Master stability function for piecewise smooth Filippov networks

This is a pre-print of the paper published in AUTOMATICA Volume 152, 2023, DOI: 10.1016/j.automatica.2023.110939.

Luca Dieci [a], Cinzia Elia [b]

[a] *School of Mathematics, Georgia Tech, Atlanta, GA 30332, US*

[b] *Dipartimento di Matematica, Univ. of Bari, I-70100, Bari, Italy*

Abstract

We consider a network of identical piecewise smooth bimodal systems, also known as systems of Filippov type, that synchronizes along the asymptotically stable periodic orbit of a single agent. We explicitly characterize the fundamental matrix solution of the network along the synchronous solution and extend the *Master Stability Function* tool to the present case of non-smooth dynamics of Filippov type.

Key words: Piecewise smooth networks, synchronization, fundamental matrix solution, Filippov, master stability function, Floquet mutlipliers, Floquet exponents.

1 Introduction

Synchronization of dynamical networks is a fascinating, widely studied, and impactful phenomenon; e.g., see [24], [4], [18] for early applications in the applied sciences, and the review [13] –and the many references there– for a thorough account on the topic of oscillators synchronization. In the most typical and studied circumstance, one has a network of N nodes, the so-called *agents*, that obey N identical nonlinear differential equations with vector field f, coupled through linear anti-symmetric coupling. The key concern is to find conditions that tie together the strength of the coupling, the structure of the network, and the agent's dynamics, in such a way that the network synchronizes. This problem has been extensively studied, under a number of different scenarios, for networks of smooth systems; e.g., see the work of Pecora and coauthors [3], [22] for a study exploiting Lyapunov exponents ideas, and see [16] for a study more along the lines of the theory of dissipative attractors.

After the cited works of Pecora and coauthors, probably the most widely adopted and successful tool to infer convergence to, and/or stability of, a synchronized solution in networks of smooth dynamical systems has been that of the master stability function (MSF). However, as remarked in [7], this *"approach requires some degree of smoothness in the agents' vector fields ... and extensions need to be found"* when dealing with piecewise smooth (PWS) systems. Our goal in this work is to provide such extension for Filippov systems, that is when the agents satisfy a differential system with discontinuous right-hand side.

There are several types of PWS differential equations, all being characterized by a change occurring when the solution crosses a specific set (termed the discontinuity set) and the discontinuity set itself is typically assumed to have a smooth manifold structure. The three most common types of PWS systems are the following.

(i) The state changes across the discontinuity set, i.e. the solution jumps to a different value as the discontinuity set is reached, even though the vector field does not change. Important examples are spiking neuronal networks, so-called *integrate and fire* systems, or also impact oscillators in mechanics. We may call these *impacting PWS systems*.
(ii) The right-hand side (the vector field) changes discontinuously as the solution reaches the discontinuity boundaries. These are more properly called differential systems with discontinuous right-hand-sides, and are usually known as *Filippov*, or *bimodal*, systems. This is the class of PWS on which we will focus our

Email addresses: `dieci@math.gatech.edu` (Luca Dieci), `cinzia.elia@uniba.it` (Cinzia Elia).

attention in this work.

(iii) The discontinuities take place in higher derivative terms, and for these reasons these are often called *PWS continuous systems*. The well studied Chua's circuit is in this class.

Of course, the above cases present distinct characteristics and difficulties, and need not be mutually exclusive. However, in the present work, we only study the synchronization problem for networks where each agent satisfies a nonlinear piecewise smooth system of Filippov type. We will call these piecewise smooth Filippov networks, or just Filippov networks, or simply piecewise smooth (PWS) networks if it is clear from the context. As it turns out, Filippov networks are the type of networks for which the extension of the MSF is lacking, whereas the standard theory and numerical methods hold unchanged for networks where the agents satisfy a PWS continuous system, and the case of networks where the agents satisfy an impacting PWS system have been considered before, at least in some cases; see below.

To reiterate, we consider the network

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij} E(x_j - x_i), \quad x_i \in \mathbb{R}^n \ , \tag{1}$$

for $i = 1, \dots, N$, where f is only piecewise smooth:

$$f(x_i) = \begin{cases} f^+(x_i), \ h(x_i) > 0 \\ f^-(x_i) \ \ h(x_i) < 0 \end{cases},$$

$i = 1, \dots, N$, and $h(x) = 0$ describes the discontinuity manifold. In (1), $A \in \mathbb{R}^{N\times N} = (a_{ij})_{i,j=1,\dots,N}$ is the adjacency matrix of the graph describing the network, and recall that $a_{ij} = 1$ if there is an arc connecting i-th and j-th node, and $a_{ij} = 0$ otherwise; as usual, we assume that the graph is undirected, simple and connected, so that A is symmetric. Also, $\sigma \geq 0$ is the the coupling strength, and $E \in \mathbb{R}^{n\times n}$ is the coupling matrix describing which components of the two agents x_i and x_j, $i \neq j$, are connected to one another. Next, we let D be the diagonal matrix with elements $d_{ii} = \sum_{j=1}^{N} a_{ij}$, and let $L = -(D - A)$, $L \in \mathbb{R}^{N\times N}$, be the negative of the graph Laplacian. Then, using Kronecker product notation, we rewrite (1) as

$$\begin{aligned} &\dot{\boldsymbol{x}} = F(\boldsymbol{x}) + \sigma M \boldsymbol{x}, \qquad \text{where} \\ &\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{nN},\ F(\boldsymbol{x}) = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \in \mathbb{R}^{nN}, \\ &M = L \otimes E \in \mathbb{R}^{nN\times nN} \ . \end{aligned} \tag{2}$$

As noted, our present interest is in the case when, taking $\sigma = 0$ in (2), each agent obeys identical piecewise smooth bimodal dynamics:

$$\dot{x}_i = f(x_i) = \begin{cases} f^+(x_i), \ h(x_i) > 0 \\ f^-(x_i), \ h(x_i) < 0 \end{cases}, \qquad i = 1 : N, \tag{3}$$

with $f^\pm : \mathbb{R}^n \to \mathbb{R}^n$ smooth vector fields (say, $\mathcal{C}^1$), and the scalar function $h : \mathbb{R}^n \to \mathbb{R}$ defines the discontinuity boundaries and is assumed to be at least $\mathcal{C}^2$. For each agent, the manifold of discontinuity is the zero set $\{x \in \mathbb{R}^n \ : \ h(x) = 0\}$, and we will use the following notation:

$$\Sigma = \{x \in \mathbb{R}^n \mid h(x) = 0\}, \ R^\pm = \{x \in \mathbb{R}^n \mid h(x) \gtrless 0\}. \tag{4}$$

Remark 1 *In general, the set Σ is a co-dimension 1 manifold embedded in $\mathbb{R}^n$. For example, in $\mathbb{R}^2$, it would be given by (the union of) non-intersecting curves. A typical case we have seen in many applications is when Σ is a hyperplane, $h(x) = c^T x - b$, but this is not necessary in our work. Finally, it should be appreciated that there may be several functions h whose 0-sets define distinct discontinuity manifolds, in which case the meaning of $f^\pm$ has to be understood to hold with respect to the discontinuity manifold $h = 0$ provided; for example, the situation $h(x) = |c^T x| - b = 0$ gives two distinct discontinuity planes.*

As customary, we say that a point $x \in \Sigma$ is a *transversal crossing point* if

$$(\nabla h(x)^T f^-(x))(\nabla h(x)^T f^+(x)) > 0, \tag{5}$$

and it is an *attractive sliding point* if

$$\nabla h(x)^T f^-(x) > 0, \quad \nabla h(x)^T f^+(x) < 0. \tag{6}$$

On Σ, sliding will be assumed to take place in the sense of Filippov, whereby on Σ the dynamics of an agent is given by

$$\begin{aligned} &\dot{x} = f_\Sigma(x) := (1 - \alpha) f^-(x) + \alpha f^+(x) \ , \\ &\alpha = \frac{\nabla h(x)^T f^-(x)}{\nabla h(x)^T (f^-(x) - f^+(x))} \ . \end{aligned} \tag{7}$$

Finally, a point $\bar{x} \in \Sigma$ is called *tangential exit point into* R^- if a trajectory $x(t)$ sliding on Σ reaches it at some value $\bar{t}$ and there it holds that

$$\begin{aligned} &\nabla h(\bar{x})^T f^-(\bar{x}) = 0, \quad \nabla h(\bar{x})^T f^+(\bar{x}) < 0, \\ &\left[\frac{d}{dt} \nabla h(x(t))^T f^-(x(t)) \right]_{t=\bar{t}} < 0, \end{aligned}$$

and similarly for a tangential exit point into R^+. The combination of transversal crossings, transversal entries on Σ, and tangential exits from Σ, are called *generic events*, or simply *events*. To illustrate these occurrences, referring to Figure 1, the point s_1 is a crossing point, s_2 is a transversal entry point, and s_3 is a tangential exit point.

Let the single agent (3) have a limit cycle with a finite number of events, and not entirely contained in Σ (see [15], [11], [5] for examples of self sustained oscillations in discontinuous systems with partial sliding along the discontinuity manifold). Let $x_S(t)$ be the corresponding T-periodic solution. Then, given the structure of M, the function $\boldsymbol{x}_S(t) = \begin{bmatrix} x_S(t) \\ \vdots \\ x_S(t) \end{bmatrix}$ is a periodic solution of (2) of period T; we will call this the *synchronous solution*. However, even if $x_S(t)$ happened to be asymptotically stable for the single agent, there is no guarantee that $\boldsymbol{x}_S$ be stable for the network dynamics for all values of σ; further, when N is large, the numerical study of the stability of $\boldsymbol{x}_S(t)$ may be prohibitively expensive. This issue can be overcome by extending the *Master Stability Function* (MSF) tool of Pecora and Carroll, see [22], to PWS networks.

The MSF technique relies on exploiting the structure of the fundamental matrix solution of the network, and for this reason in the present work our goal is two-fold. When the network synchronizes on $\boldsymbol{x}_S(t)$, first we will give the explicit expression of the fundamental matrix solution along the synchronous solution. Then, we will extend the *Master Stability Function* (MSF) to piecewise smooth Filippov networks. A rigorous justification of the **use of the MSF** for general, **nonlinear, PWS Filippov networks** appears to be lacking, and it is our purpose to give it in this work.

Other authors before us have considered piecewise smooth networks, and some important studies have been made to resolve the outstanding concern of how to infer asymptotic convergence in networks of piecewise-smooth systems. An example is the recent work of [8] where discontinuous diffusive coupling is adopted. But also the MSF tool has been used in some special type of PWS networks. To witness, in the works [5], [6], the authors use the MSF approach to study limit cycles in piecewise-linear integrate-and-fire systems; the chosen vector field is sufficiently simple that the authors can achieve some analytical progress. The work of Ladenbauer et al., [19] also considers use of the MSF on a special class of PWS networks. The authors are concerned with impacting PWS networks, in which the state –but not the vector field– changes at so-called reset points. In particular, their model is free of sliding regions, and only crossing discontinuity can occur. More precisely, they consider a planar, nonlinear, integrate-and-fire agent, and modify the smooth MSF tool to study a synchronous periodic orbit of a spiking neuronal network with delays, by adopting the methodology of saltation matrices to form the monodromy matrix through impact points (reset points, in their case). To do so, they derive the saltation matrices at the (intersection of) discontinuities. In the case studied in [19], these saltation matrices can be computed without ambiguity since a unique vector field must be evaluated at the discontinuity point regardless of the perturbation of the synchronous solution. But, in the case of PWS Filippov networks (the case we consider), several vector fields (precisely $2^N + \sum_{k=1}^{N} \binom{N}{k}$) are defined in the neighborhood of a discontinuity point of a synchronous solution and this in general produces a severe ambiguity of the saltation matrix. Overcoming this ambiguity is an important theoretical achievement in our work and it allows for use of the MSF on Filippov type systems with crossings and sliding regions.

Remark 2 *An alternative to considering the PWS system as such (monitoring events, switching vector fields, and so on) consists in replacing the original discontinuous vector field $f^{\pm}$ of the Filippov system* (3) *with a globally smooth one, which reduces to $f^{\pm}$ away from a small neighborhood of Σ. This approach is known as* regularization *and one possible regularization technique, the one that has been mostly used in the literature, was originally introduced in [23]; a basic version of it consists in replacing* (3) *with*

$$x' = (1 - g_\epsilon(z))f^-(x) + g_\epsilon(z)f^+(x) \ , \quad z = h(x) \ , \ (8)$$

where g_ϵ is a C^k transition function, $k \geq 1$, such that $g_\epsilon(z) = \begin{cases} 1 & z \geq \epsilon \\ 0 & z \leq -\epsilon \end{cases}$ *and $g'(z) > 0$ in $(-\epsilon, \epsilon)$. If we do this, then –for each given $\epsilon > 0$– we are left with a smooth problem. Of course, the trade-off is that this smooth problem depends on ϵ and we would need to study the limit as $\epsilon \to 0$ of the solution of the regularized system. Now, it is well understood that, for ϵ sufficiently small, there is a periodic solution of* (8)*, and that, in the limit as $\epsilon \to 0$, this periodic trajectory of the regularized problem converges to the Filippov periodic trajectory of* (8) *(e.g., see [2], [12]). In short, there is a non-ambiguous limit, and the limit is known. However, there is a (big) difference between inferring that as the regularization parameter goes to* 0 *the solution of the regularized problem converges to the Filippov solution, and implying that the monodromy matrix along the regularized solution converges to the one evaluated along the Filippov solution. A proof of this last statement is less clear to us unless some more restrictive assumptions apply. As a consequence of this, for a network of Filippov agents the convergence of the regularized solution to the Filippov one does not imply convergence of the Master Stability Function as well. Moreover, even willing to assume that there is a convergence of some type of the MSF of the regularized problem, where does this "regularized MSF" converge? There are no results available the literature on the monodromy matrix along a syn-*

chronous periodic orbit of a network of piecewise smooth agents. In this paper, we derive the expression for the monodromy matrix and then compute the MSF of the network. It would certainly be desirable that the monodromy matrix along the synchronous periodic orbit of the regularized problem, and hence the MSF of the regularized network, converges to the monodromy matrix of the Filippov network. As appealing as this sounds, this result is not easy to prove and it is beyond the scope of this paper.

A plan of the paper is as follows. In Section 2, we derive the precise form of the monodromy matrix along the synchronous solution. In Section 3, we extend the MSF tool to piecewise smooth networks. Finally, in Section 4 we give detailed numerical study of a network arising in mechanical vibrations and infer that, for a range of values of σ, the synchronous solution is stable.

Notation.
$e \in \mathbb{R}^N$ is the vector with all elements equal to 1, so that $\boldsymbol{x}_S = e \otimes x_S$ is the synchronous solution in $\mathbb{R}^{nN}$. I_p will always indicate the (p,p) identity matrix, and $e_1, \dots$ will be the standard unit vectors.
$h_i(\boldsymbol{x}) = h(x_i)$, $i = 1, \dots, N$. $\Sigma_i = \{\boldsymbol{x} \in \mathbb{R}^{nN} \mid h_i(\boldsymbol{x}) = 0\}$, $i = 1, \dots, N$, and $\boldsymbol{\Sigma} = \cap_{i=1}^N \Sigma_i$. We will also write $h_{i,j}(\boldsymbol{x})$ to mean one of $h_i(\boldsymbol{x})$ or $h_j(\boldsymbol{x})$, and similarly for $\Sigma_{i,j}$.

2 Fundamental matrix solution for synchronous periodic solutions

The main difficulties we need to address in this section are the following.

(i) The network (2) has N discontinuity manifolds and solutions might slide on the intersection of two or more manifolds (in fact, as we will see, a synchronous periodic solution $\boldsymbol{x}_S$ with x_S having a sliding portion, will necessarily slide on the intersection of all N manifolds). But, in general, the sliding vector field on the intersection of the discontinuity manifolds is not uniquely defined and we need to address how this impacts the form of the fundamental matrix of the linearized system. In the specific case we consider here, there is no such ambiguity, see Lemma 8 and Theorem 17.

ii) The monodromy matrix along a periodic solution of the piecewise system (3) is not continuous: it has jumps at the entry points (crossing or sliding) on the discontinuity manifold. These jumps are taken into account via so called jump or saltation matrices, whose scope is to transform the vector field at the entry time, say t^-, into the vector field at the exiting time, t^+. The correct expression for such matrices is well known in the literature in the case of a single discontinuity manifold (see [1], [21], [20]). However, synchronous sliding solutions have to slide on the intersection of N discontinuity manifolds and the fundamental matrix solution along a synchronous solution must take into account jumps at this intersection. In the literature, there are results about these jump matrices relative to the intersection of two discontinuity manifolds, see [17] for the case of crossing and [10] for the case of sliding, but no result exist for the intersection of more than two manifolds. Surely this must be because, in the case of sliding solutions, there is no uniquely defined Filippov sliding vector field on the intersection of discontinuity manifolds, as noted in i) above. However, this is not the only issue. Indeed, in general, on the intersection of discontinuity manifolds, the jump matrix itself is not uniquely defined, even if we are willing to select a specific sliding vector field (again, see [17] for the case of crossing and [10] for the case of sliding). This being the case, the fundamental matrix solution cannot be defined in a unique way. Theorems 13 and 15 deal with this aspect in case of the synchronous periodic solution $\boldsymbol{x}_S$ of (2).

After the expression for the monodromy matrix is arrived at, in Section 3 we will see how to extend the MSF tool to PWS networks.

For the above reasons, hereafter we derive the monodromy matrix along the synchronous solution $\boldsymbol{x}_S$ of (2). The main results are given in Theorem 13 and 15, where we show that the saltation matrices can be represented in a unique way. Recalling that $\boldsymbol{x}_S = \begin{bmatrix} x_S \\ \vdots \\ x_S \end{bmatrix}$, where x_S is the periodic solution of a single agent (3), we will assume that x_S has a finite number of generic events. Because of this, we will make the following convenient assumption on the dynamics of x_S.

Assumption 3 *We assume that* (3) *has a periodic solution* $x_S(t)$ *of period* T *that:*

0) At $t = 0$, $x_S(t) = s_0$ *is in* R^-*;*
1) At $t = t_1$, x_S *crosses* Σ *transversally at the point* $s_1 = x_S(t_1)$ *to enter* R^+*;*
2) At $t = t_2$, x_S *reaches transversally the attractive sliding point* $s_2 = x_S(t_2) \in \Sigma$ *and* x_S *begins sliding on* Σ*;*
3) At $t = t_3$, x_S *reaches the tangential exit point* $s_3 = x_S(t_3)$*, and it leaves* Σ *to enter into* R^-*;*
4) At $t = T$, x_S *is back at* s_0*:* $x_S(T) = s_0$.

A sketch of the above situation is in Figure 1. Under the above scenario 0)-4), the expression of the monodromy matrix $\Phi(T, 0)$ of (3) along $x_S(t)$ can be derived with the help of classical results in the theory of piecewise smooth systems (e.g., see [1], [10], [21]), combining the fundamental matrix solutions in smooth regions with appropriate *j*ump matrices connecting different vector fields at discontinuity points. Namely, $\Phi(T, 0)$ is given

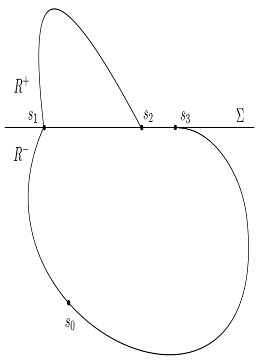

Fig. 1. Schematic of Assumption 3.

by the expression

$$\Phi(T,0) = \Phi^-(T,t_3)\Phi_\Sigma(t_3,t_2)S_{+,\Sigma}\Phi^+(t_2,t_1)S_{-,+}\Phi^-(t_1,0), \quad (9)$$

where $\Phi^\pm$ and Φ_Σ are fundamental matrix solutions in $R^\pm$ and Σ respectively, $S_{-,+} = I_n + \frac{(f^+-f^-)\nabla h(s_1)^T}{\nabla h^T f^-(s_1)}$ is the jump matrix from R^- to R^+ and $S_{+,\Sigma} = I_n + \frac{(f_\Sigma - f^+)\nabla h^T(s_2)}{\nabla h^T f^-(s_2)}$ is the jump matrix from R^+ to Σ; here, f_Σ is defined in (7). There is no jump matrix from Σ into R^- at $s = s_3$ since $f_\Sigma(s_3) = f^-(s_3)$. Note that $S_{-,+}f^-(s_1) = f^+(s_1)$ and $S_{+,\Sigma}f^+(s_2) = f_\Sigma(s_2)$.

Remark 4 *Other than the need for a* **finite number** *of generic events, the results in this section do not depend on the particular structure of $x_S(t)$ given in Assumption 3 and can be immediately extended to any finite number of generic crossings, sliding segments, and tangential exits, of the periodic orbit of* (3). *Of course, the number of generic events occurring, and their ordering in time, does impact the form of the associated monodromy matrix, but as long as the events and the times where they occur are known, the monodromy matrix can be built according to the same building blocks of what one does for the monodromy matrix of $x_S(t)$ satisfying Assumption 3. See also Remark 22.*

Now, for N agents, there are 2^N subregions (and corresponding vector fields), and we can represent them using a tree diagram with 2^N branches. We number the regions, and the vector fields, from 1 to 2^N following the branches of the tree.

Example 5 *For $N = 3$, we have the following correspondence between region numbering and signs of h_1, h_2 and h_3:*

1	2	3	4
$(---)$	$(--+)$	$(-+-)$	$(-++)$
5	6	7	8
$(+--)$	$(+-+)$	$(++-)$	$(+++)$

In each subregion R_j, the vector field in (2) *is* $F(\boldsymbol{x}) = F_j(\boldsymbol{x}) = \begin{bmatrix} f^\pm(x_1) \\ f^\pm(x_2) \\ f^\pm(x_3) \end{bmatrix}$, *$j = 1,\dots,8$, where in $f^\pm$ we select the sign in agreement with the region numbering above. For example* $F_3(\boldsymbol{x}) = \begin{bmatrix} f^-(x_1) \\ f^+(x_2) \\ f^-(x_3) \end{bmatrix}$.

Remark 6 *It is simple, but important, to observe that if $x \in \Sigma$ is an attractive sliding point for the single agent, then $F_j(\boldsymbol{x})$, $\boldsymbol{x} = \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix}$, points toward Σ_i for all $i = 1,\dots,N$, and $j = 1,\dots,2^N$, i.e.,*

$$\nabla h_i(\boldsymbol{x})^T F_j(\boldsymbol{x}) > 0, \quad \nabla h_i(\boldsymbol{x})^T F_j(\boldsymbol{x}) < 0.$$

The inequalities above imply that all solutions in a neighborhood of $\boldsymbol{x}$ reach $\boldsymbol{\Sigma}$ in finite time. Then $\boldsymbol{x} = e \otimes x \in \boldsymbol{\Sigma}$ is an attractive sliding point on $\boldsymbol{\Sigma}$ for the full network. Similarly, if $x \in \Sigma$ is a tangential exit point into R^- (respectively, R^+) for the single agent, then $\boldsymbol{x} = e \otimes x \in \boldsymbol{\Sigma}$ is a tangential exit point into R_1 (respectively, R_{2^N}) for the full network.

Remark 6 justifies the following fact. Let $x_S(t)$ satisfy Assumption 3 and let $\boldsymbol{s}_j = e \otimes s_j$, where s_j is defined in Assumption 3, $j = 1,\dots,4$. Then, the synchronous solution $\boldsymbol{x}_S(t) = (e \otimes x_S(t))$ obeys the following evolution (see Figure 2):

0) At $t = 0$, $\boldsymbol{x}_S(0) = \boldsymbol{s}_0$ is in R_1;
1) At $t = t_1$, $\boldsymbol{x}_S$ crosses R_1 at $\boldsymbol{s}_1$ and enters into R_{2^N};
2) At $t = t_2$, $\boldsymbol{x}_S$ reaches the attractive sliding point $\boldsymbol{s}_2$ and starts sliding [1] along $\boldsymbol{\Sigma}$;
3) At $t = t_3$, $\boldsymbol{x}_S$ exits $\boldsymbol{\Sigma}$ at the tangential exit point $x = \boldsymbol{s}_3$ and enters into R_1;
4) At $t = T$, $\boldsymbol{x}_S$ reaches $\boldsymbol{s}_0$.

[1] Although, in general, sliding along $\boldsymbol{\Sigma}$ is not unambiguously defined, presently this is not a concern, since we are just describing the evolution of the specific $\boldsymbol{x}_S$.

Remark 7 *Note that the synchronous solution $\boldsymbol{x}_S$ satisfies the following : i) it can only evolve in the regions R_1 or R_{2^N}; ii) it can only cross the discontinuity manifolds at points on $\boldsymbol{\Sigma}$, and iii) if it slides, it can only slide on the intersection of all N discontinuity manifolds, i.e., on $\boldsymbol{\Sigma}$.* **However***, the solution of a problem relative to* **perturbed initial conditions** *in general will not satisfy the restricted motion described by points i)-iii) above, and it* **may slide on, or cross, some of the** Σ_i**'s** *and not just $\boldsymbol{\Sigma}$. This fact must be taken into account when deriving the expression of the fundamental matrix solution of the linearized dynamics, in particular of the saltation matrices.*

Next, we study the case $N = 2$ in detail. Appropriate modifications required to generalize to the case $N > 2$ are given below. Following the tree diagram for $N = 2$, we have the following four subregions of phase space and corresponding vector fields, for $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{2n}$:

$$R_1 = \{\boldsymbol{x} \mid h_1(\boldsymbol{x}) < 0,\ h_2(\boldsymbol{x}) < 0\},\ F_1(\boldsymbol{x}) = \begin{bmatrix} f^-(x_1) \\ f^-(x_2) \end{bmatrix},$$

$$R_2 = \{\boldsymbol{x} \mid h_1(\boldsymbol{x}) < 0,\ h_2(\boldsymbol{x}) > 0\},\ F_2(\boldsymbol{x}) = \begin{bmatrix} f^-(x_1) \\ f^+(x_2) \end{bmatrix},$$

$$R_3 = \{\boldsymbol{x} \mid h_1(\boldsymbol{x}) > 0,\ h_2(\boldsymbol{x}) < 0\},\ F_3(\boldsymbol{x}) = \begin{bmatrix} f^+(x_1) \\ f^-(x_2) \end{bmatrix},$$

$$R_4 = \{\boldsymbol{x} \mid h_1(\boldsymbol{x}) > 0,\ h_2(\boldsymbol{x}) > 0\},\ F_4(\boldsymbol{x}) = \begin{bmatrix} f^+(x_1) \\ f^+(x_2) \end{bmatrix},$$

and $F(\boldsymbol{x})$ in (2) is equal to $F_i(\boldsymbol{x})$ for $\boldsymbol{x} \in R_i$. Moreover we have $\Sigma_{1,2} = \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid h_{1,2}(\boldsymbol{x}) = 0\}$ and we consider also the sets $\Sigma^{\pm}_{1,2}$, defined as follows: $\Sigma^{\pm}_1 = \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid h_1(\boldsymbol{x}) = 0$, and $h_2(\boldsymbol{x}) \gtrless 0\}$, and similarly for $\Sigma^{\pm}_2$. The synchronous solution is $\boldsymbol{x}_S(t) = \begin{bmatrix} x_S(t) \\ x_S(t) \end{bmatrix}$, and (under Assumption 3) it evolves schematically as in Figure 2.

As already pointed out in Remark 7, while $\boldsymbol{x}_S(t)$ can only evolve in R_1 and R_4, and can only cross/slide-on $\boldsymbol{\Sigma} = \Sigma_1 \cap \Sigma_2$, a perturbed solution might instead cross just Σ_1 or Σ_2, and evolve in R_2 or R_3 and it might slide along Σ_1 and/or Σ_2. Hence, in order to compute the fundamental matrix solution, we will need the expressions of the sliding vector fields on Σ_1, Σ_2 and $\boldsymbol{\Sigma}$. Let $f_\Sigma : \mathbb{R}^n \to \mathbb{R}^n$ denote the sliding vector field of (3) on Σ (see (7)) and let $F_{\mathcal{S}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote the sliding vector field on $\mathcal{S}$, where $\mathcal{S}$ is any of the following: $\mathcal{S} = \Sigma^{\pm}_1, \Sigma^{\pm}_2, \boldsymbol{\Sigma}$.

The next result provides –at a point $\boldsymbol{x}$ on $\boldsymbol{\Sigma}$– the four vector fields on $\Sigma^{\pm}_{1,2}$.

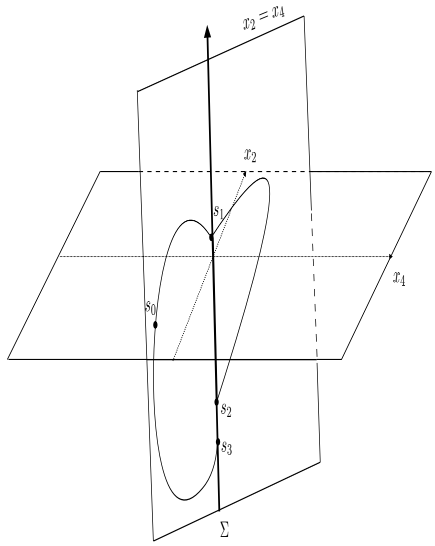

Fig. 2. Schematic of the periodic orbit for the network.

Lemma 8 *Let $\boldsymbol{x} \in \boldsymbol{\Sigma}$, $\boldsymbol{x} = \begin{bmatrix} x \\ x \end{bmatrix}$, and suppose that $\boldsymbol{x}$ is an attractive point with respect to both Σ_1 and Σ_2: cfr* (6) *and Remark 6. That is, we have*

$$0 < \frac{\nabla h(x)^T f^-(x)}{\nabla h(x)^T (f^-(x) - f^+(x))} < 1\ . \qquad (10)$$

Then, we have

$$F_{\Sigma^{\pm}_1}(\boldsymbol{x}) = \begin{pmatrix} f_\Sigma(x) \\ f^{\pm}(x) \end{pmatrix},\ F_{\Sigma^{\pm}_2}(\boldsymbol{x}) = \begin{pmatrix} f^{\pm}(x) \\ f_\Sigma(x) \end{pmatrix},$$
$$F_{\boldsymbol{\Sigma}}(\boldsymbol{x}) = \begin{pmatrix} f_\Sigma(x) \\ f_\Sigma(x) \end{pmatrix} = e \otimes f_\Sigma(x),$$

where with $f_\Sigma(x)$ we denote the sliding vector field of (3) *on Σ, as defined in* (7)*.*

PROOF. We prove the statement for $F_{\boldsymbol{\Sigma}}$. The proofs for the other sliding vector fields are analogous. Let $\boldsymbol{x}^k$ be a sequence of points in $\boldsymbol{\Sigma}$, converging to the synchronous point $\boldsymbol{x}$: $\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x} \in \boldsymbol{\Sigma}$. Since $\boldsymbol{x}^k \in \boldsymbol{\Sigma}$, we have $\boldsymbol{x}^k = \begin{bmatrix} x^k_1 \\ x^k_2 \end{bmatrix}$ and $h(x^k_i) = 0$ for all k and $i = 1, 2$. Now, let

$\hat{F}_j(\boldsymbol{x}^k) = F_j(\boldsymbol{x}^k) + \sigma M\boldsymbol{x}^k = \begin{bmatrix} f^{\pm}(x_1^k) \\ f^{\pm}(x_2^k) \end{bmatrix} + \sigma M\boldsymbol{x}^k$ with $j = 1, \dots, 4$ and the sign in $f^{\pm}$ is taken in accordance with the region R_j. Then the Filippov sliding vector field on $\boldsymbol{\Sigma}$ is given by the convex combination

$$
\begin{aligned}
&F_{\boldsymbol{\Sigma}}(\boldsymbol{x}^k) = \sum_{j=1}^{4} \lambda_j^k \hat{F}_j(\boldsymbol{x}^k) = \\
&\begin{pmatrix} (\lambda_1^k + \lambda_2^k) f^-(x_1^k) + (\lambda_3^k + \lambda_4^k) f^+(x_1^k) \\ (\lambda_1^k + \lambda_3^k) f^-(x_2^k) + (\lambda_2^k + \lambda_4^k) f^+(x_2^k) \end{pmatrix} = \\
&\begin{pmatrix} \alpha_1^k f^-(x_1^k) + (1-\alpha_1^k) f^+(x_1^k) \\ \alpha_2^k f^-(x_2^k) + (1-\alpha_2^k) f^+(x_2^k) \end{pmatrix}
\end{aligned}
$$

and α_i^k, $i = 1, 2$, must be found from the requirement that $F_{\boldsymbol{\Sigma}}$ is tangent to $\boldsymbol{\Sigma}$, that is

$$
(\nabla h^T, \nabla h^T) F_{\boldsymbol{\Sigma}}(x^k) \;=\; 0 \, .
$$

Therefore,

$$
\begin{aligned}
\alpha_1^k &= \frac{\nabla h^T f^-(x_1^k) + \sigma(\nabla h^T, 0) M\boldsymbol{x}^k}{\nabla h^T (f^-(x_1^k) - f^+(x_1^k))}, \\
\alpha_2^k &= \frac{\nabla h^T f^-(x_2^k) + \sigma(0, \nabla h^T) M\boldsymbol{x}^k}{\nabla h^T (f^-(x_2^k) - f^+(x_2^k))} \, .
\end{aligned}
$$

Now, for any $\epsilon > 0$, there exists K_ϵ such that if $k > K_\epsilon$, then $\|\boldsymbol{x}^k - \boldsymbol{x}\| < \epsilon$. This means that (for k sufficiently large) all points $\boldsymbol{x}^k$ are attractive sliding points relative to $\boldsymbol{\Sigma}$, since –because of (10)– $0 < \alpha_i^k < 1$ for k sufficiently large. Thus, the sequence of sliding vector fields on $\boldsymbol{\Sigma}$ is well defined. In the limit as $k \to \infty$, $\alpha_i^k \to \frac{\nabla h^T f^-(x)}{\nabla h^T (f^- - f^+)(x)}$ (since $M\boldsymbol{x} = 0$ for $\boldsymbol{x} \in \boldsymbol{\Sigma}$) and hence (see (7)) we obtain

$$
F_{\boldsymbol{\Sigma}}(\boldsymbol{x}) \;=\; \begin{bmatrix} f_\Sigma(x) \\ f_\Sigma(x) \end{bmatrix} .
$$

The argument for $F_{\Sigma_1^\pm}$ and $F_{\Sigma_2^\pm}$ is completely analogous to the one above, with a few differences due to the way the vector fields are constructed. For example, for $F_{\Sigma_2^-}$ we have $F_{\Sigma_2^-} = (1 - \alpha^k)\hat{F}_1(\boldsymbol{x}^k) + \alpha^k \hat{F}_2(\boldsymbol{x}^k)$, and α^k must be found from the requirement that $F_{\Sigma_2^-}$ is on the tangent plane to Σ_2^-, that is

$$
(0, \nabla h^T) \left[(1-\alpha^k) \begin{bmatrix} f^-(x_1^k) \\ f^-(x_2^k) \end{bmatrix} + \alpha^k \begin{bmatrix} f^-(x_1^k) \\ f^+(x_2^k) \end{bmatrix} + \sigma M\boldsymbol{x}^k \right] \;=\; 0
$$

Solving for α^k and letting $k \to \infty$, we obtain that $\alpha^k \to \frac{\nabla h^T f^-(x)}{\nabla h^T (f^- - f^+)(x)}$ and hence (see (7)) we obtain

$$
F_{\Sigma_2^-}(\mathbf{x}) \;=\; \begin{bmatrix} f^-(x) \\ f_\Sigma(x) \end{bmatrix} .
$$

Remark 9 *Lemma 8* **extends** *to the case $N > 2$ as follows. The sliding vector field $F_{\Sigma_i}(\boldsymbol{x})$ along a single Σ_i, has $f_\Sigma(x)$ in the i-th block while the components of the vector field in the j-th block, $j \neq i$, are equal to $f^+(x)$ or $f^-(x)$ in agreement with the sign of $h_j(x)$. Moreover,*

$$
F_{\boldsymbol{\Sigma}}(\boldsymbol{x}) = \begin{pmatrix} f_\Sigma(x) \\ \vdots \\ f_\Sigma(x) \end{pmatrix} .
$$

Next, in the case of $N = 2$, let $X(T, 0)$ denote the monodromy matrix along $\boldsymbol{x}_S(t)$. Then, using the same arguments as in [21] equation (9) formally generalizes as follows:

$$
X(T,0) = X_1(T, t_3) X_{\boldsymbol{\Sigma}}(t_3, t_2) S_{4,\boldsymbol{\Sigma}} X_4(t_2, t_1) S_{14} X_1(t_1, 0), \tag{11}
$$

where

$$
\begin{aligned}
\frac{dX_i(t,\tau)}{dt} &= (DF_i(\boldsymbol{x}_S(t)) + \sigma M) X_i(t,\tau), \\
X_i(\tau,\tau) &= I_{2n}, \quad i = 1, 4,
\end{aligned}
$$

$$
\begin{aligned}
\frac{dX_{\boldsymbol{\Sigma}}(t,\tau)}{dt} &= (DF_{\boldsymbol{\Sigma}}(\boldsymbol{x}_S(t)) + \sigma M) X_{\boldsymbol{\Sigma}}(t,\tau), \\
X_{\boldsymbol{\Sigma}}(\tau,\tau) &= I_{2n},
\end{aligned}
$$

where DF_i, $i = 1, 4$, and $DF_{\boldsymbol{\Sigma}}$ are the Jacobian matrices of the vector fields F_i, $i = 1, 4$, and $F_{\boldsymbol{\Sigma}}$, respectively, and in (11), the matrix S_{14} would be the jump matrix from R_1 into R_4, and the matrix $S_{4,\boldsymbol{\Sigma}}$ the jump matrix from R_4 into $\boldsymbol{\Sigma}$. However, in general, see [10], [17], the jump matrix on the intersection of two discontinuity manifolds (here, $\boldsymbol{\Sigma}$) is ambiguous and hence (11) is not uniquely defined. In Theorems 13 and 15 we show that for our problem we can use a unique expression for the jump matrix.

Remark 10 *Formally, the* **extension** *of* (11) **for** *$N > 2$, is easily obtained replacing S_{14} and $S_{4,\boldsymbol{\Sigma}}$ with $S_{1,2^N}$ and $S_{2^N,\boldsymbol{\Sigma}}$. See Remark 7. The formal derivation of the monodromy matrix also in this case requires the same arguments as for a single agent, though of course we will need to ensure lack of ambiguity in the jump matrix $S_{2^N,\boldsymbol{\Sigma}}$, which we will do below for our problem.*

2.1 Jump matrix at the crossing point

Now, let $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ be the solution of (2) with perturbed initial conditions $\boldsymbol{y}(0) = \boldsymbol{s}_0 + \Delta\boldsymbol{s}$, $\|\Delta\boldsymbol{s}\| \ll 1$. Recall that the monodromy matrix expresses, at first order in $\Delta\boldsymbol{s}$, the evolution of these perturbed initial conditions after one period. Below we give details on how to compute S_{14}. To go from R_1 to R_4, $\boldsymbol{y}(t)$ might either cross directly $\boldsymbol{\Sigma}$ at time $t_1 + \Delta t$ or it might instead cross Σ_1 and Σ_2 at two different times Δt_1 and Δt_2 before entering R_4 (see Remark 7). In [17], the form of S_{14} is given and the author points out that the jump matrix is ambiguous. In Lemma 11 and 12 we give the expression of S_{14} for the two different possibilities when $\boldsymbol{y}(t)$ crosses Σ_1 and Σ_2 at two different times, or it crosses directly $\boldsymbol{\Sigma}$; but, then Theorem 13 states that only one expression of S_{14} is needed to assess perturbations of synchronous solutions and we give this unique expression of S_{14} as a Kronecker product of the identity matrix I_2 with the jump matrix of the single agent.

Lemma 11 *Let $\boldsymbol{x}_S = \begin{bmatrix} x_S \\ x_S \end{bmatrix}$ where x_S satisfies Assumption 3. Let $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ be the solution of* (2) *with initial conditions $\boldsymbol{y}(0) = \boldsymbol{s}_0 + \Delta\boldsymbol{s}$, $\boldsymbol{s}_0 = \begin{bmatrix} s_0 \\ s_0 \end{bmatrix}$ and $\|\Delta\boldsymbol{s}\| \ll 1$. Let $(t_1 + \Delta t_1)$ and $(t_1 + \Delta t_2)$ be the times at which $\boldsymbol{y}(t)$ crosses respectively Σ_1 and Σ_2 before entering R_4. Assume $\Delta t_1 \neq \Delta t_2$. Then, the jump matrix S_{14} is given by*

$$S_{14} = I_2 \otimes I_n + I_2 \otimes \frac{(f^+(s_1) - f^-(s_1))\nabla h(s_1)^T}{\nabla h(s_1)^T f^-(s_1)} = I_2 \otimes S_{-,+},$$

with $S_{-,+} = I_n + \frac{(f^+(s_1) - f^-(s_1))\nabla h(s_1)^T}{\nabla h(s_1)^T f^-(s_1)}$ being the jump matrix for the single agent in (3) *from R^- to R^+.*

PROOF. We examine below the case $\Delta t_1 < \Delta t_2$. The proof for the case $\Delta t_2 > \Delta t_1$ is similar. Following [17] and [10], we can rewrite S_{14} as the composition of two jump matrices: $S_{14} = S_{34}S_{13}$, with S_{ij} being the jump matrix from R_i to R_j. Using standard results on jump matrices for one discontinuity manifold, we have that (note that $\sigma M \boldsymbol{x} = 0$ along a synchronous solution)

$$S_{13} = \begin{pmatrix} S_{-,+} & 0 \\ 0 & I_n \end{pmatrix}, \quad S_{34} = \begin{pmatrix} I_n & 0 \\ 0 & S_{-,+} \end{pmatrix},$$

with $S_{-,+}$ defined in the statement. Then $S_{34}S_{13} = I_2 \otimes S_{-,+}$. The case $\Delta t_1 > \Delta t_2$, in which the perturbed solution first crosses Σ_2 to enter into R_2 and then crosses Σ_1 to enter into R_4, gives S_{14} as the product $S_{24}S_{12}$. It is easy to verify that $S_{24}S_{12} = S_{34}S_{13}$.

The case $N > 2$ can be treated in a similar way, and see also [17, comments following Proposition 1] for an explanation on how to extend the result to the intersection of $N > 2$ discontinuity manifolds. Using same notations and same reasoning of the proof above, for $\Delta t_1 < \ldots < \Delta t_N$, the jump matrix is given by the product of the following N matrices

$$S_{1,2^N} = S_{2^N-1,2^N} \ldots S_{2^{N-1}+1,2^{N-1}+2^{N-2}+1} S_{1,2^{N-1}+1}.$$

The first factor $S_{1,2^{N-1}+1}$ is the jump matrix at Σ_1. For $k > 1$, $S_{\sum_{i=1}^{k-1} 2^{N-i}+1, \sum_{i=1}^{k} 2^{N-i}+1}$ is the jump matrix at Σ_k and it is block diagonal with the k-th block equal to $S_{-,+}$ and the other blocks equal to I_n. For $N = 3$, for example,

$$S_{18} = S_{78}S_{57}S_{15} = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & S_{-,+} \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ 0 & S_{-,+} & 0 \\ 0 & 0 & I_n \end{pmatrix} \times \begin{pmatrix} S_{-,+} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix} = I_3 \otimes S_{-,+}. \tag{12}$$

In the Lemma below we consider the case when the perturbed solution $\boldsymbol{y}(t)$ crosses Σ_1 and Σ_2 at the same time, and give a unique expression for the jump matrix S_{14} in this case.

Lemma 12 *As in Lemma 11, let $\boldsymbol{x}_S = \begin{bmatrix} x_S \\ x_S \end{bmatrix}$ where x_S satisfies Assumption 3. Let $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ be the solution of* (2) *with initial conditions $\boldsymbol{y}(0) = \boldsymbol{s}_0 + \Delta\boldsymbol{s}$, $\boldsymbol{s}_0 = \begin{bmatrix} s_0 \\ s_0 \end{bmatrix}$ and $\|\Delta\boldsymbol{s}\| \ll 1$. Let $(t_1 + \Delta t_1)$ and $(t_1 + \Delta t_2)$ be the times at which $\boldsymbol{y}(t)$ crosses respectively Σ_1 and Σ_2 before entering R_4. Assume $\Delta t_1 = \Delta t_2 = \Delta t$, so that necessarily $\boldsymbol{y}(t)$ from R_1 crosses $\boldsymbol{\Sigma}$ to enter into R_4.*

Then, without loss of generality,

$$S_{14} = \begin{pmatrix} I_n + \frac{(f^+(s_1)-f^-(s_1))\nabla h^T(s_1)}{\nabla h^T(s_1)^T f^-(s_1)} & 0 \\ \frac{(f^+(s_1)-f^-(s_1))\nabla h^T(s_1)}{\nabla h^T(s_1) f^-(s_1)} & I_n \end{pmatrix}$$
$$= I_2 \otimes I_n + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \frac{(f^+(s_1)-f^-(s_1))\nabla h^T(s_1)}{\nabla h^T(s_1) f^-(s_1)}.$$

PROOF. Following [17] and [10], there are two possible expressions for S_{14} when $\Delta t_1 = \Delta t_2$:

$$S_{14}^{(1)} = I_{2n} + (F_4(\boldsymbol{s}_1) - F_1(\boldsymbol{s}_1)) \frac{\nabla h_1^T(\boldsymbol{s}_1)}{\nabla h_1(\boldsymbol{s}_1)^T F_1(\boldsymbol{s}_1)}$$
$$= \begin{pmatrix} I_n + \frac{(f^+-f^-)\nabla h^T(s_1)}{\nabla h^T f^-(s_1)} & \mathbf{0} \\ \frac{(f^+-f^-)\nabla h^T(s_1)}{\nabla h^T f^-(s_1)} & I_n \end{pmatrix},$$
$$S_{14}^{(2)} = I_{2n} + (F_4(\boldsymbol{s}_1) - F_1(\boldsymbol{s}_1)) \frac{\nabla h_2^T(\boldsymbol{s}_1)}{\nabla h_2(\boldsymbol{s}_1)^T F_1(\boldsymbol{s}_1)}$$
$$= \begin{pmatrix} I_n & \frac{(f^+-f^-)\nabla h(s_1)^T(s_1)}{\nabla h(s_1)^T f^-(s_1)} \\ \mathbf{0} & I_n + \frac{(f^+-f^-)\nabla h^T(s_1)}{\nabla h^T(s_1) f^-(s_1)} \end{pmatrix}.$$

Below we show that the two expressions are equivalent, in the sense that their action on the input vector $\boldsymbol{x}_S(t_1) - \boldsymbol{y}(t_1)$ is identical.

Explicit computation of $S_{14}^{(1)}(\boldsymbol{x}_S(t_1) - \boldsymbol{y}(t_1))$ and $S_{14}^{(2)}(\boldsymbol{x}_S(t_1) - \boldsymbol{y}(t_1))$ gives

$$S_{14}^{(1)}(\boldsymbol{x}_S(t_1) - \boldsymbol{y}(t_1))$$
$$= \begin{pmatrix} (s_1 - y_1(t_1)) + (f^+(s_1) - f^-(s_1)) \frac{\nabla h(s_1)^T (s_1 - y_1(t_1))}{\nabla h(s_1)^T f^-(s_1)} \\ (f^+(s_1) - f^-(s_1)) \frac{\nabla h(s_1)^T (s_1 - y_1(t_1))}{\nabla h(s_1)^T f^-(s_1)} + (s_1 - y_2(t_1)) \end{pmatrix},$$
$$S_{14}^{(2)}(\boldsymbol{x}_S(t_1) - \boldsymbol{y}(t_1))$$
$$= \begin{pmatrix} (s_1 - y_1(t_1)) + (f^+(s_1) - f^-(s_1)) \frac{\nabla h(s_1)^T (s_1 - y_2(t_1))}{\nabla h(s_1)^T f^-(s_1)} \\ (f^+(s_1) - f^-(s_1)) \frac{\nabla h(s_1)^T (s_1 - y_2(t_1))}{\nabla h(s_1)^T f^-(s_1)} + (s_1 - y_2(t_1)) \end{pmatrix}.$$

The statement then follows if we show

$$\nabla h(s_1)^T (s_1 - y_1(t_1)) = \nabla h(s_1)^T (s_1 - y_2(t_1)) + \text{h.o.t.}. \tag{13}$$

We use the following Taylor expansions

$$\begin{aligned} &y_{1,2}(t_1 + \Delta t) = y_{1,2}(t_1) + f^-(y_{1,2}(t_1))\Delta t + O(\Delta t^2), \\ &h(y_{1,2}(t_1)) = h(s_1) + \nabla h^T(s_1)(y_{1,2}(t_1) - s_1) + \\ &\quad O(\|y_{1,2}(t_1) - s_1\|^2), \\ &0 = h(y_{1,2}(t_1 + \Delta t)) = \\ &h(y_{1,2}(t_1)) + \nabla h(y_{1,2}(t_1))^T f^-(y_{1,2}(t_1))\Delta t + \text{h.o.t.} \\ &= h(s_1) + \nabla h(s_1)^T (y_{1,2}(t_1) - s_1) + \\ &\quad \nabla h(s_1)^T f^-(s_1)\Delta t + \text{h.o.t.}. \end{aligned}$$

From the last equality it follows that

$$\begin{aligned} \nabla h(s_1)^T y_1(t_1) = \nabla h(s_1)^T f^-(s_1) + \text{ h.o.t.} \\ \nabla h(s_1)^T y_2(t_1) = \nabla h(s_1)^T f^-(s_1) + \text{ h.o.t.} \end{aligned} \tag{14}$$

and this implies (13).

For $N > 2$, the arguments are not different. Assume that $\Delta t_1 = \Delta t_2 < \Delta t_3 < \ldots < \Delta t_N$, i.e., the perturbed solution crosses Σ_1 and Σ_2 at the same time and then it crosses the other manifolds at subsequent times. It follows that the saltation matrix has the following expression

$$S_{1,2^N} = S_{2^N-1,2^N} \ldots S_{1,2^{N-1}+2^{N-2}+1}.$$

The perturbed solution starts in R_1, it crosses $\Sigma_1 \cap \Sigma_2$ and enters $R_{2^{N-1}+2^{N-2}+1}$. The jump matrix from R_1 into $R_{2^{N-1}+2^{N-2}+1}$ can be derived as in the proof above. Following exactly the same steps as in the proof of Lemma 12, we can use the expression

$$S_{1,2^{N-1}+2^{N-2}+1} = I_N \otimes I_n + \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \frac{(f^+-f^-)\nabla h^T(s_1)}{\nabla h^T f^-(s_1)} & \mathbf{0} \\ \mathbf{0} & I_{(N-2)n} \end{pmatrix}. \tag{15}$$

The other factors are jump matrices through the Σ_k's, $k = 3, \ldots, N$. For $N = 3$, for example, the jump matrix for $\Delta t_1 = \Delta t_2$ is given by $S_{18} = S_{78} S_{17}$, with S_{17} as in (15) and S_{78} as in (12). For clarity of exposition, we will do one more step. Assume that $\Delta t_1 = \Delta t_2 = \Delta t_3 < \Delta t_4 < \ldots < \Delta t_N$. This means that the perturbed solution crosses at the same time Σ_1, Σ_2 and Σ_3. Then

$$S_{1,2^N} = S_{2^N-1,2^N} \ldots S_{1,2^{N-1}+2^{N-2}+2^{N-3}+1},$$

with

$$S_{1,2^{N-1}+2^{N-2}+2^{N-3}+1} = I_N \otimes I_n + \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes \frac{(f^+-f^-)\nabla h^T(s_1)}{\nabla h^T f^-(s_1)} & \mathbf{0} \\ \mathbf{0} & I_{(N-3)n} \end{pmatrix}.$$

We now have the following theorem, that gives a unique expression for S_{14}.

Theorem 13 *With same notation as in Lemma 12, without loss of generality we can use the following expression for the jump matrix S_{14}:*

$$S_{14} = I_2 \otimes S_{-,+}, \tag{16}$$

where $S_{-,+}$ is the jump matrix of the single agent (3) *from R^- into R^+.*

PROOF. As a consequence of Lemmas 11 and 12, the jump matrix S_{14} can be taken as follows

$$S_{14} = \begin{cases} I_2 \otimes I_n + I_2 \otimes \frac{(f^+(s_1)-f^-(s_1))\nabla h(s_1)^T}{\nabla h(s_1)^T f^-(s_1)}, \\ \quad \Delta t_1 \neq \Delta t_2, \\ I_2 \otimes I_n + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \frac{(f^+(s_1)-f^-(s_1))\nabla h(s_1)^T}{\nabla h(s_1)^T f^-(s_1)}, \\ \quad \Delta t_1 = \Delta t_2. \end{cases}$$

Moreover, if $\Delta t_1 = \Delta t_2$, then (13) implies that at first order $\nabla h(s_1)^T y_1(t_1) = \nabla h(s_1)^T y_2(t_1)$ and similarly to the proof of Lemma 12, easy computations imply that at first order

$$\begin{aligned} &\left(I_2 \otimes \frac{(f^+(s_1) - f^-(s_1))\nabla h(s_1)^T}{\nabla h(s_1)^T f^-(s_1)}\right) \boldsymbol{y}(t_1) \\ &= \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \frac{(f^+(s_1) - f^-(s_1))\nabla h(s_1)^T}{\nabla h(s_1)^T f^-(s_1)}\right) \boldsymbol{y}(t_1). \end{aligned} \tag{17}$$

Then, at first order,

$$\boldsymbol{y}(t_1^+) - \boldsymbol{x}(t_1^+) = (I_2 \otimes S_{-,+})(\boldsymbol{y}(t_1^-) - \boldsymbol{x}(t_1^-)),$$

for any $\boldsymbol{y}(t)$ perturbed solution of $\boldsymbol{x}_S(t)$.

We stress that in general the saltation matrix at crossing points on the intersection of two or more surfaces is ambiguous, see [17,10]. However, at a synchronous solution, for $N = 2$, equation (17) guarantees that the different expressions for the saltation matrix are all equivalent. The same is true for $N > 2$. If for example $\Delta t_1 = \Delta t_2$, then equation (14) is verified also for $N > 2$. This in turn implies that at first order a formula analogous to (17) is verified, namely

$$\left(I_N \otimes \frac{(f^+ - f^-)\nabla h(s_1)^T}{\nabla h^T f^-(s_1)}\right) \boldsymbol{y}(t_1) =$$
$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \frac{(f^+-f^-)\nabla h(s_1)^T}{\nabla h^T f^-(s_1)} \boldsymbol{y}(t_1) & \boldsymbol{0} \\ \boldsymbol{0} & I_{N-2} \otimes \frac{(f^+-f^-)\nabla h(s_1)^T}{\nabla h^T f^-(s_1)} \end{pmatrix} \boldsymbol{y}(t_1).$$

Then for $N > 2$ we can use the following expression for the jump matrix at the crossing point $\boldsymbol{s}_1$

$$S_{1,2^N} = \left(I_N \otimes \left(I_n + \frac{(f^+ - f^-)\nabla h(s_1)^T}{\nabla h^T f^-(s_1)}\right)\right).$$

Remark 14 *Lemma 11 and 12 and Theorem 13* **extend** *in a straightforward way to the case $N > 2$, via the replacement of S_{14} with $S_{1,2^N} = I_N \otimes S_{-,+}$, giving $\boldsymbol{y}(t_1^+) - \boldsymbol{x}(t_1^+) = (I_N \otimes S_{-,+})(\boldsymbol{y}(t_1^-) - \boldsymbol{x}(t_1^-))$.*

2.2 Jump matrix at the sliding point

Going back to (11), we next need to analyze $S_{4,\boldsymbol{\Sigma}}$, that is the jump matrix from R_4 to $\boldsymbol{\Sigma}$. In general, the jump matrix from the region R_4 to $\boldsymbol{\Sigma}$ is not uniquely defined, which makes it not possible to give a unique expression for the mondromy matrix. However, the jump matrix for $\boldsymbol{x}_S(t)$ is an exception, as stated in Theorem 15, in the sense that (at first order) we can give a unique expression for the action of $S_{4,\boldsymbol{\Sigma}}$ on $\boldsymbol{y}(t_2^-) - \boldsymbol{x}_S(t_2^-)$. The proof in Theorem 15 sums up results analogous to the ones given in Lemmas 11 and 12 and in Theorem 13 for S_{14}.

Theorem 15 *Let $\boldsymbol{x}_s = \begin{bmatrix} x_s \\ x_s \end{bmatrix}$ where x_s satisfies Assumption 3. Then, without loss of generality we can use the following expression for the jump matrix $S_{4,\boldsymbol{\Sigma}}$:*

$$S_{4,\boldsymbol{\Sigma}} = I_2 \otimes (I_n + \frac{(f_\Sigma(s_2) - f^+(s_2))\nabla h(s_2)^T}{\nabla h(s_2)^T f^+(s_2)}) = I_2 \otimes S_{+,\Sigma},$$

where $S_{+,\Sigma}$ is the jump matrix of the single agent (3) *from R^+ into Σ, and it is explicitly given by $S_{+,\Sigma} = I + \frac{(f_\Sigma(s_2) - f^+(s_2))\nabla h(s_2)^T}{\nabla h(s_2)^T f^+(s_2)}$, with f_Σ given in* (7).

PROOF. Following [10], a perturbed solution of $\boldsymbol{x}_S(t)$ might either reach directly $\boldsymbol{\Sigma}$ in a neighborhood of $\boldsymbol{s}_2$ or it might first slide along Σ_1 or Σ_2 before reaching $\boldsymbol{\Sigma}$. Let $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ be the perturbed solution, with $\boldsymbol{y}(0) = \boldsymbol{s}_0 + \Delta \boldsymbol{s}$ and $\|\Delta \boldsymbol{s}\|\| \ll 1$.

i) We first consider the case in which the perturbed solution reaches Σ_1^+, slides along it and then it reaches $\boldsymbol{\Sigma}$. Then $S_{4,\boldsymbol{\Sigma}} = S_{\Sigma_1^+,\boldsymbol{\Sigma}} S_{4,\Sigma_1^+}$ with

$$S_{4,\Sigma_1^+} = I_{2n} + \frac{(F_{\Sigma_1^+}(\boldsymbol{s}_2) - F_4(\boldsymbol{s}_2))\nabla h_1(\boldsymbol{s}_2)^T}{\nabla h_1(\boldsymbol{s}_2)^T F_1(\boldsymbol{s}_2)}$$
$$= \begin{pmatrix} I_n + \frac{(f_\Sigma(s_2)-f^+(s_2))\nabla h(s_2)^T}{\nabla h(s_2)^T f^+(s_2)} & 0 \\ 0 & I_n \end{pmatrix},$$
$$S_{\Sigma_1^+,\boldsymbol{\Sigma}} = I_{2n} + \frac{(F_\Sigma(\boldsymbol{s}_2) - F_{\Sigma_1^+}(\boldsymbol{s}_2))\nabla h_2(\boldsymbol{s}_2)^T}{\nabla h_2(\boldsymbol{s}_2)^T F_1(\boldsymbol{s}_2)}$$
$$= \begin{pmatrix} I_n & 0 \\ 0 & I_n + \frac{(f_\Sigma(s_2)-f^+(x))\nabla h(s_2)^T}{\nabla h(s_2)^T f^+(s_2)} \end{pmatrix},$$

where the vector fields $F_{\boldsymbol{\Sigma}}$ and $F_{\Sigma_1^+}$ are as in Lemma 8. Then the statement follows in this case. The case in which $\boldsymbol{y}(t)$ reaches Σ_2^+ before reaching $\boldsymbol{\Sigma}$ is analogous and gives the same expression for the jump matrix.

ii) We next consider the case in which $\boldsymbol{y}(t)$ reaches $\boldsymbol{\Sigma}$ directly at time $t_2 + \Delta t$. As in [10], in this case $S_{4,\boldsymbol{\Sigma}}$ has two expressions, namely

$$S_{4,\boldsymbol{\Sigma}}^{(1)} = I_{2n} + \frac{(F_{\boldsymbol{\Sigma}} - F_4)(\boldsymbol{s}_2)\nabla h_1(\boldsymbol{s}_2)^T}{\nabla h_1(\boldsymbol{s}_2)^T F_4(\boldsymbol{s}_2)},$$
$$S_{4,\boldsymbol{\Sigma}}^{(2)} = I_{2n} + \frac{(F_{\boldsymbol{\Sigma}} - F_4)(\boldsymbol{s}_2)\nabla h_2(\boldsymbol{s}_2)^T}{\nabla h_2(\boldsymbol{s}_2)^T F_4(\boldsymbol{s}_2)}.$$

Using the same reasonings as in Lemma 12, at first order the perturbed solution must satisfy $\nabla h_1(\boldsymbol{s}_2)^T \boldsymbol{y}(t_2) = \nabla h_2(\boldsymbol{s}_2)^T \boldsymbol{y}(t_2)$, i.e., $\nabla h(s_2)^T y_1(t_2) = \nabla h(s_2)^T y_2(t_2)$ + h.o.t.. In this case we can use a unique expression for $S_{4,\boldsymbol{\Sigma}}$:

$$S_{4,\boldsymbol{\Sigma}} = I_2 \otimes I_n + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \frac{(f_\Sigma - f^+)(s_2)\nabla h^T}{\nabla h^T f^+(s_2)}.$$

iii) The final argument is analogous to the one in the proof of Theorem 13. We have two possible expressions for the jump matrix: in i), when $\boldsymbol{y}$ does not reach $\boldsymbol{\Sigma}$ directly and in ii) when $\boldsymbol{y}$ reaches $\boldsymbol{\Sigma}$ directly from R_4. In this last case however, $\nabla h^T(s_2)y_1(t_2) = \nabla h^T(s_2)y_2(t_2)$ + h.o.t. and this implies that at first order

$$\boldsymbol{y}(t_2^+) - \boldsymbol{x}(t_2^+) = (I_2 \otimes S_{+,\Sigma})(\boldsymbol{y}(t_2) - \boldsymbol{x}(t_2)).$$

The above points i), ii), iii) imply the statement of the theorem.

Remark 16 *The* **extension** *of Theorem 15 to the case of* $N > 2$ *is immediately achieved by taking* $S_{2^N,\boldsymbol{\Sigma}} = I_N \otimes S_{+,\Sigma}$.

2.3 Monodromy matrix at the synchronous periodic orbit

Finally, the following theorem gives the complete expression for the monodromy matrix of (2) linearized about the synchronous solution $\boldsymbol{x}_S(t)$. The proof puts together all the results previously derived in this section and is therefore omitted.

Theorem 17 *Let* $x_s(t)$ *be the limit cycle of* (3)*, and let Assumption 3 be satisfied. Let* $\boldsymbol{x}_S(t) = e \otimes x_S(t)$ *be the corresponding synchronous solution of* (2)*. Then, the monodromy matrix of* (2) *along* $\boldsymbol{x}_S$ *can be taken as*

$$X(T,0) = X_1(T,t_3)X_\Sigma(t_3,t_2)(I_N \otimes S_{+,\Sigma})$$
$$X_{2^N}(t_2,t_1)(I_N \otimes S_{-,+})X_1(t_1,0),$$

with

$$\tfrac{dX_i(t,\tau)}{dt} = (DF_i(\boldsymbol{x}_S(t)) + \sigma M)X_i(t,\tau),$$
$$X_i(\tau,\tau) = I_{Nn}, i = 1, 2^N$$

$$\tfrac{dX_{\boldsymbol{\Sigma}}(t,\tau)}{dt} = (DF_{\boldsymbol{\Sigma}}(\boldsymbol{x}_S(t)) + \sigma M)X_{\boldsymbol{\Sigma}}(t,\tau),$$
$$X_{\boldsymbol{\Sigma}}(\tau,\tau) = I_{Nn},$$

where DF_i *and* $DF_{\boldsymbol{\Sigma}}$ *are the Jacobian matrices of the vector fields* F_i, $i = 1, 2^N$ *and* $F_{\boldsymbol{\Sigma}}$ *respectively.* □

Following Remark 4, Theorem 17 can be generalized to any periodic solution with a finite number of generic events. The monodromy matrix will be given by the product of fundamental matrices in $R_{1,2^N}$ or $\boldsymbol{\Sigma}$ and jump matrices $(I_N \otimes S_{\mp,\pm})$ for each crossing event and $(I_N \otimes S_{\pm,\Sigma})$ for each sliding event. See also Remark 10.

The key implication of Theorem 17 is that the saltation matrices appearing in the expression of $X(T,0)$ can be obtained from the saltation matrices of a single agent in the network, which is a great simplification. However, computation of the matrices X_i and $X_{\boldsymbol{\Sigma}}$ involves all N agents. In case of large networks, these computations are too expensive, and an extension of the MSF theory to (2) is needed. This is the purpose of the next section.

3 Master Stability function

The MSF is a very nice technique which allows to study linearized stability of the synchronous solution of a smooth network of N agents of size n each, by working with N linearized systems of size n, rather than one linearized system of size nN, a substantial saving! The key idea in the MSF technique is to consider the variational equation along the synchronous solution and to perform a change of coordinates induced by the matrix of eigenvectors of the matrix L. For smooth systems, this change of coordinates brings the whole network into a block diagonal structure with sub-blocks of size $n \times n$ and this in turn allows one to study the stability of N systems of dimension n instead of the stability of one system of dimension nN (see [22]).

In the case of piecewise smooth vector fields for the agents, there are at least two new concerns. First, the same change of coordinates, while still bringing the variational equation along the synchronous solution into block diagonal form, will also change the equations of the discontinuity manifolds that will now in general involve more than one agent, and possibly all of them. Then, we should not expect the saltation matrices to be block diagonal. However, we will see that the transformation preserves the block structure of the saltation matrices of Theorem 17. In other words, the Kronecker products involved in the expression of the saltation matrices along a synchronous solution are left unchanged by the coordinate change induced by the eigenvectors of L (see equation (18)). The second concern is related to the portion of the fundamental matrix solution on $\mathbf{\Sigma}$, in particular to the Jacobian of $F_{\mathbf{\Sigma}}$. The general lack of uniqueness in expressing the sliding vector field on the intersection of two or more discontinuity manifolds is not a concern in this setting, since, by Lemma 8, on $\mathbf{\Sigma}$ we have a unique Flilippov sliding vector field. However, a difficulty is related to expressing the Jacobian itself, since again the coordinate change seemingly will destroy the sought block structure. We deal with this difficulty in Lemma 19 below. As a final result, in Theorem 21 we will see that we can use the MSF technique also for PWS networks, and in particular to study stability of the synchronous solution $\boldsymbol{x}_S$ by linearized analysis on N systems of size n.

As before, below $L = L^T$ is the negative of the graph Laplacian matrix, and let W be the matrix of the orthonormal eigenvectors of L: $W^T L W = \Lambda$, with Λ diagonal of eigenvalues $\lambda_1 = 0 > \lambda_2 \geq \cdots \geq \lambda_N$. Then $M = L \otimes E$ has eigenvalues $\lambda_i \mu_j$, $i = 1, \ldots N$, $j = 1, \ldots, n$, with λ_i's the eigenvalues of L, and μ_j's the eigenvalues of E. Moreover, with $V = W \otimes I$, we get $V^{-1} M V = (W^T \otimes I_n)(L \otimes E)(W \otimes I_n) = \Lambda \otimes E$. In [22], the change of variables $\boldsymbol{y} = V^{-1}\boldsymbol{x}$ reduces the variational equation along a synchronous solution $\boldsymbol{x}_S(t)$ into the following block diagonal form

$$\dot{z}_i = (Df(x_S) + \sigma \lambda_i E) z_i, \qquad i = 1, \ldots, N,$$

that is we have N systems of size n. The issue with nonsmooth agents is that the change of variables above in general changes the equations of the discontinuity manifolds as well, and this makes it impossible to study N systems independently. In particular, the new equations of the discontinuity manifold(s) might involve all the agents.

Example 18 *To illustrate the last statement, take* $L = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ *and* $M = L \otimes I_2$. *Then, using the same notations as before,* $W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *and* $V = W \otimes I_2$. *If the discontinuity surface for the single agent is the plane* $\{x \in \mathbb{R}^n | h(x) = e_1^T x = 0\}$ *, then* Σ_1 *and* Σ_2 *in the* $\boldsymbol{y}$ *coordinates become respectively* $e_1^T \boldsymbol{y} = -e_3^T \boldsymbol{y}$ *and* $e_1^T \boldsymbol{y} = e_3^T \boldsymbol{y}$ *. It follows that the agents cannot be studied independently even though the variational equations for the fundamental matrix solution are in block diagonal form.*

Nevertheless, we will now see that the special structure of the fundamental matrix solution for the case of a synchronous periodic solution of (2), allows to study the stability of $\boldsymbol{x}_S$ via N systems of dimension n.

With $V = W \otimes I$, consider the monodromy matrix for the linearization along the synchronous periodic solution. Let $\boldsymbol{y}_S(t) = V^{-1}\boldsymbol{x}_S(t)$ and $Y(T,0) = V^{-1} X(T,0) V$. Then

$$\begin{aligned} Y(T,0) = & Y_1(T,t_3) Y_{\mathbf{\Sigma}}(t_3,t_2)(V^{-1} S_{2^N,\mathbf{\Sigma}} V) \\ & Y_{2^N}(t_2,t_1)(V^{-1} S_{1,2^N} V) Y_1(t_1,0), \end{aligned}$$

with $Y_i(t,\tau) = V^{-1} X_i(t,\tau) V$, $i = 1, 2^N$, and $Y_{\mathbf{\Sigma}}(t,\tau) = V^{-1} X_{\mathbf{\Sigma}}(t,\tau) V$.

First notice that the particular structure of the jump matrices is such that

$$\begin{aligned} (V^{-1} S_{1,2^N} V) &= (W^T \otimes I_n)(I_N \otimes S_{-,+})(W \otimes I_n) \\ &= (W^T I_N W) \otimes (I_n S_{-,+} I_n) = I_N \otimes S_{-,+}, \end{aligned} \tag{18}$$

and similarly for $S_{2^N,\mathbf{\Sigma}}$.

Secondly, we show that the Y_i's can be obtained solving block diagonal systems of ODEs. Note that

$$\begin{aligned} & V^{-1} DF_i(V \boldsymbol{x}_S(t)) V \\ = & (W^T \otimes I_n)(I_N \otimes Df_*(x_S(t))(W \otimes I_n) \\ = & I_N \otimes Df_*(x_S(t)), \end{aligned}$$

with $Df_* = Df^-$ for $i = 1$, and $Df_* = Df^+$ for $i = 2^N$, and hence

$$\begin{aligned} & \frac{dY_i(t,\tau)}{dt} = [I_N \otimes Df_*(x_S(t)) + \sigma \Lambda \otimes E] \, Y_i(t,\tau), \\ & Y_i(\tau,\tau) = I_{Nn}, \quad i = 1, 2^N. \end{aligned} \tag{19}$$

The following Lemma shows how to rewrite the sliding vector field $V^{-1} DF_{\mathbf{\Sigma}}(\boldsymbol{x}_S) V$ as a Kronecker product as well.

Lemma 19 *Let $\boldsymbol{x}_S(t)$ be a synchronous periodic solution of* (2) *and let $F_{\boldsymbol{\Sigma}}(\boldsymbol{x}_S(t))$ be the sliding vector field defined as in Lemma 8. Then*

$$V^{-1}DF_{\boldsymbol{\Sigma}}(\boldsymbol{x}_S(t))V = I_N \otimes Df_\Sigma(x_S(t)) + \sigma\Lambda \otimes E + \{\frac{\sigma}{\nabla h^T(f^- - f^+)}\Lambda \otimes [(f^+ - f^-)\nabla h^T E]\}_{x_S(t)} \ . \quad (20)$$

PROOF. For simplicity, we will show the statement for $N = 2$, the generalization for $N > 2$ is immediate. The sliding vector field on Σ for (2) is defined in Lemma 8. That is, $F_{\boldsymbol{\Sigma}}$ is

$$F_{\boldsymbol{\Sigma}}(\boldsymbol{x}) = \begin{bmatrix}(1-\alpha_1(\boldsymbol{x})))f^-(x_1) + \alpha_1(\boldsymbol{x})f^+(x_1)\\(1-\alpha_2(\boldsymbol{x})))f^-(x_2) + \alpha_2(\boldsymbol{x})f^+(x_2)\end{bmatrix} + \sigma M\boldsymbol{x}_S,$$

where we have kept the term $M\boldsymbol{x}_S$ even though it is 0 (since $\boldsymbol{x}_S$ is synchronous), to clarify the computation of the Jacobian.

Now, $\alpha_1(\boldsymbol{x}_S)$ and $\alpha_2(\boldsymbol{x}_S)$ must be chosen so that: $\begin{bmatrix}\nabla h\\0\end{bmatrix}^T F_{\boldsymbol{\Sigma}}(\boldsymbol{x}_S) = \begin{bmatrix}0\\\nabla h\end{bmatrix}^T F_{\boldsymbol{\Sigma}}(\boldsymbol{x}_S) = 0$. Let $\alpha(x_S)$ be such that $f_\Sigma(x_S) = (1-\alpha(x_S))f^-(x_S) + \alpha(x_S)f^+(x_S)$ as in equation (7). Then, for $i = 1, 2$, when we impose the tangency conditions, using $M = L \otimes E$, we get:

$$\alpha_i(\boldsymbol{x}_S) = \alpha(x_S) + \frac{\sigma}{\nabla h(x_S)^T(f^- - f^+)(x_S)} \times \nabla h(x_S)^T(l_{i1}Ex_1 + l_{i2}Ex_2)_{x_{1,2}=x_S} = \alpha(x_S),$$

where the last equality follows from the definition of the matrix L ($l_{i1} = -l_{i2}$) and the fact that the solution is synchronous, i.e. $x_1 = x_2 = x_S$. The gradient of $\alpha_1(\boldsymbol{x})$ is then obtained as follows

$$D_{x_1}\alpha_1(\boldsymbol{x}_S) = [\nabla\alpha(x_1)]_{x_1=x_S} + \frac{\sigma}{\nabla h^T(f^- - f^+)(x_S)} l_{11}\nabla h(x_S)^T E + \sigma\frac{\nabla h^T(l_{11}Ex_1 + l_{12}Ex_2)(\nabla h^T(Df^+ - Df^-))}{(\nabla h^T(f^- - f^+))^2}|_{x_1=x_s}$$
$$= [\nabla\alpha(x_1)]_{x_1=x_S} + \frac{\sigma}{\nabla h^T(f^- - f^+)(x_S)} l_{11}\nabla h(x_S)^T E,$$

and similarly

$$D_{x_2}\alpha_1(\boldsymbol{x}_S) = \frac{\sigma}{\nabla h(x_S)^T(f^- - f^+)(x_S)} l_{12}\nabla h(x_S)^T E.$$

Therefore,

$$\nabla\alpha_1(\boldsymbol{x}_S) = \begin{bmatrix}[\nabla\alpha(x_1)]_{x_1=x_S} + \frac{\sigma}{\nabla h^T(f^- - f^+)} l_{11}\nabla h^T E\\ \frac{\sigma}{\nabla h^T(f^- - f^+)(x_S)} l_{12}\nabla h^T E\end{bmatrix}_{x_S}.$$

For $\alpha_2(\boldsymbol{x}_S)$ we obtain in a similar way

$$\nabla\alpha_2(\boldsymbol{x}_S) = \begin{bmatrix}\frac{\sigma}{\nabla h^T(f^- - f^+)} l_{21}\nabla h^T E\\ [\nabla\alpha(x_2)]_{x_2=x_S} + \frac{\sigma}{\nabla h^T(f^- - f^+)} l_{22}\nabla h^T E\end{bmatrix}_{x_S}.$$

Then

$$DF_\Sigma(\boldsymbol{x}_S) = \begin{bmatrix}Df_\Sigma(x_S) + \sigma l_{11}B & \sigma l_{12}B\\ \sigma l_{21}B & Df_\Sigma(x_S) + \sigma l_{22}B\end{bmatrix} + \sigma M$$
$$= I_N \otimes Df_\Sigma(x) + \sigma L \otimes B + \sigma L \otimes E, \quad (21)$$

where $B = \frac{(f^+ - f^-)(x_S)\nabla h(x_S)^T}{\nabla h(x_S)^T(f^- - f^+)(x_S)}E$. Then the statement follows at once.

Remark 20 *In the case of $N = 2$, the negative Laplacian matrix is trivial: $L = \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$. But, aside from this simplification, the proof of Lemma 19 is identical for the case of $N > 2$ except that we have to account for more entries in the Laplacian matrix. In particular, the expression* (21) *remains valid.*

From Lemma 19 and (19), with $Df_* = Df^-$ for $i = 1$, and $Df_* = Df^+$ for $i = 2^N$, we get

$$\frac{dY_i(t,\tau)}{dt} = \big(I_N \otimes Df_*(x_S(t)) + \sigma\Lambda \otimes E\big)Y_i(t,\tau),$$
$$Y_i(\tau,\tau) = I_{Nn}, \ \ i = 1, 2^N,$$
$$\frac{dY_\Sigma(t,\tau)}{dt} = \big(I_N \otimes Df_\Sigma(x_S(t)) + \sigma\Lambda \otimes E + \sigma\Lambda \otimes B\big)Y_\Sigma(t,\tau),$$
$$Y_\Sigma(\tau,\tau) = I_{Nn}. \quad (22)$$

To sum up

$$Y(T,0) = Y_1(T,t_3)Y_{\boldsymbol{\Sigma}}(t_3,t_2)(I_N \otimes S_{+,\boldsymbol{\Sigma}}) Y_{2^N}(t_2,t_1)(I_N \otimes S_{-,+})Y_1(t_1,0), \quad (23)$$

with Y_1, Y_{2^N} and $Y_{\boldsymbol{\Sigma}}$ as in the block diagonal equations (22). In conclusion, we proved the following key theorem.

Theorem 21 *Let $\lambda_1 = 0 > \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of the (negative graph Laplacian) matrix L, and let $\nu = -\sigma\lambda_i$, $i = 1, \ldots, N$. The Floquet multipliers of* (2) *along the synchronous periodic orbit $\boldsymbol{x}_S(t)$ are the nN eigenvalues of the N matrices $Z_i(T)$, $i = 1, \ldots, N$,* *satisfying the following variational equations*

$$\dot{Z}_i = \begin{cases} (Df^-(x_S(t)) - \nu E)Z_i, & 0 \le t < t_1, \\ (Df^+(x_S(t)) - \nu E)Z_i, & t_1 < t < t_2, \\ (Df_\Sigma(x_S(t)) - \nu[E+B])Z_i, & t_2 < t < t_3, \\ (Df^-(x_S(t)) - \nu E)Z_i, & t_3 < t < T, \end{cases} \tag{24}$$

where $B = \frac{(f^+-f^-)(x_S)\nabla h(x_S)^T}{\nabla h(x_S)^T(f^- - f^+)(x_S)}E$, *and subject to the initial conditions:* $Z_i(0) = I_n$, $Z_i(t_1^+) = S_{-,+}Z_i(t_1^-)$, $Z_i(t_2^+) = S_{+,\boldsymbol{\Sigma}}Z_i(t_2^-)$, $Z_i(t_3^+) = Z_i(t_3^-)$. □

Remark 22 *In agreement with Remark 4, the case of a periodic orbit* x_s *of the single agent with several generic events (transversal crossings, transversal entries on* Σ*, and tangential exits) will result in immediate adjustments of* (24)*. To exemplify, suppose that the orbit starts at* s_0 *in* R^- *(time* 0*), has a generic crossing at* s_1 *(time* t_1*) from* R^- *to* R^+*, then has a transversal entry on* Σ *at* s_2 *(time* t_2*) and sliding on* Σ *until there is a tangential exit back onto* R^+ *at* s_3 *(time* t_3*), followed by another transversal entry on* Σ *at* s_4 *(time* t_4*) and sliding on* Σ *until the tangential exit onto* R^- *at* s_5 *(time* t_5*), and finally returns to* s_0 *after the period* T*. The structure of the matrices* Z_i*'s is similar to that in* (24)*, namely now we would have*

$$\dot{Z}_i = \begin{cases} (Df^-(x_S(t)) - \nu E)Z_i, & 0 \le t < t_1, \\ (Df^+(x_S(t)) - \nu E)Z_i, & t_1 < t < t_2, \\ (Df_\Sigma(x_S(t)) - \nu[E+B])Z_i, & t_2 < t < t_3, \\ (Df^+(x_S(t)) - \nu E)Z_i, & t_3 < t < t_4, \\ (Df_\Sigma(x_S(t)) - \nu[E+B])Z_i, & t_4 < t < t_5, \\ (Df^-(x_S(t)) - \nu E)Z_i, & t_5 < t < T, \end{cases}$$

where the initial conditions are given by $Z_i(0) = I_n$, $Z_i(t_1^+) = S_{-,+}Z_i(t_1^-)$, $Z_i(t_2^+) = S_{+,\boldsymbol{\Sigma}}Z_i(t_2^-)$, $Z_i(t_3^+) = Z_i(t_3^-)$, $Z_i(t_4^+) = S_{+,\boldsymbol{\Sigma}}Z_i(t_4^-)$, $Z_i(t_5^+) = Z_i(t_5^-)$.

As usual, we call Floquet exponents $\frac{1}{T}$ times the logarithms of the multipliers. For sure there is a 0 exponent, since 1 is a multiplier, because $\boldsymbol{x}_S$ is a periodic solution of (2). Now, since our network is connected, L has only one eigenvalue equal to 0, let it be $\lambda_1 = 0$, all other eigenvalues of L being negative. With this observation, we are ready for the following definition.

Definition 23 *Let* $\lambda_1 = 0 > \lambda_2 \ge \cdots \ge \lambda_N$ *be the eigenvalues of the (negative graph Laplacian) matrix* L*. Let* $\tau_{i,j}(\sigma)$ *be the multipliers of* (24)*, for* $\nu = -\sigma\lambda_i$*, and* $i = 2, \dots, N$*, and let* $l_{i,j} = \frac{1}{T}\log|\tau_{i,j}|$*,* $j = 1, \dots, n$*. The Master Stability Function (MSF) for* (2)*, relative to the synchronous periodic solution* $\boldsymbol{x}_S$*, is the largest value* $l_{i,j}$*, call it* λ*. The synchronous solution* $\boldsymbol{x}_S$ *is transversally stable for those values of* σ*, if any, for which* $\lambda < 0$*.*

Remark 24 *In the literature for smooth networks, the MSF is defined in terms of the Lyapunov exponents of the linearized problem. Of course, in the case of periodic orbits, these are the Floquet exponents, and hence our definition is consistent with previous usage of the MSF.*

Naturally, the value of the MSF λ depends on the coupling strength σ, as well as on E and the (negative) Laplacian L. However, for a given network topology (hence, for given L and E), the MSF depends only on σ. We must further appreciate that the synchronous solution $\boldsymbol{x}_S$ is asymptotically stable if all parameters values $\sigma\lambda_k$, $k = 2, \dots, N$, give multipliers less than 1 in modulus.

4 Periodic orbit of a piecewise smooth mechanical system. Computation of the MSF

Here we study a system of identical piecewise smooth mechanical oscillators. The case of two oscillators was first studied in [15]. When they are not coupled, the single agents have a periodic solution which is stable in finite time, denote it as $x_s(t)$.

4.1 A piecewise smooth network

The equations of the single agent are

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -y_1 \pm \frac{1}{1+\gamma|y_2 - \bar{v}|} \end{aligned} \tag{25}$$

with $\gamma \ge 0$. In the notation of (3), we have

$$f^{\pm}(y) = \begin{bmatrix} y_2 \\ -y_1 \mp \frac{1}{1 \pm \gamma(y_2 - \bar{v})} \end{bmatrix}, \ y \in \mathbb{R}^2 ,$$

and the discontinuity surface is the plane $y_2 - \bar{v} = 0$, so that $\nabla h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In the computations below, we fix $\bar{v} = 0.15$, and$\gamma = 3$ as in [15]. For these parameter values, the agent (25) has a stable (in finite time) periodic solution of period T, $x_s(t) = (y_1(t), y_2(t))^T$, which we plot in Figure 3. Without loss of generality, we assume that the entry point of $x_s(t)$ on the discontinuity line $y_2 = \bar{v}$ occurs at $t = 0$. The Floquet multipliers along $x_s(t)$ must be 1 and 0. (Note that, in the presence of sliding,the system of a single agent is not reversible in time and the monodromy matrix must be singular, since the jump matrix from a region to Σ has rank 1 in this case; e.g., see [10, Lemma 2.4].)

We consider a network as in equation (2) with L being the negative of the nearest neighbor Laplacian ma-

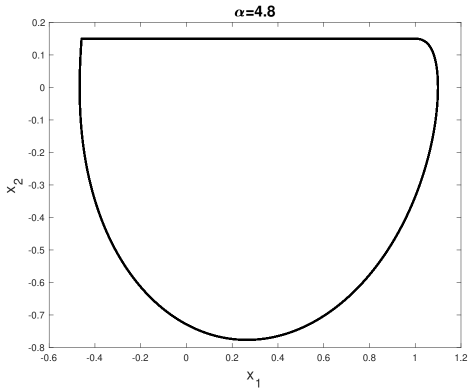

Fig. 3. Periodic orbit of (25).

trix and $E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Recall that, in our notation, $L = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}$ and it has eigenvalues $\lambda_j = -4\sin^2\left(\frac{(j-1)\pi}{2N}\right)$, $j = 1, \ldots, N$.

Now, consider N agents $x_1, \ldots, x_N$, each one satisfying equation (25) and let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$. Then, the network satisfies the differential equation

$$\dot{\boldsymbol{x}} = \begin{pmatrix} f^{\pm}(x_1) \\ \vdots \\ f^{\pm}(x_N) \end{pmatrix} + \sigma M \boldsymbol{x}, \quad M = L \otimes E, \tag{26}$$

with $\sigma \geq 0$.

We use the MSF approach to study the stability of the synchronous periodic solution. For $\sigma = 0$, the oscillators are uncoupled and the synchronous solution has N Floquet multipliers at 1 and N multipliers at 0 since x_S is a sliding periodic orbit. Of course, the synchronous solution persists for $\sigma \geq 0$, though its stability will depend on σ. We use the MSF to compute the Floquet exponents of the synchronous solution for $\sigma > 0$.

We proceed like in Section 3. Recall that $x_s(t)$ intersects $y_2 = \bar{v}$, at $t = 0$. Then, in order to study the stability of the synchronous solution $\boldsymbol{x}_S(t)$ for a given σ, instead of computing the nN Floquet multipliers of (26) along $\boldsymbol{x}_S(t)$, we employ the MSF approach and compute the Floquet multipliers of $N-1$ linear non autonomous systems of dimension $n = 2$. Let λ_i, $i = 2, \ldots, N$, be the eigenvalues of L different from 0, and for $\nu = -\sigma\lambda_i$ consider the following matrices

$$\dot{Z}_i = \begin{cases} (Df_\Sigma(x_s(t)) - \nu(E+B))Z_i, & \nabla h^T x_s(t) = \bar{v}, \\ (Df^-(x_s(t)) - \nu E)Z_i, & \nabla h^T x_s(t) < \bar{v}, \end{cases} \tag{27}$$

with $Z_i(0) = S_{-,\Sigma}$ and $S_{-,\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ being the saltation matrix from R^- into Σ. We emphasize that Theorem 15 applies also to the jump matrix $S_{1,\Sigma}$ from R_1 into Σ and that it is its rewriting as the Kronecker product $(I_N \otimes S_{-,\Sigma})$ that allows us to use the MSF approach. Finally, in (27) $B = \frac{(f^+ - f^-)(x_S)\nabla h^T(x_S)}{\nabla h(x_S)^T(f^- - f^+)(x_S)}E$, and we point out that for our problem (26), we have $E + B = 0$, and therefore in the sliding phase of (27) the linearized problem is simply $\dot{Z}_i = Df_\Sigma(x_S(t))Z_i$. We stress that for each value of $\nu = -\sigma\lambda_i$, we always obtain two Floquet multipliers, regardless of the value of N. Moreover one of these multipliers is 0 since $S_{-,\Sigma}$ is a rank 1 matrix.

4.2 Numerical experiments

Thanks to Theorem 21, the Floquet exponents of the synchronous solution of (26) for a given σ, can be computed from (27). Therefore, our task is to compute the solution of a single oscillator over one period, and then for $\nu = -\lambda_i\sigma$, $i = 2, \ldots, N$, compute the monodromy matrix $Z_i(t)$ of (27) and extract its Floquet multipliers. Computation of the periodic orbit of the single agent is done with the 4th order event technique of [9] and fixed stepsize equal to 10^{-4} (so to have a local error per step of size about `eps`) and the monodromy matrix is computed on the same mesh at once. In Figure 4 we show the multiplier different from zero in function of ν.

To confirm the results of the MSF analysis, we also integrated the full discontinuous system (25) with a variable stepsize integrator and event location techniques for sliding along the intersection of two discontinuity manifolds. For $\nu = 4.8$ the MSF plotted in Figure 4 predicts asymptotic stability of the synchronous solution. The integration of a discontinuous system with $N > 2$ discontinuity hyperplanes is not trivial and it is beyond the scope of the present paper, hence we verify correctness of our results for $N = 2$. In this case, L has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$, and so we need to verify that the synchronous solution $\boldsymbol{x}_S(t)$ is asymptotically stable for $\sigma = \frac{\eta}{2} = 2.4$. Given initial conditions that do not belong to the synchronous manifold we integrated the full network for sufficiently long time to observe convergence of the numerical solution to the synchronous periodic orbit. In Figure 5 we plot $(x_1(t) - x_3(t))$, after discarding the transient.

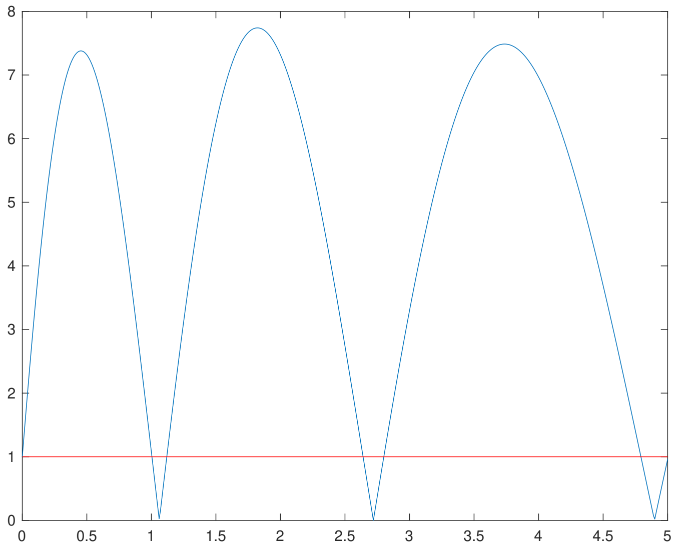

Fig. 4. Multiplier below 1 indicates stability of synchronized motion

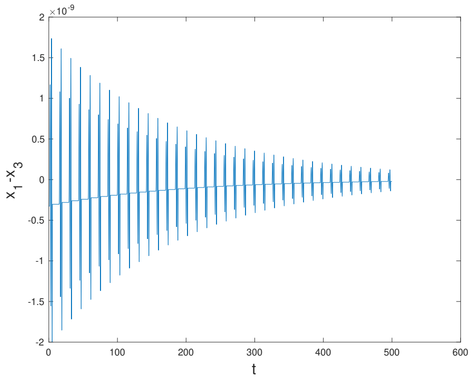

Fig. 5. Network synchronization for $\sigma = 2.4$.

We do not see synchronization for other parameter values such as $\sigma = 0.5$, or $\sigma = 1.3$, while we see synchronization for $\sigma = 1.35$, as predicted by the MSF. Finally, for the value of $\sigma = 1.2$, our analysis based on the MSF validates the observation in [15, Section 9] that the synchronous manifold is unstable.

5 Conclusions

In this work, we extended the Master Stability Function (MSF) tool to Filippov networks of identical Piecewise Smooth (PWS) differential systems, in order to infer stability of a synchronous periodic solution of the network. Our analysis rested on the appropriate extension of the fundamental matrix solution in the present PWS case. We had to overcome several difficulties, in primis the lack of uniqueness of suitable saltation matrices on the intersection of several discontinuity manifolds and the possibility to decouple the (large) linearized nN-system into N systems of size n, in order to exploit the MSF technique. We succeeded in doing this under very general assumptions, for the network synchronizing along a periodic orbit of a single agent. We complemented our analysis by a numerical illustration of the use of the MSF for a PWS system of mechanical oscillators synchronizing (for some values of the coupling parameter) on a stick-slip oscillatory regime. The case of synchronization on an orbit different from a periodic one remains to be analyzed.

Acknowledgements

This work has been partially supported by the GNCS-Indam group and the PRIN2017 research grant. The authors gratefully acknowledge the inspiration provided by a series of lectures given by Mario di Bernardo and Marco Coraggio at the University of Bari in March 2020. They were the last lectures before universities in Italy had to shut down due to the pandemic and provided a stimulating diversion in the following months.

References

[1] M.A. Aizerman and F.R. Gantmacher. On the stability of periodic motion. *Journal of Applied Mathematics*, pages 1065–1078, 1958.

[2] J. Awrejcewicz, M. Feckan, and Pawel Olejnik, *On continuous approximation of discontinuous systems*, Nonlinear Analysis 62 (2005), 1317-1331.

[3] M. Barahona and L. Pecora. Synchronization in Small-World Systems. Phys. Rev. Lett. 89-5, 054101 (2002).

[4] J. Buck. Synchronous rhythmic flashing of fireflies. II. *The Quarterly review of biology*, 63-3, pp. 265–289 (1988).

[5] S. Coombes and R. Thul. Synchrony in networks of coupled non-smooth dynamical systems: Extending the master stability function. *European Journal of Applied Mathematics*, vol. 27, pp. 904–922 (2016).

[6] S. Coombes, Y.M. Lai, M. Sayli and R. Thul. Networks of piecewise linear neural mass models. *European Journal of Applied Mathematics*, pp. 1–22 (2018).

[7] M. Coraggio, P. De Lellis, S. J. Hogan and M. di Bernardo. Synchronization of Networks of Piecewise-Smooth Systems. IEEE Control System Letters, 2(4), pp. 653–658, (2018).

[8] M. Coraggio, P. De Lellis and M. di Bernardo. Convergence and Synchronization in Networks of Piecewise-Smooth Systems via Distributed Discontinuous Coupling. July 2021, Automatica 129(3):109596.

[9] L. Dieci. Sliding integration with no projection. Applied Numerical Mathematics, Vol. 155, pp. 3-15, (2020).

[10] L. Dieci and L. Lopez. Fundamental matrix solutions of piecewise smooth dynamical systems Mathematics and Computers in Simulations, vol. 81, n. 5, pp. 932-953, (2011).

[11] L. Dieci and C. Elia. Periodic orbits for planar piecewise smooth dynamical systems with a line of discontinuity. Journal of Dynamics and Differential Equations, vol. 26 (2104), pp. 1049-1078.

[12] L. Dieci, C. Elia and D. Pi. Limit cycles for regularized discontinuous dynamical systems with a hyperplane of discontinuity. Discrete and Continuous Dynamical Systems, Series B, vol. 22-8 (2017), pp. 3091-3112.

[13] F. Dörfler and F. Bullo. Synchronization in complex networks of phase oscillators: A survey. *Automatica*, vol. 50, no. 6, pp. 1539– 1564 (2014).

[14] A.F. Filippov. Differential Equations with Discontinuous Right-Hand Sides. Mathematics and Its Applications, Kluwer Academic, Dordrecht 1988.

[15] U. Galvanetto, S.R. Bishop and L. Briseghella . Mechanical stick-slip vibrations. International Journal of Bifurcation and Chaos, vol. 5, n. 3, (1995), pp. 637-651.

[16] J. Hale. Diffusive Coupling, Dissipation, and Synchronization. Journal of Dynamics and Differential Equations, 9-1, pp. 1-52 (1997).

[17] A. P. Ivanov. The stability of periodic solutions of discontinuous systems that intersect several surfaces of discontinuity. J. Appl. Math. Mechs., vol. 62, pp. 677-685, (1998).

[18] Y. Kuramoto. Chemical oscillations, waves, and turbulence. Springer Science & Business Media, 2012, vol. 19.

[19] J. Ladenbauer, J. Lehnert, H. Rankoohi, T. Dahms, E. Scholl, and K Obermayer. Adaptation controls synchrony and cluster states of coupled threshold-model neurons. Physical Review E, 88 (4): 042713, 2013.

[20] R. I. Leine, D. H. Van Campen and B. L. Van De Vrande. Bifurcations in Nonlinear Discontinuous Systems. *Nonlinear Dynamics*, 23, 2000, pp. 105–164.

[21] P. L. Müller. Calculation of Lyapunov Exponents for Dynamic Systems with discontinuities. *Chaos Solitons & Fractals* 5 (9), 1995, pp. 1671-1681.

[22] L. Pecora and T. Carroll. Master Stability Functions for Synchronized Coupled Systems *Physical Review Letters*, vol. 80, n. 10, (1997), pp. 2109-1112.

[23] J. Sotomayor and M.A. Teixeira. Regularization of discontinuous vector fields, Proceedings of the International Conference on Differential Equations, Lisboa, (1996), pp. 207–223.

[24] A. T. Winfree. Biological rhythms and the behavior of populations of coupled oscillators. *Journal of theoretical biology*, vol. 16, no. 1, pp. 15–42 (1967).