

Y-PROPER GRADED COCHARACTERS OF THE ALGEBRA $UT_m(F)$ OF $m \times m$ UPPER TRIANGULAR MATRICES OVER F

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ABSTRACT. Let F be a field of characteristic 0 and let $UT_m(F)$ be the algebra of upper triangular matrices of order m with entries from F . In this paper we give a description of the Y -proper graded cocharacters of $UT_m(F)$ equipped with any elementary G -grading, where G is a finite group.

Keywords:

Y -proper graded cocharacters, Graded identities

doi.org/10.1016/j.jalgebra.2022.11.018

1. INTRODUCTION

The main goal of this paper is describing the space of multilinear graded polynomial identities, through the representation theory of the symmetric group, of one of the "critical" algebras appearing in the theory of algebras with polynomial identities (PI-algebras), the algebra $UT_m(F)$ of $m \times m$ upper triangular matrices with entries from a field F . From now on, any algebra mentioned herein is associative with unit while F will denote an algebraically closed field of characteristic 0.

The theory of algebras with polynomial identities has been extensively investigated since the first half of previous century (see for instance, [12], [22], [26], [27], [34] and [44]). It is well known that if F is a field of characteristic 0, then for a given algebra A the T -ideal $T(A)$ of the polynomial identities of A is generated by the multilinear ones. Let P_n denote the set of all multilinear polynomials in the variables x_1, \dots, x_n . For any $n \in \mathbb{N}$, let us consider the factor space

$$P_n(A) = \frac{P_n}{P_n \cap T(A)}.$$

The n -th codimension of A , denoted by $c_n(A)$, is the dimension of $P_n(A)$ as a vector space over F . Moreover, since $P_n(A)$ is a left S_n -module we shall denote its character by $\chi_n(A)$ and call it the n -th cocharacter of A . As a matter of fact, is more efficient to study $P_n(A)$ instead of $P_n \cap T(A)$ as showed by Regev in [39] where he proved if A is a PI -algebra, then its codimension sequence is exponentially bounded whereas $T(A) \cap P_n$ grows factorially.

Given a PI -algebra A , one may consider the limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)}.$$

Giambruno and Zaicev proved that this limit always exists and it is a non-negative integer called PI -exponent of A and denoted by $exp(A)$ (see for instance [25]).

Recall that the variety generated by an algebra A is the class

$$\mathcal{V}(A) = \{B \text{ associative algebra} \mid T(A) \subseteq T(B)\}.$$

In the language of varieties, one defines the exponent of $\mathcal{V}(A)$ as $exp(\mathcal{V}(A)) := exp(A)$. A variety \mathcal{V} is said to be *minimal* with respect to its exponent if and only if for any proper subvariety \mathcal{U} of \mathcal{V} one has $exp(\mathcal{U}) < exp(\mathcal{V})$. Accordingly, we say that a PI -algebra is minimal if it generates a minimal variety.

2020 *Mathematics Subject Classification.* 16R10, 16R40.

Key words and phrases. graded polynomial identities, upper triangular matrices.

T. F. da Silva was partially supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior -Brasil (CAPES) -Finance Code 001.

In this setting appears the algebra $UT_m(F)$ which plays an important role in the structure theory of PI -algebras. We have two remarkable reasons in mind to make this claim: the first one is that the algebras $UT_m(F)$ generate minimal varieties with a fixed exponent (see [17] for more details); the second one is due to the fact that the identities of $UT_m(F)$ may serve as a measure of the complexity of the polynomial identities of finitely generated algebras with a non-matrix identity, an identity which does not hold for the algebra $M_2(F)$, in the same way as the polynomial identities of $M_m(F)$ measure the complexity of the identities of arbitrary PI -algebras. Namely if A is an algebra satisfying a non-matrix polynomial identity, there exists a positive integer m such that

$$T(F)^m \subseteq T(A) \subseteq T(F),$$

where $T(F)$ is the commutator ideal of $F\langle X \rangle$ generated by $[F\langle X \rangle, F\langle X \rangle]$ (see [31]). Maltsev proved in [35] that

$$T(F)^m = T(UT_m(F)),$$

where $T(UT_m(F))$ denotes the T -ideal of polynomial identities of the upper triangular matrices over F . It was proved in [23] and [24] by Genov and [32] by Latyshev that every algebra satisfying the identities of $UT_m(F)$ has the finite basis property of its polynomial identities. Later, Latyshev in [33] and Popov in [37] generalized this result for PI -algebras satisfying the identity

$$[x_1, x_2, x_3] \cdots [x_{3m-2}, x_{3m-1}, x_{3m}]$$

which generates the T -ideal of $UT_m(E)$ (see also the paper [18] by Drensky).

It is worth mentioning that until Kemer developed his deep structure theory about T -ideals for resolving positively the Specht problem in characteristic 0 ([28]), the results by Genov, Latyshev and Popov together with the finite basis of the polynomial identities of $M_2(F)$ found by Razmyslov [38] covered all known results of varieties of PI -algebras with the finite basis property.

Because finding out an explicit set of polynomial identities of a given algebra is a very hard task, sometimes it is useful weakening the definition of polynomial identity adding an additional structure on the algebra. We can think to graded algebras, H -module algebras, algebras with involution, etc. Here we deal with graded identities. It turns out that many properties of the T -ideals of polynomials identities can be described in terms of graded identities and graded algebras (see for example [25] and [20]). For instance Kemer, in his remarkable work about the structure theory of T -ideals, used \mathbb{Z}_2 -graded algebras in an essential way showing also the relevance of their graded polynomial identities (see [28]). Notice also the multiplicities of the graded cocharacter sequence of a PI -algebra provides an upper bound for the multiplicities of the ordinary cocharacter sequence; in a similar way, the graded codimensions sequence of a PI -algebra provides an upper bound of the ordinary sequence.

As mentioned above, here we are interested in the study of graded cocharacters of $UT_m(F)$. We recall the explicit knowledge of its multiplicities is well known only when $m = 2$. Recently, in 2012 (see [4]), Boumova and Drensky provided an algorithm which calculates the generating function of the cocharacter sequence of $UT_m(F)$.

The main tool used in this paper is the algebra of Y -proper polynomials. This approach in the ordinary (ungraded) case was started by Specht in 1950. In the early 1980s Drensky also started using (ordinary) proper polynomials in a quantitative way to handle the T -ideals of free algebras, see [20] for a detailed exposition. In what follows, when dealing with G -graded algebras one may consider the free algebra $F\langle Y \cup Z \rangle$, where the Y 's are the variables of degree 1_G and the Z 's are the variables of homogeneous degree $g \neq 1_G$. Since proper polynomials are linear combinations of products of commutators, a polynomial $f \in F\langle Y \cup Z \rangle$ is Y -proper if all the y 's occurring in f appear in commutators only. It is well known that all graded polynomial identities of a superalgebra follow from the Y -proper ones (see for instance [14]). In [15], Di Vincenzo, Koshlukov and Valenti obtained a complete description of the Y -proper polynomials in the relatively free G -graded algebra of $UT_m(F)$, where G is a finite group. Moreover, we mention that Koshlukov and Valenti in [29] found a multilinear basis of the relatively free \mathbb{Z}_m -graded algebra of $UT_m(F)$.

In [9], L. Centrone and A. Cirrito gave a combinatorial method in order to compute the exact value of the Y -proper graded cocharacter sequence of $UT_m(F)$ endowed with the \mathbb{Z}_m -grading induced by the m -tuple $\varepsilon = (0, \dots, 0, 1, \dots, m - j)$, $j = 1$, that is with the Vasilovsky grading (see [42]). In particular, they showed the exact value of its multiplicities for upper triangular matrices of small size, i.e., $m = 2, 3, 4$. Shortly

afterwards in 2015, see [10], the same authors generalized these results for the cases $m = 2, 3, 4, 5$ and $j = 2$. Following this line of research, in this paper we are able to give a whole generalization of the aforementioned results. More precisely, we obtain a description of the Y -proper graded cocharacters of the algebra $UT_m(F)$ with the elementary G -grading induced by any m -tuple of type $(1_G^{l_1}, g_2^{l_2}, \dots, g_L^{l_L})$, where 1_G is the neutral element of G and g_2, \dots, g_L belong to G . We also give information about the rate of growth of homogeneous polynomial identities of UT_m through the Gelfand-Kirillov dimension of its relatively free graded algebra in a finite set of variables and we conjecture it is independent of the grading.

2. G -GRADED PI -ALGEBRA AND ITS RELATED GRADED STRUCTURES

Throughout this section F will be a field of characteristic 0 and all algebras we refer to will be associative with unity. In what follows, we introduce the key tools that will be used in this paper.

Definition 1. Let G be a group and A be an algebra over F . A G -grading on A is defined to be a decomposition of A into a direct sum of F -vector subspaces $A = \bigoplus_{g \in G} A^g$ such that $A^g A^h \subseteq A^{gh}$ for all $g, h \in G$.

Let A be a G -graded algebra. The vector subspaces A^g are called the G -homogeneous components of A , A^{1_G} is called the *neutral component*, where 1_G denote the neutral element of G . An element $a \in A$ is said to be *homogeneous of G -degree g* , and we denote it by $\|a\| = g$ (or $\deg a = g$), if there exists $g \in G$ such that $a \in A^g$. We always write a^g in case $a \in A^g$. It is easy to check that any element $a \in A$ can be uniquely written as a finite sum $a = \sum_{g \in G} a^g$, where $a^g \in A^g$.

Let B be a subspace of A . We say B is a G -graded subspace of A if $B = \bigoplus_{g \in G} (B \cap A^g)$. Analogously one defines G -graded subalgebras, ideals, and so on.

Let G be a group and let $X = \bigcup X^g$ be the union of the disjoint countable sets $X^g = \{x_1^g, x_2^g, \dots\}$, $g \in G$, of non-commutative variables. The free associative algebra freely generated by X over F , denoted by $F\langle X|G \rangle$, can be equipped in a natural way with a structure of G -graded algebra. Namely, set the homogeneous degree of x_i^g $\|x_i^g\| = g$ for every $x_i^g \in X^g$, then extend this grading to all monomials $m = x_{i_1}^{g_1} x_{i_2}^{g_2} \dots x_{i_r}^{g_r}$ in $F\langle X|G \rangle$, thus $\|m\| = \|x_{i_1}^{g_1}\| \|x_{i_2}^{g_2}\| \dots \|x_{i_r}^{g_r}\| = g_1 g_2 \dots g_r$. Let $F\langle X|G \rangle^g$ be the F -vector space generated by all monomials of G -degree g . Notice that

$$F\langle X|G \rangle = \bigoplus_{g \in G} F\langle X|G \rangle^g$$

and $F\langle X|G \rangle^g F\langle X|G \rangle^h \subseteq F\langle X|G \rangle^{gh}$ for all $g, h \in G$. Hence $F\langle X|G \rangle$ is a G -graded algebra and we shall refer to its elements as G -graded polynomials or, simply, *graded polynomials*.

Definition 2. Let A be a G -graded algebra and $f(x_1, \dots, x_n) \in F\langle X|G \rangle$ a graded polynomial. If

$$f(a_1, \dots, a_n) = 0$$

for all $a_1, \dots, a_n \in \bigcup_{g \in G} A^g$, such that $a_k \in A^{\deg x_k}$ for each $k = 1, \dots, n$, then $f(x_1, \dots, x_n)$ is said to be a G -graded polynomial identity of A . In this case we also write $f \equiv 0$ on A .

Given a G -graded algebra A , we can consider the set $T_G(A)$ of all graded polynomial identities of A . It is easy to check that $T_G(A)$ is an ideal invariant under all graded F -endomorphisms of $F\langle X|G \rangle$. The ideals of $F\langle X|G \rangle$ with this property are called T_G -ideals.

It is a worth mentioning that every T_G -ideal of $F\langle X|G \rangle$ is a $T_G(A)$ -ideal of some G -graded algebra A .

The development of the theory of G -graded PI -algebra passes through the representation theory of the symmetric group S_n . We start off defining the following object:

$$P_n^G = \text{span}\langle x_{\sigma(1)}^{g_1} \dots x_{\sigma(n)}^{g_n} \mid g_i \in G, \sigma \in S_n \rangle.$$

The elements of P_n^G are called G -graded multilinear polynomials of degree n .

Indeed, P_n^G has a structure of a left S_n -module under the natural left S_n -action and the same holds for $P_n^G \cap T_G(A)$. Hence the factor module

$$P_n^G(A) := \frac{P_n^G}{P_n^G \cap T_G(A)}$$

inherits a structure of a left S_n -module.

As well as in the ordinary case, when the ground field is of characteristic zero, the graded polynomial identities follow from the multilinear ones (see for instance [20] and [25]).

Definition 3. Let A be a G -graded PI -algebra. The n -th G -graded *cocharacter*, denoted by $\chi_n^G(A)$, and *codimension*, denoted by $c_n^G(A)$, of A are defined as the S_n -character and dimension (over F), respectively, of $P_n^G(A) := P_n^G / (P_n^G \cap T_G(A))$. Moreover,

$(\chi_n^G(A))_{n \in \mathbb{N}}$ is called the G -graded cocharacter sequence of A ;

$(c_n^G(A))_{n \in \mathbb{N}}$ is called the G -graded codimension sequence of A .

According to the work [13] by Di Vincenzo, in order to describe the left S_n -module $P_n^G(A)$, it is sufficient focus on smaller left S_n -submodules. Let $G = \{g_1, \dots, g_r\}$ be a finite group, then for any $l_{g_1}, \dots, l_{g_r} \in \mathbb{N}$, let us consider the space $P_{l_{g_1}, \dots, l_{g_r}}^G$ of multilinear polynomials in the indeterminates labeled as follows:

$$x_1^{g_1}, \dots, x_{l_{g_1}}^{g_1}, x_{l_{g_1}+1}^{g_2}, \dots, x_{l_{g_1}+l_{g_2}}^{g_2}, \dots$$

The space $P_{l_{g_1}, \dots, l_{g_r}}^G$ is naturally endowed with a structure of a left $(S_{l_{g_1}} \times \dots \times S_{l_{g_r}})$ -module in the following way: the group $S_{l_{g_1}} \times \dots \times S_{l_{g_r}}$ acts on the left on $P_{l_{g_1}, \dots, l_{g_r}}^G$ by permuting the variables of the same homogeneous degree, i.e., $S_{l_{g_1}}$ permutes the variables of homogeneous degree g_1 , $S_{l_{g_2}}$ permutes those of homogeneous degree g_2 and so on. Moreover such action is inherited by $P_{l_{g_1}, \dots, l_{g_r}}^G \cap T_G(A)$ and it induces a structure of a left $(S_{l_{g_1}} \times \dots \times S_{l_{g_r}})$ -module on the factor

$$P_{l_{g_1}, \dots, l_{g_r}}^G(A) = \frac{P_{l_{g_1}, \dots, l_{g_r}}^G}{(P_{l_{g_1}, \dots, l_{g_r}}^G \cap T_G(A))}.$$

We shall denote by $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A)$ and by $c_{l_{g_1}, \dots, l_{g_r}}^G(A)$ the character and the dimension, respectively, of $P_{l_{g_1}, \dots, l_{g_r}}^G(A)$. We have the following result due to Di Vincenzo (see [13]).

Theorem 4. Let A be G -graded algebra with $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A)$ graded cocharacter sequences . Then,

$$\chi_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \chi_{l_{g_1}, \dots, l_{g_r}}^G(A)^{\uparrow S_n}.$$

Moreover

$$c_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \binom{n}{l_{g_1}, \dots, l_{g_r}} c_{l_{g_1}, \dots, l_{g_r}}^G(A)^{\uparrow S_n}.$$

As mentioned in the introduction, in this paper we will focus on the so called *Y-proper graded cocharacters*.

From now on, when dealing with a G -graded algebra, we shall adopt the following notation: Y stands for X^{1_G} the set of variables of homogeneous degree 1_G , and $Z = \bigcup_{g \in G, g \neq 1} X^g$. Then we shall write $F\langle Y \cup Z \rangle$ instead of $F\langle X|G \rangle$. In this case, we shall write: y_i for the variables in Y and we refer to as *even variables*; z_j for the variables in Z and we refer to as *odd variables*. The *Y-proper polynomials* are defined as the elements of the *F-unitary subalgebra* B of $F\langle Y \cup Z \rangle$ generated by the elements of Z and by all non-zero commutators. It turns out that if $f \in F\langle Y \cup Z \rangle$ is a Y -proper polynomial, then all $y \in Y$ occurring in f appear in commutators only. It was proved in [14] that the whole $T_G(A)$ is generated as a T_G -ideal by the set $T_G(A) \cap B$, i.e., the graded polynomials follows from the Y -proper ones. We refer to the quotient algebra $B(A) := B / (T(A) \cap B)$ as the *Y-proper relatively free algebra of A*.

Let us denote by Γ_n^G the vector space of all multilinear Y -proper polynomials of P_n^G . We get Γ_n^G is a left S_n -submodule of P_n^G and the same holds for $\Gamma_n^G \cap T_G(A)$. Accordingly, we have a structure of a left S_n -submodule of $P_n^G(A)$ on the factor module

$$\Gamma_n^G(A) = \frac{\Gamma_n^G}{(\Gamma_n^G \cap T_G(A))}.$$

Definition 5. Let A be a G -graded PI -algebra. The n -th Y -proper G -graded cocharacter, denoted by $\xi_n^G(A)$, and codimension, denoted by $\gamma_n^G(A)$, of A are defined as the S_n -character and dimension (over F), respectively, of $\Gamma_n^G(A) = \Gamma_n^G / (\Gamma_n^G \cap T_G(A))$. Moreover,

$(\xi_n^G(A))_{n \in \mathbb{N}}$ is called the G -graded proper cocharacter sequence of A ;

$(\gamma_n^G(A))_{n \in \mathbb{N}}$ is called the G -graded proper codimension sequence of A .

Now let us denote by $\Gamma_{m_1, \dots, m_r}^G$ the set of all multilinear Y -proper polynomials of P_{n_1, \dots, n_r}^G such that $\sum_{i=1}^r n_i = n$. Likewise, $\Gamma_{n_1, \dots, n_r}^G$ is a left $(S_{n_1} \times \dots \times S_{n_r})$ -submodule of P_{n_1, \dots, n_r}^G and the same holds for $\Gamma_{n_1, \dots, n_r}^G \cap T_G(A)$. Hence we have a structure of a left $(S_{n_1} \times \dots \times S_{n_r})$ -submodule of $P_n^G(A)$ on the factor module

$$\Gamma_{n_1, \dots, n_r}^G(A) = \frac{\Gamma_{n_1, \dots, n_r}^G}{(\Gamma_{n_1, \dots, n_r}^G \cap T_G(A))}.$$

We shall denote the $(S_{n_1} \times \dots \times S_{n_r})$ -character and the $(S_{n_1} \times \dots \times S_{n_r})$ -codimension of $\Gamma_{n_1, \dots, n_r}^G(A)$ by $\xi_{n_1, \dots, n_r}^G(A)$ and $\gamma_{n_1, \dots, n_r}^G(A)$, respectively. When we refer to A without any ambiguity we simply write ξ_{n_1, \dots, n_r}^G and $\gamma_{n_1, \dots, n_r}^G$ instead of $\xi_{n_1, \dots, n_r}^G(A)$ and $\gamma_{n_1, \dots, n_r}^G(A)$, respectively.

Following word by word the proof of Di Vincenzo in [13], we have the following version of the Theorem 4 for the Y -proper graded cocharacters and codimensions.

Theorem 6. Let A be a PI G -graded algebra with graded cocharacter sequence $\xi_{l_{g_1}, \dots, l_{g_r}}^G(A)$. Then,

$$\xi_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \xi_{l_{g_1}, \dots, l_{g_r}}^G(A)^{\uparrow S_n}.$$

Moreover

$$\gamma_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \binom{n}{l_{g_1}, \dots, l_{g_r}} \gamma_{l_{g_1}, \dots, l_{g_r}}^G(A)^{\uparrow S_n}.$$

Let $n \geq 1$ be a non-negative integer. Remind that a partition of n is an r -tuple $\lambda = (\lambda_1, \dots, \lambda_r)$ of integers such that

$$\lambda_1 \geq \dots \geq \lambda_r \text{ and } \lambda_1 + \dots + \lambda_r = n.$$

We shall use the notation $\lambda \vdash n$ to say λ is a partition of n and $|\lambda| = r$ to say that λ is a partition of height r .

We say that two partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ are equal if $r = s$ and $\lambda_1 = \mu_1, \dots, \lambda_r = \mu_s$.

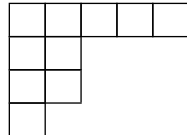
When $\lambda = (\lambda_1, \dots, \lambda_{k_1}, \dots, \lambda_{k_p})$ and

$$\lambda_1 = \dots = \lambda_{k_1} = \mu_1, \dots, \lambda_{k_1 + \dots + k_{p-1} + 1} = \dots = \lambda_{k_1 + \dots + k_p} = \mu_p,$$

we shall adopt the notation

$$\lambda = (\mu_1^{k_1}, \dots, \mu_p^{k_p}).$$

Consider a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of a certain non-negative integer. We associate to λ its *Young Diagram* $[\lambda]$ having r rows such that the i -th row contains λ_i square. For instance, if $\lambda = (4, 2, 2, 1)$ we have



We shall denote the "strip" partition (k) by \boxed{k} .

Recall from the classical representation theory of groups that if G is a finite group and F is a field of characteristic 0 or $p \nmid |G|$, then the *Maschke's Theorem* says that every finite dimensional representation is completely reducible, i.e., the group algebra $F[G]$ is semisimple and isomorphic to the direct sum of matrix algebras with entries from a division ring. Moreover, every finite dimensional left G -module is a direct

sum of irreducible G -modules that are isomorphic to a minimal left ideals of $F[G]$. Putting $G = S_n$, the left irreducible S_n -modules, and therefore their related irreducible characters, may be described in terms of partitions and Young Diagrams. Indeed, this applies for ordinary, graded and Y -proper cocharacters of PI -algebras. For simplicity we shall denote the irreducible S_n -character related to the partition λ by its Young diagram $[\lambda]$. When no ambiguity occurs, we shall denote the irreducible character associated to λ by λ as well. We shall call *arm* any horizontal strip of a partition. We say a cocharacter sequence has l arms of infinite width if in that sequence we can always find partitions $\lambda(n) = (\lambda_1(n), \lambda_2(n), \dots, \lambda_l(n), \lambda_{l+1}(n), \dots)$ such that the sequences of integers $\{\lambda_i(n)\}$ are not limited if $i \leq l$ but $\{\lambda_{l+1}(n)\}$ and l is the maximal integer with this property.

We come back to polynomial identities for a while. Let S be any subset of $F\langle X|G \rangle$ and denote by $\langle S \rangle^{T_G}$ the smallest T_G -ideal containing the set S (the T_G -ideal generated by S). The reason to consider Y -proper polynomials is the following: the proper polynomials in a T_G -ideal uniquely determine it. More explicitly, we have the following result.

Proposition 7. *Let I be the T_G -ideal of G -graded identities of a unitary G -graded algebra A over an infinite field, then $(B_Y \cap I)^{T_G} = I$.*

The Y -proper graded-codimensions (Y -proper graded-cocharacters) are strictly related to graded-codimensions (graded-cocharacters). For we state the next result which proof can be found for instance in the book by Drensky (Theorem 4.3.12 for codimensions and Theorem 12.5.4 for cocharacters) and can be written verbatim in the language of graded-polynomials via the link of Proposition 7.

Theorem 8. *Let A be a unitary G -graded algebra satisfying a graded polynomial identity and let $\chi_n^G(A) = \sum_{\lambda \vdash n} m_\lambda^G(A) \chi_\lambda$, $c_n^G(A)$ and $\xi_n^G(A) = \sum_{\lambda \vdash n} k_\nu^G(A) \chi_\nu$, $c_{n,Y}^G(A)$ be, respectively, the n -th H -cocharacter and codimension and the Y -proper H -cocharacter and codimension of the T_G -ideal $T_G(A)$. Then the multiplicities $m_\lambda^G(A)$ and $k_\nu^G(A)$ are related by*

$$m_\lambda^G(A) = \sum k_\nu^G(A),$$

where for $\lambda = (\lambda_1, \dots, \lambda_n)$ the summation runs on all partitions $\nu = (\nu_1, \dots, \nu_n)$ such that $\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \dots \geq \lambda_n \geq \nu_n$. Moreover

$$c_n^G(A) = \sum_{k \leq n} \binom{n}{k} \gamma_k^G(A).$$

3. GRADINGS AND GRADED IDENTITIES OF $UT_m(F)$

In this section, G will denote a finite group. We shall recall some results about the graded polynomial identities of $UT_m(F)$ equipped with the so-called *elementary G -grading* appearing in the references [15] and [16]. Moreover, we shall present the main ingredients in the description of Y -proper cocharacters.

Definition 9. Let G be a (possibly infinite) group and consider the algebra $UT_m(F)$ of $m \times m$ upper triangular matrices with entries from F . Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ be an m -tuple of elements of G . For every $k = 1, \dots, m$, let

$$UT_m(F)^g = \text{span} \left\{ e_{ij} \mid \varepsilon_i \varepsilon_j^{-1} = g, \quad i \leq j \right\}.$$

It turns out that $UT_m(F) = \bigoplus_{g \in G} UT_m(F)^g$, then $UT_m(F)$ is G -graded by the so called *elementary grading* induced by ε . Here e_{ij} denote the matrix units whose (i, j) -th entry equals 1 and all other entries equal 0.

Zaicev and Valenti proved, firstly for finite abelian groups and algebraically closed fields and shortly afterwards for the general case, that every grading on $UT_m(F)$ is isomorphic to an elementary one (see [40],[41], for more details).

From now on, we shall focus on the upper triangular matrices endowed with any elementary G -grading, where G is a finite group. Let us consider the following definitions (see [15]).

Definition 10. We call *normal* any Y -commutator c of Z -degree at most 1 such that $c = z$ or $c = [z, y_{i_1}, \dots, y_{i_t}]$ for some $t \geq 1$. Moreover we say:

- (1) a normal commutator $[y_{j_1}, y_{j_2}, \dots, y_{j_p}]$ of Z -degree 0 is *semistandard* if the indexes j_1, j_2, \dots, j_p satisfy the inequalities $j_1 \geq j_2 \geq \dots \geq j_p$;

- (2) a normal commutator $[z_{j_1}, y_{j_2}, \dots, y_{j_p}]$ of Z -degree 1 and length $p \geq 1$ is *semistandard* if the indexes j_2, \dots, j_p satisfy the inequalities $j_2 \leq \dots \leq j_p$.

Definition 11. Let $\tilde{\eta} = (\eta_1, \dots, \eta_n)$ be a sequence of elements of G . We say that $\tilde{\eta}$ is an ε -good sequence of $(UT_m(F), \varepsilon)$ if there exists a sequence of n matrix units (r_1, \dots, r_n) in the Jacobson radical of $UT_m(F)$ such that the product $r_1 \cdots r_n$ is not zero and also the homogeneous degree of r_i is η_i for all $i = 1, \dots, n$. Otherwise we say that $\tilde{\eta}$ is an ε -bad sequence.

Now, let us show the notations we will use further: given a sequence $\tilde{\eta} = (\eta_1, \dots, \eta_n)$ of elements of G , we consider the polynomial

$$f_{\tilde{\eta}} = f_{\tilde{\eta},1} \cdots f_{\tilde{\eta},n},$$

where $f_{\tilde{\eta},i} := [y_{2i-1}, y_{2i}]$ if $\eta_i = 1_G$ and $f_{\tilde{\eta},i} := x_i^{\eta_i}$ if $\eta_i \neq 1_G$.

Moreover, let us consider a product of the type $c = c_1 \cdots c_n$, where each c_i is a normal commutator. We shall consider the sequence $\tilde{\eta}_c = (\eta_1, \dots, \eta_n) \in G^n$, where $\eta_i = \deg_G c_i$, for all $i = 1, \dots, n$. It is easy to see that, if the sequence $\tilde{\eta}_c$ is ε -bad, then the product $c = c_1 \cdots c_n$ is a G -graded polynomial identity of $(UT_m(F), \varepsilon)$.

We have the following results which are true over infinite fields (see [15]).

Theorem 12. Let G be a group and let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ be an elementary G -grading on the algebra $UT_m(F)$ of $m \times m$ upper triangular matrices over an infinite field F . Then

- (1) The ideal $T_G(UT_m(F), \varepsilon)$ is generated by the multilinear polynomials $f_{\tilde{\eta}}$, where $\tilde{\eta} = (\eta_1, \dots, \eta_n)$ is ε -bad and $n \leq m$.
- (2) A linear basis for the Y -proper polynomials in the relatively free graded algebra $F\langle X \rangle / T_G(UT_m(F), \varepsilon)$ consists of 1 and of the polynomials $c = c_1 \cdots c_n$, where each c_i is a semistandard commutator and the sequence $\tilde{\eta}_c$ is ε -good.

Definition 13. Given an ε -grading on the algebra $UT_m(F)$. We define the *diagonal sequence* associated to ε as the sequence $d(\varepsilon) = (\eta_1, \dots, \eta_{m-1})$, where $\eta_i = \varepsilon_i^{-1} \varepsilon_{i+1} = \deg e_{i,i+1}$ for all $i = 1, \dots, m$.

The following result is a useful tool to find out the ε -good sequences in $(UT_m(F), \varepsilon)$ (see the details in [15]).

Theorem 14. Let $d(\varepsilon) = (\eta_1, \dots, \eta_{m-1})$ be the diagonal sequence associated to a given ε -grading on $UT_m(F)$. If $\tilde{\gamma} = (\gamma_1, \dots, \gamma_n)$ be a sequence of elements of G , then $\tilde{\gamma}$ is ε -good if and only if there exists $n+1$ positive integers $1 \leq t_1 \leq \dots \leq t_{n+1} \leq m$ such that

$$\gamma_i = \eta_{t_i} \cdots \eta_{t_{i+1}-1}$$

for all $i = 1, \dots, n$.

Remark 15. Given a sequence $\tilde{\gamma} = (\gamma_1, \dots, \gamma_n)$ of elements of G , we say that n is the *length* of $\tilde{\gamma}$. Notice that the diagonal sequence $d(\varepsilon) = (\eta_1, \dots, \eta_{m-1})$ is the unique ε -good sequence of length $m-1$. Actually, according to Theorem 12, the diagonal sequence $d(\varepsilon)$ is the longest ε -good sequence in the ε -grading.

We shall consider the m -tuple $(g_1^{l_1}, g_2^{l_2}, \dots, g_L^{l_L})$, where $l_1 = \max\{l_1, \dots, l_L\}$ and $g_1 = 1_G$. Moreover, in what follows we shall write $[Y]$ to denote the multilinear commutator of Z -degree 0, and $[z, Y]$ to denote the multilinear commutator of Z -degree 1 with the variable z occurring in the first entry of the commutator. In both cases the indexes of the y 's are not necessarily ordered.

Lemma 16. Let us consider $(UT_m(F), \varepsilon)$, where $\varepsilon = (g_1^{l_1}, g_2^{l_2}, \dots, g_L^{l_L})$. Then

$$\underbrace{[Y] \cdots [Y]}_s [z, Y] \underbrace{[Y] \cdots [Y]}_r$$

is a graded identity of $(UT_m(F), \varepsilon)$ if $s \geq l_1$ or $r \geq l_1$.

Proof. Remark that

$$d(\varepsilon) = (\underbrace{1_G, \dots, 1_G}_{l_1-1}, g_1^{-1} g_2, \underbrace{1_G, \dots, 1_G}_{l_2-1}, g_2^{-1} g_3, \dots, g_{L-1}^{-1} g_L, \underbrace{1_G, \dots, 1_G}_{l_L-1}).$$

Now consider the following types of sequences

$$\alpha_{s,r} = (\underbrace{1_G, \dots, 1_G}_s, g, \underbrace{1_G, \dots, 1_G}_r),$$

where $g \neq 1_G$ and $s \geq l_1$ or $r \geq l_1$. Notice that there are no positive integers $1 \leq t_1 \leq \dots \leq t_{(s+r+1)+1} \leq m$ satisfying the hypothesis of Theorem 14. Hence $\alpha_{s,r}$ is an ε -bad sequence and the claim follows. \square

In what follows we shall use the next well known fact.

Lemma 17. *Let A be an algebra and let $a, b, x_1, \dots, x_n \in A$. Then*

$$[ab, x_1, \dots, x_n] = \sum [a, x_{i_1}, \dots, x_{i_k}] [b, x_{j_1}, \dots, x_{j_s}],$$

where $x_{i_1}, \dots, x_{i_k}, x_{j_1}, \dots, x_{j_s} \in \{x_1, \dots, x_n\}$ and the commutators $[a, x_{i_1}, \dots, x_{i_k}]$ and $[b, x_{j_1}, \dots, x_{j_s}]$ are allowed to be of length 1, too.

The following proposition is one of the key tools in the description of the Y -proper graded cocharacters of $UT_m(F)$. Moreover, we shall use the symbol $\equiv_{T_G(UT_m(F))}$ to denote the equivalence modulo $T_G(UT_m(F))$.

Proposition 18. *Let $UT_m(F)$ be equipped with the elementary G -grading induced by the m -tuple $\varepsilon = (g_1^{l_1}, g_2^{l_2}, \dots, g_L^{l_L})$. Then*

$$i) [y_{i_1}, \dots, y_{i_h}, z, y_{j_1}, \dots, y_{j_k}] \equiv_{T_G(UT_m(F))} \sum_{i \in I} \alpha_i f_i, \text{ where } I \text{ is a finite set of indexes, } h \geq 0, k \geq 2, \\ \alpha_i \in F \text{ and } f_i = \underbrace{[Y] \cdots [Y]}_{< l_1} [z, Y] \underbrace{[Y] \cdots [Y]}_{< l_1}.$$

$$ii) [z, y_{i_1}, \dots, y_{i_h}, y_2, y_1, y_{j_1}, \dots, y_{j_k}] \equiv_{T_G(UT_m(F))} [z, y_{i_1}, \dots, y_{i_h}, y_1, y_2, y_{j_1}, \dots, y_{j_k}] + \sum_{i \in I} \alpha_i f_i, \\ \text{where } i_1, \dots, i_h \text{ are not necessarily ordered indexes, } h, k \geq 0, \alpha_i \in F, I \text{ is a finite set of indexes} \\ \text{and } f_i = \underbrace{[Y] \cdots [Y]}_{< l_1} [z, Y] \underbrace{[Y] \cdots [Y]}_{< l_1}.$$

iii)

$$[z, y_{i_1}, \dots, y_{i_h}, \bar{y}_2, y_{j_1}, \dots, y_{j_l}, \bar{y}_1, y_{t_1}, \dots, y_{t_m}] \equiv_{T_G(UT_m(F))} \sum_{i \in I} \alpha_i f_i,$$

where $h, l, m \geq 0$, I is a finite set of indexes, $h \geq 0, k \geq 2, \alpha_i \in F$ and $f_i = \underbrace{[Y] \cdots [Y]}_{< l_1} [z, Y] \underbrace{[Y] \cdots [Y]}_{< l_1}$.

Proof. i) If $h = 0, 1$ and $k \geq 0$, then the result follows trivially. If $h \geq 2$, let $c = [y_{i_1}, y_{i_2}, \dots, y_{i_h}]$ and $k = 0$, then $[c, z] = cz - zc$ and we are done. Now suppose $k \geq 1$ and we apply the Jacobi identity. Then, using the linearity of the commutator, we get

$$\begin{aligned} [c, z, y_{j_1}, y_{j_2}, \dots, y_{j_k}] &= -[z, y_{j_1}, c, y_{j_2}, \dots, y_{j_k}] - [y_{j_1}, c, z, y_{j_2}, \dots, y_{j_k}] \\ &= [c, y_{j_1}, z, y_{j_2}, \dots, y_{j_k}] - [[z, y_{j_1}], c, y_{j_2}, \dots, y_{j_k}] \\ &= [[c, y_{j_1}], z, y_{j_2}, \dots, y_{j_k}] + [c, [z, y_{j_1}], y_{j_2}, \dots, y_{j_k}] \\ &= [[c, y_{j_1}]z - z[c, y_{j_1}], y_{j_2}, \dots, y_{j_k}] + [c[z, y_{j_1}] \\ &\quad - [z, y_{j_1}]c, y_{j_2}, \dots, y_{j_k}] \\ &= [[c, y_{j_1}]z, y_{j_2}, \dots, y_{j_k}] - [z[c, y_{j_1}], y_{j_2}, \dots, y_{j_k}] \\ &\quad + [c[z, y_{j_1}], y_{j_2}, \dots, y_{j_k}] - [[z, y_{j_1}]c, y_{j_2}, \dots, y_{j_k}] \end{aligned}$$

Now, applying Lemma 17 in the last equality we get

$$\begin{aligned} [c, z, y_{j_1}, y_{j_2}, \dots, y_{j_k}] &= \sum [[c, y_{j_1}], \dots, [z, \dots]] - \sum [z, \dots] [[c, y_{j_1}], \dots] \\ &\quad + \sum [c, \dots] [[z, y_{j_1}], \dots] - \sum [[z, y_{j_1}], \dots] [c, \dots] \\ &= \sum [[c, y_{j_1}], \dots, [z, \dots]] + \sum [y_{j_t}, z, \dots] [[c, y_{j_1}], \dots] \\ &\quad + \sum [c, \dots] [z, y_{j_1}, \dots] + \sum [y_{j_1}, z, \dots] [c, \dots]. \end{aligned}$$

Since $h \geq 2$, the commutators $[z, \dots], [y_{j_t}, z, \dots], [z, y_{j_1}, \dots]$ and $[y_{j_1}, z, \dots]$ are of length strictly less than that of $[c, z, y_{j_1}, y_{j_2}, \dots, y_{j_k}] = [y_{i_1}, \dots, y_{i_h}, z, y_{j_1}, \dots, y_{j_k}]$, then the required result follows by using an induction process jointly with Lemma 16.

ii) We will apply an induction process on the length of $[z, y_{i_1}, \dots, y_{i_h}, y_2, y_1, y_{j_1}, \dots, y_{j_k}]$.

First suppose $k = 0$ and $h \geq 0$. Putting $c = [z, y_{i_1}, \dots, y_{i_h}]$, we obtain, by using the Jacobi identity,

$$[z, y_{i_1}, \dots, y_{i_h}, y_2, y_1] = [c, y_2, y_1] = -[y_2, y_1, c] - [y_1, c, y_2] = [c, y_1, y_2] + [y_1, y_2]c - c[y_1, y_2].$$

In this case the desired result follows by applying the induction argument on c jointly with Lemma 16.

Now suppose $k \geq 1$ and $h \geq 0$. Putting $c = [z, y_{i_1}, \dots, y_{i_h}]$, from the Jacobi identity, we have

$$\begin{aligned} & [z, y_{i_1}, \dots, y_{i_h}, y_2, y_1, y_{j_1}, \dots, y_{j_k}] = [c, y_2, y_1, y_{j_1}, \dots, y_{j_k}] \\ &= -[y_2, y_1, c, y_{j_1}, \dots, y_{j_k}] - [y_1, c, y_2, y_{j_1}, \dots, y_{j_k}] \\ &= [c, y_1, y_2, y_{j_1}, \dots, y_{j_k}] + [y_1, y_2, c, y_{j_1}, \dots, y_{j_k}] \\ &= [c, y_1, y_2, y_{j_1}, \dots, y_{j_k}] + [[y_1, y_2]c, y_{j_1}, \dots, y_{j_k}] - [c[y_1, y_2], y_{j_1}, \dots, y_{j_k}]. \end{aligned}$$

Once again, we achieve our result by using Lemma 17 in the same way as in the previous item, and an induction process on the length of c jointly with Lemma 16.

iii) Let us put $c = [z, y_{i_1}, \dots, y_{i_h}]$ and, for each $1 \leq i \leq l$, $c_i = [c, y_{j_1}, \dots, y_{j_i}]$. Notice that

$$\begin{aligned} & [z, y_{i_1}, \dots, y_{i_h}, \bar{y}_2, y_{j_1}, \dots, y_{j_l}, \bar{y}_1, y_{t_1}, \dots, y_{t_m}] \\ &= [c, y_2, y_{j_1}, \dots, y_{j_l}, y_1, y_{t_1}, \dots, y_{t_m}] - [c, y_1, y_{j_1}, \dots, y_{j_l}, y_2, y_{t_1}, \dots, y_{t_m}] \\ &= [c, y_2, y_{j_1}, \dots, y_{j_l}, y_1, y_{t_1}, \dots, y_{t_m}] + [y_1, y_{j_1}, c, \dots, y_{j_l}, y_2, y_{t_1}, \dots, y_{t_m}] \\ &+ [y_{j_1}, c, y_1, \dots, y_{j_l}, y_2, y_{t_1}, \dots, y_{t_m}] \\ &\equiv [c, y_2, y_{j_1}, \dots, y_{j_l}, y_1, y_{t_1}, \dots, y_{t_m}] + \underbrace{[c, [y_{j_1}, y_1], \dots, y_{j_l}, y_2, y_{t_1}, \dots, y_{t_m}]}_{\sum \alpha_i f_i} \\ &- [c_1, y_1, y_{j_2}, \dots, y_2, y_{t_1}, \dots, y_{t_m}] \\ &\equiv [c, y_2, y_{j_1}, \dots, y_{j_l}, y_1, y_{t_1}, \dots, y_{t_m}] + \sum \alpha_i f_i + [y_1, y_{j_2}, c_1, \dots, y_2, y_{t_1}, \dots, y_{t_m}] \\ &+ [y_{j_2}, c_1, y_1, \dots, y_2, y_{t_1}, \dots, y_{t_m}] \\ &\equiv [c, y_2, y_{j_1}, \dots, y_{j_l}, y_1, y_{t_1}, \dots, y_{t_m}] + \sum \alpha_i f_i + [c_1, [y_{j_2}, y_1], \dots, y_2, y_{t_1}, \dots, y_{t_m}] \\ &- [c_2, y_1, \dots, y_2, y_{t_1}, \dots, y_{t_m}]. \end{aligned}$$

Now the result follows by using the Jacobi identity $2l + 1$ times jointly with item i) of this proposition. □

We are going to see an useful tool to check if the multilinear polynomial $f_{\tilde{\gamma}}$ is in the list of generators of the graded identities of $UT_m(F)$. It turns out that is convenient to look at its Z -degree (see [15]).

Definition 19. Given a G -grading on $UT_m(F)$ induced by the m -tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, we say that there is a *variation* in the position i in ε if $\varepsilon_1 \neq \varepsilon_{i+1}$.

Remark 20. Consider a G -grading $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ on the algebra $UT_m(F)$. There is a variation in the position i in ε if and only if $\eta_i \neq 1_G$ in the diagonal sequence $d(\varepsilon) = (\eta_1, \dots, \eta_{m-1})$ related to ε .

The following result is a direct consequence of Theorem 14.

Proposition 21. *Let t be the number of variations in the G -grading on $UT_m(F)$ induced by $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ and let $\tilde{\gamma} = (\gamma_1, \dots, \gamma_n)$ be a sequence of elements of G . If the polynomial $f_{\tilde{\gamma}}$ has Z -degree greater than $t + 1$, then it is a graded polynomial identity of $(UT_m(F), \varepsilon)$.*

Notice that if $f_{\tilde{\gamma}}$ has Z -degree 0, then $f_{\tilde{\gamma}}$ is a graded identity of $UT_m(F)$ if and only if it is a polynomial identity of the neutral component $UT_m(F)^{1_G}$.

Remark 22. Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ be a G -grading on $UT_m(F)$ and let h_1, \dots, h_s be the distinct elements of G that appear in ε . Suppose that for each $i = 1, \dots, s$, the element h_i appears m_i times. Then we have the following isomorphism of F -algebras

$$UT_m(F)^{1_G} \cong UT_{m_1}(F) \oplus \dots \oplus UT_{m_s}(F).$$

Moreover, setting $M = \max\{m_1, \dots, m_s\}$ we have the polynomial identities of Z -degree 0 follow from the polynomials identities of $UT_M(F)$, i.e., by

$$[y_1, y_2] \cdots [y_{2M-1}, y_{2M}].$$

4. Y -PROPER GRADED COCHARACTERS OF $UT_m(F)$

In what follows we will give a description of the Y -proper graded cocharacters of $UT_m(F)$ equipped with an elementary G -grading, where G is a finite group. We would like to mention in the ordinary case partial results were obtained by one of the authors in [6].

We fix a finite group G and we consider $UT_m(F)$ endowed with the elementary G -grading induced by the m -tuple $\varepsilon = (g_1^{l_1}, g_2^{l_2}, \dots, g_L^{l_L})$. Let us denote by S the support of the G -grading induced by ε and let us reorder the elements of G this way: $G = \{g_1, \dots, g_{|S|}, g_{|S|+1}, \dots, g_{|G|}\}$, where $g_i \in S$ if $i \leq |S|$ whereas $g_i \notin S$ elsewhere. Given a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of elements of G , we associate to α the following $|G|$ -tuple

$$\mu(\alpha) = (\mu_1, \dots, \mu_{|G|}),$$

where for each $i = 1, \dots, |G|$ we put

$$\mu_i = \text{the number of } \alpha_j \text{ such that } \alpha_j = g_i.$$

Indeed $\mu_i = 0$ when $i > |S|$. Let us take $\tilde{n} = (n_1, \dots, n_{|G|}) \in \mathbb{N}_0^{|G|}$ and consider the set

$$S_{\tilde{n}} = \{\alpha = (\alpha_1, \dots, \alpha_k) \mid \mu(\alpha) = (\mu_1, n_2, \dots, n_{|G|})\}.$$

Given an $F[G]$ -module M and $m \in M$ we denote by $F[G]m$ the action of $F[G]$ on m . We also suggest that in what follows we implicitly use the fact that two representations afford the same G -character if they represent isomorphic $F[G]$ -modules. With this in mind, we have the following:

Theorem 23. *Let G be a finite group and consider the algebra $UT_m(F)$ equipped with the elementary G -grading induced by the m -tuple $\varepsilon = (g_1^{l_1}, g_2^{l_2}, \dots, g_L^{l_L})$. Then for any $n_1, \dots, n_{|G|}$ we have*

$$\begin{aligned} & \xi_{n_1, \dots, n_{|G|}}^G(UT_m(F)) \\ = & \sum_{(\alpha_1, \dots, \alpha_k) \in S_{\tilde{n}}} \sum_{\sum_i r_i + \sum_j s_j = n_1} \left(\sum_{\lambda_1 \vdash r_1} m_{\lambda_1} \chi_{\lambda_1} \otimes \sum_{\lambda_2 \vdash r_2} m_{\lambda_2} \chi_{\lambda_2} \otimes \dots \otimes \sum_{\lambda_{k+1} \vdash r_{k+1}} m_{\lambda_{k+1}} \chi_{\lambda_{k+1}} \otimes \boxed{s_1} \otimes \dots \otimes \boxed{s_k} \right)^{\uparrow S_{n_1}} \\ & \otimes \left(\square \otimes \dots \otimes \square \right)^{\uparrow S_{n_2}} \otimes \dots \otimes \left(\square \otimes \dots \otimes \square \right)^{\uparrow S_{n_{|G|}}}, \end{aligned}$$

where $\sum_{\lambda_i \vdash r_i} m_{\lambda_i} \chi_{\lambda_i}$ belong to the r_i -th Y -proper cocharacter of $UT_{l_1}(F)$ and the sequences α are ε -good. Moreover, the sum of the arms of infinite width of the various $\sum_{\lambda_i \vdash r_i} m_{\lambda_i} \chi_{\lambda_i}$ cannot be greater than l_1 .

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a good sequence such that $\mu(\alpha) = (\mu_1, n_2, \dots, n_{|G|})$ and $\mu_1 \leq m - (t + 1)$. Let us consider $n_1 \in \mathbb{N}$ such that $\sum_{i=1}^{|G|} n_i = n$. Put $A = UT_m(F)$ and let $f \in \Gamma_{n_1, \dots, n_{|G|}} \subseteq \Gamma_n^G(A)$. If $H = S_{n_1} \times \dots \times S_{n_{|G|}}$, then $F[H]^{\uparrow S_n}$ has dimension $|S_n : H| \dim_F F[H]f$. On the other hand, it follows from Propositions 18, 21 and Remark 22 that $F[S_n]f$ is generated by the polynomials

$$(1) \quad [Y][z^{\alpha_i}, y_{i_1}, \dots, y_{i_{s_1}}][Y][z^{\alpha_j}, y_{j_1}, \dots, y_{j_{s_2}}][Y] \cdots [Y][z^{\alpha_r}, y_{r_1}, \dots, y_{r_{s_3}}][Y],$$

where $[Y]$ denotes a product of commutators. Indeed, because the homogeneous component of degree 1_G of $UT_m(F)$ is isomorphic to $UT_{l_1}(F)$ in virtue of Remark 22, the number of $[Y]$ cannot exceed l_1 . Moreover, the $[Y]$'s can be assumed as elements of a basis of the relatively free algebra of $UT_{l_1}(F)$. This justifies the

total number of arms of infinite width cannot exceed l_1 . Furthermore, the indexes of the remaining y 's are strictly increasing and $\sum_i s_i + \sum_j r_j = n_1$. Notice that the commutators in (1) are linearity independent, thus $\dim_F F[S_n]f = |S_n : H| \dim_F F[H]f$. Finally the proof follows since the S_n -action in both of the situations is the same and the action on the z 's is the trivial one. \square

Notice that in the previous result one should have $n_1 \leq m - (t + 1)$ because, otherwise, we have a graded identity.

5. FINDING OUT GOOD SEQUENCES

The goal of this section is giving an explicit form of the good sequences in some particular cases.

Let us consider the algebra $UT_m(F)$ endowed with the G -grading induced by the m -tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$. We consider again $\tilde{n} = (n_1, \dots, n_m) \in \mathbb{N}_0^m$ and consider the set $S_{\tilde{n}} = \{\alpha = (\alpha_1, \dots, \alpha_k) \mid \mu(\alpha) = (\mu_1, n_2, \dots, n_m)\}$, where $\mu_1 \leq m - (t + 1)$ and t is the number of variations in ε . In [9] the authors considered the Vasilovsky \mathbb{Z}_m -grading on $UT_m(F)$, i.e., the grading generated by the m -tuple $v = (\bar{0}, \bar{1}, \dots, \overline{m-1})$ and they established a sequence

$$\alpha = (\alpha_1, \dots, \alpha_k) \text{ is } v\text{-good if and only if } \mu_1 = 0 \text{ and } \sum_{j=2}^m \mu_j(j-1) \leq m-1.$$

Notice that $d(v) = (\bar{1}, \dots, \bar{1})$ and $t = m - 1$.

In [9] the authors considered the \mathbb{Z}_m -grading on $UT_m(F)$ induced by the m -tuple $\varepsilon = (\bar{0}, \bar{0}, \bar{1}, \dots, \overline{m-2})$, where $d(\varepsilon) = (\bar{0}, \bar{1}, \dots, \bar{1})$ and $t = m - 2$. It was proved a sequence

$$\begin{aligned} \alpha = (\alpha_1, \dots, \alpha_k) \text{ is } \varepsilon\text{-good if and only if} \\ \mu_1 = 0 \text{ and } \sum_{j=2}^m \mu_j(j-1) \leq m-2 \text{ if } \alpha_1 \neq \bar{0}; \\ \mu_1 = 1 \text{ and } \sum_{j=2}^m \mu_j(j-1) \leq m-2 \text{ otherwise.} \end{aligned}$$

In both of the two cases considered above, the following relations hold:

$$(2) \quad \mu_1 \leq m - (t + 1) \text{ and } \sum_{j=2}^m \mu_j(j-1) \leq t.$$

Indeed the previous inequalities are far from being a characterization of good sequences as pointed out in [9]. In the \mathbb{Z}_m -grading of $UT_m(F)$ induced by $\varepsilon = (\bar{0}, \bar{0}, \bar{1}, \dots, \overline{m-2})$ the polynomial $f_1 = z^g[y_1, y_2]$ is always a graded identities whereas the polynomial $f_2 = [y_1, y_2]z^g$ is not (possibly). The interesting fact is that the polynomial f_1 affords the pair $\alpha_1 = (g, \bar{0})$ (bad sequence) while f_2 affords the pair $\alpha_2 = (\bar{0}, g)$ ((possibly good) sequence) and both α_1 and α_2 do satisfy the inequality (2). On this purpose, we get the following, more general, result.

Proposition 24. *Let G be any group and consider $g \in G$, $g \neq 1_G$. Let us consider $UT_m(F)$ endowed with the G -grading induced by the m -tuple $\varepsilon = (1_G^l, g, g^2, \dots, g^r)$, where $l + r = m$ and $o(g) > m$. Let $\alpha = (\alpha_1, \dots, \alpha_k) \in G^k$, then we have α is ε -good if and only if there exist $r_1, \dots, r_{\mu_1} \in J(UT_1(F))$ such that $\|r_i\| = \alpha_i$, $i = 1, \dots, \mu_1$, $r_1 \cdots r_{\mu_1} \neq 0$ and*

$$\sum_{j=2}^{r+1} \mu_j(j-1) \leq t.$$

Here $J(UT_1(F))$ denotes the Jacobson radical of $UT_1(F)$ whereas $\mu_j = |\{\alpha_i \mid \alpha_i = g^{(j-1)}\}|$.

Proof. Because of the fact that $o(g) > m$, we have $g^i \neq g^j$ if $i \neq j$. Let us consider the diagonal sequence

$$d(\varepsilon) = (1_G^{l_1-1}, g, \dots, g).$$

First, we observe that if $\alpha_i \neq 1_G$ and $\alpha_{i+1} = 1_G$ for some $1 \leq i \leq k$, then α is an ε -bad sequence because $f_\alpha = [y_1, y_2] \cdots [y_{2i-1}, y_{2i}] z^{\alpha_{i+1}} [y_{2i+1}, y_{2i+2}] f_{\alpha_{i+2}, 2} \cdots f_{\alpha_k, k}$ is a graded identity in the light of $z^{\alpha_{i+1}} [y_{2i+1}, y_{2i+2}]$ being a graded identity of $(UT_m(F), \varepsilon)$ too. Suppose further α does not satisfy the inequality above, then, again, α is an ε -bad sequence in the light of Theorem 14 and the proof follows.

Conversely, suppose $\alpha = (\alpha_1, \dots, \alpha_{\mu_1}, \alpha_{\mu_1+1}, \dots, \alpha_k)$ is such that there exist $r_1, \dots, r_{\mu_1} \in J(UT_l(F))$ such that $\|r_i\| = \alpha_i$, $i = 1, \dots, \mu_1$, $r_1 \cdots r_{\mu_1} \neq 0$ and $\sum_{j=2}^{r+1} \mu_j(j-1) \leq t$. We can consider the polynomial $f_\alpha = f_1 f_2$ evaluated in the relatively free graded algebra of $UT_m(F)$, where f_1 denotes the product of μ_1 commutators in the Y 's and f_2 is the remaining part. Remark we can figure out each variable $x_i^{g^j}$ as an $m \times m$ upper triangular matrix having the same non-zero entries of $UT_m^{g^j}$ and, as entries, the commutative variables from the set X_i . Indeed, the polynomial above is a graded identity if it is the 0 matrix. Now notice the G -grading above carries the property $UT_m(F)^{g^t} UT_m(F)^{g^u} = UT_m(F)^{g^{tu}}$. This means the evaluation of f_2 is an upper triangular $m \times m$ matrix having the same non-zero entries of $UT_m^{g^{\sum_{j=2}^{r+1} \mu_j}} = UT_m^g(F)$ for a certain $u \leq r$ and as the non-zero entries, polynomials in the commutative variables from $\bigcup_i X_i$. Analogously, the polynomial f_1 is an $m \times m$ upper triangular matrix having the first μ_1 diagonal filled entirely with 0 entries. Now it is easy to see the whole product is non-zero that means α is ε -good. \square

By Proposition 24 and Theorem 23 we get the next that is a generalization of the results obtained by one of the authors in [9] and [10].

Theorem 25. *Let G be a finite group and consider the algebra $UT_m(F)$ equipped with the elementary G -grading induced by the m -tuple $\varepsilon = (1_G^l, g, g^2, \dots, g^r)$, where $l + r = m$ and $o(g) > m$. Then for any $n_1, \dots, n_{|G|}$ such that $n_1 \leq l - 1$ and $\sum_{j=2}^{r+1} \mu_j(j-1) \leq t$, where $\mu_j = |\{\alpha_i | \alpha_i = g^{(j-1)}\}|$, we have:*

$$\begin{aligned} & \xi_{n_1, \dots, n_{|G|}}^G(UT_m(F)) \\ = & \sum_{(\alpha_1, \dots, \alpha_k) \in S_{\bar{n}}} \sum_{r + \sum_j s_j = n_1} \left(\sum_{\lambda \vdash r} m_{\lambda} \chi_{\lambda} \otimes \boxed{s_1} \otimes \cdots \otimes \boxed{s_k} \right)^{\uparrow S_{n_1}} \otimes \left(\square \otimes \cdots \otimes \square \right)^{\uparrow S_{n_2}} \otimes \cdots \otimes \left(\square \otimes \cdots \otimes \square \right)^{\uparrow S_{n_{|G|}}}, \end{aligned}$$

where $\sum_{\lambda \vdash r} m_{\lambda} \chi_{\lambda}$ is the r -th Y -proper cocharacter of $UT_l(F)$ and the sequences α are ε -good.

Remark 26. Explicit computations showing the whole list of cases for $m = 2, 3$ can be found in [9] and in [10].

6. GELFAND-KIRILLOV DIMENSION OF RELATIVELY FREE ALGEBRAS OF $UT_m(F)$

We would like to show how does the Gelfand-Kirillov dimension of the relatively free G -graded algebra of $UT_m(F)$ looks like.

We recall some results about the Gelfand-Kirillov dimension of a finitely generated algebra. Let A be an F -algebra generated by a finite set $\{a_1, \dots, a_m\}$. Let

$$V^n = \text{span}\langle a_{i_1} \cdots a_{i_n} \mid i_j = 1, \dots, m \rangle, n = 1, 2, \dots;$$

we assume $V^0 = F$. The function of the non-negative argument n

$$g_V(n) = \dim_F(V^0 + V^1 + \cdots + V^n), n = 1, 2, \dots,$$

is called the *growth function* of A (with respect to $V = V^1$). The Gelfand-Kirillov dimension of A is defined by

$$\text{GKdim}(A) = \limsup_{n \rightarrow \infty} (\log_n g_V(n)) = \limsup_{n \rightarrow \infty} \frac{\log g_V(n)}{\log n}.$$

Notice that the definition of Gelfand-Kirillov dimension is independent of the generating space (see for example [30], Lemma 1.1), then we are allowed to remove the dependence on V from the symbol $g_V(n)$.

Among the large amount of interesting properties the GK dimension of an algebra carries inside, we would like to highlight the fact that the GK dimension of a finitely generated commutative algebra, for instance, coincides with its Krull dimension. This is remarkable. For further details and results about GK dimension of algebras we refer to the books [30] by Krause and Lenagan and [36] by McConnell and Robson.

There are several results towards GK dimension of PI-algebras (see Section 10 of [30]) but here we want to discuss those results regarding the GK dimension of the relatively free algebra of a PI-algebra.

Definition 27. Let A be a PI-algebra and $r \geq 1$ an integer. We denote by $\text{GKdim}_r(A)$ the GK dimension of $F\langle X_r \rangle / (F\langle X_r \rangle \cap T(A))$, where $X_r := \{x_1, \dots, x_r\}$.

In [2] the author studies several properties of $\text{GKdim}_r(A)$. In particular, it can be proved $\text{GKdim}_r(A)$ is defined by the complexity type of the algebra A or by a set of semidirect products of matrix algebras over the ring of polynomials from the variety generated by A . See also the paper [3] by Berele for explicit computations of the GK dimension of some remarkable PI-algebras or the surveys [19] by Drensky and [7] by Centrone. One of the authors of the present paper generalized the definition of $\text{GKdim}_r(A)$ to graded algebras (see [5] and [8]). Here we have the full definition.

Definition 28. Let A be a G -graded PI-algebra, where G is a finite group, and $r \geq 1$ an integer. We denote by

$$\text{GKdim}_r^G(A) := \text{GKdim}(F\langle X_r | G \rangle / (F\langle X_r | G \rangle \cap T(A))),$$

where $X_r := \{x_1^{g_1}, \dots, x_r^{g_1}, \dots, x_1^{g_{|G|}}, \dots, x_r^{g_{|G|}}\}$.

In [8] the author found a deep relation between the (graded) GK dimension of a PI-algebra and its (graded) *PI-exponent* (see below).

Theorem 29. Let A be a finite dimensional G -graded algebra. Then $\text{GKdim}_r^G(A) = \exp^G(A)r + \alpha$, where $\alpha \leq 0$ is an integer.

Hence in order to compute the graded GK dimension of $UT_m(F)$, we only need its graded exponent. In [43] the authors proved the G -graded exponent of $UT_m(F)$ equals m independently of the group G and of the grading. This information can be obtained directly from the cocharacter sequence of $UT_m(F)$. In fact, by a result of Aljadeff and Giambruno (see [1]) we have the graded exponent is simply the number of arms of infinite width in the graded cocharacter sequence of A .

Coming back to $UT_m(F)$, we have by Theorem 23 the number of arms of infinite width in the Y -proper graded cocharacter sequence of $UT_m(F)$ is $m - 1$. Of course, by (the graded version of) Proposition 7, we get the number of arms of infinite width in the graded cocharacter sequence of $UT_m(F)$ is $m - 1 + 1 = m$. We immediately have the next.

Theorem 30. Let $r \geq 2$, then

$$\text{GKdim}_r^G(UT_m(F)) = mr + \alpha,$$

where $\alpha \leq 0$ is an integer.

Another consequence of the knowledge of the (graded) cocharacter sequence of a (graded) PI-algebra relies to the knowledge of the Hilbert series of its relatively free (graded) algebra. Here we have the graded generalization of a result obtained independently by Berele and Drensky.

Theorem 31. Let A be a G -graded PI-algebra and suppose its graded cocharacter is

$$\chi_{n_1, n_2, \dots, n_{|G|}} = \sum m_{\lambda_1, \dots, \lambda_{|G|}} \chi_{\lambda_1} \otimes \dots \otimes \chi_{\lambda_{|G|}},$$

then the Hilbert series of the relatively free G -graded algebra of $UT_m(F)$ is

$$H^G(A, T) = \sum_{n=0}^{\infty} \sum_{n_1 + \dots + n_{|G|} = n} \sum m_{\lambda_1, \dots, \lambda_{|G|}} S_{\lambda_1}(T) \dots S_{\lambda_{|G|}}(T),$$

where $S_{\lambda}(T)$ denotes the Schur polynomial of shape λ in the variables T .

By one of the results of [5] we get the next.

Theorem 32. Let G be any group and let us consider $UT_m(F)$ with the G -grading induced by the m -tuple $(1, g, g^2, \dots, g^{m-1})$, where g is such that $o(g) > m$. Let $r \geq 2$, then

$$\text{GKdim}_r^G(UT_m(F)) = mr.$$

We are in position to state the following.

Theorem 33. Let G be any group and let us consider $UT_m(F)$ with the G -grading induced by the m -tuple $(1, 1, g, g^2, \dots, g^{m-2})$, where g is such that $o(g) > m - 1$. Let $r \geq 2$, then

$$\text{GKdim}(B_r^G(UT_m(F))) = (m - 1)r,$$

where $B_r^G(UT_m(F))$ is the relatively free Y -proper algebra of $UT_m(F)$ in r -variables.

Proof. By [9] we get a homogeneous basis for $B_r^G(UT_m(F))$ is given by the following two kinds of homogeneous polynomials:

$$[z_b, Y_b] \cdots [z_c, Y_c], \quad [y_a, Y_a][z_b, Y_b] \cdots [z_c, Y_c],$$

where, the Y_i 's are ordered sets of variables from Y and the sequences are good. This means we only need to count those polynomials. More precisely, the dimension of the homogeneous component of degree n of $B_r^G(UT_m(F))$ is

$$c \cdot \sum_{i_1 + \dots + i_k = n} \binom{i_1 + r}{r} \cdots \binom{i_k + r}{r} + \left(\sum_{j=2}^n \sum_{a=2}^r \sum_{i=1}^{a-1} \binom{j+r-a}{r-i+1} \right) \left(\sum_{i_1 + \dots + i_k = n-j} \binom{i_1 + r}{r} \cdots \binom{i_k + r}{r} \right),$$

where

$$c \cdot \sum_{i_1 + \dots + i_k = n} \binom{i_1 + r}{r} \cdots \binom{i_k + r}{r}$$

is the number of polynomials of the first kind whereas

$$\left(\sum_{j=2}^n \sum_{a=2}^r \sum_{i=1}^{a-1} \binom{j+r-a}{r-i+1} \right) \left(\sum_{i_1 + \dots + i_k = n-j} \binom{i_1 + r}{r} \cdots \binom{i_k + r}{r} \right)$$

is the number of polynomials the second kind. Moreover, we denoted by c the number of good sequences $\alpha = (\alpha_1, \dots, \alpha_k)$. It turns out both of the summands are polynomials in n of degree $\max_k \{kr - 1\}$ that is as a polynomial in n of degree $(m - 1)k - 1$. This forces the growth function of $B_r^G(UT_m(F))$ being a polynomial in n of degree $(m - 1)k - 1 + 1 = (m - 1)k$ which means $\text{GKdim}(B_r^G(UT_m(F))) = (m - 1)k$. \square

In the light of the previous experimental results, we make the following conjecture.

Conjecture 34. Let G be any group and let us consider $UT_m(F)$ with a G -grading. Let $r \geq 2$, then

$$\text{GKdim}_r^G(UT_m(F)) = mr.$$

7. ACKNOWLEDGEMENTS

The authors want to thank the referee for her/his useful comments.

The first author wants to thank INDAM, Gruppo Nazionale per le Strutture Algebriche e Geometriche (GNSAGA).

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