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Non existence of compact homogeneous P-Sasakian manifolds

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Abstract. We show that there exist no compact homogeneous P-Sasakian manifolds whose structure vector field is invariant under the given Lie group action.

1. Introduction

Starting from the foundational papers [10], [11] of Sato and of Matsumoto-Sato [6], many interesting features of P-Sasakian manifolds have been studied in the last decades from the differential geometric point of view, cf. for instance [7], [8], [5], [2]. These contributions to the theory are selected from a vast list of references. Recall that these manifolds are endowed with an almost paracontact structure (φ, ξ, η), consisting of a (1,1) tensor field φ, a one form η, and a globally defined vector field ξ, such that:

$$\varphi^2 X = X - \eta(X)\xi,\ \varphi\xi = 0,\ \ \eta(\xi) = 1,\ \eta(\varphi X) = 0,$$

and with a compatible Riemannian metric g such that:

$$(\nabla_X \varphi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1}$$

for every X, Y vector fields. Here ∇ is the Levi-Civita connection. The compatibility condition is:

$$g(\xi, X) = \eta(X), \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2}$$

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According to (1), such a geometric structure is formally similar to a Sasakian structure (cf. e.g. [1]), providing a possible translation of this notion in the context of paracontact geometry, keeping the requirement that g be a positive definite metric. However, an essential difference between Sasakian and P-Sasakian manifolds is that while on a Sasakian manifold η is a contact form, for P-Sasakian ones it is a closed one form. Our aim is to prove the following fact, putting in evidence another striking difference between Sasakian and P-Sasakian geometry:

Theorem. *Let (M,φ,ξ,η,g) be a P-Sasakian manifold. Assume that there exists a Lie group G acting transitively, smoothly on M and preserving the structure vector field ξ. Then M is not compact.*

2. Proof of the result

Let (M,φ,ξ,η,g) be a P-Sasakian manifold on which a Lie group G acts transitively, preserving ξ. We prove by contradiction that M is not compact. Assuming the contrary, according to a result Sasaki [9], the P-Sasakian stucture under consideration must be of type (k,k), with $k \geq 1$; this means that at each point $p \in M$ the spectrum of the operator φ_p is $\{0,1,-1\}$, 0 is a simple eigenvalue of φ_p, and the eigenvalues 1 and -1 have the same multiplicity equal to k.

Now, fix a point $p \in M$ and a unit vector $v \in T_pM$ such that $\varphi v = v$ and let $\gamma : \mathbb{R} \to M$ be the maximal integral curve of ξ passing through p. Then v can be extended to a parallel vector field $X = X(t)$ along γ, such that $X(0) = v$. Since according to (1) $\nabla_\xi \varphi = 0$, we also have

$$\varphi X(t) = X(t), \tag{3}$$

for every $t \in \mathbb{R}$. On the other hand, due to the assumption, v can also be extended to a vector field $Y \in \mathfrak{X}(M)$, such that

$$[Y,\xi] = 0. \tag{4}$$

Indeed, denoting by $\mathfrak{g}$ the Lie algebra of G, since the given action is transitive, the mapping

$$Z \in \mathfrak{g} \mapsto Z^*_p \in T_pM$$

is surjective. Here Z^* denotes the fundamental vector field generated by Z by means of the action, cf. e.g. [4, 3]. So it suffices to take $Y = Z^*$, where $Z \in \mathfrak{g}$ is chosen so that $Z^*_p = v$. Since by construction the flow of Z^* preserves ξ, we have $[Y,\xi] = 0$.

Now, consider the smooth function $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(t) = g(Y_{\gamma(t)}, X(t)).$$

Then $f(0) = g(v, v) = 1$. Since M is assumed to be compact, the norm $||Y||$ must be bounded on M, and thus f also must be bounded; indeed:

$$f(t) \leq ||Y_{\gamma(t)}|| \cdot ||X(t)|| \leq ||Y_{\gamma(t)}||.$$

But using (4), being X parallel along γ, computing f' we get:

$$f'(t) = g(\nabla_{\xi_{\gamma(t)}} Y, X(t)) = g(\nabla_{Y_{\gamma(t)}} \xi, X(t)) = g(\varphi Y_{\gamma(t)}, X(t)) = g(Y_{\gamma(t)}, \varphi X(t)),$$

where we have also used (2) and the fact that $\nabla\xi = \varphi$. Taking into account (3) this means $f' = f$, so that $f(t) = e^t$, yielding a contradiction.

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