Elliptic differential operators and positive semigroups associated with generalized Kantorovich operators

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Abstract

Deepening the study of a new approximation sequence of positive linear operators we introduced and studied in [12], in this paper we disclose its relationship with the Markov semigroup (pre)generation problem for a class of degenerate second-order elliptic differential operators which naturally arise through an asymptotic formula, as well as with the approximation of the relevant Markov semigroups in terms of the approximating operators themselves.

The analysis is carried out in the context of the space $\mathscr{C}(K)$ of all continuous functions defined on an arbitrary compact convex subset K of $\mathbf{R}^d$, $d \geq 1$, having non-empty interior and a not necessarily smooth boundary, as well as, in some particular cases, in $L^p(K)$ spaces, $1 \leq p < +\infty$. The approximation formula also allows to infer some preservation properties of the semigroup such as the preservation of the Lipschitz-continuity as well as of the convexity. We finally apply the main results to some noteworthy particular settings such as balls and ellipsoids, the unit interval and multidimensonal hypercubes and simplices. In these settings the relevant differential operators fall into the class of Fleming-Viot operators.

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Introduction

In [12] we introduced and studied a new sequence $(C_n)_{n\geq 1}$ of positive linear operators acting on function spaces defined on a convex compact subset K of some locally convex Hausdorff space. Their construction depends on a given Markov operator $T : \mathscr{C}(K) \to \mathscr{C}(K)$, a real number $a \geq 0$ and a sequence $(\mu_n)_{n\geq 1}$ of probability Borel measures on K.

More precisely, for every $n \geq 1$, they are defined by setting

$$C_n(f)(x)=\int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n + a x_{n+1}}{n+a}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n) d\mu_n(x_{n+1})$$

for every $x \in K$ and for every $f \in C(K)$, where $(\tilde{\mu}_x^T)_{x\in K}$ is the continuous selection of probability Borel measures on K corresponding to T via the Riesz representation theorem.

For particular choices of these parameters and for particular convex compact subsets, such as the unit interval or the multidimensional hypercube and simplex, these operators turn into the Kantorovich operators and in several of their wide-ranging generalizations.

In [12] we mainly investigated and studied the approximation properties of these operators in the space $\mathscr{C}(K)$ and, in some cases, in L^p-spaces, $1 \leq p < +\infty$.

Under the influence of a series of researches developed during the last two decades, which are concerned with the relationship between degenerate differential operators, Markov semigroups and approximation processes (see, e.g., [4] and [11]), it has been quite natural to investigate, in the special case when $K \subset \mathbf{R}^d$, $d \geq 1$, whether, by using the theory of one-parameter semigroups, the new approximation process can also be used to solve some classes of initial-boundary value problems associated with suitable degenerate differential operators as well as to approximate the solutions of such differential problems.

From an operator theoretical point of view this problem corresponds to determine the differential operator generated by an asymptotic formula for the approximating operators C_n and to investigate whether it (pre)generates a (positive) C_0-semigroup which, in turn, can be approximated in terms of suitable iterates of them.

In the present setting, assuming that the sequence $(\mu_n)_{n\geq 1}$ is weakly convergent to some (probability) Borel measure μ on K, then the differential operators which arise through such a method are of the form

$$V(u)(x) = \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} a(b_i - x_i)\frac{\partial u}{\partial x_i}(x)$$

($u \in \mathscr{C}^2(K), x = (x_1, \ldots, x_d) \in K$), where

$$\alpha_{ij} := T(pr_i pr_j) - pr_i pr_j \qquad (i, j = 1, \ldots, d),$$

each pr_i denoting the i^{th} coordinate function ($i = 1, \ldots, d$), and $b = (b_1, \ldots, b_d) \in K$ stands for the barycenter of the measure μ. Moreover, the coefficients α_{ij} vanish on a subset of the boundary of K which contains the extreme points of K.
Under the assumptions that T leaves invariant the continuous affine functions on K and maps polynomials into polynomials of at most the same degree, we show, indeed, that $(V, \mathscr{C}^2(K))$ is the pregenerator of a Markov semigroup on $\mathscr{C}(K)$ which is approximated in terms of suitable iterates of the C_n's. This semigroup is referred to as the limit semigroup of the C_n's.

Specializing the convex compact set K and the other parameters, we obtain several classes of differential operators which are of current interest in the research area of evolution equations. Among them we quote the degenerate diffusion operators on balls and ellipsoids ([11], [28]) and the Fleming-Viot type operators on the unit interval and on the multidimensional hypercube and simplex ([2], [3], [5], [7], [11], [15], [21], [24]).

Our approach allows to study all these particular cases in an unifying manner and also to obtain some extensions of the existing generation results.

However, the main feature of the paper rests not only on the study of the generation results for the differential operators as above in the framework of convex compact domains with not necessarily smooth boundary, but also on the approximation/representation of the relevant semigroup in terms of constructively defined linear positive operators; this kind of approximation allows to infer some preservation properties of it, such as the preservation of Lipschitz-continuity as well as of the convexity, which correspond to some spatial regularity properties of the solutions of the initial-boundary value differential problems associated with the generators.

Finally, in the particular case of the d-dimensional hypercube Q_d we show that the limit semigroup can be extended to a contraction semigroup on $L^p(Q_d)$, $1 \leq p < +\infty$, which in turn is approximated by iterates of the natural extension of the C_n's to $L^p(Q_d)$.

1 Generalized Kantorovich operators for compact convex subsets

Throughout this section we shall fix a locally convex Hausdorff space X and a convex compact subset K of X.

We denote by $\mathscr{C}(K)$ the space of all real-valued continuous functions on K; $\mathscr{C}(K)$ is a Banach lattice if endowed with the natural (pointwise) ordering and the sup-norm $\|\cdot\|_\infty$.

In particular, if $K \subset \mathbf{R}^d$, $d \geq 1$, and if it has non-empty interior, we denote by $\mathscr{C}^2(K)$ the space of all real-valued (continuous) functions on K which are twice-continuously differentiable on the interior $\mathrm{int}(K)$ of K and whose partial derivatives up to the order 2 can be continuously extended to K. For $u \in \mathscr{C}^2(K)$ and $i, j = 1, \ldots, d$, we shall continue to denote by $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ the continuous extensions to K of $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$.

For every $i = 1, \ldots, d$, pr_i will stand for the i^{th} coordinate function on K, i.e., $pr_i(x) := x_i$ for every $x = (x_1, \ldots, x_d) \in K$.

Moreover, we shall denote by $\|\cdot\|_2$ the Euclidean norm on $\mathbf{R}^d$.

Coming back to an arbitrary convex compact subset K of a locally convex Hausdorff space, let B_K be the σ-algebra of all Borel subsets of K and $M^+(K)$ (resp., $M_1^+(K)$) the cone of all regular Borel measures on K (resp., the cone of all regular probability Borel measures on K).

If $\mu \in M^+(K)$ and $1 \leq p < +\infty$, we denote by $L^p(K, \mu)$ the space of all (equivalence classes of) μ-integrable in the p-th power measurable functions on K; in particular, if $\mu = \lambda_d$, λ_d being the Borel-Lebesgue measure on $K \subset \mathbf{R}^d$, then we shall use the symbol $L^p(K)$ instead of $L^p(K, \lambda_d)$.

We denote by $A(K)$ the space of all continuous affine functions on K. For every $m \geq 1$, the symbol $P_m(K)$ stands for the linear subspace generated by products of m continuous affine functions on K, i.e.,

$$P_m(K) := \mathrm{span}\left(\left\{\prod_{i=1}^{m} h_i \;\middle|\; h_1, \ldots, h_m \in A(K)\right\}\right). \tag{1.1}$$

Clearly, $P_m(K) \subset P_{m+1}(K)$ and

$$P_\infty(K) := \bigcup_{m \geq 1} P_m(K) \tag{1.2}$$

is a subalgebra of $\mathscr{C}(K)$ which separates the points of K and contains the constants; hence, by the Stone-Weierstrass theorem, it is dense in $\mathscr{C}(K)$.

From now on let $T : \mathscr{C}(K) \to \mathscr{C}(K)$ be a Markov operator on $\mathscr{C}(K)$, i.e., a positive linear operator on $\mathscr{C}(K)$ such that $T(\mathbf{1}) = \mathbf{1}$, where the symbol $\mathbf{1}$ stands for the function of constant value 1 on K. Moreover, let $(\tilde{\mu}_x^T)_{x \in K}$ be the continuous selection in $M_1^+(K)$ corresponding to T via the Riesz representation theorem, i.e.,

$$\int_K f \, d\tilde{\mu}_x^T = T(f)(x) \qquad (f \in \mathscr{C}(K), x \in K). \tag{1.3}$$

Further, we assume that T satisfies the following condition

$$T(h) = h \qquad \text{for every } h \in A(K). \tag{1.4}$$

Fix $a \geq 0$ and a sequence $(\mu_n)_{n\geq 1}$ in $M_1^+(K)$. Then, for every $n \geq 1$, we consider the positive linear operator C_n defined by setting

$$C_n(f)(x)=\int_K\cdots\int_K f\left(\frac{x_1+\ldots+x_n+ax_{n+1}}{n+a}\right)d\tilde{\mu}_x^T(x_1)\cdots d\tilde{\mu}_x^T(x_n)d\mu_n(x_{n+1}) \tag{1.5}$$

for every $x \in K$ and for every $f \in \mathscr{C}(K)$.

The operators C_n were introduced in [12], where the authors studied their approximation properties as well as some preservation properties. These operators generalize the Kantorovich operators on the unit interval, on hypercubes and on simplices together with several wide-ranging extensions of theirs (see [6], [12], [14], [16] and the references quoted therein) (see also [8]).

Introducing the auxiliary continuous function

$$I_n(f)(x) := \int_K f\left(\frac{n}{n+a}x+\frac{a}{n+a}t\right)d\mu_n(t) \qquad (f\in\mathscr{C}(K),\, x\in K), \tag{1.6}$$

for every $n \geq 1$, then

$$C_n(f) = B_n(I_n(f)), \tag{1.7}$$

where, for every $n\geq 1$, $f\in\mathscr{C}(K)$ and $x\in K$, the operator B_n is defined as

$$B_n(f)(x) = \int_K\cdots\int_K f\left(\frac{x_1+\ldots+x_n}{n}\right)d\tilde{\mu}_x^T(x_1)\cdots d\tilde{\mu}_x^T(x_n). \tag{1.8}$$

Note that, for every $n \geq 1$, B_n is a positive linear operator from $\mathscr{C}(K)$ into $\mathscr{C}(K)$, $B_n(\mathbf{1}) = \mathbf{1}$ and hence $\|B_n\| = 1$. Moreover, $B_1 = T$.

The sequence $(B_n)_{n\geq 1}$ was introduced in [11, Chapter 3] (see also [4, Chapter 6]) and the B_n's are called the Bernstein-Schnabl operators associated with the Markov operator T. The operators B_n generalize the classical Bernstein operators on the unit interval, on multidimensional simplices and hypercubes and they share with them several preservation properties which have been investigated in [4] and in [11].

In particular, under assumption (1.4), the sequence $(B_n)_{n\geq 1}$ is a (positive) approximation process in $\mathscr{C}(K)$, i.e., for every $f \in \mathscr{C}(K)$,

$$\lim_{n\to\infty} B_n(f) = f \quad \text{uniformly on } K. \tag{1.9}$$

Clearly, if $a = 0$, the operators C_n correspond to the B_n's.

Therefore $C_n(f) \in \mathscr{C}(K)$ for any $n \geq 1$ and the C_n's are positive linear operators on $\mathscr{C}(K)$; hence, each C_n is continuous and $\|C_n\| = 1$, since $C_n(\mathbf{1}) = \mathbf{1}$.

Note that assumption (1.4) is not essential in defining the operators C_n, but it's needed (see [12, Theorem 3.2]) in order to prove that $(C_n)_{n\geq 1}$ is an approximation process on $\mathscr{C}(K)$, i.e., that, for every $f \in \mathscr{C}(K)$,

$$\lim_{n\to\infty} C_n(f) = f \quad \text{uniformly on } K. \tag{1.10}$$

By specifying the Markov operator T, i.e., the family of representing measures $(\tilde{\mu}_x^T)_{x\in K}$, the parameter $a \geq 0$ and the sequence of measures $(\mu_n)_{n\geq 1}$, we obtain several classes of approximating operators which have been investigated in several papers.

For the convenience of the reader, below we show some examples which, among other things, will be useful to describe some particular cases where our results have rather striking applications. For additional examples we refer to [12, Section 2].

Examples 1.1.
1. Assume $K = [0,1]$ and consider the Markov operator $T_1 : \mathscr{C}([0,1]) \to \mathscr{C}([0,1])$ defined by setting

$$T_1(f)(x) := (1-x)f(0) + xf(1) \tag{1.11}$$

$(f \in \mathscr{C}([0,1]),\ 0 \leq x \leq 1)$.

The Bernstein-Schnabl operators (1.8) *associated with T_1 are the classical Bernstein operators*

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{1.12}$$

$(n \geq 1, f \in \mathscr{C}([0,1]),\ x \in [0,1])$.
Fix $a \geq 0$ and $(\mu_n)_{n\geq 1}$ in $M_1^+([0,1])$; then, from (1.6) *and* (1.7) *we get*

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} \int_0^1 f\left(\frac{k+as}{n+a}\right) d\mu_n(s) \tag{1.13}$$

$(n \geq 1,\ f \in \mathscr{C}([0,1]),\ 0 \leq x \leq 1)$.
For particular choices of the measures μ_n $(n \geq 1)$ we get more specific examples such as, for instance,

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k+b_n}{n+a}\right) \tag{1.14}$$

$(n \geq 1,\ f \in \mathscr{C}([0,1]),\ x \in [0,1])$ where $a > 0$ and, for every $n \geq 1$, $\mu_n := \epsilon_{b_n/a}$ denotes the Dirac measure concentrated at b_n/a with $b_n \leq a$. These operators have been first considered in [27].
As another example, we consider $a > 0$ and two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of real numbers satisfying $0 \leq a_n < b_n \leq 1$ $(n \geq 1)$. If we denote by μ_n the image measure of the Borel-Lebesgue measure λ_1 on $[0,1]$ under the mapping $T_n(x) = (b_n - a_n)x + a_n$ $(0 \leq x \leq 1)$, then from (1.13) *we get*

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} \left(\frac{n+a}{a(b_n-a_n)} \int_{\frac{k+aa_n}{n+a}}^{\frac{k+ab_n}{n+a}} f(t)dt \right) \tag{1.15}$$

These operators have been first considered when $a = 1$ in the paper [14] to which we refer the reader for more details and additional examples.

2. *Let* $Q_d := [0,1]^d$, $d \geq 1$, *and consider the Markov operator* $S_d : \mathscr{C}(Q_d) \to \mathscr{C}(Q_d)$ *defined by*

$$S_d(f)(x) := \sum_{h_1,\ldots,h_d=0}^{1} f(\delta_{h_1 1},\ldots,\delta_{h_d 1}) x_1^{h_1}(1-x_1)^{1-h_1}\cdots x_d^{h_d}(1-x_d)^{1-h_d} \tag{1.16}$$

($f \in \mathscr{C}(Q_d)$, $x = (x_1,\ldots,x_d) \in [0,1]^d$), *where* δ_{ij} *stands for the Kronecker symbol.*

Fix $a \geq 0$ *and a sequence* $(\mu_n)_{n\geq 1}$ *in* $M_1^+(Q_d)$*; then, taking* (1.7) *and [11, (3.1.30)] into account, the operators* C_n *given by* (1.5) *become*

$$\begin{aligned} C_n(f)(x) := &\sum_{h_1,\ldots,h_d=0}^{n} \prod_{i=1}^{d} \binom{n}{h_i} x_i^{h_i}(1-x_i)^{n-h_i} \\ &\times \int_{Q_d} f\left(\frac{h_1+as_1}{n+a},\ldots,\frac{h_d+as_d}{n+a}\right) d\mu_n(s_1,\ldots,s_d) \end{aligned} \tag{1.17}$$

($n \geq 1$, $f \in \mathscr{C}(Q_d)$, $x = (x_1,\ldots,x_d) \in Q_d$).

3. *Denote by* K_d *the canonical simplex in* $\mathbf{R}^d$, $d \geq 1$, *i.e.,*

$$K_d := \left\{(x_1,\ldots,x_d) \in \mathbf{R}^d \,|\, x_i \geq 0\,(i=1,\ldots,d) \text{ and } \sum_{i=1}^{d} x_i \leq 1\right\} \tag{1.18}$$

and consider the canonical Markov operator $T_d : \mathscr{C}(K_d) \to \mathscr{C}(K_d)$ *defined by*

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(0) + \sum_{i=1}^{d} x_i f(e_i) \tag{1.19}$$

($f \in \mathscr{C}(K_d)$ *and* $x = (x_1,\ldots,x_d) \in K_d$) *where, for every* $i = 1,\ldots,d$, $e_i := (\delta_{ij})_{1\leq j\leq d}$, δ_{ij} *being the Kronecker symbol.*

Let $a \geq 0$ *be fixed and consider a sequence* $(\mu_n)_{n\geq 1}$ *in* $M_1^+(K_d)$*; by means of* (1.7) *and [11, (3.1.18)], we obtain*

$$\begin{aligned} &C_n(f)(x) := \\ &\sum_{\substack{h_1,\ldots,h_d=0\\ h_1+\ldots+h_d\leq n}}^{n} P_{n,h}^*(x) \int_{K_d} f\left(\frac{h_1+as_1}{n+a},\ldots,\frac{h_d+as_d}{n+a}\right) d\mu_n(s_1,\ldots,s_d) \end{aligned} \tag{1.20}$$

($n \geq 1, f \in \mathscr{C}(K_d)$ *and* $x = (x_1,\ldots,x_d) \in K_d$), *where, for every* $n \geq 1$, $h =$

$(h_1,\dots,h_d)\in\{0,\dots,n\}^d$, $|h|:=h_1+\ldots+h_d\le n$ *and* $x=(x_1,\dots,x_d)\in K_d$,

$$P^*_{n,h}(x):=\frac{n!}{h_1!\ldots h_d!(n-h_1-\cdots-h_d)!}x_1^{h_1}\ldots x_d^{h_d}\left(1-\sum_{i=1}^d x_i\right)^{n-\sum\limits_{i=1}^d h_i}. \tag{1.21}$$

We end this section with a result that allows us to evaluate the C_n's on each $P_m(K)$ (see (1.1)), $m\ge 1$.

From now on, for any $m,q\ge 1$, $1\le q\le m$, we set

$$N_m(q):=\{(i_1,\dots,i_q)\in\{1,\dots,m\}^q\mid i_r\ne i_s \text{ for } r\ne s\} \tag{1.22}$$

and

$$\widetilde{N}_m:=\{((i_1,\dots,i_q),(j_1,\dots,j_{m-q}))\in N_m(q)\times N_m(m-q)\mid i_h\ne j_k \\ \text{for every } h=1,\dots,q, \text{ and } k=1,\dots,m-q\}. \tag{1.23}$$

Lemma 1.2. *Let $h_1,\dots,h_m\in A(K)$, $m\ge 1$. Then, for every $n\ge 1$,*

$$C_n\left(\prod_{j=1}^m h_j\right)=\frac{1}{(n+a)^m}\left[\left(a^m\int_K\prod_{j=1}^m h_j\;d\mu_n\right)\mathbf{1}+n^mB_n\left(\prod_{j=1}^m h_j\right)\right.$$
$$\left.+\sum_{q=1}^{m-1}a^qn^{m-q}\sum_{((i_1,\dots,i_q),(j_1,\dots,j_{m-q}))\in\widetilde{N}_m}\left(\int_K h_{i_1}\cdots h_{i_q}\;d\mu_n\right)B_n\left(h_{j_1}\cdots h_{j_{m-q}}\right)\right], \tag{1.24}$$

where B_n is defined by (1.8).

Therefore, if

$$T(P_m(K))\subset P_m(K)\quad \textit{for every } m\ge 1, \tag{1.25}$$

then

$$C_n(P_m(K))\subset P_m(K)\quad \textit{for every } n,m\ge 1. \tag{1.26}$$

Proof. Fix $n,m\ge 1$, $h_1,\dots,h_m\in A(K)$ and $x_1,\dots,x_{n+1}\in K$; then

$$\begin{aligned}&\prod_{j=1}^m h_j\left(\frac{x_1+\ldots+x_n+ax_{n+1}}{n+a}\right)\\ =&\frac{1}{(n+a)^m}\prod_{j=1}^m\left(nh_j\left(\frac{x_1+\ldots+x_n}{n}\right)+ah_j(x_{n+1})\right)\\ =&\frac{1}{(n+a)^m}\left(a^m\prod_{j=1}^m h_j(x_{n+1})+n^m\prod_{j=1}^m h_j\left(\frac{x_1+\ldots+x_n}{n}\right)\right.\end{aligned}$$

$$+\sum_{q=1}^{m-1} a^q n^{m-q} \sum_{((i_1,\ldots,i_q),(j_1,\ldots,j_{m-q}))\in\widetilde{N}_m} \Big(h_{i_1}(x_{n+1})\cdots h_{i_q}(x_{n+1})$$
$$\times h_{j_1}\left(\frac{x_1+\ldots+x_n}{n}\right)\cdots h_{j_{m-q}}\left(\frac{x_1+\ldots+x_n}{n}\right)\Big)\Big).$$

Accordingly, we get (1.24) and this completes the proof of (1.26), since, under assumption (1.25), $B_n(P_m(K)) \subset P_m(K)$ for every $n, m \geq 1$ (see [11, Lemma 4.1.1]). □

2 On the differential operators associated with the C_n's

In this section we present an asymptotic formula for the operators C_n in the finite-dimensional setting. In particular, this asymptotic formula involves an elliptic second-order differential operator which is of concern in the study of diffusion problems arising from different areas such as biology, mathematical finance, physics.

From now on, we shall assume that K is a convex compact subset of $\mathbf{R}^d$, $d \geq 1$, having non-empty interior $\mathrm{int}(K)$. Moreover, we fix a Markov operator T on $\mathscr{C}(K)$ satisfying (1.4), $a \geq 0$ and $(\mu_n)_{n\geq 1}$ in $M_1^+(K)$ as in Section 1.

In this setting we shall consider a second-order differential operator which is a first-order perturbation of the elliptic second-order differential operator W_T associated with T, introduced and studied in [9] and [11, Chapter 4].

We begin by recalling the definition of W_T, i.e.,

$$W_T(u) := \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \qquad (u \in \mathscr{C}^2(K)), \tag{2.1}$$

where

$$\alpha_{ij} := T(pr_i pr_j) - pr_i pr_j \qquad (i, j = 1, \ldots, d). \tag{2.2}$$

The differential operator W_T is elliptic and it degenerates on the subset

$$\partial_T K := \{x \in K \mid T(f)(x) = f(x) \text{ for every } f \in \mathscr{C}(K)\} \tag{2.3}$$

which contains the subset of extreme points of K (see [11, (3.1.4)]).

Since $M_1^+(K)$ is weakly compact (see [20]), unless replacing $(\mu_n)_{n\geq 1}$ with a suitable subsequence, we may assume that it converges weakly to some $\mu \in M_1^+(K)$, i.e.,

$$\lim_{n\to\infty}\int_K f\,d\mu_n = \int_K f\,d\mu \quad \text{for every } f \in \mathscr{C}(K). \tag{2.4}$$

Let $b = (b_1, \ldots, b_d) \in K$ be the barycenter of μ , i.e.,

$$\int_K pr_i \, d\mu = b_i \quad \text{for every } i = 1, \ldots, d \tag{2.5}$$

(see, e.g., [4, p. 55]), and let us set

$$\beta_i := a(b_i - pr_i) \qquad (i = 1, \ldots, d); \tag{2.6}$$

then the differential operator we are interested in studying is defined by

$$\begin{aligned} V_T(u)(x) &:= W_T(u)(x) + \sum_{i=1}^{d} \beta_i(x) \frac{\partial u}{\partial x_i}(x) \\ &= \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} \beta_i(x) \frac{\partial u}{\partial x_i}(x) \end{aligned} \tag{2.7}$$

$(u \in \mathscr{C}^2(K),\ x \in K)$.

On account of the explicit examples of operators W_T described in [11, Section 4.2], below we can detail some examples.

Examples 2.1.
1. ***The unit interval.*** *Setting* $e_2(x) := x^2$ $(0 \le x \le 1)$ *and* $\alpha = T(e_2) - e_2$, *then the differential operator* (2.7) *turns into*

$$V_T(u)(x) = \frac{\alpha(x)}{2} u''(x) + a(b - x)u'(x) \tag{2.8}$$

$(u \in \mathscr{C}^2([0,1]), 0 \le x \le 1)$, *with* $a \ge 0$ *and* $b \in [0,1]$.

Note that, in this particular case, $0 \le \alpha(x) \le x(1-x)$ *for every* $0 < x < 1$. *Conversely, if* $\alpha \in \mathscr{C}([0,1])$ *satisfies a similar inequality, then there always exists a Markov operator* T *on* $\mathscr{C}(K)$ *satisfying* (1.4) *such that* $\alpha = T(e_2) - e_2$ *([11, Example 4.2.1, 1]).*

For instance, if $p \in \mathscr{C}([0,1])$ *is a polynomial of degree not greater than 2 such that* $0 \le p(x) \le \frac{x(1-x)}{2}$ $(0 \le x \le 1)$, *and if we consider the Markov operator* $T(f)(x) := (1 - x - 2p(x))f(0) + 4p(x)f(\frac{1}{2}) + (x - 2p(x))f(1)$, $(f \in \mathscr{C}([0,1]),\ 0 \le x \le 1)$, *then*

$$V_T(u)(x) = \frac{x(1-x) - p(x)}{2} u''(x) + a(b - x)u'(x) \tag{2.9}$$

$(u \in \mathscr{C}^2([0,1]), 0 \le x \le 1)$.

As another example, given $\lambda \in [0,1]$, *consider the Markov operator* $T(f)(x) := \lambda[xf(1) + (1-x)f(0)] + (1-\lambda)f(x)$ $(f \in \mathscr{C}([0,1]),\ 0 \le x \le 1)$. *Then*

$$V_T(u)(x) = \lambda \frac{x(1-x)}{2} u''(x) + a(b - x)u'(x) \tag{2.10}$$

$(u \in \mathscr{C}^2([0,1]), 0 \le x \le 1)$.

2. ***The unit hypercube*** Q_d ***of*** $\mathbf{R}^d$***,*** $d \ge 1$***.*** *Consider the particular case where* $T = S_d$ *(see* (1.16)*). Then*

$$V_T(u)(x) = \frac{1}{2}\sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + a\sum_{i=1}^{d}(b_i - x_i)\frac{\partial u}{\partial x_i}(x) \tag{2.11}$$

$(u \in \mathscr{C}^2(Q_d)$, $x = (x_1, \ldots, x_d) \in Q_d)$, *where* $b = (b_1, \ldots, b_d) \in Q_d$ *and* $a \ge 0$.

3. ***The canonical simplex*** K_d ***of*** $\mathbf{R}^d$***,*** $d \ge 1$***.*** *Assume that* $T = T_d$ *(see* (1.19)*). Then*

$$V_T(u)(x)=\frac{1}{2}\sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x)-\sum_{1\le i<j\le d} x_i x_j\frac{\partial^2 u}{\partial x_i \partial x_j}(x)+a\sum_{i=1}^{d}(b_i-x_i)\frac{\partial u}{\partial x_i}(x) \tag{2.12}$$

$(u \in \mathscr{C}^2(K_d)$, $x = (x_1, \ldots, x_d) \in K_d)$, *where* $b = (b_1, \ldots, b_d) \in K_d$ *and* $a \ge 0$.

4. ***Ellipsoids and balls in*** $\mathbf{R}^d$***,*** $d \ge 2$***.*** *Assume that the boundary* ∂K *of* K *is an ellipsoid, i.e., there exists a real symmetric and positive-definite matrix* $R = (r_{ij})_{1 \le i,j \le d}$ *and* $\overline{x} = (\overline{x}_i)_{1 \le i \le d} \in \mathbf{R}^d$ *such that*

$$K = \left\{ x \in \mathbf{R}^d \;\middle|\; Q(x - \overline{x}) := \sum_{i,j=1}^{d} r_{ij}(x_i - \overline{x}_i)(x_j - \overline{x}_j) \le 1 \right\}. \tag{2.13}$$

Furthermore, consider a strictly elliptic differential operator

$$L(u)(x) := \sum_{i,j=1}^{d} c_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \tag{2.14}$$

$(u \in \mathscr{C}^2(\mathrm{int}(K))$, $x \in \mathrm{int}(K))$ *associated with a real symmetric and positive-definite matrix* $C = (c_{ij})_{1 \le i,j \le d}$ *and denote by* T_L *the relevant Poisson operator on* $\mathscr{C}(K)$, *defined by assigning to every* $f \in \mathscr{C}(K)$ *the unique solution of the Dirichlet problem*

$$\begin{cases} Lu = 0 & \text{on } \mathrm{int}(K), \quad u \in \mathscr{C}(K) \cap \mathscr{C}^2(\mathrm{int}(K)); \\ u = f & \text{on } \partial K. \end{cases} \tag{2.15}$$

With no loss of generality we also assume that $\sum_{i,j=1}^{d} r_{ij}c_{ij} = 1$ *(see [11, Remark 4.2.4]).*

In this case, the differential operator (2.7) *for* $T = T_L$ *turns into*

$$V_T(u)(x) = \frac{(1-Q)(x-\overline{x})}{2} L(u)(x) + a \sum_{i=1}^{d} (b_i - x_i) \frac{\partial u}{\partial x_i}(x) \tag{2.16}$$

$(u \in \mathscr{C}^2(K), x = (x_1, \ldots, x_d) \in K)$, *where* $b = (b_1, \ldots, b_d) \in K$ *and* $a \geq 0$.

In particular, if K *denotes the closed ball (with respect to the Euclidean norm* $\|\cdot\|_2$*) of center* $\overline{x} \in \mathbf{R}^d$ *and radius* $r > 0$ *and* L *is the Laplacian* Δ, *then*

$$V_T(u)(x) = \frac{r^2 - \|x-\overline{x}\|_2}{2d} \Delta(u)(x) + a \sum_{i=1}^{d} (b_i - x_i) \frac{\partial u}{\partial x_i}(x) \tag{2.17}$$

$(u \in \mathscr{C}^2(K), x \in K)$, *where* $b = (b_1, \ldots, b_d) \in K$ *and* $a \geq 0$.

From now on, for a given $x \in K$, we denote by $\Psi_x \in \mathscr{C}(K)$ the function defined by

$$\Psi_x(y) := y - x \tag{2.18}$$

for every $y \in K$, and by $d_x \in \mathscr{C}(K)$ the function defined by

$$d_x(y) := \|y - x\|_2 \qquad (y \in K). \tag{2.19}$$

Note that, since for every $y = (y_1, \ldots, y_d) \in K$ and $i = 1, \ldots, d$,

$$(pr_i \circ \Psi_x)(y) = pr_i(y-x) = y_i - x_i = pr_i(y) - x_i,$$

then (see (2.2)), for every $x \in K$ and $i, j = 1, \ldots, d$,

$$\alpha_{ij}(x) = T((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x). \tag{2.20}$$

The link between the differential operator V_T and the operators C_n is enlightened by the next result.

Theorem 2.2. *Under assumptions* (2.4), *for every* $u \in \mathscr{C}^2(K)$,

$$\lim_{n\to\infty} n(C_n(u) - u) = V_T(u) \quad \textit{uniformly on } K.$$

Proof. According to Theorem 1.5.2 of [11] (see also [13, Theorem 3.5]), the claim will be proved if we show that, for every $i, j = 1, \ldots, d$,

(a) $\lim_{n\to\infty} nC_n(pr_i \circ \Psi_x)(x) - \beta_i(x) = 0$ (see (2.6)) uniformly w.r.t. $x \in K$,

(b) $\lim_{n\to\infty} nC_n((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x) - \alpha_{ij}(x) = 0$ (see (2.2)) uniformly w.r.t. $x \in K$,

(c) $\sup_{n\geq 1, x\in K} nC_n(d_x^2)(x) < +\infty$, and

(d) $\lim_{n\to\infty} nC_n(d_x^4)(x) = 0$ uniformly w.r.t. $x \in K$,

where, for a fixed $x \in K$, d_x ad Ψ_x are given by (2.19) and (2.18), respectively.

We proceed to verify (a). To this end, fix $i = 1, \dots, d$ and $x \in K$; since the function $pr_i \circ \Psi_x \in A(K)$, according to formula (1.24) for $m = 1$, we get

$$
\begin{aligned}
&C_n(pr_i \circ \Psi_x)(x) = \frac{a}{n+a}\left[\int_K (pr_i \circ \Psi_x)\ d\mu_n\right] + \frac{n}{n+a}(pr_i \circ \Psi_x)(x)\\
&= \frac{a}{n+a}\left[\int_K pr_i(y-x)\ d\mu_n(y)\right] = \frac{a}{n+a}\left[\int_K pr_i\ d\mu_n - x_i\right].
\end{aligned}
$$

Hence, for any $i = 1, \dots, d$ and $x \in K$,

$$
\begin{aligned}
0 &\le |nC_n(pr_i \circ \Psi_x)(x) - \beta_i(x)|\\
&\le \left|\frac{na}{n+a}\int_K pr_i\ d\mu_n - ab_i\right| + \left|ax_i - \frac{na}{n+a}x_i\right|\\
&\le a\left|\frac{n}{n+a}\int_K pr_i\ d\mu_n - b_i\right| + a\left(1 - \frac{n}{n+a}\right)\sup_{x\in K}\|x\|_2
\end{aligned}
$$

and we get the required assertion thanks to (2.4) and (2.5).

To prove statement (b) we preliminary notice that, by virtue of formula (1.24) for $m = 2$ and [11, formula (3.2.3)] (see, also, [9, Proposition 3.2]), for every $x \in K$ and $i, j = 1, \dots, d$,

$$
\begin{aligned}
&C_n((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x) = \frac{1}{(n+a)^2}\\
&\times\left\{a^2\int_K (pr_i \circ \Psi_x)(pr_j \circ \Psi_x)\ d\mu_n + n^2 B_n((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x)\right\}\\
&= \frac{1}{(n+a)^2}\left\{a^2\int_K (pr_i \circ \Psi_x)(pr_j \circ \Psi_x)\ d\mu_n + n\alpha_{ij}(x)\right\}.
\end{aligned}
$$

Hence, for every $x \in K$,

$$
\begin{aligned}
&0 \le |nC_n((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x) - \alpha_{ij}(x)|\\
&\le \frac{na^2}{(n+a)^2}\int_K |(pr_i \circ \Psi_x)(pr_j \circ \Psi_x)|\ d\mu_n + \left(1 - \frac{n^2}{(n+a)^2}\right)|\alpha_{ij}(x)|\\
&\le \frac{na^2}{(n+a)^2}\sup_{x\in K}||(pr_i \circ \Psi_x)(pr_j \circ \Psi_x)||_\infty + \left(1 - \frac{n^2}{(n+a)^2}\right)||\alpha_{ij}||_\infty
\end{aligned}
$$

and this completes the proof of (b).

Finally, we proceed to verify conditions (c) and (d).

To this end, we first recall that $r_0 := \sup_{n\geq 1, x\in K} nB_n(d_x^2)(x) < +\infty$ and $\lim_{n\to\infty} nB_n(d_x^4)(x) = 0$ uniformly w.r.t. $x \in K$ ([9, Theorem 4.2]; see also [11, Theorem 4.1.5]).

Moreover, for every $n \geq 1$, $q \geq 2$ and $x, x_1, \dots, x_{n+1} \in K$,

$$
\begin{aligned}
&d_x^q\left(\frac{x_1 + \ldots + ax_{n+1}}{n+a}\right) = \left|\left|\frac{x_1 + \ldots + ax_{n+1}}{n+a} - x\right|\right|_2^q \\
&\leq \left(\frac{a}{n+a}\right)^q 2^{q-1} d_x^q(x_{n+1}) + \left(\frac{n}{n+a}\right)^q 2^{q-1} d_x^q\left(\frac{x_1 + \ldots + x_n}{n}\right).
\end{aligned}
$$

Hence

$$
C_n(d_x^q)(x) \leq \left(\frac{a}{n+a}\right)^q 2^{q-1} \int_K d_x^q \, d\mu_n + \left(\frac{n}{n+a}\right)^q 2^{q-1} B_n(d_x^q)(x).
$$

Accordingly, for any $n \geq 1$ and $x \in K$,

$$
\begin{aligned}
nC_n(d_x^2)(x) &\leq \frac{2na^2}{(n+a)^2} \int_K d_x^2 \, d\mu_n + 2\left(\frac{n}{n+a}\right)^2 nB_n(d_x^2)(x) \\
&\leq \frac{2na^2}{(n+a)^2} \rho(K)^2 + 2r_0
\end{aligned}
$$

and

$$
\begin{aligned}
0 \leq nC_n(d_x^4)(x) &\leq \frac{8na^4}{(n+a)^4} \int_K d_x^4 \, d\mu_n + 8\left(\frac{n}{n+a}\right)^4 nB_n(d_x^4)(x) \\
&\leq \frac{8na^4}{(n+1)^4} \rho(K)^4 + 8\left(\frac{n}{n+a}\right)^4 nB_n(d_x^4)(x),
\end{aligned}
$$

where $\rho(K) := \sup\{||y-x||_2 \mid x, y \in K\}$.

This completes the proof. $\square$

3 The associated Markov semigroup

The main aim of this section is to show that the differential operator $(V_T, \mathscr{C}^2(K))$ (see (2.7)) is closable and its closure is the generator of a Markov semigroup on $\mathscr{C}(K)$ which in turn may be approximated by suitable iterates of the operators C_n.

These results allow us to represent the solutions to the initial-boundary value differential problems governed by such a semigroup in terms of the C_n's and to deduce some spatial regularity properties of the relevant solutions. For unexplained terminology concerning semigroup theory, we refer, e.g., to [11, Chapter 2].

Theorem 3.1. *Consider the sequence $(C_n)_{n\geq 1}$ of operators as in* (1.5) *associated with a Markov operator $T : \mathscr{C}(K) \to \mathscr{C}(K)$ satisfying* (1.4) *and* (1.25)*, $a \geq 0$ and a sequence $(\mu_n)_{n\geq 1}$ of probability Borel measures on K such that* (2.4) *holds true.*

Then the differential operator $(V_T, \mathscr{C}^2(K))$ defined by (2.7) *is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}(K)$ such that, if $t \geq 0$, $f \in \mathscr{C}(K)$ and $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, then*

$$T(t)(f) = \lim_{n\to\infty} C_n^{k(n)}(f) \quad \textit{uniformly on } K, \tag{3.1}$$

where each $C_n^{k(n)}$ denotes the iterate of C_n of order $k(n)$.

Moreover, $P_\infty(K)$ (and hence $\mathscr{C}^2(K)$ too) is a core for $(A_T, D(A_T))$ and $T(t)(P_m(K)) \subset P_m(K)$ for every $t \geq 0$ and $m \geq 1$.

Proof. First of all we remark that each subspace $P_m(K)$, $m \geq 1$, of $\mathscr{C}^2(K)$ is finite dimensional, it is invariant under every operator C_n by virtue of Lemma 1.2 and assumptions (1.4) and (1.25), and $P_\infty(K)$ is dense in $\mathscr{C}(K)$.

Moreover, from Theorem 2.2, we get

$$\lim_{n\to\infty} n(C_n(u) - u) = V_T(u) \quad \text{uniformly on } K,$$

for every $u \in \mathscr{C}^2(K)$, and hence for every $u \in P_\infty(K)$.

Then, from Corollary 2.2.11 of [11], it follows that $(V_T, \mathscr{C}^2(K))$ is closable and its closure $(A_T, D(A_T))$ is the generator of a contraction C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}(K)$ such that for every $t \geq 0$ and $f \in \mathscr{C}(K)$,

$$T(t)(f) = \lim_{n\to\infty} C_n^{k(n)}(f) \quad \text{uniformly on } K$$

for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty} k(n)/n = t$. Moreover, $P_\infty(K)$ is a core for $(A_T, D(A_T))$.

Formula (3.1) implies that each $T(t)$ is a Markov operator and hence the semigroup is Markovian.

From (1.26) it also follows that, if $f \in P_m(K)$ for some $m \geq 1$, then $C_n^k \in P_m(K)$ for every $n, k \geq 1$; hence, for every $t \geq 0$ and every sequence $(k(n))_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty} k(n)/n = t$, we get

$$T(t)(f) = \lim_{n\to\infty} C_n^{k(n)}(f) \in P_m(K),$$

since $P_m(K)$ is closed, and this completes the proof. □

Remarks 3.2.

1. A simple sequence $(k(n))_{n\geq 1}$ to which formula (3.1) can be applied is given by $k(n) := [nt]$, $[nt]$ denoting the integer part of nt $(n \geq 1)$.

2. According to [11, Remark 2.2.12], if $u, v \in \mathscr{C}(K)$ and $\lim_{n\to\infty} n(C_n(u) - u) = v$ uniformly on K, then $u \in D(A_T)$ and $A_T(u) = v$.

In particular, if $\lim_{n\to\infty} n(C_n(u) - u) = 0$ uniformly on K, then $u \in D(A_T)$ and $A_T(u) = 0$ (a saturation result for the operators C_n, $n \geq 1$).

3. In [10] and in [11, Section 4.3] condition (1.25) is carefully analyzed and several examples are furnished. In particular, all the Markov operators (1.11), (1.16) and (1.19) as well as the Poisson operator considered in Example 2.1.4 verify (1.25). Accordingly, Theorem 3.1 applies to the differential operator (2.8) (with $\alpha(x) = x(1-x), 0 \leq x \leq 1$), (2.11), (2.12), (2.16) and (2.17).

Let us now consider the Cauchy problem associated with the operator $(A_T, D(A_T))$ defined in Theorem 3.1, namely

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A_T(u(\cdot,t))(x) & x \in K, \quad t \geq 0; \\ \\ u(x,0) = u_0(x) & u_0 \in D(A_T), \quad x \in K. \end{cases} \tag{3.2}$$

Since $(A_T, D(A_T))$ generates a Markov semigroup, such a Cauchy problem admits a unique solution $u : K \times [0, +\infty[\to \mathbf{R}$ given by $u(x,t) = T(t)(u_0)(x)$ for every $x \in K$ and $t \geq 0$ (see, e.g., [26, Chapter A-II]). Hence, by Theorem 3.1, that solution may be approximated in terms of suitable iterates of the C_n's, i.e.,

$$u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} C_n^{[nt]}(u_0)(x), \tag{3.3}$$

where the limit is uniform with respect to $x \in K$.

Moreover, A_T coincides with the elliptic second-order differential operator V_T defined by (2.7) on $\mathscr{C}^2(K)$.

Therefore, if $u_0 \in P_m(K)$, then $u(x,t)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{2}\sum\limits_{i,j=1}^{d} \alpha_{ij}(x)\dfrac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + a\sum\limits_{i=1}^{d}(b_i - x_i)\dfrac{\partial u(x,t)}{\partial x_i} & x \in K, t \geq 0; \\ \\ u(x,0) = u_0(x) & x \in K \end{cases}$$

and $u(\cdot,t) \in P_m(K)$ for every $t \geq 0$.

By investigating some preservation properties of the operators C_n, by means of formula (3.3) we may infer some spatial regularity properties of the solutions u, i.e., properties of the functions $u(\cdot,t)$, $t \geq 0$.

Below we show some results in this direction which are revolved around an analysis on the C_n's we carried out in [12, Section 5].

For a given norm $\|\cdot\|$ on $\mathbf{R}^d$ and for every $M \geq 0$ and $0 < \alpha \leq 1$ we denote by $\mathrm{Lip}(M, \alpha)$ the class of all Hölder continuous functions f on K

with exponent α and constant M, i.e., such that $|f(x)-f(y)| \leq M\|x-y\|^\alpha$ for every $x, y \in K$.

Corollary 3.3. *Under the same assumptions of Theorem 3.1, if* $T(\mathrm{Lip}(1,1)) \subset \mathrm{Lip}(1,1)$ *and* $u_0 \in \mathrm{Lip}(M,1)$ *for some* $M \geq 0$*, then* $u(\cdot,t) \in \mathrm{Lip}(M,1)$ *for every* $t \geq 0$*.*

Proof. In [12, Proposition 5.1] we proved that $C_n(\mathrm{Lip}(M,1)) \subset \mathrm{Lip}(M,1)$ for every $n \geq 1$, provided that $T(\mathrm{Lip}(1,1)) \subset \mathrm{Lip}(1,1)$. By iterating this inclusion, since $\mathrm{Lip}(M,1)$ is closed under uniform norm, we get the result by applying (3.3). $\square$

Remark 3.4. The operators T_1, S_d and T_d defined by (1.11), (1.16) and (1.19), respectively, verify the hypotheses of Corollary 3.3 which then applies to the context of the Examples 2.1.1, 2.1.2 and 2.1.3.

We pass now to present some results about the convexity of $u(\cdot,t)$, $t \geq 0$. To this end, for a given $f \in \mathscr{C}(K)$, we set

$$\Delta(f;x,y) := B_2(f)(x) + B_2(f)(y) - 2\iint_{K^2} f\left(\frac{s+t}{2}\right) d\tilde{\mu}_x^T(s)d\tilde{\mu}_y^T(t) \tag{3.4}$$

for every $x, y \in K$, where the operator B_2 is defined as in (1.8).

Then, under suitable assumptions on the sign of the quantity (3.4), taking [12, Theorem 5.4] into account, we may infer some information about the convexity of $u(\cdot,t)$, as the next result shows.

Corollary 3.5. *Suppose that* T *satisfies the following assumptions:*

(c_1) T *maps continuous convex functions into (continuous) convex functions;*

(c_2) $\Delta(f;x,y) \geq 0$ *for every convex function* $f \in \mathscr{C}(K)$ *and for every* $x, y \in K$*.*

If $u_0 \in D(A_T)$ *is convex, then* $u(\cdot,t)$ *is convex for every* $t \geq 0$*.*

Remark 3.6. Conditions (c_1) and (c_2) are discussed in detail in [11, Remark 3.4.4 and Examples 3.4.5-3.4.11]. In the case of the unit interval they are satisfied for $T = T_1$ and when T is the Bernstein operator (1.12) of order $n \geq 1$ (see [1] and [23] for some recent advances).

The next result is concerned with the simplex K_d and the Markov operator T_d. As we showed in [12, Corollary 5.6], the relevant operators C_n defined by (1.20) preserve continuous axially convex functions, i.e., those continuous functions which are convex on each segment parallel to a segment joining two vertices of the simplex.

The class of such functions is closed under the uniform norm and hence, again by (3.3), we get the following further result.

Corollary 3.7. *If $K = K_d$ denotes the canonical simplex of $\mathbf{R}^d$, $d \geq 1$, and $T = T_d$ (see* (1.19)*), by referring to the differential operator* (2.12) *and to the solution $u(\cdot, t)$ of the relevant Cauchy problem, then $u(\cdot, t)$ is axially convex for every $t \geq 0$, provided that $u_0 \in D(A_T)$ is axially convex.*

Remarks 3.8.

1. The differential operators (2.11) and (2.12) are particular cases of the so-called Fleming-Viot operators which appear in the theory of Fleming-Viot processes involved in the description of a stochastic process associated with a diffusion approximation of a gene frequency model in population genetics. Their generators have been object of several papers. For more details we refer to [11, Subsection 2.3.4 and Section 5.8, together with the relevant Notes and Comments]. In particular, in [11, Section 5.8] an approximation of the semigroup $(T(t))_{t\geq 0}$ in terms of other sequences of positive operators is also discussed.
2. In the context of Example 2.1.4 the differential operator (2.17) is referred to as a diffusion operator which describes analytically a strong Markov process with continuous path in K (see [28] for more details). Moreover, in [28] the domain $D(A_{T_L})$ is described in terms of the so-called Ventcel' boundary conditions.

We deepen now the previous results in the particular case where $K = [0, 1]$; in this setting it is possible to explicitly describe the domain $D(A_T)$ which, as recalled by (3.2), consists of all initial data for which the Cauchy problem (3.2) has a unique solution given by (3.3).

Consider a Markov operator $T : \mathscr{C}([0, 1]) \to \mathscr{C}([0, 1])$ such that $T(e_1) = e_1$ and which not necessarily maps polynomials into polynomials of the same degree and set

$$\alpha := T(e_2) - e_2, \tag{3.5}$$

where $e_i(x) := x^i$ $(0 \leq x \leq 1, i = 1, 2)$. Then

$$0 \leq \alpha(x) \leq x(1 - x) \qquad (0 \leq x \leq 1). \tag{3.6}$$

In [11, Example 4.2.1, 1] we pointed out that, conversely, if $\alpha \in \mathscr{C}([0, 1])$ satisfies (3.6), then it can constructively be furnished a Markov operator T on $\mathscr{C}([0, 1])$ such that $T(e_1) = e_1$ and $\alpha = T(e_2) - e_2$.

Furthermore, fix $a \geq 0$ and $b \in [0, 1]$ and let $(\mu_n)_{n\geq 1}$ be an arbitrary sequence of probability Borel measures on [0, 1] satisfying (2.4) and (2.5). Therefore, from Theorem 2.2 it follows that

$$\lim_{n\to\infty} n(C_n(u) - u) = V_T(u) \quad \text{uniformly on } [0, 1] \tag{3.7}$$

for every $u \in \mathscr{C}^2([0, 1])$, where (see (2.8))

$$V_T(u)(x) = \frac{\alpha(x)}{2} u''(x) + a(b - x)u'(x) \quad (0 \leq x \leq 1).$$

We now consider the differential operator

$$Z_T(u)(x) = \frac{\alpha(x)}{2} u''(x) + a(b-x)u'(x) \quad (0 < x < 1) \tag{3.8}$$

defined for every $u \in \mathscr{C}^2(]0,1[)$ and set

$$D_M(Z_T) := \left\{ u \in \mathscr{C}([0,1]) \cap \mathscr{C}^2(]0,1[) \;\middle|\; \lim_{\substack{x\to 0^+ \\ x \to 1^-}} Z_T(u)(x) \in \mathbf{R} \right\}, \tag{3.9}$$

$$D_{VM}(Z_T) := \left\{ u \in D_M(Z_T) \;\middle|\; \lim_{x\to 0^+} Z_T(u)(x) = 0 \right\} \tag{3.10}$$

and

$$D_{MV}(Z_T) := \left\{ u \in D_M(Z_T) \;\middle|\; \lim_{x\to 1^-} Z_T(u)(x) = 0 \right\}. \tag{3.11}$$

For every $u \in \mathscr{C}([0,1])$ belonging to such domains, $Z_T(u)$ can be continuously extended to $[0,1]$ and this extension will still be denoted by $Z_T(u)$.

From now on we shall assume that

(i) $0 < \alpha(x)$ for each $0 < x < 1$;

(ii) α is differentiable at 0 and at 1 and $\alpha'(0) \neq 0 \neq \alpha'(1)$;

(iii) The function

$$r(x) := \begin{cases} \dfrac{ab}{2\alpha'(0)} & \text{if } x = 0; \\[2ex] \dfrac{a(b-x)x(1-x)}{2\alpha(x)} & \text{if } 0 < x < 1; \\[2ex] -\dfrac{a(b-1)}{2\alpha'(1)} & \text{if } x = 1 \end{cases} \tag{3.12}$$

is Hölder continuous at 0 and at 1.

Condition (iii) is satisfied, for instance, if α is differentiable in [0, 1].

Furthermore, assume $a > 0$ and set

$$D(Z_T) := \begin{cases} D_M(Z_T) & \text{if } ab \geq \frac{1}{2}\alpha'(0) \text{ and } a(b-1) \leq \frac{1}{2}\alpha'(1); \\[2ex] D_{MV}(Z_T) & \text{if } a \geq \frac{1}{2}\alpha'(0) \text{ and } b = 1; \\[2ex] D_{VM}(Z_T) & \text{if } a \geq -\frac{1}{2}\alpha'(1) \text{ and } b = 0. \end{cases} \tag{3.13}$$

Thus, Z_T is a linear operator from $D(Z_T)$ into $\mathscr{C}([0,1])$. Moreover,

$$Z_T = V_T \qquad \text{on } \mathscr{C}^2([0,1]) \cap D(Z_T). \tag{3.14}$$

Theorem 3.9. *Under assumptions (i)-(iii), assume further that $a > 0$ and that one of the following statements holds true:*

(1) $ab \geq \frac{1}{2}\alpha'(0)$ *and* $a(b-1) \leq \frac{1}{2}\alpha'(1)$*;*

(2) $a \geq \frac{1}{2}\alpha'(0)$ *and* $b = 1$*;*

(3) $a \geq -\frac{1}{2}\alpha'(1)$ *and* $b = 0$*.*

Then $(Z_T, D(Z_T))$ is the generator of a Markov semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}([0,1])$ and $\mathscr{C}^2([0,1]) \cap D(Z_T)$ is a core for $(Z_T, D(Z_T))$. Moreover, the approximation formula (3.1) *holds true for every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$.*

Finally, $(Z_T, D(Z_T))$ coincides with the closure of $(V_T, \mathscr{C}^2([0,1]) \cap D(Z_T))$. In particular, under the hypothesis (1), $(Z_T, D_M(Z_T))$ coincides with the closure of $(V_T, \mathscr{C}^2([0,1]))$.

Proof. The first part of the result follows from Theorems 5.7.7 and 5.7.13 of [11]. The second part is a consequence of (3.7), (3.14) and Corollary 2.2.3 of [11] (see, also [11, Remark 2.2.4, 2]).

Finally, from (3.14) it also turns out that $(V_T, \mathscr{C}^2([0,1]) \cap D(Z_T))$ is closable and its closure $(B_T, D(B_T))$ satisfies $D(B_T) \subset D(Z_T)$ and $Z_T = B_T$ on $D(B_T)$. Accordingly, from Proposition 2.1.7 of [11], part (g), we infer that, actually, $(Z_T, D(Z_T)) = (B_T, D(B_T))$. □

Remarks 3.10.
1. If $a = 0$, then the differential operator $(V_T, \mathscr{C}^2([0,1]))$ is closable and its closure coincides with the differential operator $(Z_T, D_V(Z_T))$, where

$$D_V(Z_T) := \left\{ u \in D_M(Z_T) \ \mid \ \lim_{\substack{x\to 0^+ \\ x\to 1^-}} Z_T(u)(x) = 0 \right\}. \tag{3.15}$$

Moreover, $(Z_T, D_V(Z_T))$ generates a Markov semigroup on [0, 1] that can be approximated by iterates of Bernstein-Schnabl operators associated with T (see [11, Section 4.5]).
2. A special case of Theorem 3.9, part (1), has been previously obtained in [15, Section 3]. Moreover, in the particular case where $\alpha(x) = x(1-x)$ $(0 \leq x \leq 1)$, the generation properties of the differential operator (2.8) have been largely studied in several papers (see, e.g., [17], [18], [19], [24]).

In [18, Theorem 3.3], among other things, the authors showed that, in the case (1) of Theorem 3.9, i.e., $ab \geq \frac{1}{2}$ and $a(1-b) \geq \frac{1}{2}$, for every $f \in \mathscr{C}([0,1])$,

$$\lim_{t\to+\infty} T(t)(f) = \frac{1}{B(\gamma+1, \delta+1)} \int_0^1 x^\gamma (1-x)^\delta f(x)\, dx \tag{3.16}$$

uniformly on $[0,1]$, where $\gamma = 2ab-1$, $\delta = 2a(1-b)-1$ and $B(\gamma+1,\delta+1) := \int_0^1 t^\gamma (1-t)^\delta \, dt$ denotes the familiar Euler beta function.

In the case (2) (resp. (3)) of Theorem 3.9, i.e., $a \geq \frac{1}{2}$ and $b = 1$ (resp. $b = 0$), then, on account of Theorem 4.2 of [14], it follows that, for every $f \in C([0,1])$,

$$\lim_{t \to +\infty} T(t)f = f(1) \quad \text{uniformly on } [0,1]$$

(resp.

$$\lim_{t \to +\infty} T(t)f = f(0) \quad \text{uniformly on } [0,1]).$$

4 Generation results in L^p-spaces

We end this paper by presenting some generation results in L^p-spaces, $1 \leq p < +\infty$, in the setting of the hypercube Q_d of $\mathbf{R}^d$.

Assume, in particular, that $K = Q_d$ and, given $a \geq 0$, for every $h = (h_1, \ldots, h_d) \in \{0, \ldots, n\}^d$ and $n \geq 1$, set

$$Q_{n,h}(a) := \prod_{i=1}^{d} \left[\frac{h_i}{n+a}, \frac{h_i+a}{n+a} \right] \subset Q_d; \tag{4.1}$$

then

$$\bigcup_{h \in \{0,\ldots,n\}^d} Q_{n,h}(a) = Q_d.$$

Moreover, assume that all the μ_n coincide with the Borel-Lebesgue measure λ_d on Q_d.

In such a case the operators C_n in (1.17) are well defined on $L^1(Q_d)$ and, for every $n \geq 1$, $f \in L^1(Q_d)$ and $x = (x_1, \ldots, x_d) \in Q_d$,

$$C_n(f)(x) = \sum_{h_1,\ldots,h_d=0}^{n} \prod_{i=1}^{d} \binom{n}{h_i} x_i^{h_i} (1-x_i)^{n-h_i} \int_{Q_d} f\left(\frac{h+au}{n+a}\right) du \tag{4.2}$$

and, if $a > 0$,

$$C_n(f)(x) = \sum_{h_1,\ldots,h_d=0}^{n} \prod_{i=1}^{d} \binom{n}{h_i} x_i^{h_i} (1-x_i)^{n-h_i} \left(\frac{n+a}{a}\right)^d \int_{Q_{n,h}(a)} f(v)\, dv. \tag{4.3}$$

In particular, if $d = 1$, then (4.2) turns into

$$C_n(f)(x) = \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \int_0^1 f\left(\frac{h+as}{n+a}\right) ds \tag{4.4}$$

and, if $a > 0$,

$$C_n(f)(x) = \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \left(\frac{n+a}{a}\right) \int_{\frac{h}{n+a}}^{\frac{h+a}{n+a}} f(t)\, dt \tag{4.5}$$

$(n \geq 1, f \in L^1([0,1]), x \in [0,1])$.

The sequence $(C_n)_{n\geq 1}$ is an approximation process in L^p-spaces, $1 \leq p < +\infty$. In fact, as it was shown in [12, Theorem 4.1]), for every $n \geq 1$, $1 \leq p < +\infty$ and $f \in L^p(Q_d)$,

$$\|C_n(f)\|_p \leq M^{1/p}\|f\|_p$$

where

$$M := \sup_{n\geq 1}\left(\frac{n+a}{a(n+1)}\right)^d, \tag{4.6}$$

and, for every $f \in L^p(Q_d)$ and $1 \leq p < +\infty$,

$$\lim_{n\to\infty} C_n(f) = f \qquad \text{in } L^p(Q_d). \tag{4.7}$$

The operator V_{S_d} associated with the Markov operator S_d on Q_d defined by (1.16), according to (2.5), is given by

$$V_{S_d}(u)(x) = \frac{1}{2}\sum_{i=1}^d x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + a\sum_{i=1}^d \left(\frac{1}{2} - x_i\right)\frac{\partial u}{\partial x_i}(x) \tag{4.8}$$

($a \geq 0$, $u \in \mathscr{C}^2(Q_d)$, $x = (x_1, \dots, x_d) \in Q_d$), since the barycenter of λ_d on Q_d is $b = (1/2, \dots, 1/2)$.

S_d satisfies (1.4) and (1.25), so that, by Theorem 3.1, $(V_{S_d}, \mathscr{C}^2(Q_d))$ is closable and its closure $(A_{S_d}, D(A_{S_d}))$ generates a Markov semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}(Q_d)$ such that, for every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $k(n)/n \to t$, one has that, for every $f \in \mathscr{C}(K)$,

$$T(t)(f) = \lim_{n\to\infty} C_n^{k(n)}(f) \quad \text{uniformly on } Q_d. \tag{4.9}$$

The next result shows that, if $a \geq 1$, the semigroup $(T(t))_{t\geq 0}$ may be extended to a positive contraction semigroup on $L^p(Q_d)$ and that the representation formula (4.9) extends to $L^p(Q_d)$.

Theorem 4.1. *Consider the sequence $(C_n)_{n\geq 1}$ defined by (4.3) along with a parameter $a \geq 1$. Then, for every $p \geq 1$, $(T(t))_{t\geq 0}$ extends to a positive contraction C_0-semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $L^p(Q_d)$.*

Moreover, the generator $(\widetilde{A}, D(\widetilde{A}))$ of the semigroup $(\widetilde{T}(t))_{t\geq 0}$ is an extension of $(A_{S_d}, D(A_{S_d}))$ and $\mathscr{C}^2(Q_d)$ is a core for $(\widetilde{A}, D(\widetilde{A}))$, so that $(\widetilde{A}, D(\widetilde{A}))$ is the closure of $(V_{S_d}, \mathscr{C}^2(Q_d))$ in $L^p(Q_d)$ as well.

Finally, if $t \geq 0$ and if $(k(n)_{n\geq 1}$ is a sequence of positive integers such that $\lim\limits_{n\to\infty} k(n)/n = t$, then for every $f \in L^p(Q_d)$,

$$\lim_{n\to\infty} C_n^{k(n)}(f) = \widetilde{T}(t)(f) \qquad \textit{in } L^p(Q_d). \tag{4.10}$$

Proof. The proof is similar to that one of Theorem 3.4 of [7], so that here we only sketch some details for the sake of clarity.
First of all, we notice that, taking (4.6) into account, if $a \geq 1$, then $\|C_n\|_{L^p,L^p} \leq 1$ for any $n \geq 1$.

Fix now $t \geq 0$ and consider an arbitrary sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$. Then, for every $f \in \mathscr{C}(Q_d)$,

$$\|T(t)f\|_p = \lim_{n\to\infty} \|C_n^{k(n)}(f)\|_p \leq \|f\|_p.$$

Therefore, there exists a unique linear continuous extension $\widetilde{T}(t) : L^p(Q_d) \to L^p(Q_d)$ of $T(t)$. Moreover, $\|\widetilde{T}(t)\|_{L^p,L^p} \leq 1$ for every $t \geq 0$.

For every $t \geq 0$ the operator $\widetilde{T}(t)$ is positive and the family $(\widetilde{T}(t))_{t\geq 0}$ is a strongly continuous semigroup.

Let $(\widetilde{A}, D(\widetilde{A}))$ be the generator of $(\widetilde{T}(t))_{t\geq 0}$. Then, it is easily seen that $D(A_{S_d}) \subset D(\widetilde{A})$ and $\widetilde{A} = A_{S_d}$ on $D(A_{S_d})$. Moreover, $D(A_{S_d})$ is a core for $(\widetilde{A}, D(\widetilde{A}))$, since

$$\widetilde{T}(t)(D(A_{S_d})) = T(t)(D(A_{S_d})) \subset D(A_{S_d})$$

for every $t \geq 0$ (see, e. g., [22, Chapter II, Proposition 1.7]).

As a consequence, thanks also to Theorem 3.1, $\mathscr{C}^2(Q_d)$ is a core for $(\widetilde{A}, D(\widetilde{A}))$.

Finally, formula (4.10) follows from (4.9).

□

Remarks 4.2.

1. In the special case $d = 1$ and $a = 1$, Theorem 4.1 has been already proven in [25, Theorem 1], with a completely different method. Moreover, in the same paper a representation of the semigroup in terms of the Legendre polynomials is also given.
2. The differential operator $(V_{S_d}, \mathscr{C}^2(Q_d))$ falls within a more general class of second-order differential operators that have been investigated in [11, Chapter 5] (see, also, [5, Section 4, formula (4.1) and Example 2.2, 2]). In particular, in [11, Section 5.6] it has already been shown that $(V_{S_d}, \mathscr{C}^2(Q_d))$ is closable and its closure is the generator of a Markov semigroup on $\mathscr{C}(Q_d)$ that can be approximated, as in (4.9), by iterates of modified Bernstein-Schnabl operators. However, in general, these approximating operators are not defined on $L^p(Q_d)$, so that formula (4.10) doesn't apply for them.
3. The generation properties of the operator $(V_{S_d}, \mathscr{C}^2(Q_d))$ in the space $L^p(Q_d)$ have been also investigated in [24, Theorem 2.5] (see also [7]). In particular, it is shown that the semigroup $(\widetilde{T}(t))_{t\geq 0}$ is analytic and a description of the domain $D(\widetilde{A})$ in terms of weighted Sobolev spaces is given.

Finally, in [18, Theorem 3.1] it is shown that, if $a = 1$, then for every $f \in \mathscr{C}(Q_d)$ (resp., $f \in L^p(Q_d), 1 \leq p < +\infty$)

$$\lim_{t \to +\infty} T(t)(f) = \int_{Q_d} f(x)\, dx \quad \text{uniformly on } Q_d \tag{4.11}$$

(resp.,

$$\lim_{t \to +\infty} \widetilde{T}(t)(f) = \int_{Q_d} f(x)\, dx \quad \text{in } L^p(Q_d)). \tag{4.12}$$

A result similar to Theorem 4.1 in the context of the simplex K_d should be highly interesting because of the importance of the generation properties of the Fleming-Viot type operators, like (2.12), in $L^p(K_d)$ (see, e.g., [2], [21]). However, it seems that the approach we follow to derive Theorem 4.1 does not work in $L^p(K_d)$, mainly because the relevant operators defined by (1.20) are not contractive on $L^p(K_d)$ (see [12, Theorem 5.4]). Nevertheless, the differential operator (2.12) generates a contraction semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $L^p(K_d)$ (see, e.g., [2, pp. 1259-1260]) which extends the semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}(K_d)$ given by Theorem 3.1.

So, it seems that, in order to approximate the semigroup $(\widetilde{T}(t))_{t\geq 0}$ by iterates of positive operators as in (4.10), it is necessary to replace the operators C_n with other approximating operators like Bernstein-Durrmeyer type ones (see [2, Section 2]).

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