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EXISTENCE AND MULTIPLICITY OF SCHRÖDINGER-BORN-INFELD SYSTEM WITH A GENERAL NONLINEARITY

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ABSTRACT. In this paper, we study the following Schrödinger-Born-infeld system with a general nonlinearity

$$\begin{cases} -\Delta u + u + \phi u = f(u) + \mu |u|^4 u & \text{in } \mathbb{R}^3, \\ -\text{div}\Big(\frac{\nabla \phi}{\sqrt{1-|\nabla \phi|^2}}\Big) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases}$$

where $\mu \geq 0$ and $f \in C(\mathbb{R}, \mathbb{R})$. This system arises from a suitable coupling of the nonlinear Schrödinger equation and the Born-infeld theory. We use a perturbation approach to prove the existence and multiple of nontrivial solutions of the above system when f satisfies a general nonlinear growth. We emphasize that our results obtained in this paper cover the case $f(u) = |u|^{p+1}$ for $p \in (2, \frac{5}{2})$ which is seem as an open problem in *[Azzollini, Pomponio and Siciliano, Bull. Braz. Math. Soc, 2019]*.

1. Introduction and results

1.1. **Overview.** We are concerned with the following Schrödinger-Born-Infeld system

$$\text{(SBI)} \qquad \begin{cases} -\Delta u + u + \phi u = f(u) + \mu |u|^4 u & \text{in } \mathbb{R}^3, \\ -\text{div}\Big(\frac{\nabla \phi}{\sqrt{1-|\nabla \phi|^2}}\Big) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ is a suitable nonlinearity and $\mu \geq 0$. Such a system arise from the coupling of the nonlinear Schrödinger equations and the Born-Infeld theory, and can be proposed to provide a mathematical description of the interaction between a charged particle and the electromagnetic field generated by itself. The Born-Infeld theory was firstly developed by Born and Infeld and introduced a idea that both the matter and the electromagnetic field were expression of a unique physical entity.

Note that, Yu [18] studied the following dualistic model obtained by coupling Klein-Gordon and Born-Infeld lagrangians,

$$\begin{cases} -\Delta u + (m^2 - (\omega + \phi)^2) u = |u|^{p-1} u & \text{in } \mathbb{R}^3, \\ -\text{div}\Big(\frac{\nabla \phi}{\sqrt{1-|\nabla \phi|^2}}\Big) = u^2(\omega + \phi) & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases}$$

which is used to express the electrostatic case. Due to the form of the differential operator in the second equation, We can not use variation approach to deal with this problem by restricting the functional at the usual function spaces. The reason is that the quantity $1/\sqrt{1-|\nabla \phi(x)|^2}$ makes

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sense only when $x \in \mathbb{R}^3$ is such that $|\nabla\phi(x)| < 1$. And so this inequality has to be considered in the functional setting as a necessary constraint.

Inspired by Yu [18], Azzollini, Pomponio and Siciliano [4] proposed a new model which represents a variant of the well-known Schrödinger-Maxwell system as it was introduced in D'Aprile and Mugnai. By replacing the usual Maxwell Lagrangian with the Born-Infeld one, they [18] studied the existence of electrostatic solutions for the following system

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\text{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty. \end{cases} \tag{1.1}$$

More precisely, Azzollini, Pomponio and Siciliano [4] proposed a new model and use a slightly modified version of the monotonicity trick due to Jeanjean [8] and Struwe [17] to prove the existence of radial ground state solutions of (1.1) for $p \in (5/2, 5)$. They left as an open problem the case of smaller p and the existence of non-radial solutions.

1.2. **Main result.** In this paper, we aim to establish a novel variational approach to study the existence of positive solutions to (SBI) with a general nonlinearity and partly answer the open problem mentioned in Azzollini, Pomponio and Siciliano [4]. We recall that (SBI) comes variationally from the action functional F defined by

$$F(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{2}\int_{\mathbb{R}^3}\phi u^2 - \frac{1}{2}\int_{\mathbb{R}^3}(1 - \sqrt{1-|\nabla\phi|^2}) - \int_{\mathbb{R}^3}F(u) - \frac{\mu}{6}\int_{\mathbb{R}^3}|u|^6.$$

An evident difficulty should be overcame in dealing with functional F. Firstly, as introduced in [18], the presence of the term $\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi|^2})$ forces us to restrict the setting of admissible function ϕ and to define a suitable function set. Define

$$X := D^{1,2}(\mathbb{R}^3) \cap \{\phi \in C^{0,1}(\mathbb{R}^3) : \ \|\nabla\phi\|_\infty \leq 1\}$$

where $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\nabla \cdot \|_2$. We will study functional F by restricting on the function setting $H^1(\mathbb{R}^3) \times X$. Observe that the setting $H^1(\mathbb{R}^3) \times X$ is not a vector space. This fact brings us some difficulties in dealing with functional G via variational approach. To compute variations with respect to ϕ along the direction established by a generic smooth and compactly supported function, a direct restriction is to require in advance that $\|\nabla\phi\|_\infty < 1$. This fact prevents us from using directly the standard method to deal with the strongly indefinite nature of G. Indeed, the strongly indefinite nature of the functional can be classically removed by the usual reduction method. Precisely, for any radial $u \in H_r^1(\mathbb{R}^3)$ fixed, there exists a unique $\phi_u \in X$ solution of the second equation of system (SBI) and reducing the problem to that of finding critical points of the (no more strongly indefinite) one variable functional $I(u) = F(u, \phi_u)$, defined on $H^1(\mathbb{R}^3)$.

For this purpose, we make the following assumptions on the nonlinearity f:

- (f1) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(s) = o(s)$ as $s \to 0$;
- (f2) $|f(s)| \leq C(1 + |s|^p)$ for $p \in (2, 5)$;
- (f3) for any $s > 0$, $0 < \varrho F(s) \leq f(s)s$, where $\varrho > 3$ and $F(s) = \int_0^s f(\tau)d\tau$.

The following is our main result in this paper.

Theorem 1.1. *Assume that (f1)-(f3) hold. Then system (SBI) has at least a radial ground state solution, namely, a solution $(u, \phi) \in H_r^1(\mathbb{R}^3) \times X_r$ minimizing the energy functional F among all the nontrivial radial solutions.*

Theorem 1.2. *Assume that* (f1)-(f3) *hold and f is odd. Then system* (SBI) *has infinite many radial high energy solutions, namely, solutions $(u_j, \phi_j) \subset H_r^1(\mathbb{R}^3) \times X_r$ such that the energy functional tends to infinity.*

In what follows, we turn our attention to the case where the nonlinearity f has critical growth. For this purpose, we consider the following assumption on the nonlinearity f:

(f4) there exist $D > 0$ and $2 < r < 6$ such that $f(t) \geq Dt^r$ for $t \geq 0$.

Theorem 1.3. *Assume that* (f1)-(f4) *hold. Then system* (SBI) *admits a radial ground state solution $(u, \phi) \in H_r^1(\mathbb{R}^3) \times X_r$ for $r \in (2, 4]$ with D sufficiently large, or $q \in (4, 6)$.*

We remark that the solutions found are of class $C^2(\mathbb{R}^3)$, hence classical.

Hereafter, the letter C will be repeatedly used to denote various positive constants whose exact values are irrelevant. This paper is organized as follows. Firstly, some preliminaries are given in Section 2, and Section 3 is devoted to the existence of ground state solutions and multiple of high energy solutions to equation (SBI) with a general subcritical nonlinearity. In Section 4, we prove the existence of radial ground state solutions of SBI) with a general critical nonlinearity.

2. Preliminary results

Let us fix some notation. For every $1 \leq s \leq +\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^2)$.

We recall some properties of the ambient space X.

Lemma 2.1. *The following conclusions hold:*

(i) *X is continuously embedded in $W^{1,p}(\mathbb{R}^3)$ for all $p \in [6, +\infty)$;*
(ii) *X is continuously embedded in $L^\infty(\mathbb{R}^3)$;*
(iii) *if $\phi \in X$, then $\lim_{|x|\to\infty} \phi(x) = 0$;*
(iv) *X is weakly closed;*
(v) *if $\{\phi_n\} \subset X$ is bounded, there exists $\phi \in X$ such that, up to subsequence, $\phi_n \rightharpoonup \phi$ weakly in X and uniformly on compact sets.*

We define weak solutions to (SBI) in the following way.

Definition 2.2. *A weak solution of* (SBI) *is a couple $(u, \phi) \in H^1(\mathbb{R}^3) \times X$ such that for all $(v, \psi) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$, we have*

$$\frac{1}{2}\int_{\mathbb{R}^3} (\nabla u \nabla v + uv) + \int_{\mathbb{R}^3} \phi_u uv = \int_{\mathbb{R}^3} f(u)v + \mu \int_{\mathbb{R}^3} |u|^4 u\psi,$$
$$\int_{\mathbb{R}^3} \frac{\nabla\phi\nabla\psi}{\sqrt{1 - |\nabla\phi|^2}} = \int_{\mathbb{R}^3} u^2\psi.$$

The next fact are also known.

Lemma 2.3. *For any $u \in H^1(\mathbb{R}^3)$ fixed, there exists a unique $\phi_u \in X$ such that the following properties hold:*

(i) *ϕ_u is the unique minimizer of $E_u(\phi) : X \to \mathbb{R}$ and $E_u(\phi_u) \leq 0$, that is,*

$$\int_{\mathbb{R}^3} \phi_u u^2 \geq \int_{\mathbb{R}^3} (1 - \sqrt{1 - |\nabla\phi_u|^2});$$

(ii) *$\phi_u \geq 0$ and $\phi_u = 0$ if and only if $u = 0$;*

(iii) *if ϕ is a weak solution of the second equation of* (SBI)*, then $\phi = \phi_u$ it satisfies the following equality*

$$\int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1-|\nabla \phi_u|^2}} = \int_{\mathbb{R}^3} \phi_u u^2.$$

Moreover, if $u \in H_r^1(\mathbb{R}^3)$, then $\phi_u \in X_r$ is the unique weak solution of the second equation of system (SBI)

(iv) *X is weakly closed;*

(v) *if $\{\phi_n\} \subset X$ is bounded, there exists $\phi \in X$ such that, up to subsequence, $\phi_n \rightharpoonup \phi$ weakly in X and uniformly on compact sets.*

Let us set

$$H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u \text{ is radially symmetric}\}$$

and

$$X_r = \{\phi \in X \,|\, \phi \text{ is radially symmetric}\}.$$

It is known that the radial setting is a natural constraint for the problem.

We recall also a Pohozaev type identity associated with (SBI) whose proof can be obtained as in [4].

Lemma 2.4. *If $(u, \phi) \in H_r^1(\mathbb{R}^3) \times X_r$ is a solution of* (SBI)*, then the following Pohozaev type identity is satisfied:*

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 + 2\int_{\mathbb{R}^3} \phi u^2 - \frac{3}{2}\int_{\mathbb{R}^3} (1-\sqrt{1-|\nabla \phi|^2}) = 3\int_{\mathbb{R}^3} F(u) + \frac{\mu}{2}\int_{\mathbb{R}^3} |u|^6. \tag{2.1}$$

In view of Lemma 2.3, we know that the associated energy functional can be written in the following form

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{2}\int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} F(u) - \frac{1}{2}\int_{\mathbb{R}^3} (1-\sqrt{1-|\nabla \phi_u|^2}) - \frac{\mu}{6}\int_{\mathbb{R}^3} |u|^6,$$

and is of class C^1. It is known that if u is the critical point of I, then (u, ϕ_u) is a weak solution of (SBI). For this reason we will speak simply of u as a solution of (SBI).

We recall a technical lemma which is of use in studying the geometry of the functional.

Lemma 2.5. *Let $s \in [2, 3)$. Then there exist positive constants C and C' such that for any $u \in H^1(\mathbb{R}^3)$, we have*

$$\|\nabla \phi_u\|_2^{\frac{s-1}{s}} \le C\|u\|_{2(s^*)'} \le C'\|u\|,$$

where s^* is the critical Sobolev exponent related to s and $(s^*)'$ is its conjugate exponent, namely

$$s^* = \frac{3s}{3-s} \quad \text{and} \quad (s^*)' = \frac{3s}{4s-3}.$$

3. Subcritical case

3.1. **The modified equation.** In this section, we are planning to study the subcritical case where $\mu = 0$. Here, we introduce *a perturbation technique* to overcome the lack of geometry structure of Mountain-Pass and the difficulty related to the boundedness of Palais-Smale sequence by modifying system (SBI). We consider the following modified problem

$$\begin{cases} -\Delta u + u + \phi u + \lambda \|u\|_2^{2\alpha} u = f(u) + \lambda |u|^q & \text{in } \mathbb{R}^3, \\ -\text{div}\left(\frac{\nabla \phi}{\sqrt{1-|\nabla \phi|^2}}\right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases} \tag{3.1}$$

where $\lambda \in (0,1]$, $\alpha \in (0,1)$ and $q \in (\max\{p,4\},5)$. Thus, its associated functional is

$$I_\lambda(u) = I(u) + \frac{\lambda}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)} - \frac{\lambda}{q+1}\int_{\mathbb{R}^3}|u|^{q+1}.$$

3.2. **Existence.** We now verify that the functional I_λ has the Mountain Pass geometry.

Lemma 3.1. *Suppose that (f1)-(f3) hold. Then for fixed $\lambda \in (0,1]$, the following conclusions hold:*

(i) there exist $\rho, \delta_0 > 0$ such that $I_\lambda(u) \geq \delta_0$ for every $u \in S_\rho = \{u \in E : \|u\| = \rho\}$;

(ii) there is $e_0 \in H_r^1(\mathbb{R}^3)$ with $\|e_0\| > \rho$ such that $I_\lambda(e_0) < 0$.

Proof. (i) It follows from (f1)-(f2) that (in correspondence of the above q) for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C_\varepsilon|s|^q. \tag{3.2}$$

Then for any $u \in H_r^1(\mathbb{R}^3)$, by the definition of I_λ and Lemma 2.3-(i), one has

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} F(u) - \frac{\lambda}{q+1}\int_{\mathbb{R}^3}|u|^{q+1} \\ &\geq \frac{1}{2}\|u\|^2 - \varepsilon\|u\|_2^2 - \frac{1+C_\varepsilon}{q+1}\int_{\mathbb{R}^3}|u|^{q+1} \\ &\geq \frac{1-\varepsilon}{2}\|u\|^2 - \frac{1+C_\varepsilon}{q+1}C\|u\|^{q+1}. \end{aligned} \tag{3.3}$$

Taking $\varepsilon = 1/2$, and $\|u\| = \rho > 0$ small enough, it is easy to check that there exists $\delta_0 > 0$ such that $I_\lambda(u) \geq \delta_0$ for every $u \in S_\rho$.

(ii) For fixed $\lambda \in (0,1]$, take $e \in H_r^1(\mathbb{R}^3) \setminus \{0\}$, then it follows from the definition of I_λ and Lemma 2.5 that by letting $s = 2$, we have

$$\begin{aligned} I_\lambda(e) &\leq \frac{1}{2}\|e\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}\phi_e e^2 + \frac{\lambda}{2(1+\alpha)}\|e\|_2^{2(1+\alpha)} - \int_{\mathbb{R}^2} F(e) - \frac{\lambda}{q+1}\int_{\mathbb{R}^3}|e|^{q+1} \\ &\leq \frac{1}{2}\|e\|^2 + \frac{1}{2}\|\phi_e\|_6\|e\|_{\frac{12}{5}}^2 + \frac{\lambda}{2(1+\alpha)}\|e\|_2^{2(1+\alpha)} - \frac{\lambda}{q+1}\int_{\mathbb{R}^3}|e|^{q+1} \\ &\leq \frac{1}{2}\|e\|^2 + C\|\nabla\phi_e\|_2\|e\|^2 + C_1\|e\|^{2(1+\alpha)} - C_2\lambda\|u\|^{q+1} \\ &\leq \frac{1}{2}\|e\|^2 + C\|e\|^4 + C_1\|e\|^{2(1+\alpha)} - C_2\lambda\|e\|^{q+1}, \end{aligned} \tag{3.4}$$

ALTERNATIVE PROOF of (ii) TO AVOID THE DEPENDENCE ON λ
Take $e \in C_c^\infty(\mathbb{R}^3) \setminus \{0\}$ positive, radial, satisfying $e \leq 1$ and with support in the compact set K.

Then it follows from (f3), the definition of I_λ and Lemma 2.5 with $s = 2$,

$$
\begin{aligned}
I_\lambda(te) &\le \frac{t^2}{2}\|e\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{te}(te)^2 + \frac{t^{2(1+\alpha)}}{2(1+\alpha)}\|e\|_2^{2(1+\alpha)} - \int_K F(te) \\
&\le \frac{t^2}{2}\|e\|^2 + \frac{t^2}{2}\|\phi_{te}\|_6\|e\|_{12/5}^2 + \frac{t^{2(1+\alpha)}}{2}|K|^{1+\alpha} - C_1 t^\varrho \int_K e^\varrho + C_2|K| \\
&\le \frac{t^2}{2}\|e\|^2 + C_3 t^2\|\nabla\phi_{te}\|_2\|e\|^2 + C_4 t^{2(1+\alpha)} - C_1 t^\varrho\|e\|_\varrho^\varrho + C_2|K| \\
&\le \frac{t^2}{2}\|e\|^2 + C_5 t^4\|e\|^4 + C_4 t^{2(1+\alpha)} - C_1 t^\varrho\|e\|_\varrho^\varrho + C_2|K| \\
&=: g(t)
\end{aligned}
\tag{3.5}
$$

If $\boxed{\varrho > 4}$, there exists $s_0 > 0$ large enough such that $g(s_0) < 0$. But this contrasts with the first line in blue color at page 9.

Remark 3.2. *Note that since by* (f2) *and* (f3) *it holds*

$$c_1 s^\varrho - c_2 \le c(|s| + |s|^{p+1})$$

it has to be $p \ge \varrho - 1 > 3$. But if $p > 3$ we have nothing new: we want to allow $p < 5/2$!

Remark 3.3. *A further tentative is to take the curve $t \mapsto t^2 e(tx)$ in* (3.5) *(instead of $t \mapsto te(x)$). In this case, by computations like in* (3.5) *we arrive at $\varrho > 15/4$, so $p > \varrho - 1 > 15/4 - 1 = 11/4$: but this is still greater then $5/2$.*

□

By the well-known Mountain-Pass theorem (see [19]), there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset H_r^1(\mathbb{R}^3)$, that is,

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_n) \to 0. \tag{3.6}$$

Here c_λ is the Mountain Pass level characterized by

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \tag{3.7}$$

with

$$\Gamma_\lambda := \left\{\gamma \in C^1([0,1], H_r^1(\mathbb{R}^3)) : \gamma(0) = 0 \quad \text{and} \quad I_\lambda(\gamma(1)) < 0\right\}.$$

Remark 3.4. *Observe from Lemma 3.1 that there exist two constants $a, b > 0$ independently of λ such that $a < c_\lambda < b$.*

We state the following lemma to ensure that Palais-Smale sequences of I_λ at c_λ have at least a convergence subsequence.

Lemma 3.5. *Let $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ be the $(PS)_{c_\lambda}$ sequence of I_λ for fixed $\alpha \in (0,1]$, then there exists $u_0 \in H_r^1(\mathbb{R}^3)$ such that $u_n \to u_0$ in $H_r^1(\mathbb{R}^3)$.*

Proof We first claim that the sequence $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. For $\theta \in (4, q+1)$, by Lemma 2.5, there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} C_1 + C_2\|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\theta} I'_\lambda(u_n)u_n \\ &\geq \frac{\theta-2}{2\theta}\|u_n\|^2 + \frac{\theta-2}{2\theta}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\lambda\|u_n\|_2^{2(1+\alpha)} \\ &\quad + \int_{\mathbb{R}^3}(\frac{1}{\theta}f(u_n)u_n - F(u_n)) - \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_n}|^2}) + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u_n|^{q+1} \\ &\geq \frac{\theta-2}{2\theta}C_3\|u_n\|^2 + \frac{\theta-2}{2\theta}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\|u_n\|_2^{2(1+\alpha)} \\ &\quad - C_4\int_{\mathbb{R}^3}|u_n|^{p+1} - \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_n}|^2}) + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u_n|^{q+1}. \end{aligned}$$

Since for all $t \in [0,1)$, the following holds:

$$1-\sqrt{1-t} \leq \frac{1}{2}\frac{t}{\sqrt{1-t}}, \tag{3.8}$$

by recalling (iii) in Lemma 2.3, it follows that

$$\begin{aligned} C_1 + C_2\|u_n\| &\geq \frac{\theta-2}{2\theta}C_3\|u_n\|^2 + \frac{\theta-4}{4\theta}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\|u_n\|_2^{2(1+\alpha)} \\ &\quad - C_4\int_{\mathbb{R}^3}|u_n|^{p+1} + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u_n|^{q+1} \\ &\geq \frac{\theta-2}{2\theta}C_3\|u_n\|^2 + \frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\|u_n\|_2^{2(1+\alpha)} \\ &\quad - C_4\int_{\mathbb{R}^3}|u_n|^{p+1} + \frac{q+1-\theta}{\theta(q+1)}\lambda\int_{\mathbb{R}^3}|u_n|^{q+1}. \end{aligned} \tag{3.9}$$

Observe that for any large $B_1 > 0$, there exists $B_2 > 0$ such that

$$\frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\|u_n\|_2^{2(1+\alpha)} \geq B_1\|u_n\|_2^2 - B_2,$$

which, together with (3.9), implies that

$$C_1 + B_2 + C_2\|u_n\| \geq \frac{\theta-2}{2\theta}C_3\|u_n\|^2 + \int_{\mathbb{R}^3}\Big(B_1|u_n|^2 - C_4|u_n|^{p+1} + \frac{q+1-\theta}{\theta(q+1)}\lambda|u_n|^{q+1}\Big). \tag{3.10}$$

We note that $B_1|t|^2 - C_4|t|^{p+1} + \frac{q+1-\theta}{\theta(q+1)}\lambda|t|^{q+1} \geq 0$ for $t \geq 0$, since B_1 is a number large enough. Thus, it follows from (3.10) that $\|u_n\| \leq C$ for some C independently of n.

Up to a subsequence, we suppose that there exist $u_0 \in H_r^1(\mathbb{R}^3)$ such that

$$\begin{aligned} &u_n \rightharpoonup u_0 \quad \text{weakly in } H_r^1(\mathbb{R}^3), \\ &u_n \to u_0 \quad \text{in } L^p(\mathbb{R}^3),\ 2 < p < 6, \\ &u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{3.11}$$

So (f2) implies by (3.11) that

$$\int_{\mathbb{R}^3}F(u_n) \to \int_{\mathbb{R}^3}F(u_0), \quad \int_{\mathbb{R}^3}f(u_n)u_n \to \int_{\mathbb{R}^3}f(u_0)u_0. \tag{3.12}$$

Form (3.11) and (3.12), it is easy to conclude that $u_n \to u_0$ in $H_r^1(\mathbb{R}^3)$. □

Now we are attempt to prove Theorem 1.1

Proof of Theorem 1.1 It follows from Lemma 3.5 that for each $\lambda \subset (0,1]$, there exists $u_\lambda \in H_r^1(\mathbb{R}^3)$ such that $I_\lambda(u_\lambda) = c_\lambda$ and $I'_\lambda(u_\lambda) = 0$. By $I'_\lambda(u_\lambda)u_\lambda = 0$, we have

$$\int_{\mathbb{R}^3}(|\nabla u_\lambda|^2 + u_\lambda^2) + \int_{\mathbb{R}^3}\phi_{u_\lambda}u_\lambda^2 + \lambda\|u_\lambda\|_2^{2(1+\alpha)} - \int_{\mathbb{R}^3} f(u_\lambda)u_\lambda - \lambda\int_{\mathbb{R}^3}|u_\lambda|^{q+1} = 0. \tag{3.13}$$

Recalling hypothesis (f3), it follows from (3.13) that for $b > 0$,

$$b\int_{\mathbb{R}^3} F(u_\lambda) \le \frac{b}{\varrho}\int_{\mathbb{R}^3}(|\nabla u_\lambda|^2 + u_\lambda^2) + \frac{b}{\varrho}\int_{\mathbb{R}^3}\phi_{u_\lambda}u_\lambda^2 + \frac{b\lambda}{\varrho}\|u_\lambda\|_2^{2(1+\alpha)} - \frac{b\lambda}{\varrho}\int_{\mathbb{R}^3}|u_\lambda|^{q+1}. \tag{3.14}$$

Moreover, similarly to Lemma 2.4, we obtain the associated Pohozaev type identity with the modified problem as follows

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_\lambda|^2 + \frac{3}{2}\int_{\mathbb{R}^3}u_\lambda^2 + 2\int_{\mathbb{R}^3}\phi_{u_\lambda}u_\lambda^2 + \frac{3}{2}\|u_\lambda\|_2^{2(1+\alpha)}\\ - \frac{3}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_\lambda}|^2}) = 3\int_{\mathbb{R}^3}F(u_\lambda) + \frac{3}{q+1}\int_{\mathbb{R}^3}|u_\lambda|^{q+1}.\end{aligned} \tag{3.15}$$

Combining (3.14) with (3.15), we have for $a \in \mathbb{R}$

$$\begin{aligned}(a+b)\int_{\mathbb{R}^3}F(u_\lambda) \le &\left(\frac{a}{6}+\frac{b}{\varrho}\right)\int_{\mathbb{R}^3}|\nabla u_\lambda|^2 + (\frac{a}{2}+\frac{b}{\varrho})\int_{\mathbb{R}^3}|u_\lambda|^2\\ &+\left(\frac{2a}{3}+\frac{b}{\varrho}\right)\int_{\mathbb{R}^3}\phi_{u_\lambda}u_\lambda^2 - \frac{a}{2}\int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla u|^2}\right)\\ &+\left(\frac{a}{2}+\frac{b}{\varrho}\right)\lambda\|u_\lambda\|_2^{2(1+\alpha)} - \left(\frac{a}{q+1}+\frac{b}{\varrho}\right)\lambda\|u_\lambda\|_{q+1}^{q+1}.\end{aligned} \tag{3.16}$$

By letting $a = 1-b$ in (3.16), using the definition of I_λ and that

$$\frac{b}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla u|^2}) \le \frac{b}{4}\int_{\mathbb{R}^3}\phi_u u^2,$$

it follows that

$$\begin{aligned}c_\lambda = I_\lambda(u_\lambda) \ge &\left(\frac{1}{3}+\frac{b(\varrho-6)}{6\varrho}\right)\int_{\mathbb{R}^3}|\nabla u_\lambda|^2 + \left(\frac{b}{2}-\frac{b}{\varrho}\right)\int_{\mathbb{R}^3}|u_\lambda|^2\\ &+\left(\frac{1}{2}-\frac{2(1-b)}{3}-\frac{b}{\varrho}\right)\int_{\mathbb{R}^3}\phi_{u_\lambda}u_\lambda^2 - \frac{b}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla u_\lambda|^2})\\ &+\left(\frac{1}{2(1+\alpha)}-\frac{1-b}{2}-\frac{b}{\varrho}\right)\lambda\|u_\lambda\|_2^{2(1+\alpha)} + \left(\frac{1-b}{q+1}+\frac{b}{\varrho}-\frac{1}{q+1}\right)\lambda\|u_\lambda\|_{q+1}^{q+1}\\ \ge &\left(\frac{1}{3}+\frac{b(\varrho-6)}{6\varrho}\right)\int_{\mathbb{R}^3}|\nabla u_\lambda|^2 + \left(\frac{b}{2}-\frac{b}{\varrho}\right)\int_{\mathbb{R}^3}|u_\lambda|^2\\ &+\left(\frac{1}{2}-\frac{2(1-b)}{3}-\frac{b}{\varrho}-\frac{b}{4}\right)\int_{\mathbb{R}^3}\phi_{u_\lambda}u_\lambda^2\\ &+\left(\frac{1}{2(1+\alpha)}-\frac{1-b}{2}-\frac{b}{\varrho}\right)\lambda\|u_\lambda\|_2^{2(1+\alpha)} + \left(\frac{1-b}{q+1}+\frac{b}{\varrho}-\frac{1}{q+1}\right)\lambda\|u_\lambda\|_{q+1}^{q+1}\end{aligned} \tag{3.17}$$

Choosing $b = 2$ it is easy to check that all the coefficients are positive. Indeed

- $\frac{1}{3}+\frac{b(\varrho-6)}{6\varrho} > 0$, $\frac{b}{2}-\frac{b}{\varrho} > 0$;
- $\frac{1-b}{q+1}+\frac{b}{\varrho}-\frac{1}{q+1} > 0$, since $q > p \ge \varrho - 1$;

- $\dfrac{1}{2(1+\alpha)} - \dfrac{1-b}{2} - \dfrac{b}{\varrho} > 0$, since $\alpha < \frac{2(\varrho-2)}{4-\varrho}$ (this is the restriction on α we need. Note that it has to be $\varrho < 4$);
- $\dfrac{1}{2} - \dfrac{2(1-b)}{3} - \dfrac{b}{\varrho} - \dfrac{b}{4} > 0$.

Based on the above facts, it follows from (3.17) and Remark 3.3 that $\{u_\lambda\}_{\lambda\in(0,1]}$ is bounded in $H_r^1(\mathbb{R}^3)$ uniformly for $\lambda \in (0,1]$. By letting $\lambda \to 0^+$, then there exists $c_0 > 0$ such that $c_\lambda \to c_0$ and $I(u_\lambda) = c_0 + o(1)$. Moreover, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$I'(u_\lambda)\varphi = I'_\lambda(u_\lambda)\varphi - \lambda\|u_\lambda\|_2^\alpha u_\lambda\varphi - \int_{\mathbb{R}^3}\lambda|u_\lambda|^{p-1}u_\lambda\varphi = o(1)\|\varphi\|.$$

Thus, $\{u_\lambda\}$ is a Palais-Smale sequence of I with level c_0. Arguing similarly as in the proof of Lemma 3.5, we prove that there exists a nontrivial $u_0 \in H_r^1(\mathbb{R}^3)$ such that $I'(u_0) = 0$ and $I(u_0) = c$. Define

$$\mathcal{N} := \{u \in H_r^1(\mathbb{R}^3) \setminus \{0\} : I'(u) = 0\}.$$

Obviously, $\mathcal{N} \neq \emptyset$. For any $u \in \mathcal{N}$, by ($\mathrm{f_1}$) and ($\mathrm{f_2}$), we have for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|u\|^2 \le \|u\|^2 + \int_{\mathbb{R}^3}\frac{|\nabla\phi_u|^2}{\sqrt{1-|\nabla\phi_u|^2}} \le \varepsilon\int_{\mathbb{R}^3}u^2 + C_\varepsilon\int_{\mathbb{R}^3}|u|^{p+1},$$

which implies by taking $\varepsilon = \frac{1}{2}$ and Sobolev's inequality that $\|u\| \ge C$ for any $u \in \mathcal{N}$ and for some $C > 0$ independently of u. Arguing similarly as above, we can obtain one associated estimate as (3.17) such that there exists some $c > 0$ satisfying $I(u) \ge c\|u\|^2$ for all $u \in \mathcal{N}$. Based on the above facts, we infer that

$$c_* := \inf_{u\in\mathcal{N}} I(u) > 0.$$

Take $\{u_n\} \subset \mathcal{N}$ so that $I(u_n) \to c_*$. The same arguments as before, $\{u_n\}$ is bounded. Similar to Lemma 3.5, there exists $u \in H_r^1(\mathbb{R}^3)$ so that $u_n \to u$ in $H_r^1(\mathbb{R}^3)$ and $I'(u) = 0$. Then (u, ϕ_u) is a radial ground state solution of (1.1). □

3.3. **Multiplicty.** In this subsection, we aim at using the perturbation approach together with the symmetric Mountain-Pass theorem to prove that equation (SBI) has infinitely many high energy solutions.

Proof of Theorem 1.3 Choose a sequence of finite dimensional subspaces E_j on $H_r(\mathbb{R}^3)$ with $\dim E_j = j$, and $R_j > 0$ such that $I_\lambda(u) < 0$ for $u \in E_j \cap \partial B_{R_j}$. Then by the conclusion (i) of Lemma 3.1, for fixed λ, the functional I_λ, has a sequence of critical values $c_j(\lambda)$ satisfying $c_j(\lambda) \to +\infty$ as $j \to +\infty$. Here

$$c_j(\lambda) = \inf_{B\in\Gamma_j}\sup_{u\in B} I_\lambda(u),$$

where

$$\Gamma_j = \Big\{B = \phi(E_j \cap B_{R_j}), \phi \in C(E_j \cap B_{R_j}, H_r(\mathbb{R}^3)),\ \phi \text{ is odd } \phi = Id, \text{ on } E_j \cap \partial B_{R_j}\Big\}.$$

For any fixed j, by the definition of $c_j(\lambda)$, we have

$$\begin{aligned} c_j(\lambda) &\le \sup_{u\in E_j\cap B_{R_j}} I_\lambda(u) \\ &\le \sup_{u\in E_j\cap B_{R_j}}\left\{\frac{1}{2}\|u\|^2 + \frac{1}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)}\right\} := C_{R_j}. \end{aligned}$$

Here, C_{R_j} is independent of $\lambda \in (0,1]$. It follows from Lemma 3.5 that equation (SBI) has a solution u_j satisfying

$$I(u_j) = c_j := \lim_{\lambda \to 0^+} c_j(\lambda).$$

Observe that if $c_j \to +\infty$, then the problem (SBI) has infinitely many solutions. Now we show $c_j \to +\infty$, as $j \to +\infty$. Recalling Lemma 2.3, we estimate I_λ as follows

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{2}\int_{\mathbb{R}^3}\phi_u u^2 - \int_{\mathbb{R}^3}F(u) - \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_u|^2}) \\ &+ \frac{\lambda}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)} - \frac{\lambda}{q+1}\int_{\mathbb{R}^3}|u|^{q+1} \\ \geq& \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) - \int_{\mathbb{R}^3}F(u) - \frac{1}{q+1}\int_{\mathbb{R}^3}|u|^{q+1} := J(u). \end{aligned}$$

Define the set $\Theta \subset H_r^1(\mathbb{R}^3)$ by

$$\Theta := \Big\{ \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) \geq \int_{\mathbb{R}^3} f(u)u + \int_{\mathbb{R}^3}|u|^{q+1} \Big\}.$$

If $B \in \Gamma_j$, then an intersection property holds so that $\gamma(B \cap \partial\Theta) \geq j$ (see []), here $\gamma(\cdot)$ is the genus of a symmetric set. So,

$$c_j(\lambda) = \inf_{B\in\Gamma_j}\sup_{u\in B} I_\lambda(u) \geq \inf_{A\subset\partial\Theta,\gamma(A)\geq j}\sup_{u\in A} J(u) := b_j.$$

It is not hard to verify that functional J satisfies all conditions of the well-known symmetric mountain-pass theorem, and so there exists critical point $\tilde{u}_j$ of J such that $J(\tilde{u}_j) := b_j \to +\infty$ as $j \to +\infty$. Therefore,

$$c_j = \lim_{\lambda\to 0^+} c_j(\lambda) \geq b_j \to +\infty.$$

That is to say, equation (SBI) has infinitely many high energy solutions. The proof is complete. □

4. Critical Case

In this section we consider Schrödinger-Born-field system in the critical case. We aim at establish the existence of ground state solutions to (SBI) with a general critical nonlinear term. In order to overcome the difficulties with the lack of Mini-Max geometry of energy functional I. Without loss of generality, we assume $\mu = 1$. Motivated by [9], we also introduce a perturbation technique to overcome this difficulty by modifying system (SBI). We state the following modified problem

$$\begin{cases} -\Delta u + u + \phi u + \lambda\|u\|_2^\alpha u = f(u) + |u|^4 u & \text{in } \mathbb{R}^3, \\ -div\Big(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\Big) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases} \tag{4.1}$$

where $\lambda \in (0,1]$, $\alpha = (0,1)$. Obviously, its associated energy functional is

$$J_\lambda(u) := I(u) + \frac{\lambda}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)}.$$

Arguing as in Lemma 4.1, it is not hard to check that J_λ also satisfies the Mountain-Pass geometry.

Lemma 4.1. *Suppose* (f$_1$) -(f$_3$) *hold, then for fixed* $\lambda \in (0,1]$, *the following conclusions hold:*
(i) there exist $\rho, \delta_0 > 0$ such that $J_\lambda|_{S_\rho}(u) \geq \delta_0$ for every $u \in S_\rho = \{u \in H_r^1(\mathbb{R}^3) : \|u\| = \rho\}$;
(ii) there is $e_0 \in H_r^1(\mathbb{R}^3)$ with $\|e_0\| > \rho$ such that $J_\lambda(e_0) < 0$.

Let us stress that there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset H_r^1(\mathbb{R}^2)$ (see [19]), that is,

$$J_\lambda(u_n) \to c_\lambda \quad \text{and} \quad J'_\lambda(u_n) \to 0, \tag{4.2}$$

where c_λ is the mountain pass level characterized by

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)) \tag{4.3}$$

with

$$\Gamma := \{\gamma \in C^1([0,1], H^1(\mathbb{R}^2)) : \gamma(0) = 0 \quad \text{and} \quad J_\lambda(\gamma(1)) < 0\}.$$

Remark 4.2. *It follows from Lemma 4.1 and the definition of* c_λ *that there exist two constants* $c, d > 0$ *independently of* λ *such that* $c < c_\lambda < d$.

In the following, we will give an upper bounded estimate on c_λ which will be of use in proving the convergence of Palais-Smale sequences.

Lemma 4.3. *Assume* (f$_1$) -(f$_4$) *hold and if* $q \in (4,6)$ *or* $q \in (2,4]$ *and* D *is sufficiently large, t hen* $c_\lambda < \frac{1}{3} S^{\frac{3}{2}}$.

Proof Let $\phi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with support in $\phi \in C_0^\infty(B_{2r}(0))$ so that $0 \leq \phi(x) \leq 1$ and $\phi(x) \equiv 1$ on $B_r(0)$, where $r > 0$. It is well-known that S is attained by the functions $\frac{\varepsilon^{1/4}}{(\varepsilon + |x|^2)^{1/2}}$ for $\varepsilon > 0$. Define $U_\varepsilon(x) = \frac{\phi(x)\varepsilon^{1/4}}{(\varepsilon + |x|^2)^{1/2}}$, then by direct calculation, one has

$$\int_{\mathbb{R}^3} |\nabla U_\varepsilon|^2 dx = K_1 + O(\varepsilon^{\frac{1}{2}}), \qquad \int_{\mathbb{R}^3} |U_\varepsilon|^6 dx = K_2 + O(\varepsilon^{\frac{3}{2}}) \tag{4.4}$$

and

$$\int_{\mathbb{R}^3} |U_\varepsilon|^t dx = \begin{cases} O(\varepsilon^{\frac{t}{4}}), & t \in [2,3); \\ O(\varepsilon^{\frac{3}{4}} |\ln \varepsilon|), & t = 3; \\ O(\varepsilon^{\frac{6-t}{4}}), & t \in (3,6), \end{cases} \tag{4.5}$$

where K_1, K_2 are positive constants. Moreover, $S = \frac{K_1}{K_2^{1/3}}$. Combining (4.4) and (4.5), we have

$$\frac{\int_{\mathbb{R}^3} |\nabla U_\varepsilon|^2 dx}{(\int_{\mathbb{R}^3} |U_\varepsilon|^6 dx)^{1/3}} = S + O(\varepsilon^{\frac{1}{2}}). \tag{4.6}$$

According to the definition of c_λ, we obtain $c_\lambda \leq \max_{t \geq 0} J_\lambda(tU_\varepsilon)$. Define function $y(t) := \frac{t^2}{2}\|U_\varepsilon\|^2 - \frac{t^6}{6}\int_{\mathbb{R}^3} |U_\varepsilon|^6$. It is easy to check that $y(t)$ attains its maximum at

$$t_0 = \left(\frac{\|U_\varepsilon\|^2}{\int_{\mathbb{R}^3} |U_\varepsilon|^6} \right)^{1/4}$$

and $y(t_0) = \frac{1}{3} \frac{\|U_\varepsilon\|^3}{\|U_\varepsilon\|_6^3}$. Recalling Lemma 2.5, we estimate

$$\int_{\mathbb{R}^3} \phi_{U_\varepsilon} U_\varepsilon^2 \leq C \|\nabla \phi_{U_\varepsilon}\| \cdot \|U_\varepsilon\|^2_{\frac{12}{5}} \leq C \|U_\varepsilon\|^4_{\frac{12}{5}}. \tag{4.7}$$

So that there exists $t' \in (0,1)$ such that for $\varepsilon < 1$, we have

$$\begin{aligned}\max_{t'\geq t\geq 0} J_\lambda(tU_\varepsilon(x)) &\leq \max_{t'\geq t\geq 0}\left(\frac{t^2}{2}\|U_\varepsilon\|^2 + \frac{t^2}{2}\int_{\mathbb{R}^3}\phi_{tU_\varepsilon}U_\varepsilon^2 + \frac{\lambda t^{2(1+\alpha)}}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)}\right)\\ &\leq \max_{t'\geq t\geq 0}\left(\frac{t^2}{2}\|U_\varepsilon\|^2 + C\frac{t^4}{2}\|U_\varepsilon\|^4_{\frac{12}{5}} + \frac{\lambda t^{2(1+\alpha)}}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)}\right) < \frac{1}{3}S^{3/2}.\end{aligned} \tag{4.8}$$

Using hypothesis (f_4), one has

$$J_\lambda(tU_\varepsilon(x)) \leq y(t) + C\frac{t^4}{2}\|U_\varepsilon\|^4_{\frac{12}{5}} + \frac{t^{2(1+\alpha)}}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)} - \frac{D}{q}t^q\int_{\mathbb{R}^3}|U_\varepsilon|^q. \tag{4.9}$$

Now we claim that there exists $\varepsilon_0 \in (0,1)$ such that $\lim_{t\to+\infty} J_\lambda(tU_\varepsilon(x)) < 0$ uniformly in $\varepsilon \in (0,\varepsilon_0)$. Indeed, It follows from Lemma 4.1 and (4.9) that $\lim_{t\to+\infty} J_\lambda(tU_\varepsilon(x)) = -\infty$ and $J_\lambda(tU_\varepsilon(x)) > 0$ as t is close to 0. Define

$$e(t) := \frac{t^2}{2}\|U_\varepsilon\|^2 - \frac{t^6}{6}\int_{\mathbb{R}^3}|U_\varepsilon|^6 + C\frac{t^4}{2}\|U_\varepsilon\|^4_{\frac{12}{5}} + \frac{t^{2(1+\alpha)}}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)} - \frac{D}{q}t^q\int_{\mathbb{R}^3}|U_\varepsilon|^q,$$

then there exists $t_\varepsilon > 0$ such that $e(t_\varepsilon) = 0$ and $e(t) < 0$ for $t > t_\varepsilon$. From

$$e(t_\varepsilon) = t_\varepsilon^2\left(\frac{1}{2}\|U_\varepsilon\|^2 + C\frac{t^2}{2}\|U_\varepsilon\|^4_{\frac{12}{5}} + \frac{t^{2\alpha}}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)} - \frac{t_\varepsilon^4}{6}\int_{\mathbb{R}^3}|U_\varepsilon|^6 - \frac{D}{q}t^{q-2}\int_{\mathbb{R}^3}|U_\varepsilon|^q\right) = 0,$$

we have

$$\begin{aligned}\frac{1}{2}\|U_\varepsilon\|^2 + C\frac{t_\varepsilon^2}{2}\|U_\varepsilon\|^4_{\frac{12}{5}} + \frac{t_\varepsilon^{2\alpha}}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)} &= \frac{t_\varepsilon^4}{6}\int_{\mathbb{R}^3}|U_\varepsilon|^6 + \frac{D}{q}t^{q-2}\int_{\mathbb{R}^3}|U_\varepsilon|^q\\ &\geq \frac{t_\varepsilon^4}{6}\int_{\mathbb{R}^3}|U_\varepsilon|^6,\end{aligned}$$

which, together with (4.4), (4.5), implies that

$$\begin{aligned}\frac{\lambda t_\varepsilon^4}{6} &\leq \frac{1}{2}\frac{\|U_\varepsilon\|^2}{\int_{\mathbb{R}^3}|U_\varepsilon|^6} + C\frac{t_\varepsilon^2}{2}\frac{\|U_\varepsilon\|^4_{\frac{12}{5}}}{\int_{\mathbb{R}^3}|U_\varepsilon|^6} + \frac{t_\varepsilon^{2\alpha}}{2(1+\alpha)}\frac{\|u\|_2^{2(1+\alpha)}}{\int_{\mathbb{R}^3}|U_\varepsilon|^6}\\ &\leq \frac{1}{2}\frac{K_1 + O(\varepsilon_0^{\frac{1}{2}})}{K_2 + O(\varepsilon_0^{\frac{3}{2}})} + C_{\varepsilon_0}(t_\varepsilon^2 + t_\varepsilon^{2\alpha})\end{aligned} \tag{4.10}$$

for $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 > 0$ small enough. (4.10) implies that t_ε is bounded from above by some $t^* > 0$ uniformly for $\varepsilon \in (0,\varepsilon_0)$, where t^* is independent of ε. Combining the above fact and (4.9), we know that there exists $\varepsilon_0 \in (0,1)$ such that $\lim_{t\to+\infty} J_\lambda(tU_\varepsilon(x)) < 0$ uniformly in $\varepsilon \in (0,\varepsilon_0)$. Thus there exists $t'' > t^*$ such that for $\varepsilon \in (0,\varepsilon_0)$,

$$\max_{t\geq t''} J_\lambda(tU_\varepsilon(x)) < \frac{1}{3}S^{3/2}. \tag{4.11}$$

From (4.5), (4.6) and (4.9), we infer that

$$\begin{aligned}&\max_{t''\geq t\geq t'} J_\lambda(tU_\varepsilon(x))\\ &\leq y(t_0)+C\frac{t^4}{2}\|U_\varepsilon\|^4_{\frac{12}{5}}+\frac{\lambda t^{2(1+\alpha)}}{2(1+\alpha)}\|u\|_2^{2(1+\alpha)}-CD\int_{\mathbb{R}^3}|U_\varepsilon|^q\\ &=\frac{1}{3}S^{3/2}+O(\varepsilon^{\frac{1}{2}})-CD\int_{\mathbb{R}^3}|U_\varepsilon|^q.\end{aligned}\tag{4.12}$$

For $q\in(2,4]$ and D sufficiently large, $\varepsilon\in(0,\varepsilon_0)$ fixed, we derive from (4.5) and (4.12) that

$$\max_{t''\geq t\geq t'} J_\lambda(tU_\varepsilon(x))<\frac{1}{3}S^{3/2}.\tag{4.13}$$

For $q\in(4,6)$, observe that $\frac{6-q}{4}<\frac{1}{2}$, then it follows from (4.5) and (4.12) that, there exists $\varepsilon_1\in(0,\varepsilon_0)$ small enough such that for $\varepsilon\in(0,\varepsilon_1)$,

$$\max_{t''\geq t\geq t'} J_\lambda(tU_\varepsilon(x))<\frac{1}{3}S^{3/2}.\tag{4.14}$$

Combining (4.8), (4.11) and (4.13), (4.14), we deduce that $c_\lambda<\frac{1}{3}S^{3/2}$. □

Lemma 4.4. *Let $\{u_n\}\subset H^1_r(\mathbb{R}^3)$ be the $(PS)_{c_\lambda}$ sequence of J_λ for fixed $\alpha\in(0,1]$, then there exists $u_0\in H^1_r(\mathbb{R}^3)$ such that $u_n\to u_0$ in $H^1_r(\mathbb{R}^3)$.*

Proof We first show that sequence $\{u_n\}$ is bounded in $H^1_r(\mathbb{R}^3)$. For $\theta\in(4,6)$, by Lemma 2.5, there exist $C_1,C_2>0$ such that

$$\begin{aligned}C_1+C_2\|u_n\|\geq& I_\lambda(u_n)-\frac{1}{\theta}I'_\lambda(u_n)u_n\\ \geq&\frac{\theta-2}{2\theta}\|u_n\|^2-\frac{\theta-2}{2\theta}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+\frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\lambda\|u_n\|_2^{2(1+\alpha)}\\ &+\int_{\mathbb{R}^3}(\frac{1}{\theta}f(u_n)u_n-F(u_n))-\frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_n}|^2})+\frac{6-\theta}{6\theta}\lambda\int_{\mathbb{R}^3}|u_n|^6\\ \geq&\frac{\theta-2}{2\theta}C_3\|u_n\|^2-\frac{\theta-2}{2\theta}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+\frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\|u_n\|_2^{2(1+\alpha)}\\ &-C_4\int_{\mathbb{R}^3}|u_n|^{p+1}-\frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_n}|^2})+\frac{6-\theta}{6\theta}\int_{\mathbb{R}^3}|u_n|^6,\end{aligned}\tag{4.15}$$

which implies by (3.8) and Lemma 2.3 that

$$\begin{aligned}C_1+C_2\|u_n\|\geq&\frac{\theta-2}{2\theta}C_3\|u_n\|^2+\frac{\theta-4}{4\theta}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+\frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\|u_n\|_2^{2(1+\alpha)}\\ &-C_4\int_{\mathbb{R}^3}|u_n|^{p+1}+\frac{6-\theta}{6\theta}\int_{\mathbb{R}^3}|u_n|^{q+1}\\ \geq&\frac{\theta-2}{2\theta}C_3\|u_n\|^2+\frac{\theta-2(1+\alpha)}{2\theta(1+\alpha)}\|u_n\|_2^{2(1+\alpha)}\\ &-C_4\int_{\mathbb{R}^3}|u_n|^{p+1}+\frac{6-\theta}{6\theta}\int_{\mathbb{R}^3}|u_n|^{q+1}.\end{aligned}\tag{4.16}$$

Based on above, arguing similarly to in Lemma 3.5, we deduce that $\|u_n\|\leq C$ for some C independently of n. The claim immediately follows. Thus, there exists a subsequence of $\{u_n\}$

(still denoted by $\{u_n\}$, without loss of generality) such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{weakly in } H_r^1(\mathbb{R}^3), \\ u_n &\to u_0 \quad \text{in } L^p(\mathbb{R}^3),\ 2<p<6, \\ u_n &\to u_0 \quad \text{a.e. in } \mathbb{R}^3 \end{aligned} \tag{4.17}$$

for some $u_0 \in H_r^1(\mathbb{R}^3)$. Define $w_n := u_n - u_0$. We note by (4.17) and the similar argument as Lemma 3.5 that $J'_\lambda(u_0) = 0$ and $J_\lambda(u_0) \geq 0$, respectively. It remains to prove that $u_n \to u_0$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$. Indeed, on the one hand, observe by (4.17), (3.12) and the well-known Brezis-Lieb Lemma that

$$o(1) = J'_\lambda(u_n)u_n - J'_\lambda(u_0)u_0 = \|w_n\|^2 - \|w_n\|_6^6. \tag{4.18}$$

On the other hand, note by (4.17), (3.12) and the well-known Brezis-Lieb Lemma that

$$J_\lambda(u_n) - J_\lambda(u_0) = \frac{1}{2}\|w_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_n}|^2}) + \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_0}|^2}) - \frac{1}{6}\|w_n\|^2 \tag{4.19}$$

Recalling (3.8), one has

$$0 \leq \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_n}|^2}) \leq \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2.$$

In the spirit of the fact that $\phi_{u_n}u_n^2 \to \phi_{u_0}u_0^2$ in $L^1(\mathbb{R}^3)$, there exists $h(x) \in L^1(\mathbb{R}^3)$ such that $|\phi_{u_n(x)}u_n^2(x)| \leq h(x)$ a.a. $x \in \mathbb{R}^3$ and uniformly for n. Based on above, Lebesgue dominated convergence theorem implies that

$$\frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_n}|^2}) \to \frac{1}{2}\int_{\mathbb{R}^3}(1-\sqrt{1-|\nabla\phi_{u_0}|^2}), \quad \text{as } n\to\infty. \tag{4.20}$$

Put (4.20) into (4.19), then the following holds

$$c_\lambda \geq J_\lambda(u_n) - J_\lambda(u_0) = \frac{1}{2}\|w_n\|^2 - \frac{1}{6}\|w_n\|_6^6 + o(1) \tag{4.21}$$

Combining (4.18) with (4.21), we get

$$\frac{1}{3}S^{3/2} > c_\lambda \geq \frac{1}{3}\|w_n\|^2 + o(1). \tag{4.22}$$

Recalling $\|w_n\|^2 \geq S\|w_n\|_6^2$, then $\|w_n\|^{\frac{4}{3}} \geq S$ for $w_n \not\equiv 0$ due to (4.18). It contradicts with (4.22), and therefore, $w_n \equiv 0$. That is, $u_n \to u_0$ in $H_r^1(\mathbb{R}^3)$ which completes the proof. □

Now we are attempt to prove Theorem 1.1

Proof of Theorem 1.3 It follows from Lemma 3.5 that for fixed $\lambda \in (0,1]$, there exists $u_\lambda \in H_r^1(\mathbb{R}^3)$ such that $J_\lambda(u_\lambda) = c_\lambda$ and $J'_\lambda(u_\lambda) = 0$. Choosing a sequence $\{\lambda_n\} \subset (0,1]$ satisfying $\lambda_n \to 0$, then we find a sequence of nontrivial critical points $\{u_{\lambda_n}\}$(still denoted by $\{u_n\}$) of J_{λ_n} with $J_{\lambda_n}(u_n) = c_{\lambda_n}$. Arguing similarly in the proof of Theorem 1.1, we obtain that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$ uniformly for n. By letting $n \to +\infty$, then there exists $c_0 > 0$ such that $c_{\lambda_n} \to c_0$ and $J_{\lambda_n}(u_n) = c_0 + o(1)$. Moreover, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$J'(u_n)\varphi = J'_{\lambda_n}(u_n)\varphi - \lambda_n\|u_n\|_2^\alpha u_n\varphi = o(1)\|\varphi\|.$$

Thus, $\{u_n\}$ is a Palais-Smale sequence of J with level c_0. Arguing similarly as in the proof of Lemma 3.5, we prove that there exists a nontrivial $u_0 \in H_r^1(\mathbb{R}^3)$ such that $J'(u_0) = 0$ and $J(u_0) = c$. Arguing Similarly as in the Theorem 1.1, we can get that (1.1) has at least a radial ground state solution (u, ϕ_u). □

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