_i|^2 - \frac{1}{2^\star}\int_\Omega (P\mathcal{U}_i)^{2^\star}\right)}_{:=(I)}$$
$$+ \underbrace{\sum_{i>h}\int_\Omega (\nabla P\mathcal{U}_i \nabla P\mathcal{U}_h - f(P\mathcal{U}_i)P\mathcal{U}_h)}_{:=(II)} - \underbrace{\sum_{h<i}\int_\Omega f(P\mathcal{U}_i)P\mathcal{U}_h}_{:=(III)}$$
$$\underbrace{- \frac{\varepsilon}{2}\sum_{i=1}^k \int_\Omega (P\mathcal{U}_i)^2}_{(IV)} \underbrace{- \varepsilon \sum_{h<i}\int_\Omega P\mathcal{U}_i P\mathcal{U}_h}_{:=(V)} \underbrace{- \varepsilon \sum_{i=1}^k \int_\Omega u_0 P\mathcal{U}_i}_{(VI)}$$
$$\underbrace{- \sum_{i=1}^k \int_\Omega u_0 (P\mathcal{U}_i)^{2^\star - 1}}_{:=(VII)} \underbrace{- \sum_{i=1}^k \int_\Omega (\nabla u_0 \nabla P\mathcal{U}_i - f(u_0)P\mathcal{U}_i)}_{=0}$$
$$\underbrace{- \int_\Omega \left[F(u_0 - \sum_{i=1}^k P\mathcal{U}_i) - F(u_0) - \sum_{i=1}^k F(P\mathcal{U}_i) - \sum_{i\neq h} f(P\mathcal{U}_i)P\mathcal{U}_h\right]}_{(VIII)}$$
$$\underbrace{- \int_\Omega \left[\sum_{i=1}^k f(u_0)P\mathcal{U}_i - \sum_{i=1}^k u_0 f(P\mathcal{U}_i)\right]}_{:=(IX)}$$

where $f(u) = |u|^{p-1}u$ and $F(s) = \int_0^s f(t)dt$. Now we evaluate each term. Recalling that $\eta_i := \operatorname{dist}(\xi_i, \partial\Omega) = \mathcal{O}(\varepsilon^\beta)$, we have

$$(I) \ = \left(\frac{1}{2} - \frac{1}{2^\star}\right)\sum_i \int_\Omega (\mathcal{U}_i)^{2^\star} + \frac{1}{2}\sum_i \int_\Omega (\mathcal{U}_i)^{2^\star - 1}\varphi_{\delta_i, \xi_i}$$

$$
\begin{aligned}
&+\mathcal{O}\left(\int_\Omega (\mathcal{U}_i)^{2^\star-2}\varphi_{\delta_i,\xi_i}^2+\int_\Omega \varphi_{\delta_i,\xi_i}^{2^\star}\right)\\
&=\frac{k}{N}\int_{\mathbb{R}^N}\mathcal{U}^{2^\star}+\frac{1}{2}\alpha_N\sum_i\delta_i^{N-2}H(\xi_i,\xi_i)\int_{\mathbb{R}^N}\mathcal{U}^{2^\star-1}+\mathcal{O}\left(\frac{\delta_i^N}{\eta_i^N}\right)\\
&=\frac{k}{N}\int_{\mathbb{R}^N}\mathcal{U}^{2^\star}+\frac{1}{2}\alpha_N\sum_i\delta_i^{N-2}H(\xi_i,\xi_i)\int_{\mathbb{R}^N}\mathcal{U}^{2^\star-1}+\mathcal{O}\left(\varepsilon^{\hat\theta+\sigma}\right)
\end{aligned}
$$

$$
\begin{aligned}
(IV)\ &=\frac{\varepsilon}{2}\sum_i\delta_i^2\int_{\mathbb{R}^N}\mathcal{U}^2+\mathcal{O}(\varepsilon\sum_i\|\varphi_{\delta_i,\xi_i}\|_{L^2(\Omega)}^2)\\
&+\varepsilon\sum_i\|\varphi_{\delta_i,\xi_i}\|_{L^\infty(\Omega)}\int_{B_{\eta_i}(\xi_i)}\mathcal{U}_i+\mathcal{O}(\varepsilon\sum_i\|\varphi_{\delta_i,\xi_i}\|_{L^{\frac{2N}{N+2}}(\Omega)}\|\mathcal{U}_i\|_{L^{2^\star}(\Omega\setminus B_{\eta_i}(\xi_i))})\\
&=\frac{\varepsilon}{2}\sum_i\delta_i^2\int_{\mathbb{R}^N}\mathcal{U}^2+\mathcal{O}\left(\varepsilon\frac{\delta_i^{N-2}}{\eta_i^{N-2}}\right)\\
&=\frac{\varepsilon}{2}\sum_i\delta_i^2\int_{\mathbb{R}^N}\mathcal{U}^2+\mathcal{O}\left(\varepsilon^{\hat\theta+\sigma}\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
(VII)\ &=\sum_i\int_\Omega(u_0\mathcal{U}_i)^{2^\star-1}+\sum_i\int_\Omega u_0\left((P\mathcal{U}_i)^{2^\star}-(\mathcal{U}_i)^{2^\star}\right)\\
&=\sum_i\delta_i^{\frac{N-2}{2}}\int_{\frac{\Omega-\xi_i}{\eta_i}}u_0(\delta_i y+\xi_i)U^{2^\star-1}+\mathcal{O}\left(\sum_i\int_{B_{\eta_i}(\xi_i)}u_0(x)\varphi_{\delta_i,\xi_i}^{2^\star-1}\right)\\
&+\mathcal{O}\left(\sum_i\int_{B_{\eta_i}(\xi_i)}u_0(x)(\mathcal{U}_i)^{2^\star-2}\varphi_{\delta_i,\xi_i}\right)+\mathcal{O}\left(\sum_i\int_{\Omega\setminus B_{\eta_i}(\xi_i)}u_0(x)(\mathcal{U}_i)^{2^\star-1}\right)\\
&=\sum_i\delta_i^{\frac{N-2}{2}}u_0(\xi_i)\int_{\mathbb{R}^N}\mathcal{U}^{2^\star-1}+\mathcal{O}\left(\sum_i\delta_i^{\frac{N-2}{2}}u_0(\xi_i)\int_{\frac{\eta_i}{\delta_i}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^{\frac{N+2}{2}}}\right)\\
&+\mathcal{O}\left(\sum_i\|\varphi_{\delta_i,\xi_i}\|_{L^\infty(\Omega)}^{2^\star-1}\eta_i^N u_0(\xi_i)\right)+\mathcal{O}\left(\sum_i\delta_i^2\|\varphi_{\delta_i,\xi_i}\|_{L^\infty(\Omega)}\eta_i^N u_0(\xi_i)\int_{B_1(0)}\frac{1}{|y|^4}\right)\\
&+\mathcal{O}\left(\sum_i\delta_i^{\frac{N-2}{2}}u_0(\xi_i)\int_{\frac{\eta_i}{\delta_i}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^{\frac{N+2}{2}}}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
(VII)&=\textstyle\sum_i\delta_i^{\frac{N-2}{2}}u_0(\xi_i)\int_{\mathbb{R}^N}\mathcal{U}^{2^\star-1}+\mathcal{O}\left(\sum_i\delta_i^{\frac{N-2}{2}}u_0(\xi_i)\frac{\delta_i^2}{\eta_i^2}\right)\\
&=\textstyle\sum_i\delta_i^{\frac{N-2}{2}}u_0(\xi_i)\int_{\mathbb{R}^N}\mathcal{U}^{2^\star-1}+\mathcal{O}\left(\varepsilon^{\hat\theta+\sigma}\right).
\end{aligned}
$$

The other important term is (III). Indeed

$$
\begin{aligned}
(III)\ &=\sum_{h<i}\int_\Omega(P\mathcal{U}_i)^{2^\star-1}P\mathcal{U}_h=\sum_{h<i}\int_\Omega(\mathcal{U}_i)^{2^\star-1}\mathcal{U}_h-\sum_{h<i}\int_\Omega(\mathcal{U}_i)^{2^\star-1}\varphi_{\delta_h,\xi_h}\\
&+\sum_{h<i}\int_\Omega\left((P\mathcal{U}_i)^{2^\star-1}-(\mathcal{U}_i)^{2^\star-1}\right)P\mathcal{U}_h
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{h<i} \frac{\delta_i^{\frac{N-2}{2}} \alpha_N \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}} \int_{\mathbb{R}^N} \mathcal{U}^{2^\star-1}(1+o(1)) + \mathcal{O}\left(\sum_{h<i} \frac{\delta_h^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}}{\eta_h^{N-2}}\right)\\
&\quad +\mathcal{O}\left(\sum_{h<i} \int_{B_{\eta_i}(\xi_i)} (\mathcal{U}_i)^{2^\star-2} \varphi_{\delta_i,\xi_i} \mathcal{U}_h\right) + \mathcal{O}\left(\sum_{h<i} \int_{\Omega \setminus B_{\eta_i}(\xi_i)} (\mathcal{U}_i)^{2^\star-1} \mathcal{U}_h\right)\\
&= \sum_{h<i} \frac{\delta_i^{\frac{N-2}{2}} \alpha_N \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}} \int_{\mathbb{R}^N} \mathcal{U}^{2^\star-1}(1+o(1)) + \mathcal{O}\left(\sum_{h<i} \frac{\delta_h^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}}{\eta_h^{N-2}}\right)\\
&\quad +\mathcal{O}\left(\sum_{h<i} \frac{\delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}}}{|\xi_h - \xi_h|^{N-2}} \frac{\delta_i^2}{\eta_i^2}\right)\\
&= \sum_{h<i} \frac{\delta_i^{\frac{N-2}{2}} \alpha_N \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}} \int_{\mathbb{R}^N} \mathcal{U}^{2^\star-1}(1+o(1))\\
&= \sum_{h<i} \frac{\delta_i^{\frac{N-2}{2}} \alpha_N \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}} \int_{\mathbb{R}^N} \mathcal{U}^{2^\star-1} + \mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right)
\end{aligned}$$

We have to show that the other terms are of higher order.
Reasoning as the third term of (III) we have

$$\begin{aligned}
(II) &= \sum_{h>i} \int_{\Omega} \left((P\mathcal{U}_i)^{2^\star-1} - (\mathcal{U}_i)^{2^\star-1}\right) P\mathcal{U}_h = \mathcal{O}\left(\sum_{h>i} \frac{\delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}}}{|\xi_h - \xi_h|^{N-2}} \frac{\delta_i^2}{\eta_i^2}\right)\\
&= \mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right)
\end{aligned}$$

Now

$$\begin{aligned}
(V) &\lesssim \varepsilon \sum_{i<h} \int_{\Omega} \mathcal{U}_i \mathcal{U}_h \lesssim \varepsilon \sum_{i<h} \delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}} \int_{B_{\eta_i}(\xi_i)} \frac{1}{|x-\xi_i|^{N-2}|x-\xi_h|^{N-2}}\\
&\quad +\varepsilon \sum_{i<h} \int_{\Omega \setminus B_{\eta_i}(\xi_i)} \mathcal{U}_i \mathcal{U}_h\\
&\lesssim \varepsilon \sum_{i<h} \delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}} \int_{B_{\eta_i}(0)} \frac{1}{|y|^{N-2}|y+\xi_i-\xi_h|^{N-2}} + \varepsilon \sum_{i<h} \frac{\delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}} \delta_i^2\\
&\quad \int_{\frac{\eta_i}{\delta_i}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-2}}\\
&\lesssim \varepsilon \sum_{i<h} \frac{\delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}} \int_{B_{\eta_i}(0)} \frac{1}{|y|^{N-2}}\\
&\quad +\varepsilon \sum_{i<h} \frac{\delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}} \eta_i^2\\
&\lesssim \varepsilon \sum_{i<h} \frac{\delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}} = o\left(\sum_{i<h} \frac{\delta_i^{\frac{N-2}{2}} \delta_h^{\frac{N-2}{2}}}{|\xi_i - \xi_h|^{N-2}}\right) = \mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right)
\end{aligned}$$

Now

$$(VI) \;\lesssim \varepsilon \sum_i \delta_i^{\frac{N-2}{2}} u_0(\xi_i)\eta_i^2 = \mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right)$$

Now we can split the term $(VIII)+(IX)$ into the sum of integrals on the balls $B_{\eta_h}(\xi_h)$ and the integral on $\Omega\backslash \cup B_{\eta_h}(\xi_h)$ that we call A_h and B respectively,

where we can evaluate the integral B on $\Omega\backslash \cup B_{\eta_h}(\xi_h)$ as

$$\begin{aligned}
|B| \leqslant & \int_{\Omega\backslash\bigcup_h B_{\eta_h}(\xi_h)} \left| F(u_0 - \sum_h P\mathcal{U}_h) - F(u_0) + f(u_0)\sum_h P\mathcal{U}_h \right| \\
& + \int_{\Omega\backslash\bigcup_h B_{\eta_h}(\xi_h)} u_0 \sum_h f(P\mathcal{U}_h) + \sum_{i\neq h} \int_{\Omega\backslash\bigcup_h B_{\eta_h}(\xi_h)} f(P\mathcal{U}_i)P\mathcal{U}_h \\
& + \sum_h \int_{\Omega\backslash\bigcup_h B_{\eta_h}(\xi_h)} F(P\mathcal{U}_h) \\
\lesssim & \sum_h \int_{\Omega\backslash\bigcup_h B_{\eta_h}} u_0^{2^\star-2}(\mathcal{U}_h)^2 + \sum_h \int_{\Omega\backslash\bigcup_h B_{\eta_h}} u_0(\mathcal{U}_h)^{2^\star-1} \\
& + \sum_{i\neq h} \int_{\Omega\backslash\bigcup_h B_{\eta_h}} \mathcal{U}_i^{2^\star-1}\mathcal{U}_h + \sum_h \int_{\Omega\backslash\bigcup_h B_{\eta_h}} \mathcal{U}_h^{2^\star} \\
\lesssim & \sum_h u_0^{2^\star-2}(\xi_h)\delta_h^2 \int_{\frac{\eta_h}{\delta_h}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-2}} + \sum_h u_0(\xi_h)\delta_h^{\frac{N-2}{2}} \int_{\frac{\eta_h}{\delta_h}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N+2}{2}}} \\
& + \sum_{i\neq h} \frac{\delta_i^{\frac{N-2}{2}}\delta_h^{\frac{N-2}{2}}}{|\xi_i-\xi_h|^{N-2}} \int_{\frac{\eta_h}{\delta_h}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N+2}{2}}} + \sum_h \int_{\frac{\eta_h}{\delta_h}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^N} \\
\lesssim & \sum_h u_0^{2^\star-2}(\xi_h)\delta_h^2 \left(\frac{\delta_h}{\eta_h}\right)^{N-4} + \sum_h u_0(\xi_h)\delta_h^{\frac{N-2}{2}} \left(\frac{\delta_h}{\eta_h}\right)^2 \\
& + \sum_{i\neq h} \frac{\delta_i^{\frac{N-2}{2}}\delta_h^{\frac{N-2}{2}}}{|\xi_i-\xi_h|^{N-2}} \left(\frac{\delta_h}{\eta_h}\right)^2 + \sum_h \left(\frac{\delta_h}{\eta_h}\right)^N \\
= & \; \mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right)
\end{aligned}$$

and in a similar way for the integral A_h on a ball $B_{\eta_h}(\xi_h)$ we get that

$$|A_h| = \mathcal{O}(\varepsilon^{\hat{\theta}+\sigma}).$$

At the end we get

$$\begin{aligned}
J_\varepsilon(u_0 - \sum_h P\mathcal{U}_i) = & \underbrace{J_0(u_0) - \varepsilon\tilde{J}_0(u_0) + \frac{k}{N}\int_{\mathbb{R}^N} \mathcal{U}^{2^\star}}_{:=\mathbb{A}} \\
& + \frac{1}{2}\alpha_N \sum_i \delta_i^{N-2} H(\xi_i,\xi_i) \int_{\mathbb{R}^N} \mathcal{U}^{2^\star-1} - \frac{\varepsilon}{2}\sum_i \delta_i^2 \int_{\mathbb{R}^N} \mathcal{U}^2 \\
& - \sum_i \delta_i^{\frac{N-2}{2}} u_0(\xi_i) \int_{\mathbb{R}^N} U^{2^\star-1}
\end{aligned}$$

$$-\sum_{h<i}\frac{\delta_i^{\frac{N-2}{2}}\alpha_N\delta_h^{\frac{N-2}{2}}}{|\xi_i-\xi_h|^{N-2}}\int_{\mathbb{R}^N}U^{2^\star-1}+\mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right)$$

We put

$$\mathtt{B}:=\frac{1}{2}\int_{\mathbb{R}^N}\mathcal{U}^2,\quad \mathtt{C}:=\int_{\mathbb{R}^N}\mathcal{U}^{2^\star-1}.$$

Then

$$J_\varepsilon(u_0-\sum_i P\mathcal{U}_i)=\mathtt{A}+\frac{1}{2}\alpha_N\mathtt{C}\sum_i\delta_i^{N-2}H(\xi_i,\xi_i)-\varepsilon\mathtt{B}\sum_i\delta_i^2-\mathtt{C}\sum_i\delta_i^{\frac{N-2}{2}}u_0(\xi_i)$$
$$-\alpha_N\mathtt{C}\sum_{h<i}\frac{\delta_i^{\frac{N-2}{2}}\delta_h^{\frac{N-2}{2}}}{|\xi_i-\xi_h|^{N-2}}+\mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right).$$

Now

$$-\varepsilon\mathtt{B}\sum_h\delta_i^2=-\varepsilon\mathtt{B}\sum_i\left(\varepsilon^\alpha d_0+\varepsilon^{\hat{\alpha}}d_i\right)^2$$
$$=-\varepsilon^{1+2\alpha}\mathtt{B}kd_0^2-2\varepsilon^{1+\alpha+\hat{\alpha}}\mathtt{B}d_0\sum_i d_i-\varepsilon^{1+2\hat{\alpha}}\mathtt{B}\sum_i d_i^2$$

Moreover

$$\frac{1}{2}\alpha_N\mathtt{C}\sum_i\delta_i^{N-2}H(\xi_i,\xi_i)=\frac{1}{2}\alpha_N\mathtt{C}\sum_i\delta_i^{N-2}$$
$$\left[\frac{1}{2^{N-2}\mathrm{dist}(\xi_i,\partial\Omega)^{N-2}}+o\left(\frac{1}{\mathrm{dist}(\xi_i,\partial\Omega)^{N-2}}\right)\right]$$
$$=\frac{1}{2^{N-1}}\alpha_N\sum_i\varepsilon^{\alpha(N-2)}(d_0+\varepsilon^{\hat{\alpha}-\alpha})^{N-2}\left[\frac{1}{\varepsilon^{\beta(N-2)}t_0^{N-2}}-\frac{N-2}{\varepsilon^{\beta(N-1)}t_0^{N-1}}\varepsilon^{\tilde{\beta}}t_i\right.$$
$$+\frac{(N-2)(N-1)}{2\varepsilon^{\beta N}t_0^N}\varepsilon^{2\tilde{\beta}}t_i^2$$
$$\left.+\mathcal{O}\left(\varepsilon^{3\tilde{\beta}-\beta(N+1)}\right)\right]+o\left(\sum_i\frac{\delta_i^{N-2}}{\mathrm{dist}(\xi_i,\partial\Omega)^{N-2}}\right)$$
$$=\frac{\alpha_N}{2^{N-1}}\mathtt{C}\sum_i\varepsilon^{\alpha(N-2)}\left(d_0^{N-2}+(N-2)d_0^{N-3}d_i\varepsilon^{\hat{\alpha}-\alpha}\right.$$
$$\left.+\frac{(N-2)(N-3)}{2}d_0^{N-4}d_i^2\varepsilon^{2\hat{\alpha}-2\alpha}+\mathcal{O}\left(\varepsilon^{3\hat{\alpha}-3\alpha}\right)\right)$$
$$*\left[\frac{1}{\varepsilon^{\beta(N-2)}t_0^{N-2}}-\frac{N-2}{\varepsilon^{\beta(N-1)}t_0^{N-1}}\varepsilon^{\tilde{\beta}}t_i+\frac{(N-2)(N-1)}{2\varepsilon^{\beta N}t_0^N}\varepsilon^{2\tilde{\beta}}t_i^2\right.$$
$$\left.+\mathcal{O}\left(\varepsilon^{3\tilde{\beta}-\beta(N+1)}\right)\right]$$
$$+o\left(\sum_i\frac{\delta_i^{N-2}}{\mathrm{dist}(\xi_i,\partial\Omega)^{N-2}}\right)$$
$$=\varepsilon^{(\alpha-\beta)(N-2)}\frac{\alpha_N}{2^{N-1}}\mathtt{C}k\frac{d_0^{N-2}}{t_0^{N-2}}$$

$$-\varepsilon^{\alpha(N-2)-\beta(N-1)+\tilde{\beta}}\frac{\alpha_N(N-2)}{2^{N-1}}\frac{d_0^{N-2}}{t_0^{N-1}}\mathrm{C}\sum_i t_i+\varepsilon^{\alpha(N-3)-\beta(N-2)+\hat{\alpha}}\frac{\alpha_N}{2^{N-1}}\frac{d_0^{N-3}}{t_0^{N-2}}\mathrm{C}\sum_i d_i$$

$$+\varepsilon^{\alpha(N-2)-\beta N+2\tilde{\beta}}\frac{\alpha_N(N-2)(N-1)}{2^N}\frac{d_0^{N-2}}{t_0^N}\mathrm{C}$$

$$\sum_i t_i^2-\varepsilon^{\alpha(N-3)+\hat{\alpha}-\beta(N-1)+\tilde{\beta}}\frac{\alpha_N(N-2)^2}{2^{N-1}}\frac{d_0^{N-3}}{t_0^{N-1}}\mathrm{C}\sum_i t_i d_i$$

$$+\varepsilon^{\alpha(N-4)+2\hat{\alpha}-\beta(N-2)}\frac{\alpha_N(N-2)(N-3)}{2^N}\frac{d_0^{N-4}}{t_0^{N-2}}\mathrm{C}$$

$$\sum_i d_i^2+\underbrace{\mathcal{O}\left(\varepsilon^{\alpha(N-2)+3\tilde{\beta}-\beta(N+1)}\right)}_{:=\mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right)}$$

$$+o\left(\sum_i\frac{\delta_i^{N-2}}{\operatorname{dist}(\xi_i,\partial\Omega)^{N-2}}\right)$$

Now by (2.10) we have that

$$-\alpha_N\mathrm{C}\sum_{h<i}\frac{\delta_i^{\frac{N-2}{2}}\delta_h^{\frac{N-2}{2}}}{|\xi_i-\xi_h|^{N-2}}=-\varepsilon^{\alpha(N-2)-\hat{\beta}(N-2)}\alpha_N\mathrm{C}\sum_{h<i}\frac{d_0^{N-2}}{|\tau_i-\tau_h|^{N-2}}(1+o(1)).$$

At the end, observing that

$$\hat{\xi}_i-\xi_0=(\varepsilon^{\hat{\beta}}\tau_i,\vartheta(\xi_0'+\varepsilon^{\hat{\beta}}\tau_i)-\vartheta(\xi_0'))=(\varepsilon^{\hat{\beta}}\tau_i,\varepsilon^{\hat{\beta}}\underbrace{\nabla_{N-1}\vartheta(\xi_0')^T}_{=0}\cdot\tau_i+\mathcal{O}(\varepsilon^{2\hat{\beta}}))$$

and

$$\begin{aligned}\nabla u_0(\hat{\xi}_i)^T\cdot\nu(\hat{\xi}_i)=\partial_\nu u_0(\hat{\xi}_i)&=\partial_\nu u_0(\xi_0)+\nabla_N\partial_\nu u_0(\xi_0)^T\cdot(\xi_i-\xi_0)\\&\quad+\frac{1}{2}(D_N^2\partial_\nu u_0(\xi_i)(\xi_i-\xi_0))^T\cdot(\xi_i-\xi_0)+\mathcal{O}(|\hat{\xi}_i-\xi_0|^3)\\&=\partial_\nu u_0(\xi_0)+\varepsilon^{\hat{\beta}}\underbrace{\nabla_{N-1}\partial_\nu u_0(\xi_0)^T}_{\text{We need this equal to zero}}\cdot\tau_i+\mathcal{O}(\varepsilon^{2\hat{\beta}})\partial_{x_N}\partial_\nu u_0(\xi_0)\\&\quad+\frac{1}{2}\varepsilon^{2\hat{\beta}}(D_{N-1}^2\partial_\nu u_0(\xi_0)\tau_i)^T\cdot\tau_i+\mathcal{O}(\varepsilon^{3\hat{\beta}}),\end{aligned}$$

as $\hat{\xi}_i\in\Omega$ and ξ_0 is a critical point of $\partial_\nu u_0(\xi)$, we have

$$\begin{aligned}u_0(\xi_i)&=u_0(\hat{\xi}_i+(\varepsilon^{\beta}t_0+\varepsilon^{\tilde{\beta}}t_i)\nu(\hat{\xi}_i))\\&=\underbrace{u_0(\hat{\xi}_i)}_{=0}+(\varepsilon^{\beta}t_0+\varepsilon^{\tilde{\beta}}t_i)\nabla u_0(\hat{\xi}_i)^T\cdot\nu(\hat{\xi}_i)+\mathcal{O}((\varepsilon^{\beta}t_0+\varepsilon^{\tilde{\beta}}t_i)^2)\\&=(\varepsilon^{\beta}t_0+\varepsilon^{\tilde{\beta}}t_i)\partial_\nu u_0(\xi_0)+\frac{1}{2}(\varepsilon^{\beta}t_0+\varepsilon^{\tilde{\beta}}t_i)\varepsilon^{2\hat{\beta}}(D_{N-1}^2\partial_\nu u_0(\xi_0)\tau_i)^T\cdot\tau_i\\&\quad+\mathcal{O}((\varepsilon^{\beta}t_0+\varepsilon^{\tilde{\beta}}t_i)^2)+\mathcal{O}(\varepsilon^{2\hat{\beta}}(\varepsilon^{\beta}t_0+\varepsilon^{\tilde{\beta}}t_i)).\end{aligned}$$

Then

$$
\begin{aligned}
&-\mathtt{C}\sum_i \delta_i^{\frac{N-2}{2}} u_0(\xi_i)\\
&=-\mathtt{C}\sum_i \varepsilon^{\alpha\frac{N-2}{2}}\left(d_0^{\frac{N-2}{2}}+\frac{N-2}{2}\varepsilon^{\hat\alpha-\alpha}d_0^{\frac{N-4}{2}}d_i+\frac{(N-2)(N-4)}{4}\varepsilon^{2(\hat\alpha-\alpha)}d_0^{\frac{N-6}{2}}d_i^2\right)u_0(\xi_i)\\
&=-\mathtt{C}\varepsilon^{\alpha\frac{N-2}{2}+\beta}d_0^{\frac{N-2}{2}}t_0k\partial_\nu u_0(\xi_0)+\mathcal{O}(\varepsilon^{\alpha\frac{N-2}{2}+2\beta}t_0)\\
&\quad-\mathtt{C}\varepsilon^{\alpha\frac{N-2}{2}+\tilde\beta}d_0^{\frac{N-2}{2}}\partial_\nu u_0(\xi_0)\sum_i t_i+\sum\mathcal{O}(\varepsilon^{\alpha\frac{N-2}{2}+\beta+\tilde\beta}t_i)\\
&\quad-\mathtt{C}\frac{N-2}{2}\varepsilon^{\alpha\frac{N-2}{2}+\hat\alpha-\alpha+\beta}d_0^{\frac{N-4}{2}}t_0\partial_\nu u_0(\xi_0)\sum_i d_i+\sum\mathcal{O}(\varepsilon^{\alpha\frac{N-2}{2}+2\beta+\hat\alpha-\alpha}d_i)\\
&\quad-\frac{\mathtt{C}}{2}\varepsilon^{\alpha\frac{N-2}{2}+\beta+2\hat\beta}d_0^{\frac{N-2}{2}}t_0\sum_i(D^2_{N-1}\partial_\nu u_0(\xi_0)\tau_i)^T\cdot\tau_i+\sum\mathcal{O}_{\tau_i}(\varepsilon^{\alpha\frac{N-2}{2}+\tilde\beta+2\hat\beta})\\
&\quad-\mathtt{C}\frac{N-2}{2}\varepsilon^{\alpha\frac{N-2}{2}+\hat\alpha-\alpha+\tilde\beta}d_0^{\frac{N-4}{2}}\partial_\nu u_0(\xi_0)\sum_i d_it_i+\sum\mathcal{O}(\varepsilon^{\alpha\frac{N-2}{2}+\hat\alpha-\alpha+\beta+\tilde\beta}t_id_i)\\
&\quad-\mathtt{C}\frac{(N-2)(N-4)}{4}\varepsilon^{\alpha\frac{N-2}{2}+2\hat\alpha-2\alpha+\beta}d_0^{\frac{N-6}{2}}t_0\partial_\nu u_0(\xi_0)\sum_i d_i^2\\
&\quad+\sum\mathcal{O}(\varepsilon^{\alpha\frac{N-2}{2}+2\beta+2\hat\alpha-2\alpha}d_i^2)+\sum\mathcal{O}(\varepsilon^{\alpha\frac{N-2}{2}+2\tilde\beta}t_i^2)
\end{aligned}
$$

where the lower order terms are $\mathcal{O}\left(\varepsilon^{\hat\theta+\sigma}\right)$. Now the zero order terms are

$$
\begin{aligned}
&\mathtt{A}+\varepsilon^{(\alpha-\beta)(N-2)}\frac{\alpha_N}{2^{N-1}}\mathtt{C}k\frac{d_0^{N-2}}{t_0^{N-2}}-\varepsilon^{1+2\alpha}\mathtt{B}kd_0^2-\mathtt{C}\varepsilon^{\alpha\frac{N-2}{2}+\beta}d_0^{\frac{N-2}{2}}t_0k\partial_\nu u_0(\xi_0)\\
&\quad+\varepsilon^{\theta+\sigma}k\mathfrak{f}(d_0,t_0)\\
&\quad=\mathtt{A}+\varepsilon^{\theta}k\left(\frac{\alpha_N}{2^{N-1}}\mathtt{C}\frac{d_0^{N-2}}{t_0^{N-2}}-\mathtt{B}d_0^2-\mathtt{C}d_0^{\frac{N-2}{2}}t_0\partial_\nu u_0(\xi_0)\right)+\varepsilon^{\theta+\sigma}k\mathfrak{f}(d_0,t_0).
\end{aligned}
$$

The first order terms are

$$
\begin{aligned}
&-2\varepsilon^{1+\alpha+\hat\alpha}\mathtt{B}d_0\sum_i d_i+\varepsilon^{\alpha(N-3)-\beta(N-2)+\hat\alpha}\frac{\alpha_N}{2^{N-1}}\frac{d_0^{N-3}}{t_0^{N-2}}\mathtt{C}\\
&\quad\sum_i d_i-\mathtt{C}\frac{N-2}{2}\varepsilon^{\alpha\frac{N-2}{2}+\hat\alpha+\beta}d_0^{\frac{N-4}{2}}t_0\partial_\nu u_0(\xi_0)\sum_i d_i\\
&\quad-\varepsilon^{\alpha(N-2)-\beta(N-1)+\tilde\beta}\frac{\alpha_N(N-2)}{2^{N-1}}\frac{d_0^{N-2}}{t_0^{N-1}}\mathtt{C}\sum_i t_i-\mathtt{C}\varepsilon^{\alpha\frac{N-2}{2}+\tilde\beta}d_0^{\frac{N-2}{2}}\partial_\nu u_0(\xi_0)\sum_i t_i\\
&=\varepsilon^{\hat\theta}\underbrace{\left(-2\mathtt{B}d_0+\frac{\alpha_N}{2^{N-1}}\frac{d_0^{N-3}}{t_0^{N-2}}\mathtt{C}-\mathtt{C}\frac{N-2}{2}d_0^{\frac{N-4}{2}}t_0\partial_\nu u_0(\xi_0)\right)}_{=0}\sum_i d_i\\
&\quad+\varepsilon^{\hat\theta}\underbrace{\left(-\frac{\alpha_N(N-2)}{2^{N-1}}\frac{d_0^{N-2}}{t_0^{N-1}}\mathtt{C}-\mathtt{C}d_0^{\frac{N-2}{2}}\partial_\nu u_0(\xi_0)\right)}_{=0}\sum_i t_i
\end{aligned}
$$

The quantities in the previous box are zero as d_0 and t_0 satisfy the system in (2.7).

Then the order zero terms are a function of (d_0, t_0), namely

$$\begin{aligned}
&\mathtt{A}+\varepsilon^{\theta}k\left(\frac{\alpha_N}{2^{N-1}}\mathtt{C}\frac{d_0^{N-2}}{t_0^{N-2}}-\mathtt{B}d_0^2-\mathtt{C}d_0^{\frac{N-2}{2}}t_0\partial_\nu u_0(\xi_0)\right)+k\varepsilon^{\theta+\sigma}\mathfrak{f}(d_0,t_0)\\
&\quad=\mathtt{A}+\varepsilon^{\theta}k\mathfrak{g}(d_0,t_0)+k\varepsilon^{\theta+\sigma}\mathfrak{f}(d_0,t_0).
\end{aligned}$$

Now the second order terms are

$$\begin{aligned}
&-\varepsilon^{1+2\hat\alpha}\mathtt{B}\sum_i d_i^2+\varepsilon^{\alpha(N-2)-\beta N+2\tilde\beta}\frac{\alpha_N(N-2)(N-1)}{2^N}\frac{d_0^{N-2}}{t_0^N}\mathtt{C}\sum_i t_i^2\\
&\quad-\varepsilon^{\alpha(N-3)+\hat\alpha-\beta(N-1)+\tilde\beta}\frac{\alpha_N(N-2)^2}{2^{N-1}}\frac{d_0^{N-3}}{t_0^{N-1}}\mathtt{C}\sum_i t_i d_i\\
&\quad+\varepsilon^{\alpha(N-4)+2\hat\alpha-\beta(N-2)}\frac{\alpha_N(N-2)(N-3)}{2^N}\frac{d_0^{N-4}}{t_0^{N-2}}\mathtt{C}\sum_i d_i^2-\varepsilon^{\alpha(N-2)-\hat\beta(N-2)}\alpha_N\mathtt{C}\\
&\quad\sum_{h<i}\frac{d_0^{N-2}}{|\tau_i-\tau_h|^{N-2}}\\
&\quad-\frac{\mathtt{C}}{2}\varepsilon^{\alpha\frac{N-2}{2}+\beta+2\hat\beta}d_0^{\frac{N-2}{2}}t_0\sum_i\left(D^2_{N-1}\partial_\nu u_0(\xi_0)\tau_i\right)^T\cdot\tau_i\\
&\quad-\mathtt{C}\frac{N-2}{2}\varepsilon^{\alpha\frac{N-2}{2}+\hat\alpha-\alpha+\tilde\beta}d_0^{\frac{N-4}{2}}\partial_\nu u_0(\xi_0)\sum_i d_i t_i\\
&\quad-\mathtt{C}\frac{(N-2)(N-4)}{4}\varepsilon^{\alpha\frac{N-2}{2}+2\hat\alpha-2\alpha+\beta}d_0^{\frac{N-6}{2}}t_0\partial_\nu u_0(\xi_0)\sum_i d_i^2\\
&=\varepsilon^{\hat\theta}\underbrace{\left(-\mathtt{B}+\frac{\alpha_N(N-2)(N-3)}{2^N}\frac{d_0^{N-4}}{t_0^{N-2}}\mathtt{C}-\mathtt{C}\frac{(N-2)(N-4)}{4}d_0^{\frac{N-6}{2}}t_0\partial_\nu u_0(\xi_0)\right)}_{:=\mathfrak{A}}\sum_i d_i^2\\
&\quad+\varepsilon^{\hat\theta}\underbrace{\left(-\frac{\alpha_N(N-2)^2}{2^{N-1}}\frac{d_0^{N-3}}{t_0^{N-1}}\mathtt{C}-\mathtt{C}\frac{N-2}{2}d_0^{\frac{N-4}{2}}\partial_\nu u_0(\xi_0)\right)}_{:=\mathfrak{B}}\sum_i t_i d_i\\
&\quad+\varepsilon^{\hat\theta}\underbrace{\left(\frac{\alpha_N(N-2)(N-1)}{2^N}\frac{d_0^{N-2}}{t_0^N}\mathtt{C}\right)}_{:=\mathfrak{C}>0}\sum_i t_i^2\\
&\quad+\varepsilon^{\hat\theta}\left(-\alpha_N\mathtt{C}\sum_{h<i}\frac{d_0^{N-2}}{|\tau_i-\tau_h|^{N-2}}-\frac{\mathtt{C}}{2}d_0^{\frac{N-2}{2}}t_0\sum_i\left(D^2_{N-1}\partial_\nu u_0(\xi_0)\tau_i\right)^T\cdot\tau_i\right)
\end{aligned}$$

By using the first equation of (2.7) we get

$$\mathfrak{A}:=\frac{\alpha_N(N^2-5N+5)}{2^N}\frac{d_0^{N-4}}{t_0^{N-2}}\mathtt{C}-\mathtt{C}\frac{(N-2)(N-5)}{4}d_0^{\frac{N-6}{2}}t_0\partial_\nu u_0(\xi_0)>0$$

while by using the second equation of (2.7) we get

$$\mathfrak{B}:=0.$$

Then at the end

$$\begin{aligned}J_\varepsilon\Big(u_0-\sum_i P\mathcal{U}_i\Big)&=\mathtt{A}+\varepsilon^{\theta}\mathfrak{g}(d_0,t_0)+k\varepsilon^{\theta+\sigma}\mathfrak{f}(d_0,t_0)\\&+\varepsilon^{\hat\theta}\Bigg(\mathfrak{A}\sum_i d_i^2+\mathfrak{C}\sum_i t_i^2-\alpha_N\mathsf{C}\sum_{h<i}\frac{d_0^{N-2}}{|\tau_i-\tau_h|^{N-2}}-\frac{\mathsf{C}}{2}d_0^{\frac{N-2}{2}}t_0\\&\quad\sum_i(D^2_{N-1}\partial_\nu u_0(\xi_0)\tau_i)^T\cdot\tau_i\Bigg)\\&+\mathcal{O}\left(\varepsilon^{\hat\theta+\sigma}\right).\end{aligned}$$

$\square$

Now standard arguments permit us to conclude that if $(\boldsymbol{d}_\varepsilon,\boldsymbol{t}_\varepsilon,\boldsymbol{\tau}_\varepsilon)$ is a critical point of $\tilde{J}_\varepsilon$, then

$$W_{\boldsymbol{\delta}_\varepsilon,\boldsymbol{\xi}_\varepsilon}+\phi_{\boldsymbol{\delta}_\varepsilon,\boldsymbol{\xi}_\varepsilon}=u_0-\sum_{h=1}^{k}P\mathcal{U}_{\delta_{h\varepsilon},\xi_{h\varepsilon}}+\phi_{\boldsymbol{\delta}_\varepsilon,\boldsymbol{\xi}_\varepsilon}$$

is a solution of (BN) where $\boldsymbol{\delta}_\varepsilon$ and $\boldsymbol{\xi}_\varepsilon$ are as in Sect. 2.

Indeed we can rewrite the problem (3.7) as

$$u-i^\star(f(u)+\varepsilon u)=\sum c_i^j P\psi_i^j \tag{4.3}$$

and our goal is to find appropriate parameters $\boldsymbol{d}=(d_1,\dots,d_k)$, $\boldsymbol{t}=(t_1,\dots,t_k)\in\mathbb{R}^k$ and points $\boldsymbol{\tau}=(\tau_1,\dots,\tau_k)\in(\mathbb{R}^{N-1})^k$ such that $c_i^j=c_i^j(\boldsymbol{d},\boldsymbol{t},\boldsymbol{\tau})=0$ for all $i=1,\dots,k$ and $j=0,\dots,N$.

Proposition 4.3 *If for all $i=1,\dots,k$ and $j=1,\dots N$*

$$DJ_\varepsilon(u)[\partial_{\delta_i}u]=DJ_\varepsilon(u)[\partial_{(\xi_i)_j}u]=0, \tag{4.4}$$

then $c_i^j=0$.

Proof It's easy to see that

$$DJ_\varepsilon(u)[f]=\langle u-i^\star(f(u)+\varepsilon)),f\rangle.$$

Then by (4.4) follows

$$\begin{cases}\langle u-i^\star(f(u)+\varepsilon)),\partial_{\delta_i}u\rangle=0\\ \langle u-i^\star(f(u)+\varepsilon)),\nabla_{\xi_i}u\rangle=0\end{cases}$$

and combining it with (4.3), for all $i=1,\dots,k$, $r=1,\dots,N$ we have

$$\sum_{h,j}c_h^j\int_\Omega\mathcal{U}_h^{p-1}\psi_h^j\partial_{\delta_i}u=\sum_{h,j}c_h^j\int_\Omega\mathcal{U}_h^{p-1}\psi_h^j\partial_{(\xi_i)_r}u=0.$$

We have the following estimates on the integrals

$$\delta_i\int_\Omega\mathcal{U}_h^{p-1}\psi_h^j\partial_{\delta_i}PU_i=\begin{cases}b_1+o(1) & \text{if } i=h,\ j=0\\ o(1) & \text{otherwise}\end{cases};$$

$$\delta_i\int_\Omega\mathcal{U}_h^{p-1}\psi_h^j\partial_{(\xi_i)_r}PU_i=\begin{cases}b_2+o(1) & \text{if } i=h,\ r=j\\ o(1) & \text{otherwise}\end{cases}$$

where $b_1, b_2 \neq 0$. Recalling that ϕ satisfies (3.21), by the Implicit Function Theorem we can prove that there exist $\partial_{\delta_i}\phi, \nabla_{\xi_i}\phi \in H_0^1(\Omega)$. In particular using (3.13) and proceeding as in Proposition 3.5, it is possible to show that

$$\delta_i \|\partial_{\delta_i}\phi\|_{H_0^1(\Omega)} = o(1); \qquad \delta_i \|\partial_{(\xi_i)_j}\phi\|_{H_0^1(\Omega)} = o(1) \tag{4.5}$$

for all $i = 1, \dots, k$ and $j = 1, \dots, N$. Then

$$\delta_i \int_\Omega \mathcal{U}_h^{p-1} \psi_h^j \partial_{\delta_i}\phi = o(1);$$
$$\delta_i \int_\Omega \mathcal{U}_h^{p-1} \psi_h^j \partial_{(\xi_i)_r}\phi = o(1).$$

The linear system in the c_i^j's has the only possible solution $c_i^j = 0$ for all $i = 1, \dots, k$ and $j = 0, \dots, N$. □

Now we are able to conclude our proof.

Proof of Theorem 2.1 By Proposition 4.2, we have that

$$\tilde{J}_\varepsilon(\boldsymbol{d}, \boldsymbol{t}, \boldsymbol{\tau}) = \varepsilon^{\hat{\theta}} \Phi(\boldsymbol{\delta}, \boldsymbol{t}, \boldsymbol{\tau}) + \mathbb{A}_1 + \mathcal{O}\left(\varepsilon^{\hat{\theta}+\sigma}\right)$$

where A_1 depends only on (d_0, t_0, ξ_0) and

$$\Phi(\boldsymbol{d}, \boldsymbol{t}, \boldsymbol{\tau}) = \mathfrak{A} \sum_i d_i^2 + \mathfrak{C} \sum_i t_i^2 - \alpha_N \mathrm{C} \sum_{h<i} \frac{d_0^{N-2}}{|\tau_i - \tau_h|^{N-2}} - \frac{\mathrm{C}}{2} d_0^{\frac{N-2}{2}} t_0$$
$$\sum_i (D_{N-1}^2 \partial_\nu u_0(\xi_0)\tau_i)^T \cdot \tau_i.$$

Since the constants $\mathfrak{A}$, $\mathfrak{C}$ are strictly positive, $\boldsymbol{d} = \boldsymbol{t} = (0, \dots, 0) \in \mathbb{R}^{2k}$ is the unique critical point for Φ in $\boldsymbol{d}$ and $\boldsymbol{t}$. Since the matrix $D_{N-1}^2 \partial_\nu u_0(\xi_0)$ is positive definite there also exists a critical point $\boldsymbol{\tau}_0$ in $\boldsymbol{\tau}$. The point $(\mathbf{0}, \mathbf{0}, \boldsymbol{\tau})$ is a critical point for Φ, which is stable under small perturbation of the function. Hence for all ε sufficiently small there exists $(\boldsymbol{\delta}_\varepsilon, \boldsymbol{t}_\varepsilon, \boldsymbol{\tau}_\varepsilon)$, satisfying (2.9), critical point for $\tilde{J}_\varepsilon$, which is close to $(\mathbf{0}, \mathbf{0}, \boldsymbol{\tau})$.

Now observe that for all $i = 1, \dots, k$ and $r = 1, \dots, N-1$

$$\begin{cases} \frac{\partial}{\partial_{\delta_i}} P\mathcal{U}_i = \frac{\partial}{\partial_{d_i}} P\mathcal{U}_i \frac{\partial}{\partial_{\delta_i}} d_i; \\ \frac{\partial}{\partial_{(\xi_i)_r}} P\mathcal{U}_i = \frac{\partial}{\partial_{t_i}} P\mathcal{U}_i \frac{\partial}{\partial_{(\xi_i)_r}} t_i + \sum_{r=1}^{N-1} \frac{\partial}{\partial_{(\tau_i)_r}} P\mathcal{U}_i \frac{\partial}{\partial_{(\xi_i)_r}} (\tau_i)_r. \end{cases}$$

Hence the assumptions of Proposition 4.3 are satisfied and we conclude that $u_\varepsilon = W_{\boldsymbol{\delta}_\varepsilon, \boldsymbol{\xi}_\varepsilon} + \phi_{\boldsymbol{\delta}_\varepsilon, \boldsymbol{\xi}_\varepsilon}$ is a solution of (BN). □

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