CRITICAL SCHRÖDINGER-BOPP-PODOLSKY SYSTEMS: SOLUTIONS IN THE SEMICLASSICAL LIMIT

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Abstract. In this paper we consider the following critical Schrödinger-Bopp-Podolsky system

$$\begin{cases}-\epsilon^2\Delta u+V(x)u+Q(x)\phi u=h(x,u)+K(x)|u|^4u & \text{in } \mathbb{R}^3\\ -\Delta\phi+a^2\Delta^2\phi=4\pi Q(x)u^2 & \text{in } \mathbb{R}^3\end{cases}$$

in the unknowns $u,\phi:\mathbb{R}^3\to\mathbb{R}$ and where $\varepsilon,a>0$ are parameters. The functions V,K,Q satisfy suitable assumptions as well as the nonlinearity h which is subcritical. For any fixed $a>0$, we show existence of "small" solutions in the semiclassical limit, namely whenever $\varepsilon\to 0$. We give also estimates of the norm of this solutions in terms of ε. Moreover, we show also that fixed ε suitably small, when $a\to 0$ the solutions found strongly converge to solutions of the Schrödinger-Poisson system.

Contents

1. Introduction

In the pioneering paper [5] a new system of elliptic partial differential equations has been introduced for the first time in the mathematical literature. It describes the stationary solutions of a charged particle in the electromagnetic field developed by Bopp-Podolsky in the 1930s which is considered better then the classical Maxwell Theory of the electromagnetism for short range interactions. The system studied is the following one in the whole space $\mathbb{R}^3$:

$$\begin{cases}-\Delta u+\omega u+q^2\phi u=|u|^{p-2}u,\\ -\Delta\phi+a^2\Delta^2\phi=4\pi u^2,\end{cases}$$

in the unknowns $u,\phi:\mathbb{R}^3\to\mathbb{R}$ and where $a>0$ is the parameter of the Bopp-Podolski theory, $\omega>0$ is the frequency of the standing wave $\psi(x,t)=u(x)e^{-i\omega t}$, $q\neq 0$ is the charge of the

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particle and $p \in (2,6)$ is a suitable parameter related to the interaction between many particles. In particular the power nonlinearity is subcritical. For the derivation of the system and the mathematical approach in order to treat it with variational methods, we refer the reader to the mentioned paper.

Motivated by [6], where the existence of semiclassical solutions for the Schrödinger equation

$$-\epsilon^2\Delta u + V(x)u = h(x,u) + K(x)|u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N$$

has been studied whenever $\epsilon \to 0$, in this paper we consider the following critical and semiclassical Schrödinger-Bopp-Podolsky system in $\mathbb{R}^3$

$$\begin{cases} -\epsilon^2\Delta u + V(x)u + Q(x)\phi u = h(x,u) + K(x)|u|^4 u, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi Q(x)u^2. \end{cases}$$

Note that by setting $\lambda = 1/\epsilon^2$, we get the following equivalent system

$$\begin{cases} -\Delta u + \lambda V(x)u + \lambda Q(x)\phi u = \lambda h(x,u) + \lambda K(x)|u|^4 u \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi Q(x)u^2 \end{cases} \tag{1.1}$$

whose study, whenever $\lambda \to +\infty$, is then the aim of this paper. Here $u, \phi : \mathbb{R}^3 \to \mathbb{R}$ are the unknowns and a is a suitable parameter.

We assume that the real functions V, Q, h, K satisfy the following:

(V0) $V \in C(\mathbb{R}^3)$, $V \geq 0$, there is $x_0 \in \mathbb{R}^3$ such that $V(x_0) = 0$ and there is $b > 0$ such that $\text{meas}\{x \in \mathbb{R}^3 : V(x) < b\} < +\infty$;

(Q0) it is $Q \geq 0$ ($\not\equiv 0$) and one of the following holds:
  (q1) $Q \in L^s(\mathbb{R}^3)$ for some $s \in (3, +\infty)$,
  (q2) $Q \in L^s(\mathbb{R}^3)$ for some $s \in [3/2, 3]$ and there exists $\lim_{x\to x_0} Q(x) = \ell \in \mathbb{R}$ (where x_0 is the one in (V0)),
  (q3) $Q \in L^\infty(\mathbb{R}^3)$ and for every $t > 0, \text{meas}\{x \in \mathbb{R}^3 : Q(x) \geq t\} < +\infty$;

(K0) $K \in L^\infty(\mathbb{R}^3)$ and $0 < K_{\min} := \inf K \leq \sup K =: K_{\max} < \infty$;

(H0) (h1) $h \in C(\mathbb{R}^3 \times \mathbb{R})$ and $h(x,u) = o(|u|)$ uniformly in x as $u \to 0$;
  (h2) there exist $C_0 > 0$ and $q \in (2,6)$ such that for any $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$ it holds
$$|h(x,u)| \leq C_0(|u| + |u|^{q-1});$$
  (h3) there exist $\nu > 0, p > 4$ and $4 < \mu < 6$ such that for any $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$ it holds
$$\nu\mu|u|^p \leq \mu H(x,u) \leq h(x,u)u,$$
  where $H(x,u) = \int_0^u h(x,s)ds$;

Few comments on these assumptions are made at the end of this section.

1.1. Preliminaries.

Before stating our result some preliminaries are in order. Let us introduce the space

$$E := \left\{ u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} V(x)u^2 < \infty \right\}$$

which is a Hilbert space equipped with the following inner product and (squared) norm:

$$\langle u, v\rangle := \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv), \quad \|u\|^2 = \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(x)u^2.$$

This norm is equivalent to the norm $\|\cdot\|_\lambda$ deduced by the inner product

$$\langle u, v\rangle_\lambda := \int_{\mathbb{R}^3} (\nabla u \nabla v + \lambda V(x) uv),$$

for each $\lambda > 0$. By condition (V0) using Sobolev and Gagliardo-Nirenberg inequalities, it follows that E embeds continuously in $H^1(\mathbb{R}^3)$. It is thus clear that, for each $s \in [2,6]$, there is $c_s > 0$ such that

$$|u|_s \leq c_s \|u\| \leq c_s \|u\|_\lambda, \quad \forall \lambda \geq 1,\ u \in E.$$

From now on $|\cdot|_p$ will denote the usual L^p norm and we will implicitly assume $\lambda \geq 1$.

Let $\mathcal{D}$ be the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|.\|_{\mathcal{D}}$ induced by the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} \nabla\varphi\nabla\psi + a^2 \int_{\mathbb{R}^3} \Delta\varphi\Delta\psi.$$

Then $\mathcal{D}$ is an Hilbert space continuously embedded in $D^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$. As proved in [5] it also embeds continuously in $L^\infty(\mathbb{R}^3)$ and moreover it is

$$\mathcal{D} = \{\phi \in D^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}.$$

It is easy to see that the critical points of the functional

$$\begin{aligned} F_{\lambda,a}(u,\phi) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} Q(x)\phi u^2 - \frac{\lambda}{16\pi}|\nabla\phi|_2^2 - \frac{\lambda a^2}{16\pi}|\Delta\phi|_2^2 \\ &- \lambda \int_{\mathbb{R}^3}\left(H(x,u) + \frac{1}{6}K(x)|u|^6\right) \end{aligned}$$

in $E \times \mathcal{D}$ are weak solutions of (1.1); indeed, if $(u,\phi) \in E \times \mathcal{D}$ is a critical point of $F_{\lambda,a}$ then for every $(v,\xi) \in E \times \mathcal{D}$ we have

$$\begin{aligned} 0 &= \partial_u F_{\lambda,a}(u,\phi)[v] = \langle u, v\rangle_\lambda + \lambda\int_{\mathbb{R}^3} Q(x)\phi uv - \lambda\int_{\mathbb{R}^3}(h(x,u) + K(x)|u|^4 u)v, \\ 0 &= \partial_\phi F_{\lambda,a}(u,\phi)[\xi] = \frac{\lambda}{2}\int_{\mathbb{R}^3} Q(x)u^2\xi - \frac{\lambda}{8\pi}\int_{\mathbb{R}^3}\nabla\phi\nabla\xi - \frac{\lambda}{8\pi}a^2\int_{\mathbb{R}^3}\Delta\phi\Delta\xi, \end{aligned}$$

which say, by definition, that (u,ϕ) is a weak solution of (1.1).

By means of a standard arguments by now, we can reduce ourselves to find the critical points of a functional of a single variable. Let

$$\mathcal{K}(x) = \frac{1 - e^{-|x|/a}}{|x|}, \quad x \in \mathbb{R}^3 \setminus \{0\}, a > 0$$

and recall the following result given in [5, Lemma 3.3].

Lemma 1.1. *For all $y \in \mathbb{R}^3$, $\mathcal{K}(\cdot - y)$ solves in the sense of distributions*

$$-\Delta\phi + a^2\Delta^2\phi = \delta_y.$$

Furthermore:

- *(i) if $f \in L^1_{loc}(\mathbb{R}^3)$ and, for a.e. $x \in \mathbb{R}^3$, the map $y \in \mathbb{R}^3 \mapsto \frac{f(y)}{|x-y|}$ is summable, then $\mathcal{K} * f \in L^1_{loc}(\mathbb{R}^3)$;*
- *(ii) if $f \in L^p(\mathbb{R}^3)$ with $1 \leq p < \frac{3}{2}$, then $\mathcal{K} * f \in L^q(\mathbb{R}^3)$ for $q \in (\frac{3p}{3-2p}, +\infty]$.*

In both cases, $\mathcal{K} * f$ solves

$$-\Delta\phi + a^2\Delta^2\phi = f,$$

in the sense of distributions and we have the following distributional derivatives

$$\nabla(\mathcal{K} * f) = (\nabla\mathcal{K}) * f \quad \textit{and} \quad \Delta(\mathcal{K} * f) = (\Delta\mathcal{K}) * f \quad \textit{a.e. in } \mathbb{R}^3.$$

If $u \in H^1(\mathbb{R}^3)$, then $u^2 Q \in L^1(\mathbb{R}^3)$ as soon as $Q \in L^s(\mathbb{R}^3)$ with $s \in [3/2, +\infty]$. By the lemma there is a unique solution in $\mathcal{D}$ of the second equation in (1.1), which we denote by ϕ_u^a and has an explicit formula given by

$$\phi_u^a(x) = (\mathcal{K} * Qu^2)(x) = \int_{\mathbb{R}^3} \frac{1 - e^{-|x-y|/a}}{4\pi|x-y|} Q(y)u^2(y)dy \in \mathcal{D}. \tag{1.2}$$

Some properties involving ϕ_u^a will be needed in the sequel, so we collect them here. We denote with $C_0(\mathbb{R}^3)$ the space of continuous functions vanishing at infinity.

Lemma 1.2. *Assume (Q0). For all $u \in H^1(\mathbb{R}^3)$ we have:*

- *(i) for all $y \in \mathbb{R}^3, \phi^a_{u(.+y)} = \phi_u^a(. + y)$;*
- *(ii) $\phi_u^a > 0$;*
- *(iii) for every $s \in (3, +\infty], \phi_u^a \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;*
- *(iv) for every $s \in (3/2, +\infty], \nabla\phi_u^a \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$;*
- *(v) there is $C > 0$ such that for all $a > 0$, it is $|\phi_u^a|_6 \le C\|u\|^2$ and then*

$$\|\phi_u^a\|^2_{\mathcal{D}} = \int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 \le C\|u\|^4;$$

- *(vi) $\phi^a_{tu} = t^2\phi_u^a$;*
- *(vii) if $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, then $\phi^a_{v_n} \to \phi^a_v$ in $\mathcal{D}$ and*

$$\int_{\mathbb{R}^3} Q(x)\phi^a_{v_n} v_n^2 \to \int_{\mathbb{R}^3} Q(x)\phi^a_v v^2,$$

$$\int_{\mathbb{R}^3} Q(x)\phi^a_{v_n} v_n\varphi \to \int_{\mathbb{R}^3} Q(x)\phi^a_v v\varphi, \ \forall\varphi \in C_c^\infty(\mathbb{R}^3);$$

- *(viii) ϕ_u^a is the unique minimizer of the functional*

$$\mathcal{E}_{u,a}(\phi) = \frac{1}{2}|\nabla\phi|_2^2 + \frac{a^2}{2}|\Delta\phi|_2^2 - \int_{\mathbb{R}^3} Q(x)\phi u^2, \quad \phi \in \mathcal{D}.$$

Proof. We just need to prove the first convergence of item (vii), since all the other facts are done as in [5]. In [16] it has been proved that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$ implies that $\phi^a_{v_n} \to \phi^a_v$ in $\mathcal{D}$ whenever $Q \in L^2(\mathbb{R}^3)$. However the same proof holds for $Q \in L^s(\mathbb{R}^3)$ with $s \in [3/2, +\infty)$ (without any other assumption on Q). So it remains to prove the convergence when (q3) holds, namely when $Q \in L^\infty(\mathbb{R}^3)$ and it vanishes at infinity.

Given $v \in H^1(\mathbb{R}^3)$ define the linear operator

$$\mathcal{L}_v : \xi \in \mathcal{D} \mapsto \int_{\mathbb{R}^3} Q(x)v^2\xi \in \mathbb{R},$$

and the Lax-Milgram Theorem guarantees that $\|\mathcal{L}_v\|_{\mathcal{L}(\mathcal{D};\mathbb{R})} = \|\phi^a_v\|_{\mathcal{D}}$. Then it is sufficient to show that $\mathcal{L}_{v_n} \to \mathcal{L}_v$ in the operator norm. Now, given an arbitrary $t > 0$, setting $A_t := \{x \in \mathbb{R}^3 : Q(x) \ge t\}$, which has finite measure, we have

$$\begin{aligned}\int_{\mathbb{R}^3} Q(x)|v_n^2 - v^2||\xi| &\le \Big(\int_{\mathbb{R}^3\setminus A_t} Q(x)|v_n^2 - v^2| + \int_{A_t} Q(x)|v_n^2 - v^2|\Big)|\xi|_\infty \\ &\le \Big(tC + |Q|_\infty \int_{A_t} |v_n^2 - v^2|\Big)\|\xi\|_{\mathcal{D}} \\ &\le (tC + |Q|_\infty\epsilon_n)\|\xi\|_{\mathcal{D}}\end{aligned}$$

where $\epsilon_n \to 0$. This gives the result. □

Then usual arguments (see [5]) show that, defining the energy functional of a single variable $u \in E$, by

$$\begin{aligned}\Phi_{\lambda,a}(u) &:= F_{\lambda,a}(u,\phi_u^a)\\ &= \frac{1}{2}\|u\|_\lambda^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 - \lambda\int_{\mathbb{R}^3}\left(H(x,u)+\frac{1}{6}K(x)|u|^6\right),\end{aligned} \tag{1.3}$$

its critical points give the solutions we are looking for, and solving system (1.1) is equivalent to solve the equation

$$-\Delta u + \lambda V(x)u + \lambda Q(x)\phi_u^a u = \lambda h(x,u) + \lambda K(x)|u|^4 u \quad \text{in } \mathbb{R}^3 \tag{1.4}$$

which is the equation we will consider from now on. In fact it happens that the following statements are equivalent:

- the pair $(u_{\lambda,a}, \phi_{\lambda,a}) \in E \times \mathcal{D}$ is a critical point of $F_{\lambda,a}$,
- $u_{\lambda,a} \in E$ is a critical point of $\Phi_{\lambda,a}$ and $\phi_{\lambda,a} = \phi^a_{u_{\lambda,a}}$ given in (1.2).

Then speaking of solutions $u_{\lambda,a}$ of (1.4) has to be understood as speaking of solutions $(u_{\lambda,a}, \phi^a_{u_{\lambda,a}})$ of (1.1). Note also that since $\phi^a_{u_{\lambda,a}} > 0$, any statement about the sign of the solutions of (1.1) refers to the function $u_{\lambda,a}$.

It will be useful to define in E also the functional

$$\Psi_{\lambda,a}(u) := \frac{1}{2}\|u\|_\lambda^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 - \lambda\int_{\mathbb{R}^3}\left(H(x,u^+)+\frac{1}{6}K(x)|u^+|^6\right)$$

which is of class C^1 and clearly for all $u, v \in E$ it is

$$\Psi'_{\lambda,a}(u)[v] = \int_{\mathbb{R}^3}(\nabla u\nabla v + \lambda V(x)uv) + \lambda\int_{\mathbb{R}^3} Q(x)\phi_u^a uv - \lambda\int_{\mathbb{R}^3}(h(x,u^+)+K(x)|u^+|^4u^+)v$$

where as usual $u^\pm := \max\{\pm u, 0\}$. The critical points of $\Psi_{\lambda,a}$ give rise to positive solutions of (1.4), then of (1.1).

In view of the study of the limit behaviour of the solutions of (1.1) we consider the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \lambda V(x)u + \lambda Q(x)\phi u = \lambda h(x,u) + \lambda K(x)|u|^4 u\\ -\Delta\phi = 4\pi Q(x)u^2.\end{cases} \tag{1.5}$$

If we set

$$\phi_u^0 := (\frac{1}{|\cdot|} * Qu^2)(x) = \int_{\mathbb{R}^3}\frac{1}{|x-y|}Q(y)u^2(y)dy \in D^{1,2}(\mathbb{R}^3)$$

the usual reduction argument lead to study the equation

$$-\Delta u + \lambda V(x)u + \lambda Q(x)\phi_u^0 u = \lambda h(x,u) + \lambda K(x)|u|^4 u \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

We see then that the Schrödinger-Poisson system can be seen as a limit problem of the Schrödinger-Bopp-Podolsky system, since it is obtained formally by setting $a = 0$. Its solutions are characterised as critical points of the functional

$$\Phi_{\lambda,0}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} Q(x)\phi_u^0 u^2 - \lambda\int_{\mathbb{R}^3}\left(H(x,u)+\frac{1}{6}K(x)|u|^6\right).$$

Now we can state the main results of the paper.

1.2. **Existence and behaviour of the solutions.** Our results deal with existence of ground state, multiplicity and limit behaviour of the solutions in the semiclassical limit, namely as $\lambda \to +\infty$.

Theorem 1.3. *Let (V0), (Q0), (K0) and (H0) be satisfied. Then for any* $\sigma > 0$*, there is* $\Lambda_\sigma > 0$ *such that for all* $\lambda \geq \Lambda_\sigma$ *and* $a > 0$ *equation (1.1) has a positive solution of least energy* $u_{\lambda,a} \in E$*.*

Under the odd condition on the nonlinearity a multiplicity result is obtained. It is clear that in this case if $u_{\lambda,a}$ is a solution also $-u_{\lambda,a}$ is, so the solutions will always appear in pairs.

Theorem 1.4. *Let (V0), (Q0), (K0) and (H0) be satisfied and assume also that* h *is odd with respect to* u*. For any* $m \in \mathbb{N}$ *and* $\sigma > 0$*, there is* $\Lambda_\sigma^m > 0$ *such that for all* $\lambda \geq \Lambda_\sigma^m$ *and* $a > 0$ *equation (1.1) has at least* m *(pairs of) solutions* $u_{\lambda,a}$*.*

The next result, which is an interesting consequence of some computations, gives the behaviour of the solutions in the semiclassical limit.

Theorem 1.5. *Under the assumptions of Theorem 1.3, respectively Theorem 1.4, for any* $\sigma > 0$ *the solutions* $u_{\lambda,a}$ *found there satisfy for* $\lambda \geq \Lambda_\sigma$*, respectively* $\lambda \geq \Lambda_\sigma^m$*, for any* $a > 0$*:*

$$0 < \Phi_{\lambda,a}(u_{\lambda,a}) \leq \sigma\lambda^{-1/2} \quad \text{and} \quad |\nabla u_{\lambda,a}|_2^2 \leq \frac{2\mu}{\mu-2}\sigma\lambda^{-1/2}$$

as well

$$\begin{aligned} \int_{\mathbb{R}^3} V(x)u_{\lambda,a}^2 &\leq \frac{2\mu}{\mu-2}\sigma\lambda^{-3/2}, & \int_{\mathbb{R}^3} Q(x)\phi_{u_{\lambda,a}}^a u_{\lambda,a}^2 &\leq \frac{4\mu}{\mu-4}\sigma\lambda^{-3/2}, \\ \int_{\mathbb{R}^3} H(x,u_{\lambda,a}) &\leq \frac{4}{\mu-4}\sigma\lambda^{-3/2}, & \int_{\mathbb{R}^3} K(x)|u_{\lambda,a}|^6 &\leq 12\sigma\lambda^{-3/2}. \end{aligned}$$

In particular the solutions squeeze to zero uniformly in $a > 0$ *as* $\lambda \to +\infty$*, in the sense that*

$$\lim_{\lambda\to+\infty} \sup_{a>0} \|u_{\lambda,a}\|_\lambda = \lim_{\lambda\to+\infty} \sup_{a>0} \Phi_{\lambda,a}(u_{\lambda,a}) = 0$$
$$\lim_{\lambda\to+\infty} \sup_{a>0} |\phi_{u_{\lambda,a}}^a|_\infty = \lim_{\lambda\to+\infty} \sup_{a>0} \|\phi_{u_{\lambda,a}}^a\|_{\mathcal{D}} = 0.$$

By analysing the proofs of the above theorems, we see that indeed the results hold for the limit problem, namely

Theorem 1.6. *Analogous results to Theorem 1.3 and Theorem 1.4, as well as the behaviour as* $\lambda \to +\infty$ *stated in Theorem 1.5, are true for the solutions of the Schrödinger-Poisson system (1.5).*

Finally the following behaviour with respect to a of the solutions is obtained.

Theorem 1.7. *Let (V0), (Q0), (K0) and (H0) be satisfied and let* $\{u_{\lambda_*,a}\}_{a\in(0,1)}$ *be a family of solutions of (1.4) for a suitable fixed parameter* $\lambda_* > 0$*. Then*

$$\lim_{a\to0} u_{\lambda_*,a} = u_{\lambda_*,0} \quad \text{in } E \quad \text{and} \quad \lim_{a\to0} \Phi_{\lambda_*,a}(u_{\lambda_*,a}) = \Phi_{\lambda_*,0}(u_{\lambda_*,0}),$$

where $u_{\lambda_*,0}$ *is a solution of (1.6) with parameter* λ_**.*

Then the Schrödinger-Poisson system can be seen actually as a limit of the Schrödinger-Bopp-Podolsky system.

It is worth to recall the literature which developed recently, and is still increasing, on this kind of elliptic systems. In fact the Schrödinger-Bopp-Podolsky system has been object of many investigations under different assumptions, or even in bounded domain, obtaining existence, multiplicity and qualitative properties of the solutions depending on the parameters involved, as well as the behaviour of the solutions whenever these parameters tend to a limit value.

In particular in [1, 11, 14] the problem has been treated in bounded domains proving the existence of solutions under Neumann and Dirichlet boundary conditions and in the whole space: in both cases normalised solutions are found. In [7] the existence of multiple solutions has been shown under the restriction of the "interacting energy": namely solutions u satisfying

$$\int_{\mathbb{R}^3} \phi_u^a u^2 = \text{constant}$$

are found. Other interesting papers, where the problem has been addressed also on manifolds and by means of the Ljusternick-Schnirelamnn theory are [4, 8–10, 13, 15].

We remark finally that the case of a critical nonlinearity has been studied also in [3, 12, 17] but with different assumptions on the potentials and not in the semiclassical limit. Different results from ours are found.

1.3. **Few comments.** Let us spend few words on our assumptions.

1. The assumptions on K and h are quite natural when one wants to use variational methods and a comparison energy functional.

2. As it will be evident by the proof, actually it is sufficient that V is continuous just in x_0, a point of minimum. The assumption that $V(x_0) = 0$ is used in just one point in the proofs, more specifically, in order to estimate the mountain pass value and make it falls down in a region where the compactness condition holds.

3. As for Q, which can be seen as a charge density, we can allow it is in $L^\infty(\mathbb{R}^3)$ under the assumption it vanishes at infinity. This is also quite reasonable from a physical point of view, and to the best of our knowledge it has never been considered before. Previous paper on Schrödinger-Poisson systems or Schrödinger-Bopp-Podolski systems consider $Q \in L^2(\mathbb{R}^3)$.

4. It is interesting noticing that (maintaining the general assumptions) if the problem is invariant under an orthogonal involution, namely the following condition is satisfied:

(S0) V, Q, h and K are Hölder continuous, and there is an orthogonal involution τ such that $V(\tau x) = V(x), Q(\tau x) = Q(x)$ and $H(\tau x, .) = H(x, .)$ for all $x \in \mathbb{R}^3$,

then, again the main results are still valid and the solutions found share the same symmetry. See the key Remark 3.

Of course, under the radial symmetry, again all our results are true (where in (Q0) we just ask that $Q \in L^s_{\mathrm{rad}}(\mathbb{R}^3), s \in [3/2, +\infty]$ and $Q \geq 0$ ($\not\equiv 0$), since item (vii) of Lemma 1.2 is automatically true, see [5]). In this case the solutions will be radial.

Note finally that from (H0) and (K0) the following holds: given any $\delta > 0$ there exists a constant $C(\delta) > 0$ such that for all $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$ it is

$$h(x,u)u + K(x)|u|^6, H(x,u) + \frac{1}{6}K(x)|u|^6 \ \leq \delta u^2 + C(\delta)|u|^6. \tag{1.7}$$

This estimate will be used in the following.

Organisation of the paper. The paper is organised as follows. In the next Section 2 the variational framework of the problem is given. In particular the compactness condition and the Mountain Pass structure for the energy functional are shown. Then in Section 3 the results are proved. We finish with Section 4 were we spend few words whenever condition (S0) is satisfied.

Notation. We use $\epsilon_n, \epsilon_\delta, \epsilon'_\delta, \dots$ to denote generic vanishing sequences and C', C'' for generic constant (usually depending just of the embedding constants, but not on the functions or the parameters involved) and whose value may change also from line to line.

Given a positive number μ, we will use the notation $B_\mu(0)$ to denote the closed ball centered in zero with radius μ in $\mathbb{R}^3$ and with B_μ the "same" ball in E: it will be clear form the context that no confusion should arise.

Other notations will be introduced whenever we need.

2. VARIATIONAL FRAMEWORK

As we said before, we are reduced to find critical points of the functional Φ_λ in (1.3). Let us first recall some facts.

We say that a sequence $\{u_n\} \subset E$ is called a Palais-Smale sequence at the level $c \in \mathbb{R}$ for Φ_λ, for short $(PS)_c$ sequence, if

$$\Phi_{\lambda,a}(u_n) \to c \quad \text{and} \quad \Phi'_{\lambda,a}(u_n) \to 0.$$

The functional $\Phi_{\lambda,a}$ is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence contains a convergent subsequence. If the value c is not relevant, we speak of (PS) sequences or (PS) condition.

In the next results (V0), (Q0), (K0) and (H0) are tacitly assumed.

2.1. The compactness condition.

Let us start to show that Palais-Smale sequences may exists just at positive levels of Φ_λ, and in this case they are automatically bounded.

Lemma 2.1. *Let $a > 0$ and $\{u_n\}$ be a $(PS)_c$ sequence for $\Phi_{\lambda,a}$. Then $c \geq 0$ and $\{u_n\}$ is bounded in E. In particular $c = 0$ if and only if $u_n \to 0$.*

Proof. If $\{u_n\}$ is a $(PS)_c$ sequence, then by the condition (H0) and Lemma 1.2 we have

$$\begin{aligned}\Phi_{\lambda,a}(u_n) - \frac{1}{\mu}\Phi'_{\lambda,a}(u_n)[u_n] &= \frac{\mu-2}{2\mu}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + \lambda V(x)u_n^2) + \frac{6-\mu}{6\mu}\lambda\int_{\mathbb{R}^3}K(x)|u_n|^6 \\ &+ \lambda\int_{\mathbb{R}^3}\left(\frac{1}{\mu}h(x,u_n)u_n - H(x,u_n)\right) + \frac{\mu-4}{4\mu}\lambda\int_{\mathbb{R}^3}Q(x)\phi^a_{u_n}u_n^2 \\ &\geq \frac{\mu-2}{2\mu}\|u_n\|^2_\lambda.\end{aligned}$$

On the other hand,

$$\Phi_{\lambda,a}(u_n) - \frac{1}{\mu}\Phi'_{\lambda,a}(u_n)[u_n] \leq c + \epsilon_n + \epsilon_n\|u_n\|_\lambda,$$

where $\epsilon_n \to 0$ as $n \to \infty$. From both inequalities it follows that $\frac{\mu-2}{2\mu}\|u_n\|^2_\lambda \leq c + \epsilon_n + \epsilon_n\|u_n\|_\lambda$, and the conclusion follows. □

Note that by the proof it follows that if $c = 0$, then $u_n \to 0$. By the above lemma without loss of generality, we can assume that the (PS) sequence $\{u_n\}$ is such that $u_n \rightharpoonup u$ in E, $L^p(\mathbb{R}^3)$ with $2 \leq p \leq 6$, $u_n \to u$ in $L^s_{loc}(\mathbb{R}^3)$ for $1 \leq s < 6$ and $u_n(x) \to u(x)$ a.e. for $x \in \mathbb{R}^3$. Moreover it is easy to check that u is a critical point of $\Phi_{\lambda,a}$ and hence $\Phi_{\lambda,a}(u) \geq 0$.

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying

$$\eta(t) = \begin{cases} 1, & \text{if } t \leq 1, \\ 0, & \text{if } t \geq 2, \end{cases}$$

and consider the following truncation of the weak limit, $(T_n u)(x) := \eta(2|x|/n)u(x)$. Clearly,

$$\|u - T_n u\|_\lambda \to 0 \quad \text{as} \quad n \to \infty. \tag{2.1}$$

The following is a result from [6] and is a general fact holding for weakly convergent sequences in E.

Lemma 2.2. *Under the above notations, up to subsequences we have*

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^3}\left(h(x,u_n)-h(x,u_n-T_nu)-h(x,T_nu)\right)\varphi\right|=0$$

uniformly in $\varphi\in E$ *with* $\|\varphi\|\leq 1$.

Given a (PS) sequence $\{u_n\}$ with weak limit (up to subsequence) u, for convenience let us define the sequence

$$w_n := u_n - T_nu$$

where T_n is the truncation given above. By (2.1) we see that $w_n \rightharpoonup 0$ in E. Note also that $w_n\to 0$ if and only if $u_n\to u$. The next result says it is again a Palais-Smale sequence.

Lemma 2.3. *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence for* $\Phi_{\lambda,a}$ *whose weak limit is* u*, then along a subsequence we have*

$$\Phi_{\lambda,a}(w_n)\to c-\Phi_{\lambda,a}(u)\geq 0 \quad \textit{and} \quad \Phi'_{\lambda,a}(w_n)\to 0.$$

In particular $c=\Phi_{\lambda,a}(u)$ *if and only if* $u_n\to u$.

Proof. We have

$$\begin{aligned}\Phi_{\lambda,a}(w_n) &= \Phi_{\lambda,a}(u_n)-\Phi_{\lambda,a}(T_nu)-2\langle w_n,T_nu\rangle_\lambda\\ &+ \frac{\lambda}{4}\int_{\mathbb{R}^3}Q(x)\Big(\phi^a_{w_n}w_n^2+\phi^a_{T_nu}(T_nu)^2-\phi^a_{u_n}u_n^2\Big)\\ &+ \frac{\lambda}{6}\int_{\mathbb{R}^3}K(x)\left(|u_n|^6-|w_n|^6-|T_nu|^6\right)\\ &+ \lambda\int_{\mathbb{R}^3}\Big(H(x,u_n)-H(x,w_n)-H(x,T_nu)\Big).\end{aligned}$$

Let us see the convergence of each term above. Of course $\Phi_{\lambda,a}(u_n)\to c$, $\Phi_{\lambda,a}(T_nu)\to\Phi_{\lambda,a}(u)$ and $\langle w_n,T_nu\rangle_\lambda\to 0$. Moreover by Lemma 1.2 item (vii) we get

$$\begin{aligned}&\int_{\mathbb{R}^3}Q(x)\phi^a_{w_n}w_n^2\to 0,\\ &\int_{\mathbb{R}^3}Q(x)\phi^a_{T_nu}(T_nu)^2\to\int_{\mathbb{R}^3}Q(x)\phi^a_uu^2,\\ &\int_{\mathbb{R}^3}Q(x)\phi^a_{u_n}u_n^2\to\int_{\mathbb{R}^3}Q(x)\phi^a_uu^2.\end{aligned}$$

In virtue of (2.1) as in [6] it is

$$\int_{\mathbb{R}^3}K(x)\left(|u_n|^6-|w_n|^6-|T_nu|^6\right)\to 0$$

and

$$\int_{\mathbb{R}^3}\Big(H(x,u_n)-H(x,w_n)-H(x,T_nu)\Big)\to 0,$$

and we obtain the first convergence. The second convergence is also straightforward. We write for any $\varphi \in C_c^\infty(\mathbb{R}^3)$:

$$\begin{aligned}\Phi'_\lambda(w_n)[\varphi] &= \Phi'_{\lambda,a}(u_n)[\varphi] - \Phi'_{\lambda,a}(T_n u)[\varphi] + \lambda\int_{\mathbb{R}^3} Q(x)\phi^a_{w_n} w_n\varphi \\ &+ \lambda\int_{\mathbb{R}^3} Q(x)\Big(\phi^a_{T_n u}T_n u - \phi^a_{u_n}u_n\Big)\varphi \\ &+ \lambda\int_{\mathbb{R}^3} K(x)(|u_n|^4 u_n - |w_n|^4 w_n - |T_n u|^4 T_n u)\varphi \\ &+ \lambda\int_{\mathbb{R}^3} (h(x,u_n) - h(x,w_n) - h(x,T_n u))\varphi.\end{aligned}$$

Now $\Phi'_{\lambda,a}(u_n)[\varphi] \to 0$ and $\Phi'_{\lambda,a}(T_n u)[\varphi] \to \Phi'_{\lambda,a}(u)[\varphi] = 0$. Using again Lemma 1.2, we have

$$\int_{\mathbb{R}^3} Q(x)\phi^a_{T_n u}(T_n u)\varphi \to \int_{\mathbb{R}^3} Q(x)\phi^a_u u\varphi \quad \text{and} \quad \int_{\mathbb{R}^3} Q(x)\phi^a_{w_n} w_n\varphi \to 0.$$

Standard arguments show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} K(x)(|u_n|^4 u_n - |w_n|^4 w_n - |T_n u|^4 T_n u)\varphi = 0$$

and using Lemma 2.2, we get also

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} (h(x,u_n) - h(x,w_n) - h(x,T_n u))\varphi = 0.$$

Since all these convergences are uniform in $\|\varphi\| \leq 1$, the conclusion follows. ☐

Note that,

$$\begin{aligned} c - \Phi_{\lambda,a}(u) + \varepsilon_n &\geq \Phi_{\lambda,a}(w_n) - \frac{1}{4}\Phi'_{\lambda,a}(w_n)[w_n] \\ &= \frac{1}{4}\|w_n\|^2_\lambda + \frac{\lambda}{12}\int_{\mathbb{R}^3} K(x)|w_n|^6 - \lambda\int_{\mathbb{R}^3} H(x,w_n) + \frac{\lambda}{4}\int_{\mathbb{R}^3} h(x,w_n)w_n \\ &\geq \frac{\lambda K_{\min}}{12}|w_n|_6^6 + \lambda\int_{\mathbb{R}^3}\left(\frac{1}{\mu}h(x,w_n)w_n - H(x,w_n)\right) \\ &\geq \frac{\lambda K_{\min}}{12}|w_n|_6^6,\end{aligned}$$

from which we conclude that, if $u_n \not\to u$,

$$|w_n|_6^6 \leq \frac{12(c - \Phi_{\lambda,a}(u))}{\lambda K_{\min}} + \epsilon_n. \tag{2.2}$$

Let us denote with S the best Sobolev constant, so

$$S|u|_6^2 \leq |\nabla u|_2^2, \quad \forall u \in H^1(\mathbb{R}^3).$$

We have the following.

Proposition 2.4. *Let $a > 0, \{u_n\}$ be a $(PS)_c$ sequence for $\Phi_{\lambda,a}$ such that (see Lemma 2.1) up to subsequences $u_n \rightharpoonup u$. Then one of the following alternatives hold:*

$$\textit{either} \quad u_n \to u \quad \textit{or} \quad c - \Phi_{\lambda,a}(u) \geq \widetilde{S}\lambda^{-1/2}$$

with

$$\widetilde{S} := \frac{1}{3}S^{3/2}\frac{K_{min}}{4C(b)^{3/2}}$$

where b is the number in (V0) and $C(b) > 0$ is the constant in (1.7) with $\delta = b$.

In particular $\Phi_{\lambda,a}$ satisfies the $(PS)_c$ condition at any level c such that

$$c < \widetilde{S}\lambda^{-1/2}.$$

Proof. Assume that $u_n \not\to u$. Then defining as before $w_n := u_n - T_n u \rightharpoonup 0$, we have $\liminf\limits_{n\to\infty}\|w_n\|_\lambda > 0$ and $c - \Phi_{\lambda,a}(u) > 0$, $\Phi'_{\lambda,a}(w_n) \to 0$. Using the assumption (V0), setting $V^b := \{x \in \mathbb{R}^3 : V(x) < b\}$, it is $\operatorname{meas}\{V^b\} < +\infty$, and then we see that

$$\int_{\mathbb{R}^3} |b - V(x)| w_n^2 = \int_{V^b} (b - V(x)) w_n^2 + \int_{\mathbb{R}^3 \setminus V^b} (V(x) - b) w_n^2 \le \epsilon_n.$$

Then by (1.7) with $\delta = b$, we infer

$$\begin{aligned}
S|w_n|_6^2 &\le |\nabla w_n|_2^2 + \lambda \int_{\mathbb{R}^3} V(x) w_n^2 - \lambda \int_{\mathbb{R}^3} V(x) w_n^2 \\
&= -\lambda \int_{\mathbb{R}^3} Q(x)\phi^a_{w_n} w_n^2 + \lambda \int_{\mathbb{R}^3} (h(x,w_n)w_n + K(x)|w_n|^6) - \lambda \int_{\mathbb{R}^3} V(x) w_n^2 + \epsilon_n \\
&\le \lambda \int_{\mathbb{R}^3} (h(x,w_n)w_n + K(x)|w_n|^6) - \lambda \int_{\mathbb{R}^3} V(x) w_n^2 + \epsilon_n \\
&\le \lambda \int_{\mathbb{R}^3} (b w_n^2 + C(b)|w_n|^6) - \lambda \int_{\mathbb{R}^3} V(x) w_n^2 + \epsilon_n \\
&\le \int_{\mathbb{R}^3} |b - V(x)| w_n^2 + \lambda C(b)|w_n|_6^6 + \epsilon_n \\
&\le \lambda C(b)|w_n|_6^6 + \epsilon_n.
\end{aligned}$$

Now, in virtue of (2.2) we obtain

$$S \le \lambda C(b) \left(\frac{12(c - \Phi_{\lambda,a}(u))}{\lambda K_{\min}} \right)^{2/3} + \epsilon_n$$

or equivalently,

$$\widetilde{S}\lambda^{-1/2} \le c - \Phi_{\lambda,a}(u) + \epsilon_n.$$

Assume now $c < \widetilde{S}\lambda^{-1/2}$ and take a $(PS)_c$ sequence $\{u_n\}$. If the first alternative does not hold, then

$$\widetilde{S}\lambda^{-\frac{1}{2}} \le c - \Phi_{\lambda,a}(u) < \widetilde{S}\lambda^{-\frac{1}{2}} - \Phi_{\lambda,a}(u)$$

which is absurd since $\Phi_{\lambda,a}(u) \ge 0$. □

Remark 1. *It is easy to check that also $\Psi_{\lambda,a}$ satisfies the $(PS)_c$ condition if $c < \widetilde{S}\lambda^{-1/2}$. In fact let $\{u_n\}$ be a $(PS)_c$ sequence for $\Psi_{\lambda,a}$. Applying Lemma 2.1 to $\Psi_{\lambda,a}$, we have that $\{u_n\}$ is a bounded sequence. Moreover,*

$$\begin{aligned}
\Psi'_{\lambda,a}(u_n)[u_n^-] &= \|u_n^-\|_\lambda^2 + \lambda \int_{\mathbb{R}^3} Q(x)\phi^a_{u_n}(u_n^-)^2 - \lambda \int_{\mathbb{R}^3} K(x)|u_n^+|^4 u_n^+ u_n^- + h(x,u_n^+)u_n^- \\
&\ge \|u_n^-\|_\lambda^2
\end{aligned}$$

which implies that $\|u_n^-\|_\lambda \to 0$. Then $\{u_n^+\}$ is bounded and we can assume it converge weakly to a positive function. Then the proof of Proposition 2.4 can be repeated.

2.2. **The Mountain Pass structure.** In the following let $B_{\rho_\lambda} = \{u \in E : \|u\|_\lambda \le \rho_\lambda\}$.

Lemma 2.5. *There exist $\alpha_\lambda, \rho_\lambda > 0$ such that, for all $a > 0$ it holds:*

$$\Phi_{\lambda,a}(u) > 0 \quad \text{if } u \in B_{\rho_\lambda} \setminus \{0\} \qquad \text{and} \qquad \Phi_{\lambda,a}(u) \ge \alpha_\lambda \quad \text{if } u \in \partial B_{\rho_\lambda}$$

Proof. Using (1.7), we get

$$\begin{aligned} \Phi_{\lambda,a}(u) &\ge \frac{1}{2}\|u\|_\lambda^2 - \lambda \int_{\mathbb{R}^3} (\delta |u|^2 + C(\delta)|u|^6) \\ &\ge \frac{1}{4}\|u\|_\lambda^2 - \lambda\delta C'\|u\|_\lambda^2 - \lambda C''\|u\|_\lambda^6 \end{aligned}$$

and the conclusion follows by choosing δ suitably small. □

We observe that $\alpha_\lambda, \rho_\lambda$ do not depend on $a > 0$ since the term involving a has been thrown away.

Lemma 2.6. *For any finite-dimensional subspace $F \subset E$ we have*

$$\lim_{u \in F, \|u\|_\lambda \to +\infty} \Phi_{\lambda,a}(u) = -\infty.$$

Proof. Using that H is positive, $K \in L^\infty(\mathbb{R}^3)$, and (v) in Lemma 1.2 (that will give the uniformity in a of the limit) we have that, for suitable constants $C', C'' > 0$:

$$\begin{aligned} \Phi_{\lambda,a}(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} Q(x)\phi_u^a u^2 - \lambda \int_{\mathbb{R}^3} \left(H(x,u) + \frac{1}{6}K(x)|u|^6 \right) \\ &\le \frac{1}{2}\|u\|_\lambda^2 + \lambda C'\|u\|_\lambda^4 - \lambda C''|u|_6^6 \end{aligned}$$

and the conclusion follows. □

Again observe that the limit is uniform in a.

The next step is to guarantee that the mountain pass level for $\Phi_{\lambda,a}$ is in a region where the Palais-Smale condition holds, and for this we use a comparison functional.

By assumption (V0), we assume, without loss of generalities, that $x_0 = 0$, namely $V(0) = 0$. Observe that (see [6])

$$\inf\left\{ \int_{\mathbb{R}^3} |\nabla \varphi|^2 : \varphi \in C_c^\infty(\mathbb{R}^3), |\varphi|_p = 1 \right\} = 0,$$

hence for any $\delta > 0$ we can choose $\varphi_\delta \in C_c^\infty(\mathbb{R}^3)$ with

$$|\varphi_\delta|_p = 1, \quad \operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0) \quad \text{and} \quad |\nabla\varphi_\delta|_2^2 < \delta. \tag{2.3}$$

Define the rescaled function

$$e_\lambda(x) := \varphi_\delta(\lambda^{1/2}x) \in C_c^\infty(B_{\lambda^{-1/2}r_\delta}(0)), \tag{2.4}$$

We will use the following easy result, with $r = 2$ and $r = 4$.

Lemma 2.7. *Let $0 < r < p$ and consider the function $f(t) = at^r - bt^p$ where $a, b > 0$ and $t \ge 0$. Then*

$$\max_{t \ge 0} f(t) = \frac{p-r}{p}\left(\frac{r}{pb}\right)^{r/(p-r)} a^{p/(p-r)}.$$

Then we have an important estimate. This is the unique, but fundamental point, where the assumption $\min_{\mathbb{R}^3} V = 0$ is used.

Proposition 2.8. *For every $\delta > 0$ there is $\eta_\delta > 0$ such that, for all $a > 0$ and $\lambda \geq \eta_\delta$ the following holds:*

$$\max_{t\geq 0} \Phi_{\lambda,a}(te_\lambda) \leq \epsilon_\delta \lambda^{-1/2} \quad \textit{where} \quad \lim_{\delta\to 0} \epsilon_\delta = 0. \tag{2.5}$$

Proof. Preliminarly observe that by (H0) it follows that

$$\frac{1}{6}K(x)|e_\lambda|^6 + H(x, e_\lambda) \geq \nu |e_\lambda|^p \tag{2.6}$$

At this point we distinguish two cases.

Case I: (q1) or (q3) hold. The Hölder inequality gives

$$|Qe_\lambda^2|_{6/5}^{6/5} \leq |Q^{6/5}|_{5s/6} |e_\lambda^{12/5}|_{(5s/6)'} = |Q|_s^{6/5} |e_\lambda|_{12s/(5s-6)}^{12/5}. \tag{2.7}$$

We agree here that if $s = +\infty$, then $12s/(5s-6) = 12/5$. Hence, by means of the Hardy-Littlewood-Sobolev inequality and using (2.7) we obtain

$$\int_{\mathbb{R}^3} Q(x)\phi^a_{e_\lambda} e_\lambda^2 \leq \int_{\mathbb{R}^3} Q(x)\phi^0_{e_\lambda} e_\lambda^2 \leq C|Qe_\lambda^2|_{6/5}^2 \leq C|Q|_s^2 |e_\lambda|^4_{\frac{12s}{5s-6}} \tag{2.8}$$

for a suitable $C > 0$.

Let us pass to estimate $\Phi_{\lambda,a}(te_\lambda)$. For any $t \geq 0$, due to (2.6), (2.8) and then a change of variables, we have

$$\begin{aligned} \Phi_{\lambda,a}(te_\lambda) &\leq \frac{t^2}{2}\int_{\mathbb{R}^3} (|\nabla e_\lambda|^2 + \lambda V(x) e_\lambda^2) + \frac{t^4 \lambda C|Q|_s^2}{4} |e_\lambda|^4_{12s/(5s-6)} - \nu\lambda t^p |e_\lambda|_p^p \\ &\leq \frac{\lambda^{-1/2}}{2}\left(t^2 \int_{\mathbb{R}^3}\left(|\nabla\varphi_\delta|^2 + V(x/\lambda^{1/2})\varphi_\delta^2\right) + \frac{t^4 \lambda^{\frac{3-s}{s}} C|Q|_s^2}{2}|\varphi_\delta|^4_{12s/(5s-6)} - 2\nu t^p\right) \\ &= \frac{\lambda^{-1/2}}{2}\Big(M_\lambda(t\varphi_\delta) + N_\lambda(t\varphi_\delta)\Big), \end{aligned} \tag{2.9}$$

where we set for brevity

$$\begin{aligned} M_\lambda(t\varphi_\delta) &:= t^2 \int_{\mathbb{R}^3}\left(|\nabla\varphi_\delta|^2 + V(x/\lambda^{1/2})\varphi_\delta^2\right) - \nu t^p, \\ N_\lambda(t\varphi_\delta) &:= \frac{t^4 \lambda^{\frac{3-s}{s}} C|Q|_s^2}{2}|\varphi_\delta|^4_{12s/(5s-6)} - \nu t^p. \end{aligned}$$

In N_λ, if $s = +\infty$ then $(3-s)/s = -1$.

By Lemma 2.7 we see that

$$\max_{t\geq 0} M_\lambda(t\varphi_\delta) = \frac{p-2}{p}\left(\frac{2}{p\nu}\right)^{2/(p-2)} \left(\int_{\mathbb{R}^3} |\nabla\varphi_\delta|^2 + V(x/\lambda^{1/2})\varphi_\delta^2\right)^{p/(p-2)}.$$

Given $\delta > 0$ there is $\widehat{\eta}_\delta > 0$ such that for all $\lambda \geq \widehat{\eta}_\delta$, by the continuity of V in 0, it is

$$V(x/\lambda^{1/2}) \leq \frac{\delta}{|\varphi_\delta|_2^2} \quad \forall x \in B_{\lambda^{-1/2} r_\delta}(0)$$

and then, recalling (2.3),

$$\int_{\mathbb{R}^3} |\nabla\varphi_\delta|^2 + V(x/\lambda^{1/2})\varphi_\delta^2 \leq 2\delta.$$

So, for any $\lambda \geq \widehat{\eta}_\delta$ it is

$$\max_{t\geq 0} M_\lambda(t\varphi_\delta) \leq \frac{p-2}{p}\left(\frac{2}{p\nu}\right)^{2/(p-2)} (2\delta)^{p/(p-2)} =: \epsilon'_\delta. \tag{2.10}$$

Similarly, evaluating the maximum of N_λ we find

$$\max_{t\geq 0} N_\lambda(t\varphi_\delta) = \frac{p-4}{p}\Big(\frac{4}{p\nu}\Big)^{4/(p-4)}\left(\frac{\lambda^{\frac{3-s}{s}}C|Q|_s^2}{2}|\varphi_\delta|^4_{12s/(5s-6)}\right)^{p/(p-4)}.$$

Since $s>3$, there is $\widetilde{\eta}_\delta>0$ such that for $\lambda\geq\widetilde{\eta}_\delta$ it is

$$\max_{t\geq 0} N_\lambda(t\varphi_\delta)\leq \frac{p-4}{p}\Big(\frac{2}{p\nu}\Big)^{4/(p-4)}\delta^{p/(p-4)} =: \epsilon''_\delta. \tag{2.11}$$

Finally, using estimates (2.10) and (2.11) in (2.9) we obtain, for any $\lambda\geq\eta_\delta:=\max\{\widehat{\eta}_\delta,\widetilde{\eta}_\delta\}$

$$\max_{t\geq 0}\Phi_{\lambda,a}(te_\lambda)\leq\frac{1}{2}\lambda^{-1/2}(\epsilon'_\delta+\epsilon''_\delta)=:\lambda^{-1/2}\epsilon_\delta.$$

Actually, when $Q\in L^\infty(\mathbb{R}^3)$ we do not use that it vanishes at infinity.

Case II: (q2) holds. We start from

$$\int_{\mathbb{R}^3}Q(x)\phi^a_{e_\lambda}e^2_\lambda\leq\int_{\mathbb{R}^3}Q(x)\phi^0_{e_\lambda}e^2_\lambda=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}Q(x)Q(y)\frac{e^2_\lambda(x)e^2_\lambda(y)}{|x-y|}$$

and repeating the computations that led to (2.9), we get now

$$\begin{aligned}\Phi_{\lambda,a}(te_\lambda) &\leq \frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla e_\lambda|^2+\lambda V(x)e^2_\lambda)+\frac{t^4\lambda}{4}\int_{\mathbb{R}^3}Q(x)\phi^0_{e_\lambda}e^2_\lambda-\nu\lambda t^p|e_\lambda|^p_p\\ &= \frac{\lambda^{-1/2}}{2}\Big(t^2\int_{\mathbb{R}^3}\Big(|\nabla\varphi_\delta|^2+V(x/\lambda^{1/2})\varphi^2_\delta\Big)+\frac{t^4}{2\lambda}E_\lambda(\varphi_\delta)-2\nu t^p\Big)\\ &=: \frac{\lambda^{-1/2}}{2}\Big(M_\lambda(t\varphi_\delta)+\widetilde{N}_\lambda(t\varphi_\delta)\Big)\end{aligned}$$

where

$$E_\lambda(\varphi_\delta)=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}Q(x/\lambda^{1/2})Q(y/\lambda^{1/2})\frac{\varphi^2_\delta(y)\varphi^2_\delta(x)}{|x-y|}$$

and

$$\begin{aligned}M_\lambda(t\varphi_\delta) &= t^2\int_{\mathbb{R}^3}\Big(|\nabla\varphi_\delta|^2+V(x/\lambda^{1/2})\varphi^2_\delta\Big)-\nu t^p \quad \text{(as in } \textbf{Case I}\text{)},\\ \widetilde{N}_\lambda(t\varphi_\delta) &= \frac{t^4}{2\lambda}E_\lambda(\varphi_\delta)-\nu t^p.\end{aligned}$$

Then (2.10) is still valid. However, since

$$\max_{t\geq 0}\widetilde{N}_\lambda(t\varphi_\delta)=\frac{p-4}{p}\Big(\frac{4}{p\nu}\Big)^{4/(p-4)}\Big(\frac{1}{2\lambda}E_\lambda(\varphi_\delta)\Big)^{p/(p-4)}$$

by the Dominated Converge Theorem it holds

$$\lim_{\lambda\to+\infty}E_\lambda(\varphi_\delta)=\ell^2\int_{\mathbb{R}^3}\frac{\varphi^2_\delta(x)\varphi^2_\delta(y)}{|x-y|}\in\mathbb{R},$$

recalling that by assumption it is $\ell=\lim_{x\to 0}Q(x)$. Hence an estimate like (2.11) holds and we conclude as in **Case I**. ☐

Corollary 2.9. *For any $\sigma>0$ there exists $\Lambda_\sigma>0$, such that, for each $a>0,\lambda\geq\Lambda_\sigma$, there is $\overline{e}_\lambda\in E$ with $\|\overline{e}_\lambda\|>\rho_\lambda$, satisfying*

$$\Phi_{\lambda,a}(\overline{e}_\lambda)\leq 0 \quad \textit{and} \quad \max_{t\in[0,1]}\Phi_{\lambda,a}(t\overline{e}_\lambda)\leq\sigma\lambda^{-1/2}$$

where ρ_λ is given in Lemma 2.5.

Proof. Given $\sigma > 0$, choose $\delta = \delta(\sigma) > 0$ small enough such that, by (2.5) it is for $a > 0, \lambda \geq \eta_\delta$

$$\max_{t\geq 0} \Phi_{\lambda,a}(te_\lambda) \leq \sigma\lambda^{-1/2},$$

with e_λ defined as in (2.4). Since $\lim_{t\to+\infty} \Phi_{\lambda,a}(te_\lambda) = -\infty$ by Lemma 2.6, uniformly in $a > 0$, there is $\bar{t}_\lambda > 0$ such that $\|\bar{t}_\lambda e_\lambda\| \geq \rho_\lambda$ and $\Phi_{\lambda,a}(te_\lambda) < 0$ for any $t > \bar{t}_\lambda$. Hence just take $\bar{e}_\lambda := \bar{t}_\lambda e_\lambda$, proving the result with $\Lambda_\sigma := \eta_\delta$. □

Remark 2. *The above proofs work with a test function in $C_c(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. Then given φ_δ as before, it is $|\varphi_\delta| \in C_c(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ and still it satisfies (2.3). In particular the related function e_λ is nonnegative and estimate (2.5) is satisfied too. Then in Lemma 2.9 we can assume $e_\lambda \geq 0$. Moreover also Ψ_λ has the Mountain Pass geometry stated in Lemma 2.5 and Lemma 2.6 and $\Phi_\lambda(t\bar{e}_\lambda) = \Psi_\lambda(t\bar{e}_\lambda)$ for every $t \geq 0$.*

Remark 3. *Assume that also (S0) holds and denote with the same symbol τ the induced involution on E, given by $(\tau u)(x) = -u(\tau x)$; in particular $\Phi_{\lambda,a}(\tau u) = \Phi_{\lambda,a}(u)$.*
Note that if $u \in E$ then

$$\widetilde{u} := (u + \tau u)/2 \in E^\tau := \{u \in E : \tau u = u\}.$$

Moreover, given φ_δ as before, the function $\widetilde{\varphi}_\delta$ still satisfies (2.3), and then, by repeating the same proof, we can suppose that Corollary 2.9 and Proposition 2.4 hold in E^τ. In other words the Mountain Pass geometry still holds in E^τ with the estimate given in Corollary 2.9.

In general given $\delta > 0$, for any $m \in \mathbb{N}$, we can choose m functions $\varphi^j_\delta \in C^\infty_c(\mathbb{R}^3)$ such that $\operatorname{supp}\varphi^j_\delta \cap \operatorname{supp}\varphi^k_\delta = \emptyset$, if $j \neq k$, $|\varphi^j_\delta|_p = 1$ and $|\nabla\varphi^j_\delta|_2^2 < \delta$. Let $r^m_\delta > 0$ be such that $\operatorname{supp}\varphi^j_\delta \subset B_{r^m_\delta}(0)$ for $j = 1, \ldots, m$. Set

$$e^j_\lambda(x) := \varphi^j_\delta(\lambda^{1/2}x) \quad \text{for } j = 1, ..., m$$

and

$$H^m_{\lambda,\delta} := span\{e^1_\lambda, \ldots, e^m_\lambda\}.$$

Hence if $u = \sum_{j=1}^m c_j e^j_\lambda \in H^m_{\lambda,\delta}$, where $c_j \in \mathbb{R}$, it is easy to check that

$$\Phi_{\lambda,a}(u) = \sum_{j=1}^m \Phi_{\lambda,a}(c_j e^j_\lambda) + \frac{\lambda}{2}\sum_{\substack{i,j=1\\ i<j}}^m c_i^2 c_j^2 \int_{\mathbb{R}^3} Q(x)\phi^a_{e^i_\lambda}(e^j_\lambda)^2. \tag{2.12}$$

Let us estimate each term in the right hand side above.

As in the proof of Proposition 2.8, assume first that (q1) or (q3) hold. As before (see (2.9)), for every $j \in \{1, \ldots, m\}$ it is

$$\Phi_{\lambda,a}(c_j e^j_\lambda) \leq \frac{1}{2}\lambda^{-1/2}\Big(M_\lambda(|c_j|\varphi^j_\delta) + N_\lambda(|c_j|\varphi^j_\delta)\Big) \tag{2.13}$$

and setting

$$\beta_\delta := \max\{|\varphi^j_\delta|_2^2 : j = 1, ..., m\},$$

there is $\widehat{\eta}^m_\delta > 0$ such that for every $x \in B_{r^m_\delta}(0)$ we have

$$V(x/\lambda^{1/2}) \leq \frac{\delta}{\beta_\delta} \quad \forall\lambda \geq \widehat{\eta}^m_\delta.$$

Then a similar estimates to (2.10) holds for every φ^j_δ:

$$M_\lambda(|c^j|\varphi^j_\delta) \leq \max_{t\geq 0} M_\lambda(t\varphi^j_\delta) \leq \frac{p-2}{p}\Big(\frac{2}{p\nu}\Big)^{2/(p-2)}(2\delta)^{p/(p-2)} =: \epsilon'_\delta.$$

The estimate on the term $N_\lambda(|c_j|\varphi^j_\delta))$ does not require any change with respect to (2.11) and the existence of $\widetilde{\eta}^m_\delta > 0$, such that for $\lambda \geq \widetilde{\eta}^m_\delta$,

$$N_\lambda(|c^j|\varphi^j_\delta) \leq \max_{t\geq 0} N_\lambda(t\varphi^j_\delta) \leq \frac{p-4}{p}\Big(\frac{2}{p\nu}\Big)^{4/(p-2)}\delta^{p/(p-2)} =: \epsilon''_\delta,$$

is guaranteed. Finally, using the Hardy-Littlewood inequality as in (2.8) and by the change of variables, we get

$$\frac{\lambda}{2}\sum_{\substack{i,j=1\\ i<j}}^m c_i^2c_j^2\int_{\mathbb{R}^3} Q(x)\phi^a_{e^i_\lambda}(e^j_\lambda)^2 \quad\leq\quad \frac{\lambda^{\frac{6-s}{2s}}C|Q|_s^2}{2}\sum_{\substack{i,j=1\\ i<j}}^m c_i^2c_j^2|\varphi^i_\delta|^2_{12s/(5s-6)}|\varphi^j_\delta|^2_{12s/(5s-6)}. \tag{2.14}$$

Then (2.12), (2.13) and (2.14) give

$$\Phi_{\lambda,a}(u) \quad\leq\quad \frac{\lambda^{-1/2}}{2}\Big(\sum_{j=1}^m M_\lambda(|c_j|\varphi^j_\delta) + \sum_{j=1}^m N_\lambda(|c_j|\varphi^j_\delta) + \lambda^{\frac{3-s}{s}}\sum_{\substack{i,j=1\\ i<j}}^m c_i^2c_j^2|\varphi^i_\delta|^2_{12s/(5s-6)}|\varphi^j_\delta|^2_{12s/(5s-6)}\Big).$$

Again, since $s > 3$ there is $\bar{\eta}^m_\delta > 0$ such that for every $\lambda \geq \bar{\eta}^m_\delta$ it is

$$\lambda^{\frac{3-s}{s}}\sum_{\substack{i,j=1\\ i<j}}^m c_i^2c_j^2|\varphi^i_\delta|^2_{12s/(5s-6)}|\varphi^j_\delta|^2_{12s/(5s-6)} < \delta,$$

so that for any $\lambda \geq \Lambda^m_\delta := \max\{\widehat{\eta}^m_\delta, \widetilde{\eta}^m_\delta, \bar{\eta}^m_\delta\}$ we infer, for all $u \in H^m_{\lambda,\delta}$:

$$\Phi_{\lambda,a}(u) \leq \frac{1}{2}\lambda^{-1/2}(m\epsilon'_\delta + m\epsilon''_\delta + \delta) =: \frac{1}{2}\lambda^{-1/2}\epsilon_\delta.$$

We argue similarly as before if (q2) holds.

Consequently we get

Proposition 2.10. *For any $m \in \mathbb{N}$ and $\sigma > 0$ there exist $\Lambda^m_\sigma > 0$, such that, for each $a > 0, \lambda \geq \Lambda^m_\sigma$, there exists an m-dimensional subspace F^m_λ such that*

$$\max_{u\in F^m_\lambda}\Phi_{\lambda,a}(u) \leq \sigma\lambda^{-1/2}.$$

Of course it is $F^m_\lambda = H^m_{\lambda,\delta}$ constructed above.

3. Proofs of the main results

We are in a position now to prove the existence and multiplicity results.

3.0.1. *Proof of Theorem 1.3.* For any $0 < \sigma < \widetilde{S}$, by Corollary 2.9, we can choose $\Lambda_\sigma > 0$ and define for $a > 0, \lambda \geq \Lambda_\sigma$ the minimax value (recall Lemma 2.5):

$$d_{\lambda,a} := \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]}\Phi_{\lambda,a}(\gamma(t)) \in [\alpha_\lambda, +\infty) \tag{3.1}$$

where $\Gamma_\lambda := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \textit{ and } \gamma(1) = \bar{e}_\lambda\}$. Then it is

$$d_{\lambda,a} \leq \max_{t\in[0,1]}\Phi_{\lambda,a}(t\bar{e}_\lambda) \leq \sigma\lambda^{-1/2} < \widetilde{S}\lambda^{-1/2}$$

and by Proposition 2.4 and the Mountain Pass Theorem, $d_{\lambda,a}$ is a critical value obtained on a critical point $u_{\lambda,a} \in E$ which is a solution of (1.1). It is standard to see that a mountain pass solution gives a solution of least energy of (1.1).

Finally this argument also work for $\Psi_{\lambda,a}$ by Remark 1 and Remark 2, and then we get a positive solution of (1.1).

3.0.2. *Proof of Theorem 1.4.* To prove the multiplicity result in presence of symmetry, we first recall few facts. Let Σ the class of closed and symmetric subset of E, γ be the Krasnoselkii genus.

Definition 1. *Fixed $S \in \Sigma$, the pseudo-genus related to S is the triplet $(S, \mathcal{H}^*, \gamma^*)$ where $\mathcal{H}^*$ is a group of odd homeomorphism and*

$$\gamma^* : A \in \Sigma \longmapsto \min_{h \in \mathcal{H}^*} \gamma(h(A) \cap S) \in \mathbb{N} \cup \{+\infty\}.$$

Since γ is a topological invariant,

$$\gamma(h(A) \cap S) = \gamma(h^{-1}(h(A) \cap S)) = \gamma(A \cap h^{-1}(S))$$

hence, in particular, if S is a sphere in E and A is an $m-$dimensional subspace, we infer $\gamma^*(A) = m$.

We will use the following result. See [2, Theorem 2.2].

Theorem 3.1. *Let $a, b, c_0, c_\infty \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ with $-\infty \leq a < c_0 < c_\infty < b \leq +\infty, \Phi \in C^1(E)$ be an even functional, $(S, \mathcal{H}^*, \gamma^*)$ the pseudo-genus related to S, where*

$$\mathcal{H}^* := \Big\{ h \in \mathcal{H} : h \text{ is bounded and such that } h(e) = e \text{ if } e \notin \Phi^{-1}(]a, b[) \Big\}.$$

Assume that:

- *(i) Φ satisfies the $(PS)_c$ condition for every $c \in]a, b[$;*
- *(ii) $S \subseteq \Phi^{-1}([c_0, +\infty[)$;*
- *(iii) there exists $F \in \Sigma$ such that $F \subseteq \{u \in E : \Phi(u) \leq c_\infty\}$ and $\gamma^*(F) \geq m \geq 1$.*

Then, setting $\Sigma_i^* = \{A \in \Sigma : \gamma^*(A) \geq i\}$, the following numbers

$$c_i = \inf_{A \in \Sigma_i^*} \sup_{u \in A} \Phi(u), \ \ i \in \{1, \ldots, m\},$$

are critical values of Φ and

$$c_0 \leq c_1 \leq \ldots \leq c_m \leq c_\infty.$$

Furthermore, if $c := c_i = \ldots = c_{i+r}$, with $i \geq 1, r \geq 0$ and $i + r \leq m$, then

$$\gamma(\{u \in E : \Phi(u) = c, \ \ \Phi'(u) = 0\}) \geq r + 1.$$

Now we can prove our result.

By hypotheses $h(x, s)$ is odd with respect to s, so that $\Phi_{\lambda,a}$ is an even functional and also all the other assumptions of Theorem 3.1 are satisfied. In fact, given $\sigma \in (0, \widetilde{S})$ and $m \in \mathbb{N}$:

- (i) by Lemma 2.4 we have that $\Phi_{\lambda,a}$ satisfies the $(PS)_c$ condition, for all $0 < c < \widetilde{S}\lambda^{-1/2}$;
- (ii) by Lemma 2.5 it is $\Phi_{\lambda,a}(u) \geq \alpha_\lambda$ if $u \in \partial B_{\rho_\lambda}$ meaning that $\partial B_{\rho_\lambda} \subseteq \Phi^{-1}([\alpha_\lambda, +\infty[)$;
- (iii) by Proposition 2.10 there is $\Lambda_\sigma^m > 0$ such that, for each $\lambda \geq \Lambda_\sigma^m$, there is an m-dimensional subspace $F_\lambda^m \in \Sigma$ with $\max_{u \in F_\lambda^m} \Phi_{\lambda,a}(u) \leq \sigma\lambda^{-1/2}$ and $\gamma^*(F_\lambda^m) = m \geq 1$.

Then for every $\lambda \geq \Lambda_\sigma^m$ all the assumptions of Theorem 3.1 are satisfied and there are at least m pairs of critical points of $\Phi_{\lambda,a}$ with energy in $[\alpha_\lambda, \sigma\lambda^{-1/2}]$, namely at least m pairs of solutions.

3.0.3. *Proof of Theorem 1.5.* To prove the estimates with respect to λ, observe that since $u_{\lambda,a}$ is a critical point of $\Phi_{\lambda,a}$ at level $d_{\lambda,a}$, for $\xi\in[4,\mu]$ we have

$$\begin{aligned}\lambda^{-1/2}\sigma &\geq \Phi_{\lambda,a}(u_{\lambda,a})-\frac{1}{\xi}\Phi'_{\lambda,a}(u_{\lambda,a})[u_{\lambda,a}]\\ &\geq \frac{\xi-2}{2\xi}\int_{\mathbb{R}^3}(|\nabla u_{\lambda,a}|^2+\lambda V(x)u_{\lambda,a}^2)+\lambda\frac{\xi-4}{4\xi}\int_{\mathbb{R}^3}Q(x)\phi^a_{u_{\lambda,a}}u_{\lambda,a}^2\\ &+\lambda\frac{\mu-\xi}{\xi}\int_{\mathbb{R}^3}H(x,u_{\lambda,a})+\lambda\frac{6-\xi}{6\xi}\int_{\mathbb{R}^3}K(x)|u_{\lambda,a}|^6\end{aligned}$$

and all the terms in the right hand side are positive. The conclusion follows by taking $\xi=4$ and $\xi=\mu$. The limits for $\lambda\to+\infty$ follow by the estimates just found and (v) of Lemma 1.2.

3.0.4. *Proof of Theorem 1.6.* This is indeed obvious, since all the previous proofs hold also when $a=0$ with very minor changes.

3.0.5. *Proof of Theorem 1.7.* Let us first recall the following general result. See [5, Lemma 6.1].

Lemma 3.2. *Consider* $f^0\in L^{6/5}(\mathbb{R}^3)$, $\{f^a\}_{a\in(0,1)}\subset L^{6/5}(\mathbb{R}^3)$ *and let*

$$\phi^0\in D^{1,2}(\mathbb{R}^3)\text{ be the unique solution of }-\Delta\phi=f^0\text{ in }\mathbb{R}^3$$

and

$$\phi^a\in\mathcal{D}\text{ be the unique solution of }-\Delta\phi+a^2\Delta^2\phi=f^a\text{ in }\mathbb{R}^3.$$

As $a\to0$ *we have:*

(i) if $f^a\rightharpoonup f^0$ *in* $L^{6/5}(\mathbb{R}^3)$, *then* $\phi^a\rightharpoonup\phi^0$ *in* $D^{1,2}(\mathbb{R}^3)$;
(ii) if $f^a\to f^0$ *in* $L^{6/5}(\mathbb{R}^3)$, *then* $\phi^a\to\phi^0$ *in* $D^{1,2}(\mathbb{R}^3)$ *and* $a\Delta\phi^a\to0$ *in* $L^2(\mathbb{R}^3)$.

From which we have the following

Lemma 3.3. *If for a certain* λ_* *we have that* $\{u_{\lambda_*,a}\}_{a\in(0,1)}$ *is a family of solutions of* (1.4) *and* $\lim_{a\to0}u_{\lambda_*,a}=u_{\lambda_*,0}$ *in* E, *then* $u_{\lambda_*,0}$ *is a solution of* (1.6).

Proof. By (ii) of the previous Lemma 3.2 we get $\phi^a_{u_{\lambda_*,a}}\to\phi^0_{u_{\lambda_*,0}}$ in $D^{1,2}(\mathbb{R}^3)$, which solves $-\Delta\phi^0_{u_{\lambda_*,0}}=Q(x)u^2_{\lambda_*,0}$. Finally take $\varphi\in C^\infty_0(\mathbb{R}^3)$. Passing to the limit as $a\to0$ in

$$\langle\phi^a_{u_{\lambda_*,a}},\varphi\rangle_{\lambda_*}+\lambda_*\int_{\mathbb{R}^3}Q(x)\phi^a_{u_{\lambda_*,a}}u_{\lambda_*,a}\varphi=\lambda_*\int_{\mathbb{R}^3}\Big(H(x,u_{\lambda_*,a})+K(x)|u_{\lambda_*,a}|^4u_{\lambda_*,a}\Big)\varphi$$

we easily get

$$\langle\phi^0_{u_{\lambda_*,0}},\varphi\rangle_{\lambda_*}+\lambda_*\int_{\mathbb{R}^3}Q(x)\phi^0_{u_{\lambda_*,0}}u_{\lambda_*,0}\varphi=\lambda_*\int_{\mathbb{R}^3}\Big(H(x,u_{\lambda_*,0})+K(x)|u_{\lambda_*,0}|^4u_{\lambda_*,0}\Big)\varphi$$

and this means that $(u_{\lambda_*,0},\phi^0_{u_{\lambda_*,0}})$ is a solution of (1.5) with parameter λ_*, or equivalently $u_{\lambda_*,0}$ is a solution of (1.6) with parameter λ_*. ☐

Now we can prove the convergence of the solutions. Given $\sigma\in(0,\widetilde{S})$, apply Corollary 2.9: fixed $\lambda_*\geq\Lambda_\sigma$ there is $\overline{e}_{\lambda_*}\in E$ such that

$$\Phi_{\lambda_*,1}(\overline{e}_{\lambda_*})\leq0\quad\text{ and }\quad\max_{t\in[0,1]}\Phi_{\lambda_*,1}(t\overline{e}_{\lambda_*})\leq\sigma\lambda_*^{-1/2}.$$

Let $d_{\lambda_*,1}$ be the Mountain Pass level of $\Phi_{\lambda_*,1}$ defined in (3.1). Then $d_{\lambda_*,1}\leq\sigma\lambda_*^{-1/2}<\widetilde{S}\lambda_*^{-1/2}$. Recall that $\Phi_{\lambda_*,1}$ satisfies the (PS) condition in $(0,\widetilde{S}\lambda_*^{-1/2})$.

Consider the family of solutions $\{u_{\lambda_*,a}\}_{a\in(0,1)}$ of (1.4), namely critical points of $\Phi_{\lambda_*,a}$ at the level Mountain Pass level $d_{\lambda_*,a}$ defined in (3.1). Standard computations give

$$\widetilde{S}\lambda_*^{-1/2} \geq \Phi_{\lambda_*,a}(u_{\lambda_*,a}) - \frac{1}{\mu}\Phi'_{\lambda_*,a}(u_{\lambda_*,a})[u_{\lambda_*,a}] \geq \frac{\mu-2}{2\mu}\|u_{\lambda_*,a}\|^2_{\lambda_*}.$$

The boundedness of $\{u_{\lambda_*,a}\}_{a\in(0,1)}$ implies the boundedness (with respect to a) of all the other terms in $\Phi_{\lambda_*,1}(u_{\lambda_*,a})$ and then, up to subsequences, as $a\to 0$ we have that

$$\Phi_{\lambda_*,1}(u_{\lambda_*,a}) \to c \leq d_{\lambda_*,1}.$$

Moreover

$$\Phi'_{\lambda_*,1}(u_{\lambda_*,a}) \to 0 \quad \text{as } a\to 0$$

as is easy to check on $\varphi\in C_c^\infty(\mathbb{R}^3)$. Then $\{u_{\lambda_*,a}\}_{a\in(0,1)}$ is a $(PS)_c$ sequence for $\Phi_{\lambda_*,1}$ and then (up to subsequences) is strongly convergent to some $u_{\lambda_*,0}$ in E. But then by Lemma 3.3, $u_{\lambda_*,0}$ is a solution of (1.6). Finally, since by Lemma 3.3 it is also $\phi^a_{u_{\lambda_*,a}} \to \phi^0_{u_{\lambda_*,0}}$ in $D^{1,2}(\mathbb{R}^3)$, we see that

$$\int_{\mathbb{R}^3} Q(x)\phi^a_{u_{\lambda_*,a}}u^2_{\lambda_*,a} \to \int_{\mathbb{R}^3} Q(x)\phi^0_{u_{\lambda_*,0}}u^2_{\lambda_*,0}$$

and then $\Phi_{\lambda_*,a}(u_{\lambda_*,a}) \to \Phi_{\lambda_*,0}(u_{\lambda_*,0})$.

4. Final remark

We conclude the paper with few words on the “symmetric case”. Assume that also (S0) holds. Then $\Phi_{\lambda,a}$ is τ-invariant, i.e., $\Phi_{\lambda,a}(\tau u) = \Phi_{\lambda,a}(u)$ and $\Phi'_{\lambda,a}(\tau u) = \tau\Phi'_{\lambda,a}(u)$ is $\Phi'_{\lambda,a}$ is τ-equivariant.

By Remark 3 the functional $\Phi_{\lambda,a}$ has the Mountain Pass structure on E^τ and since also the (PS) condition continues to hold in E^τ we can say that: for every $\sigma>0$ there is $\Lambda_\sigma>0$ such that for all $a>0, \lambda\geq\Lambda_\sigma$ there is a critical point $v_{\lambda,a}\in E^\tau$ of $\Phi_{\lambda,a}$ and $\Phi_{\lambda,a}(u_{\lambda,a})\leq \sigma\lambda^{-1/2}$. This allow us to recover all the previous results on the space E^τ.

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