Local and nonlocal singular Liouville equations in Euclidean spaces

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Abstract

We study metrics of constant Q-curvature in the Euclidean space with a prescribed singularity at the origin, namely solutions to the equation

$$(-\Delta)^{\frac{n}{2}} w = e^{nw} - c\delta_0 \text{ on } \mathbb{R}^n,$$

under a finite volume condition. We analyze the asymptotic behaviour at infinity and the existence of solutions for every $n \geq 3$ also in a supercritical regime. Finally, we state some open problems.

1 Introduction

In this paper we will deal with the classification of the solutions to the equation

$$(-\Delta)^{\frac{n}{2}} w = e^{nw} - c\delta_0 \text{ on } \mathbb{R}^n, \tag{1}$$

where $e^{nw} \in L^1(\mathbb{R}^n)$, $c \in \mathbb{R}$, δ_0 is a Dirac mass at the origin and $n \geq 2$. Equations of this kind arise naturally in the study of the prescribed Q-curvature problem with singularities. We recall that the Q-curvature is a curvature of order n that was introduced by [3, 4, 10] and in the last decades has been intensely studied in problems of conformal geometry. If w is a solution of (1), then $e^{2w}|dx|^2$ is a metric on $\mathbb{R}^n$ conformal to the Euclidean metric $|dx|^2$ and having Q-curvature equal to 1 everywhere except at the origin, where it has a special kind of singularity. When $n = 2$, singularities of this kind are known as conical singularities and have been studied e.g. in [2, 25, 26]. The case $n = 1$, which we will not discuss here, has a different geometric interpretation, in terms of the curvature of curves in $\mathbb{R}^2$ given in conformal parametrization, see e.g. [8, 9, 11, 21, 30]. In this case the term $c\delta_0$ corresponds to a corner. In dimension higher than 2, the geometry of such singularities (in particular the shape around the singular point) has not yet been investigated to our knowledge. Indeed, in dimension 3 and higher the Q-curvature,

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being a scalar quantity, only describes the geometry of a manifold in a limited way. For instance its positivity has no implication on the positivity of the scalar or sectional curvatures, not even in symmetric situations.

In this work we shall focus on existence results for (1), both in a symmetric and non-symmetric setting, and on the asymptotic behavior at infinity of the solutions. Writing

$$w = u + \frac{c}{\gamma_n} \log|x|,$$

we see that u satisfies

$$(-\Delta)^{\frac{n}{2}} u = |x|^{\frac{nc}{\gamma_n}} e^{nu} \quad \text{on } \mathbb{R}^n, \tag{2}$$

where $\gamma_n := \frac{(n-1)!}{2}|S^n|$ is such that

$$(-\Delta)^{\frac{n}{2}}\left(\frac{1}{\gamma_n}\log\frac{1}{|x|}\right) = \delta_0 \quad \text{in } \mathbb{R}^n,$$

and it will be convenient to study equation (2), since the RHS of (2) belongs to $L^1(\mathbb{R}^n)$ and this allows to use representation formulas at least for a special class of solutions.

More precisely, for a given $\alpha > -1$, we consider the problem

$$\begin{cases} (-\Delta)^{\frac{n}{2}} u = |x|^{n\alpha} e^{nu} \text{ on } \mathbb{R}^n, \\ \Lambda := \int_{\mathbb{R}^n} |x|^{n\alpha} e^{nu}\, dx < \infty. \end{cases} \tag{3}$$

Geometrically, if u solves (3), then the metric $e^{2u}|dx|^2$ has Q-curvature $|x|^{n\alpha}$ and total Q-curvature Λ. The parameter Λ is also the volume of the metric $e^{2w}|dx|^2$, and it plays a crucial role in the existence of solutions to (3).

In order to give a good definition of weak solutions to (3), we need to define $(-\Delta)^{\frac{n}{2}}$ for a sufficiently large class of functions. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions, and for $s > 0$ set

$$L_s(\mathbb{R}^n) := \left\{ w \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|w(x)|}{1+|x|^{n+2s}} dx < \infty \right\}.$$

Given a function $v \in L_s(\mathbb{R}^n)$, $s > 0$, we define $(-\Delta)^s v$ as a tempered distribution as follows:

$$\langle(-\Delta)^s v, \varphi\rangle := \int_{\mathbb{R}^n} v(-\Delta)^s\varphi\, dx \qquad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{4}$$

where

$$(-\Delta)^{\frac{s}{2}}\varphi := \mathcal{F}^{-1}(|\xi|^s \mathcal{F}\varphi(\xi)),$$

and

$$\mathcal{F}\varphi(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\varphi(x)e^{-ix\cdot\xi} dx$$

is the Fourier transform. The right-hand side of (4) makes sense because

$$(-\Delta)^s\varphi(x) = O(|x|^{-n-2s}) \quad \text{as } |x| \to \infty,$$

see e.g. [18, Proposition 2.1].

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Definition 1.1. *Given $m \geq 1$ and a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, we say that u is a solution of*

$$(-\Delta)^{\frac{m}{2}} u = f \quad in \ \mathbb{R}^n$$

if the following holds. In case $m \geq 2$ is even, we require $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} u(-\Delta)^{\frac{m}{2}} \varphi \, dx = \langle f, \varphi \rangle \quad for \ every \ \varphi \in C^\infty_c(\mathbb{R}^n).$$

In case $m \geq 1$ is odd, we require $u \in L_{\frac{m}{2}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} u(-\Delta)^{\frac{m}{2}} \varphi \, dx = \langle f, \varphi \rangle \quad for \ every \ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

As we shall see in Theorem 2.1, weak solutions to (3) are smooth away from the origin and at least Hölder continuous near the origin.

In the classical case $n = 2$, for any $\alpha > -1$ there are explicit solutions to (3) taking the form

$$u(x) = \log\left(\frac{2(\alpha+1)}{1 + |x|^{2(\alpha+1)}}\right), \quad \Lambda = 4\pi(\alpha+1).$$

With the scaling $u_\lambda(x) = u(\lambda^{\frac{1}{1+\alpha}} x) + \log \lambda$ we obtain the family of solutions

$$u(x) = \log\left(\frac{2(\alpha+1)\lambda}{1 + \lambda^2|x|^{2(\alpha+1)}}\right), \quad \Lambda = 4\pi(\alpha+1), \quad \lambda > 0, \alpha > -1. \tag{5}$$

In fact, as shown by Chen-Li [7], when $n = 2$ for every solution to (3) we necessarily have $\Lambda = 4\pi(1 + \alpha)$. In [25] Prajapat-Tarantello proved that solutions as in (5) exhaust the set of solutions of (3) if α is not an integer, while when $\alpha \in \mathbb{N}$ instead there exist more solutions, all of which can be written using the complex notation in the form

$$u(z) = \log\left(\frac{2(\alpha+1)\lambda}{1 + \lambda^2|z^{\alpha+1} - \zeta|^2}\right), \quad \lambda > 0, \ \zeta \in \mathbb{C}. \tag{6}$$

The solutions in (6) were first introduced by Chanillo and Kiessling [6]. Interestingly, they are not radially symmetric, except in the case $\zeta = 0$ (when they reduce to (5)) or $\alpha = 0$ (in which case they are radial about the point ζ) and blow up at the $\alpha + 1$ roots of the complex equation $z^{\alpha+1} = \zeta$ as $\lambda \to \infty$.

When $n > 2$, the structure of the set of solutions of (3) becomes richer. If $\alpha = 0$ one can still identify a special family of solutions, known as *normal* (cfr. Definition 1.2) or *sperical* solutions, having an explicit expression similar to (6) (with $\alpha = 0$). These solutions have total Q-curvature $\Lambda = \Lambda_1 := (n-1)!|S^n|$ and are the only solutions of (3) with $u = o(|x|^2)$ as $|x| \to +\infty$. However, in addition to these solutions there are also non-normal solutions, behaving polynomially at infinity and with Λ not necessarily equal to Λ_1 see e.g. [5, 16, 19, 20, 28].

When $n \neq 2$ and $\alpha \neq 0$ we do not have explicit formulas for solutions to (3) (not even when α is integer, to the best of our knowledge) so that the existence of even one single solution is not obvious. We shall prove two existence results based on a Schauder fixed-point argument, in a spirit similar to that of previous works, such as [16, 17, 19, 28].

There are two crucial ingredients in this approach. One is to have a good ansatz, namely restrict the set among which we will look for solutions to functions having a particular asymptotic behaviour. The second ingredient is a precise information on the value of Λ for normal solutions to (3), namely solutions to an integral equation, see Definition 1.2. Both this properties are contained in the following theorem, which extends to the case $\alpha \neq 0$ the classification results in [22, 23, 27, 20, 18].

Theorem 1.1. *For $\alpha > -1$ let u solve (3) and define v as follows*

$$v(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1 + |y|}{|x - y|} \right) |y|^{n\alpha} e^{nu(y)} \, dy. \tag{7}$$

Then there exists an upper-bounded polynomial p of degree at most $n - 1$ such that $u = v + p$. Moreover

$$\lim_{|x| \to \infty} \frac{v(x)}{\log |x|} = -\beta := -\frac{\Lambda}{\gamma_n}. \tag{8}$$

Finally, if p is constant, then $\Lambda = \Lambda_1(1 + \alpha)$.

Notice that p being upper bounded implies that p has even degree, hence p has degree at most $n - 2$ when n is even.

Definition 1.2 (Normal solutions)**.** *We call a solution u to (3) normal if writing $u = v + p$ as in Theorem 1.1 the polynomial p is constant. Equivalently if there exists a constant $c \in \mathbb{R}$ such that u solves the integral equation*

$$u(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1 + |y|}{|x - y|} \right) |y|^{n\alpha} e^{nu(y)} \, dy + c.$$

Namely, u is a normal solution if, up to a constant, u admits an integral representation via the fundamental solution.

The proof of Theorem 1.1 is innovative in comparison to the previous works, and it also simplifies them considerably (see Remark 3.1). It is based on a Campanato-space estimate (Lemma 3.6). In fact, instead of having an L^∞-upper bound on the function v, which is usually difficult to obtain in the case $\alpha = 0$ and appears to be much more challenging in the case $\alpha \neq 0$, we content ourselves with a decay estimate on the oscillation of v, which will be sufficient to conclude.

We can summarize Theorem 1.1 by saying that all solutions to (3) have the form $v + p$, where v behaves logarithmically at infinity and p is a polynomial, and if p is constant, then $\Lambda = \Lambda_1(1 + \alpha)$. Using this information we can move towards the existence results.

In the first result we show existence of radial solutions up to the critical treshold $\Lambda_1(1 + \alpha)$.

Theorem 1.2. *Let $n \geq 3$ and $\alpha > -1$. Then for every $0 < \Lambda < \Lambda_1(1 + \alpha)$ there exists a radially symmetric solution to (3). For $\Lambda = \Lambda_1(1 + \alpha)$ there exists a radially symmetric and normal solution to (3).*

Existence for every $\Lambda < \Lambda_1(1 + \alpha)$ is obtained by a standard compactness argument. These solutions will have the form $v + p$ with $p(x) = -|x|^2$. The normal solutions obtained for the case $\Lambda = \Lambda_1(1 + \alpha)$ are the higher-dimensional counterparts of (5), whose existence is not obvious,

since we do not have explicit formulas for them. For this part we will use a blow-up argument together with a non-existence result based on a Pohozaev-type identity.

Next we show that, breaking radial symmetry, we are able to produce solutions to (3) above the critical threshold.

Theorem 1.3. *Let $n \geq 3$. Then for every $\alpha > -1$ and for every $\Lambda \in (0, \Lambda_1)$ there exists a solution to* (3).

Notice that $\Lambda_1 > \Lambda_1(1 + \alpha)$ for $\alpha \in (-1, 0)$ in Theorem 1.3, hence we have an existence result above the critical threshold of compactness.

As we shall see, the solutions given by Theorem 1.3 are non-radial, by construction. On the other hand, in dimension 3 and 4 this is also a necessity. Indeed from the Pohozaev identity we obtain:

Proposition 1.1. *For $n = 3$ and $n = 4$ every* radial *solution u to* (3) *satisfies $\Lambda \leq \Lambda_1(1 + \alpha)$ with identity if and only if u is normal.*

1.1 Some open problems

Using a variational argument as in [16] one should be able to find solutions to (3) of the form $u = v + p$ with the polynomial p prescribed. For instance one could try to prove that for $n \geq 3$, $\alpha > -1$, $0 < \Lambda < \Lambda_1 \min\{1, 1 + \alpha\}$ and a given polynomial p with $deg(p) \leq n - 1$ and satisfying

$$p(x) \xrightarrow{|x| \to \infty} -\infty, \tag{9}$$

there exists a solution u to (3) such that

$$u(x) = p(x) - \frac{2\Lambda}{\Lambda_1} \log |x| + C + o(1), \quad o(1) \xrightarrow{|x| \to \infty} 0.$$

Also the existence of solutions to (3) for arbitrarily large Λ is open, in analogy with the case $\alpha = 0$ studied in [15, 17, 24]. In the radial case, using methods from [17] it should be possible to prove that for $n \geq 5$, $\alpha > -1$ and for every $\Lambda > 0$ there exists a radially symmetric solution to (3). Notice that the condition $n \geq 5$ is necessary in view of Proposition 1.1 and the known results in dimension 1 and 2. If instead we drop the radial symmetry, it is unknown whether already in dimension 3 and 4 we can have solutions to (3) with arbitrarily large Λ, for $\alpha \neq 0$.

Notation   In what follows $B_R(x)$ will denote the ball of radius R centered at x (the dependence on x will often be omitted if $x = 0$) and C will denote a generic constant that can change from line to line.

2 Regularity of solutions

If $\Omega \subseteq \mathbb{R}^n$ is an open set, we denote

$$C^s(\Omega) := \left\{ u \in C^{\lfloor s \rfloor}(\Omega) : D^{\lfloor s \rfloor} u \in C^{0, s - \lfloor s \rfloor}(\Omega) \right\}, \quad s - 1 < \lfloor s \rfloor \leq s, \ \lfloor s \rfloor \in \mathbb{N},$$

Then, we set

$$C_{\text{loc}}^s(\mathbb{R}^n) := \{ u \in C^0(\mathbb{R}^n) : u|_\Omega \in C^s(\Omega) \text{ for every } \Omega \Subset \mathbb{R}^n \}.$$

Theorem 2.1. *Let u be a solution of* (3) *with $\alpha > -1$. If $\alpha \in (-1, 0)$ or $n\alpha - 1 \in 2\mathbb{N}$, then $u \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C_{\text{loc}}^s(\mathbb{R}^n)$, for $s < n(1 + \alpha)$. If $\alpha > 0$ with $n\alpha \notin \mathbb{N}$ we have $u \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap C_{\text{loc}}^{n(1+\alpha)}(\mathbb{R}^n)$, and if $n\alpha \in 2\mathbb{N}$ we have $u \in C^\infty(\mathbb{R}^n)$.*

Proof. First, we claim that $e^{nu} \in L_{\text{loc}}^q(\mathbb{R}^n)$ for any $q \geq 1$. Indeed, given any q, we can take $\varepsilon = \varepsilon(q)$ such that $q < \frac{\gamma_n}{\varepsilon}$ and we can split $|x|^{n\alpha} e^{nu} = f_1 + f_2$, where $f_1, f_2 \geq 0$ and

$$f_1 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad \|f_2\|_{L^1(\mathbb{R}^n)} \leq \varepsilon.$$

Let us define the functions

$$u_i(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1 + |y|}{|x - y|} \right) f_i(y) dy, \quad x \in \mathbb{R}^n, \ i = 1, 2,$$

and

$$u_3 := u - u_1 - u_2.$$

It is easy to see that $u_1 \in C^{n-1}(\mathbb{R}^n)$ and that u_3 is $\frac{n}{2}$-harmonic (that is $(-\Delta)^{\frac{n}{2}} u_3 = 0$ in $\mathbb{R}^n$, in the sense of Definition 1.1) and hence $u_3 \in C^\infty(\mathbb{R}^n)$. Moreover, using Jensen's inequality we get

$$
\begin{aligned}
\int_{B_R} e^{n q u_2} dx &= \int_{B_R} \exp \left(\int_{\mathbb{R}^n} \frac{nq \|f_2\|}{\gamma_n} \log \left(\frac{1 + |y|}{|x - y|} \right) \frac{f_2(y)}{\|f_2\|} dy \right) dx \\
&\leq \int_{B_R} \int_{\mathbb{R}^n} \exp \left(\frac{nq \|f_2\|}{\gamma_n} \log \left(\frac{1 + |y|}{|x - y|} \right) \right) \frac{f_2(y)}{\|f_2\|} dy dx \\
&= \frac{1}{\|f_2\|} \int_{\mathbb{R}^n} f_2(y) \int_{B_R} \left(\frac{1 + |y|}{|x - y|} \right)^{\frac{nq \|f_2\|}{\gamma_n}} dx dy \\
&\leq C(q, R),
\end{aligned}
\tag{10}
$$

where $\| \cdot \|$ denotes the $L^1(\mathbb{R}^n)$ norm. Hence the claim is proved.

Set $\bar{p} := -\frac{1}{\alpha}$ for $\alpha \in (-1, 0)$ and $\bar{p} := \infty$ if $\alpha > 0$. Then $| \cdot |^{n\alpha} \in L_{\text{loc}}^p(\mathbb{R}^n)$ for every $1 \leq p < \bar{p}$. It follows that $(-\Delta)^{\frac{n}{2}} u = | \cdot |^{n\alpha} e^{nu} \in L_{\text{loc}}^p(\mathbb{R}^n)$ for $p < \bar{p}$, and this implies $u \in W_{\text{loc}}^{n,p}(\mathbb{R}^n)$ for $p < \bar{p}$. Indeed, for any given $R > 0$, we can write $u = v_1 + v_2 + v_3$, where v_3 is $\frac{n}{2}$–harmonic (and thus $v_3 \in C^\infty(\mathbb{R}^n)$) and

$$v_i(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1 + |y|}{|x - y|} \right) g_i(y) dy, \quad x \in \mathbb{R}^n, \ i = 1, 2, \tag{11}$$

with

$$g_1 = | \cdot |^{n\alpha} e^{nu} \chi_{B_R}, \quad g_2 = | \cdot |^{n\alpha} e^{nu} \chi_{B_R^c}.$$

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Differentiating (11) we obtain that $v_1 \in W^{n,p}(B_{\frac{R}{2}})$ (by the Calderon-Zygmund theory) and $v_2 \in C^\infty(B_{\frac{R}{2}})$.

By the Sobolev embedding we then infer that $u \in C^s_{loc}(\mathbb{R}^n)$ for $s < n(1 + \min\{\alpha, 0\})$.

Since $|\cdot|^{n\alpha} \in C^\infty(\mathbb{R}^n \setminus \{0\})$, by bootstrapping regularity we see that $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for every $\alpha > -1$. Now, if $\alpha \geq 0$, we observe that

$$|\cdot|^{n\alpha} \in \begin{cases} C^{n\alpha}(\mathbb{R}^n) & \text{if } n\alpha \notin \mathbb{N}, \\ C^s_{loc}(\mathbb{R}^n) & \text{if } s < n\alpha, \ n\alpha - 1 \in 2\mathbb{N}, \\ C^\infty(\mathbb{R}^n) & \text{if } n\alpha \in 2\mathbb{N}. \end{cases}$$

In any case, we can conclude the proof by bootstrapping regularity using Schauder's estimates. $\qquad\square$

3 Proof of Theorem 1.1

Lemma 3.1. *Let u be a solution to Problem* (3) *and let v be as in* (7)*. Then for $|x| \geq 1$ we have*

$$v(x) \geq -\beta \log|x|, \tag{12}$$

where β is as in (8)*.*

Proof. Since $|x| \geq 1$, thanks to the triangular inequality we have

$$|x - y| \leq |x| + |y| \leq |x| + |x||y| = |x|\,(1 + |y|)\,,$$

for any $y \in \mathbb{R}^n$. Therefore

$$\log\left(\frac{1 + |y|}{|x - y|}\right) \geq -\log|x|.$$

$\qquad\square$

Lemma 3.2. *Let u be a solution to Problem* (3) *and v as in* (7)*. Then $u = v + p$ where p is a polynomial of degree at most $n - 1$.*

Proof. Set $p = u - v$ so that $(-\Delta)^{\frac{n}{2}}p = 0$. From Lemma 3.1 we have

$$p(x) \leq u(x) + \beta \log(1 + |x|) + C.$$

Recalling that $|\cdot|^{n\alpha}e^{nu} \in L^1$, it follows from a Liouville-type theorem (see Theorem A.1 in the Appendix) that p is a polynomial of degree at most $n - 1$. $\qquad\square$

Lemma 3.3. *Let p be a polynomial as in Lemma 3.2. Then*

$$\sup_{x \in \mathbb{R}^n} p(x) < +\infty. \tag{13}$$

In particular p has even degree. Moreover for every $q \geq 1$ and $0 < \rho \leq \rho_0$ there exists $C = C(q, \rho_0)$ such that

$$\int_{B_\rho(x)} e^{qp}dx \leq C|x|^{n(\beta - \alpha)}, \tag{14}$$

for any $x \in \mathbb{R}^n$ with $|x| \geq 1$.

Proof. We start by proving (13). Following [23] we define

$$f(r) = \sup_{\partial B_r} p.$$

From Theorem 3.1 in [14] it follows that if $\sup_{\mathbb{R}^n} p = +\infty$ then there exists $s > 0$ such that

$$\lim_{r \to +\infty} \frac{f(r)}{r^s} = +\infty.$$

From Lemma 3.2 p is a polynomial of degree at most $n - 1$. In particular, we have that $|\nabla p(x)| \leq C|x|^{n-2}$ for $|x|$ large. From Lemma 3.1 it follows that there exists $R > 0$ such that for every $r \geq R$ we can find x_r with $|x_r| = r$ such that

$$u(y) = v(y) + p(y) \geq r^s$$

for $|y - x_r| \leq \frac{1}{r^{n-2}}$. Now using Fubini's theorem we get

$$\int_{\mathbb{R}^n} |x|^{n\alpha} e^{nu} \, dx \geq \int_R^{+\infty} \int_{\partial B_r(0) \cap B_{r^{2-n}}(x_r)} r^{n\alpha} e^{nr^s} \, d\sigma \, dr$$

$$\geq C \int_R^{+\infty} \frac{r^{n\alpha} e^{nr^s}}{r^{(n-2)(n-1)}} \, dr = +\infty,$$

that is a contradiction, hence (13) is proven.

The proof of (14) follows at once from (13) and Lemma 3.1. Indeed, if $|x| \geq 2\rho_0$ then

$$C \geq \int_{B_\rho(x)} |y|^{n\alpha} e^{nu(y)} dy \geq C \int_{B_\rho(x)} |y|^{n(\alpha-\beta)} e^{np(y)} dy \geq C|x|^{n(\alpha-\beta)} \int_{B_\rho(x)} e^{qp(y)} dy,$$

while (14) is trivial for $|x| \in [1, 2\rho_0)$. $\square$

Lemma 3.4. *For any $\varepsilon > 0$ there exists $R > 0$ such that for $|x| \geq R$*

$$v(x) \leq (-\beta + \varepsilon) \log |x| + \frac{1}{\gamma_n} \int_{B_1(x)} \log \left(\frac{1}{|x-y|} \right) |y|^{n\alpha} e^{nu} dy. \tag{15}$$

Proof. The simple proof is similar to [22, pag. 213] and is omitted. $\square$

Lemma 3.5. *For any $q \geq 1$, $\varepsilon_1, \varepsilon_2 > 0$ there exists a constant $C = C(q, \varepsilon_1, \varepsilon_2)$ such that for $0 < \rho \leq 1$ and $x \in \mathbb{R}^n$*

$$\frac{1}{\rho^{n-\varepsilon_2}} \int_{B_\rho(x)} e^{qv(y)} dy \leq \frac{C}{|x|^{(\beta-\varepsilon_1)q}}.$$

Proof. For x in a compact set the statement is trivial. Set $f(x) = |x|^{n\alpha} e^{nu(x)}$, and fix $R > 0$ such that

$$\frac{q}{\gamma_n} \|f\|_{L^1(B_R^c)} \leq \varepsilon_2.$$

From Lemma 3.4 up to taking R larger, we have for $|z| > R + 1$

$$v(z) \leq (-\beta + \varepsilon_1) \log |z| + \frac{1}{\gamma_n} \int_{B_1(z)} \log \left(\frac{1}{|z-y|} \right) f(y) \chi_{|z-y| \leq 1} dy$$

$$\leq (-\beta + \varepsilon_1) \log |z| + \frac{1}{\gamma_n} \int_{B_R^c(0)} \log \left(\frac{1}{|z-y|} \right) f(y) \chi_{|z-y| \leq 1} dy.$$

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Applying Jensen's inequality with respect to the measure $d\mu(y) := \frac{f(y)}{\|f\|_{L^1(B_R^c)}}dy$ in B_R^c, we get for $|x| > R + 2$

$$\int_{B_\rho(x)} e^{qv(z)}dz \le \int_{B_\rho(x)} \frac{1}{|z|^{(\beta-\varepsilon_1)q}} e^{q\frac{\|f\|_{L^1(B_R^c)}}{\gamma_n}\int_{B_R^c(0)}\log\left(\frac{1}{|z-y|}\right)\chi_{|z-y|\le 1}d\mu(y)}dz$$

$$\le \frac{C}{|x|^{(\beta-\varepsilon_1)q}}\int_{B_\rho(x)}\int_{B_R^c(0)} e^{q\frac{\|f\|_{L^1(B_R^c)}}{\gamma_n}\log\left(\frac{1}{|z-y|}\right)\chi_{|z-y|\le 1}}d\mu(y)dz$$

$$\le \frac{C}{|x|^{(\beta-\varepsilon_1)q}}\int_{B_R^c(0)}\int_{B_\rho(x)} e^{\varepsilon_2\log\left(\frac{1}{|z-y|}\right)\chi_{|z-y|\le 1}}dzd\mu(y)$$

$$\le \frac{C}{|x|^{(\beta-\varepsilon_1)q}}\int_{B_R^c(0)}\int_{B_\rho(x)}\left(1 + \frac{1}{|z-y|^{\varepsilon_2}}\right)dzd\mu(y)$$

$$\le \frac{C\rho^{n-\varepsilon_2}}{|x|^{(\beta-\varepsilon_1)q}}.$$

$\square$

Remark 3.1. *Using Lemma 3.5 one gets a simpler proof of Theorem 1.1 in the classical case $\alpha = 0$, or even if $\alpha \in (-1,\beta)$. Indeed using the Hölder inequality in (15), with $\varepsilon \le \beta - \alpha$ and applying Lemma 3.5 with $\varepsilon_1 = \varepsilon$ and $\rho = 1$ we get for $|x|$ large*

$$\int_{B_1(x)}\log\left(\frac{1}{|x-y|}\right)|y|^{n\alpha}e^{nu}dy \le C|x|^{n\alpha}\int_{B_1(x)}\log\left(\frac{1}{|x-y|}\right)e^{nv}dy \le C\frac{|x|^{n\alpha}}{|x|^{n(\beta-\varepsilon)}} \le C.$$

The proof of (8) follows at once from (15), and the rest of the proof will follow easily from the Pohozaev identity as we shall see below.

Remark 3.2. *Arguing as in Lemmas 3.1-3.5 one can also obtain the following: Let v be a solution to*

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n}\log\left(\frac{1+|y|}{|x-y|}\right)K(y)e^{nv(y)}dy + c$$

for some $c \in \mathbb{R}$ and some non-negative function $K \in L^\infty(\mathbb{R}^n \setminus B_1)$ with $Ke^{nv} \in L^1(\mathbb{R}^n)$. Then

$$\lim_{|x|\to\infty}\frac{v(x)}{\log|x|} = -\frac{1}{\gamma_n}\int_{\mathbb{R}^n}Ke^{nv}dx.$$

From now on we shall assume $\alpha \ge \beta$.

Lemma 3.6. *We have*

$$\tau(x) := \sup_{\rho\in(0,4]}\frac{1}{\rho^{n+\frac{1}{\log|x|}}}\int_{B_\rho(x)}\left|v(y) - \fint_{B_\rho(x)}v(z)dz\right|dy = o(1), \tag{16}$$

with $o(1) \to 0$ as $|x| \to \infty$. As a consequence we have

$$[v]_{C^{0,\frac{1}{\log(|x_0|+1)}}(B_1(x_0))} := \sup_{\substack{x,y\in B_1(x_0)\\x\ne y}}\frac{|v(x)-v(y)|}{|x-y|^{\frac{1}{\log(|x_0|+1)}}} = o(1)\log(|x_0|+1), \tag{17}$$

with $o(1) \to 0$ as $|x_0| \to \infty$.

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Proof. We start proving (16). We have

$$\int_{B_\rho(x)} \fint_{B_\rho(x)} |v(y) - v(z)|\, dzdy \leq \int_{B_\rho(x)} \fint_{B_\rho(x)} \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \left| \log\left(\frac{|z-\xi|}{|y-\xi|}\right) \right| f(\xi) d\xi dzdy,$$

where $f(x) := |x|^{n\alpha} e^{nu(x)}$. By (14), Lemma 3.5 and Hölder's inequality we get for given $\varepsilon_1, \varepsilon_2, r > 0$, that

$$\int_{B_r(x)} f(y) dy \leq C|x|^{n\alpha} \left(\int_{B_r(x)} e^{2nv} dy \right)^{\frac{1}{2}} \left(\int_{B_r(x)} e^{2np} dy \right)^{\frac{1}{2}}$$

$$\leq C|x|^{n\alpha} \left(\frac{r^{n-\varepsilon_2}}{|x|^{2n(\beta-\varepsilon_1)}} \right)^{\frac{1}{2}} \left(|x|^{n(\beta-\alpha)} \right)^{\frac{1}{2}}$$

$$= C|x|^{c_1} r^{\frac{n}{2} - \frac{\varepsilon_2}{2}}, \quad c_1 := \frac{n(\alpha-\beta)}{2} + n\varepsilon_1.$$

Choosing $r = 2\sqrt{\rho}$, together with Lemma A.2 this yields

$$(I) := \int_{B_\rho(x)} \fint_{B_\rho(x)} \int_{B_{2\sqrt{\rho}}(x)} \left| \log\left(\frac{|z-\xi|}{|y-\xi|}\right) \right| f(\xi) d\xi dzdy$$

$$\leq C\rho^n \int_{B_{2\sqrt{\rho}}(x)} f(\xi) d\xi$$

$$\leq C\rho^{\frac{5n}{4} - \frac{\varepsilon_2}{4}} |x|^{c_1}.$$

Now choosing $\varepsilon_2 \leq \frac{n}{4}$ and taking $|x|$ sufficiently large, we have $\frac{n}{4} - \frac{\varepsilon_2}{4} - \frac{1}{\log|x|} > \frac{n}{8}$. Then, for $\rho \in (0, |x|^{-\frac{8c_1}{n}})$, we further bound

$$(I) \leq C\rho^{n + \frac{1}{\log|x|}} \rho^{\frac{n}{4} - \frac{\varepsilon_2}{4} - \frac{1}{\log|x|}} |x|^{c_1}$$

$$= o(1)\rho^{n + \frac{1}{\log|x|}} \rho^{\frac{n}{8}} |x|^{c_1}$$

$$= o(1)\rho^{n + \frac{1}{\log|x|}},$$

and with Lemma A.2 part *ii)*

$$(II) := \int_{B_\rho(x)} \fint_{B_\rho(x)} \int_{B_{2\sqrt{\rho}}(x)^c} \left| \log\left(\frac{|z-\xi|}{|y-\xi|}\right) \right| f(\xi) d\xi dzdy \leq C\rho^{n + \frac{1}{2}} = o(1)\rho^{n + \frac{1}{\log|x|}}.$$

For $\rho \in (|x|^{-\frac{8c_1}{n}}, 1)$ we write

$$\int_{B_\rho(x)} \fint_{B_\rho(x)} \int_{\mathbb{R}^n} \left| \log\left(\frac{|z-\xi|}{|y-\xi|}\right) \right| f(\xi) d\xi dzdy = I_1 + I_2,$$

where

$$I_i := \int_{B_\rho(x)} \fint_{B_\rho(x)} \int_{A_i} \left| \log\left(\frac{|z-\xi|}{|y-\xi|}\right) \right| f(\xi) d\xi dzdy, \quad A_1 := B_{\frac{|x|}{2}}(x), \ A_2 = A_1^c.$$

Using Lemma A.2 we bound the I_i's as

$$I_1 \leq C\rho^n \int_{A_1} f(\xi)d\xi = o(1)\rho^n = o(1)\rho^{n+\frac{1}{\log|x|}},$$

and

$$I_2 = o(1)\rho^n = o(1)\rho^{n+\frac{1}{\log|x|}},$$

where we have used that

$$\|f\|_{L^1(A_1)} \xrightarrow{|x|\to\infty} 0, \quad |x|^{\frac{1}{\log|x|}} = e, \quad dist(A_2, x) \xrightarrow{|x|\to\infty} \infty.$$

This proves (16).

For the proof of (17) we essentially follow Theorem 5.5 of [12]. Given $x \in \mathbb{R}^n$ and $\rho > 0$ we use the notation

$$v_{x,\rho} := \fint_{B_\rho(x)} v(y)dy.$$

Fix

$$\sigma = \frac{1}{\log(|x_0| + 1)} \in (0, 1) \quad \text{and} \quad \lambda = n + \sigma.$$

For $0 < r < R \leq 4$, $x \in B_1(x_0)$ and $z \in B_R(x)$ we have

$$|v_{x,r} - v_{x,R}| \leq |v(z) - v_{x,r}| + |v(z) - v_{x,R}|,$$

and integrating with respect to z we bound

$$\begin{aligned}
|v_{x,R} - v_{x,r}| &\leq \frac{1}{|B_r|} \left(\int_{B_r(x)} |v(z) - v_{x,r}|dz + \int_{B_R(x)} |v(z) - v_{x,R}|dz \right) \\
&\leq \frac{C}{r^n} \left(r^\lambda + R^\lambda \right) \tau(x) \\
&\leq CR^\lambda r^{-n} \tau(x).
\end{aligned}$$

Setting $R_k = \frac{R}{2^k}$ we infer

$$|v_{x,R_k} - v_{x,R_{k+1}}| \leq CR^\sigma 2^{-k\sigma} \tau(x).$$

Applying the triangular inequality for $h > k$ we bound

$$|v_{x,R_h} - v_{x,R_k}| \leq \sum_{j=k}^{h-1} |v_{x,R_{j+1}} - v_{x,R_j}| \leq CR^\sigma \sum_{j=k}^{h-1} 2^{-j\sigma} \tau(x) \leq \frac{CR^\sigma}{1 - 2^{-\sigma}} \tau(x).$$

Since the function $s \mapsto \frac{s}{1-2^{-s}}$ is increasing in $[0, 1]$ one has $\frac{1}{1-2^{-\sigma}} \leq \frac{2}{\sigma}$ and we get

$$|v_{x,R_h} - v_{x,R_k}| \leq \frac{C}{\sigma} R^\sigma \tau(x) \qquad 0 \leq k < h.$$

Taking $k = 0$ and letting $h \to \infty$ we now obtain

$$|v_{x,R} - v(x)| \leq \frac{C}{\sigma} R^\sigma \tau(x), \quad x \in B_1(x_0). \tag{18}$$

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For $x, y \in B_1(x_0)$ with $x \neq y$, take $R = |x - y|$. Then with (18) and the triangle inequality we bound

$$|v(x) - v(y)| \leq |v_{x,2R} - v(x)| + |v_{x,2R} - v_{y,2R}| + |v_{y,2R} - v(y)|$$
$$\leq \frac{CR^\sigma(\tau(x) + \tau(y))}{\sigma} + |v_{x,2R} - v_{y,2R}|. \tag{19}$$

For any $z \in \mathbb{R}^n$, we have

$$|v_{x,2R} - v_{y,2R}| \leq |v_{x,2R} - v(z)| + |v_{y,2R} - v(z)|.$$

Integrating as $z \in B_{2R}(x) \cap B_{2R}(y)$ we get

$$|v_{x,2R} - v_{y,2R}| \leq \frac{1}{|B_{2R}(x) \cap B_{2R}(y)|} \left(\int_{B_{2R}(x)} |v(z) - v_{x,2R}| dz + \int_{B_{2R}(y)} |v(z) - v_{y,2R}| dz \right)$$
$$\leq \frac{CR^\lambda(\tau(x) + \tau(y))}{R^n}$$
$$\leq CR^\sigma(\tau(x) + \tau(y)).$$

From (16) and (19) we finally infer

$$\frac{|v(x) - v(y)|}{|x - y|^\sigma} \leq \frac{C}{\sigma}(\tau(x) + \tau(y)) = o(1) \log(|x_0| + 1).$$

$\square$

Proof of Theorem 1.1. First we prove that (8) holds. From Lemma 3.1 we have

$$\liminf_{|x| \to \infty} \frac{v(x)}{\log |x|} \geq -\beta.$$

We assume by contradiction that

$$\lim_{|x| \to \infty} \frac{v(x)}{\log |x|} \neq -\beta.$$

Then there exists a sequence of points (x_k) in $\mathbb{R}^n$ such that $|x_k| \to \infty$ and

$$v(x_k) \geq (-\beta + 2\delta) \log |x_k| \quad \text{for some } \delta > 0.$$

Indeed, for $|x_k|$ large, by Lemma 3.6

$$v(x) = v(x_k) + o(1) \log |x_k| \geq (-\beta + \delta) \log |x_k| \quad \text{for } x \in B_1(x_k).$$

Hence

$$\lim_{k \to \infty} |x_k|^{\beta - \delta} \int_{B_1(x_k)} e^{v(x)} dx \geq \lim_{k \to \infty} |x_k|^{\beta - \delta} \int_{B_1(x_k)} e^{(-\beta + \delta) \log |x_k|} dx = |B_1|.$$

This contradicts Lemma 3.5 with $\rho = 1$, $q = 1$ and $0 < \varepsilon_1 < \delta$. Thus (8) is proved.

It remains to show that $\Lambda = \Lambda_1(1 + \alpha)$ if p is constant. In this case we have

$$u(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1 + |y|}{|x - y|} \right) |y|^{n\alpha} e^{nu(y)} dy + C.$$

Then, we are in position to apply the Pohozaev-type identity of Proposition A.2 to conclude that $\Lambda = \Lambda_1(1 + \alpha)$.

$\square$

4 Proof of Theorem 1.2

When $\Lambda \in (0, \Lambda_1(1 + \alpha))$ we will look for solutions of the form $u(x) = v(x) - |x|^2 + c$ where $c \in \mathbb{R}$ and v satisfies the integral equation

$$v(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x - y|}\right) |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c)} dy, \tag{20}$$

so that in particular

$$(-\Delta)^{\frac{n}{2}} v(x) = |x|^{n\alpha} e^{-n|x|^2} e^{n(v(x)+c)}.$$

Our approach will be based on Schauder's fixed-point theorem (see [13, Theorem 11.3]), an idea already exploited in several works on the construction of entire solutions. More precisely we set

$$X := \left\{ v \in C^0_{rad}(\mathbb{R}^n) : \|v\|_X < \infty \right\}, \quad \|v\|_X := \sup_{x \in \mathbb{R}^n} \frac{|v(x)|}{1 + |x|}.$$

For $v \in X$ we set $c_v \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c_v)} dy = \Lambda,$$

and define $T = T_\Lambda : X \to X$, $Tv = \bar{v}$ where

$$\bar{v}(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x - y|}\right) |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c_v)} dy.$$

Lemma 4.1. *The operator $T : X \to X$ is compact.*

Proof. Continuity follows by dominated convergence. Let now $(v_k) \subset X$ be a bounded sequence. From the definition of c_{v_k} it follows easily that $|c_{v_k}| \leq C$. Therefore,

$$|\bar{v}_k(x)| \leq C \int_{\mathbb{R}^n} |\log|x - y|| |y|^{n\alpha} e^{-|y|^2} dy \leq C \log(2 + |x|). \tag{21}$$

Moreover,

$$|\bar{v}_k(x) - \bar{v}_k(z)| \leq C \int_{\mathbb{R}^n} \left| \log\left(\frac{|z - y|}{|x - y|}\right) \right| |y|^{n\alpha} e^{-|y|^2} dy \to 0, \quad \text{as } |x - z| \to 0, \ x, z \in \mathbb{R}^n.$$

Thus, the sequence $(\bar{v}_k)$ is equicontinuous on $\mathbb{R}^n$. Hence, by the theorem of Arzelà-Ascoli, up to a subsequence, $\bar{v}_k \to v$ in $C^0_{loc}(\mathbb{R}^n)$ for some $v \in C^0(\mathbb{R}^n)$. In particular, $\bar{v}_k \to v$ in X, thanks to (21). $\square$

Lemma 4.2. *The function $\bar{v}$ is radially decreasing.*

Proof. Consider the functions

$$\bar{v}_\varepsilon(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n \setminus B_\varepsilon} \log\left(\frac{1}{|x - y|}\right) |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c_v)} dy.$$

Differentiating under the integral sign one gets $\Delta \bar{v}_\varepsilon < 0$, which implies that $\bar{v}_\varepsilon$ is radially decreasing. Letting now $\varepsilon \to 0$ we get $\bar{v}_\varepsilon \to \bar{v}$ by dominated convergence, hence $\bar{v}$ is radially decreasing. $\square$

Lemma 4.3. *There exist constants $C, C(s) > 0$, for $s > 0$, such that*

$$\int_{B_R} |\Delta \bar{v}| dx \leq C R^{n-2}, \quad \int_{B_R^c} \frac{|\Delta \bar{v}|}{|x|^{n+s}} dx \leq \frac{C(s)}{R^{s+2}}, \tag{22}$$

for any $v \in X$ and $R > 0$.

Proof. For any $y \in \mathbb{R}^n$ we have

$$\int_{B_R} \frac{1}{|x-y|^2} dx \leq \int_{B_R(y)} \frac{1}{|x-y|^2} dx + \int_{B_R \cap B_R(y)^c} \frac{1}{|x-y|^2} dx \leq C R^{n-2}.$$

Then writing by Fubini-Tonelli's theorem

$$\int_{B_R} |\Delta \bar{v}| dx \leq C \int_{\mathbb{R}^n} |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c_v)} \left(\int_{B_R} \frac{1}{|x-y|^2} dx \right) dy$$

$$\leq C R^{n-2} \Lambda$$

the first estimate in (22) follows. The second one is proven in the same way since

$$\int_{B_R^c} \frac{1}{|x-y|^2 |x|^{n+s}} dx \leq \int_{B_R(y)} \frac{1}{|x-y|^2 R^{n+s}} dx + \int_{B_R^c \cap B_R(y)^c} \frac{1}{R^2 |x|^{n+s}} dx \leq \frac{C}{R^{s+2}}.$$

$\square$

We are now in a position to prove the main a priori estimate.

Proposition 4.1. *Assume that $\Lambda \in (0, \Lambda_1(1+\alpha))$. Then there exists $C > 0$ such that for every $(v, t) \in X \times [0, 1]$ satisfying $v = tT(v)$ we have $\|v\| \leq C$.*

Proof. Assume by contradiction that there exists $(v_k, t_k) \in X \times [0, 1]$ such that

$$v_k = t_k T(v_k) \quad \text{and} \quad \|v_k\| \to \infty.$$

Then v_k satisfies the integral equation

$$v_k(x) := \frac{t_k}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) |y|^{n\alpha} e^{-n|y|^2} e^{n(v_k(y)+c_{v_k})} dy. \tag{23}$$

We observe that if

$$w_k := v_k + c_{v_k} + \frac{1}{n} \log t_k \leq C \quad \text{on } \mathbb{R}^n,$$

then from (23)

$$|v_k(x)| \leq \frac{e^{nC}}{\gamma_n} \int_{\mathbb{R}^n} |\log|x-y|| |y|^{n\alpha} e^{-n|y|^2} dy \leq C \log(2 + |x|),$$

a contradiction to our assumption $\|v_k\| \to \infty$. Thus, together with Lemma 4.2, we have

$$\max_{\mathbb{R}^n} w_k = w_k(0) \to \infty.$$

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We set
$$\eta_k(x) = w_k(r_k x) - w_k(0), \quad r_k^{1+\alpha} := e^{-w_k(0)} \to 0.$$

Then, on compact sets we get
$$(-\Delta)^{\frac{n}{2}} \eta_k(x) = |x|^{n\alpha} e^{-n|r_k x|^2} e^{n\eta_k(x)} = (1 + o(1))|x|^{n\alpha} e^{n\eta_k(x)}.$$

Moreover by Lemma 4.3 we obtain
$$\int_{B_R} |\Delta \eta_k(x)| dx \le C R^{n-2}, \quad \int_{\mathbb{R}^n \setminus B_R} \frac{|\Delta \eta_k(x)|}{|x|^{2n-2}} dx \le \frac{C}{R^n}. \tag{24}$$

Then by elliptic estimates, Proposition A.3 in the appendix, up to a subsequence, $\eta_k \to \eta$ locally uniformly $\mathbb{R}^n$ and in $C^{n-1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Note further that we have

$$\int_{\mathbb{R}^n} |x|^{n\alpha} e^{n\eta(x)} dx = \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R} |x|^{n\alpha} e^{n\eta_k(x)} dx$$
$$= \lim_{R \to \infty} \lim_{k \to \infty} t_k \int_{B_{R r_k}} |x|^{n\alpha} e^{-n|x|^2} e^{n(v_k(x) + c_{v_k})} dx$$
$$\le \Lambda t_\infty. \tag{25}$$

where, up to a subsequence, $t_\infty := \lim_{k \to \infty} t_k$. Thus $t_\infty \ne 0$ and
$$\int_{\mathbb{R}^n} |x|^{n\alpha} e^{n\eta} dx < \infty.$$

We claim that η is a normal solution to $(-\Delta)^{\frac{n}{2}} \eta = |x|^{n\alpha} e^{n\eta}$, namely

$$\eta(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1 + |y|}{|x - y|}\right) |y|^{n\alpha} e^{n\eta(y)} dy + c, \tag{26}$$

for some $c \in \mathbb{R}$. Since $n \ge 3$ we have that $\Delta \eta_k \to \Delta \eta$ locally uniformly in $\mathbb{R}^n \setminus \{0\}$ and by the first estimate in (24) we conclude that $\Delta \eta_k \to \Delta \eta$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. Then, also using the second estimate in (24) and recalling that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have
$$|(-\Delta)^{\frac{n-2}{2}} \varphi| \le C(1 + |x|)^{-2n+2},$$

we get

$$\int_{\mathbb{R}^n} (-\Delta \eta)(-\Delta)^{\frac{n-2}{2}} \varphi dx = \lim_{k \to \infty} \int_{\mathbb{R}^n} (-\Delta \eta_k)(-\Delta)^{\frac{n-2}{2}} \varphi dx$$
$$= \lim_{k \to \infty} \int_{\mathbb{R}^n} \eta_k (-\Delta)^{\frac{n}{2}} \varphi dx$$
$$= \lim_{k \to \infty} \int_{\mathbb{R}^n} |x|^{n\alpha} e^{-n|r_k x|^2} e^{n\eta_k} \varphi dx$$
$$= \int_{\mathbb{R}^n} |x|^{n\alpha} e^{n\eta} \varphi dx. \tag{27}$$

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In order to prove that (27) implies (26) (compare to Definitions 1.1 and 1.2), we set

$$\tilde{\eta}(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) |y|^{n\alpha} e^{n\eta(y)} dy, \quad p := \eta - \tilde{\eta}.$$

Then p satisfies

$$\int_{B_R} |\Delta p| dx \le CR^{n-2}, \quad \int_{\mathbb{R}^n} \frac{|\Delta p|}{1+|x|^{2n-2}} dx < \infty, \quad \int_{\mathbb{R}^n} \Delta p (-\Delta)^{\frac{n-2}{2}} \varphi \, dx = 0,$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Hence $\Delta p \in L_{\frac{n-2}{2}}(\mathbb{R}^n)$ is $\frac{n-2}{2}$-harmonic in $\mathbb{R}^n$, which implies that Δp is a polynomial (see e.g. the proof of [18, Lemma 2.4]). Now the estimate $\int_{B_R} |\Delta p| dx \le CR^{n-2}$ gives $\Delta p \equiv 0$. Since $\tilde{\eta}(x) \ge -C \log(2+|x|)$ on $\mathbb{R}^n$ (see the proof of Lemma 3.1), and $|\cdot|^{n\alpha} e^{n\eta} \in L^1(\mathbb{R}^n)$, by Theorem A.1 we get $p \equiv const$, and (26) follows.

From the Pohozaev identity of Proposition A.2 (case $\mu = 0$) and (25), we infer

$$\Lambda_1(1+\alpha) = \int_{\mathbb{R}^n} |x|^{n\alpha} e^{n\eta} dx \le \Lambda,$$

which contradicts our assumption $\Lambda < \Lambda_1(1+\alpha)$. $\qquad\square$

Proof of Theorem 1.2 (completed). Assume first $\Lambda \in (0, \Lambda_1(1+\alpha))$. Thanks to Proposition 4.1 we can apply Schauder's theorem, hence T has a fixed point v. Then the function $u(x) = v(x) - |x|^2 + c_v$ satisfies (3), as wished.

Now we consider the case $\Lambda = \Lambda_1(1+\alpha)$. We fix a sequence $\Lambda_k \uparrow \Lambda$, and for each k we apply the previous procedure to find v_k fixed point of the corresponding T_{Λ_k}. We claim that

$$v_k(0) + c_{v_k} \to \infty. \tag{28}$$

Otherwise, from (23) we would infer that v_k satisfies $|v_k(x)| \le C \log(2+|x|)$ on $\mathbb{R}^n$. Then from the definition of c_{v_k} we get $|c_{v_k}| \le C$. Moreover (v_k) is equicontinuous on $\mathbb{R}^n$, and therefore, up to a subsequence, $v_k \to v$ locally uniformly in $\mathbb{R}^n$. The limit function v satisfies

$$v(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c)} dy,$$

where $c_{v_k} \to c$ and

$$\int_{\mathbb{R}^n} |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c)} dy \le \lim_{k\to\infty} \Lambda_k = \Lambda_1(1+\alpha),$$

a contradiction to Proposition A.2 (case $\mu = n$). This proves (28).

Setting

$$\eta_k(x) = v_k(r_k x) - v_k(0), \quad r_k^{1+\alpha} := e^{-v_k(0) - c_{v_k}} \to 0,$$

as in the proof of Lemma 4.1, one obtains $\eta_k \to \eta$, where η is a normal solution to (3). $\qquad\square$

5 Proof of Theorem 1.3

We set

$$X := \left\{ v \in C^0(\mathbb{R}^n) : \|v\| < \infty \right\}, \quad \|v\| := \sup_{x \in \mathbb{R}^n} \frac{|v(x)|}{1 + |x|}.$$

We fix $\Lambda \in (0, \Lambda_1)$ and $\alpha \in (-1, \infty)$. For $v \in X$ let $c_v \in \mathbb{R}$ be determined by

$$\int_{\mathbb{R}^n} |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c_v)} dy = \Lambda, \tag{29}$$

and let $v^* \in \mathbb{R}$ be given by

$$v^* := \sup_{|x| \leq 1} e^{\frac{2}{1+\alpha}(v(x)+c_v)}.$$

We define $T : X \to X$, $v \mapsto \bar{v}$ where

$$\bar{v}(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c_v)} dy + v^* x_1.$$

Notice that

$$(-\Delta)^{\frac{n}{2}} \bar{v}(x) = |x|^{n\alpha} e^{n(v(x)-|x|^2+c_v)}.$$

We will look for solutions of the form

$$u(x) = v(x) - |x|^2 + c_v.$$

Lemma 5.1. *The operator $T : X \to X$ is compact.*

Proof. As in Lemma 4.1 continuity follows by dominated convergence. Moreover given a bounded sequence $(v_k) \subset X$, from the definition of c_{v_k} it follows that $|c_{v_k}| \leq C$. This in turn implies $|v_k^*| \leq C$. If we set $\tilde{v}_k(x) := \bar{v}_k(x) - v_k^* x_1$, we get $|\tilde{v}_k(x)| \leq C \log(2 + |x|)$, and the sequence $(\tilde{v}_k)$ is equicontinuous, with the same proof as in Lemma 4.1. In particular, up to a subsequence, $\tilde{v}_k \to \tilde{v}$ in X. Since, up to a subsequence, $\|v_k^* x_1 - c_0 x_1\| \to 0$ for some $c_0 > 0$, we conclude that $\bar{v}_k \to \tilde{v} + c_0 x_1$ in X. $\qquad\square$

The proof of Theorem 1.3 follows at once from Schauder's fixed-point theorem and the following a priori estimate.

Proposition 5.1. *There exists $C > 0$ such that*

$$\|v\| \leq C \quad \text{for every } (v, t) \in X \times (0, 1] \text{ with } v = tT(v).$$

The proof of Proposition 5.1 will be based on the following three lemmas.

Lemma 5.2. *For every $R > 0$ there exists a constant $C(R) > 0$ such that for every $(t, v) \in (0, 1] \times X$ with $v = tT(v)$, that is*

$$v(x) := \frac{t}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) |y|^{n\alpha} e^{-n|y|^2} e^{n(v(y)+c_v)} dy + tv^* x_1, \tag{30}$$

we have

$$w := v + c_v + \frac{1}{n} \log t \leq C(R) \quad \text{on } B_R.$$

Proof. Assume by contradiction that for a sequence $(t_k, v_k) \in (0, 1] \times X$ such that $v_k = t_k T(v_k)$ one has

$$\max_{\bar{B}_{2R_0}} w_k =: w_k(\xi_k) = v_k(\xi_k) + c_{v_k} + \frac{1}{n} \log t_k \to \infty$$

for some $R_0 > 0$. We set $x_k := \xi_k$ and $s_k = 0$ if $|\xi_k| \nrightarrow 2R_0$, and otherwise we let $x_k \in B_{R_0}(\xi_k)$ and $s_k \in [0, R_0]$ be such that (see [1])

$$(R_0 - s_k)e^{w_k(x_k)} = (R_0 - s_k) \max_{\bar{B}_{s_k}(\xi_k)} e^{w_k} = \max_{s \in [0, R_0]} \left((R_0 - s) \max_{\bar{B}_s(\xi_k)} e^{w_k} \right) =: L_k.$$

Then $L_k \to \infty$, and

$$w_k(x_k + \mu_k x) - w_k(x_k) \leq \log 2 \quad \text{for } |x| \leq \frac{L_k}{2}, \quad \mu_k := \frac{R_0 - s_k}{L_k}. \tag{31}$$

We distinguish the following cases.

Case 1 Up to a subsequence $|x_k|^{1+\alpha} e^{w_k(x_k)} \to c_0 \in [0, \infty)$.

In this case we have $x_k = \xi_k \to 0$, which implies that $w_k \leq w_k(x_k)$ on B_{R_0}. We set

$$\eta_k(x) := w_k(r_k x) - w_k(x_k), \quad r_k^{1+\alpha} := e^{-w_k(x_k)}.$$

It follows from the definition of r_k that $|x_k| = O(r_k)$. Therefore, on any compact set

$$(-\Delta)^{\frac{n}{2}} \eta_k(x) = |x|^{n\alpha} e^{n\eta_k(x)} e^{-n|r_k x|^2} = (1 + o(1)) |x|^{n\alpha} e^{n\eta_k(x)}. \tag{32}$$

Since $\alpha > -1$ and $\eta_k \leq \log 2$ for large k on any compact set, from (32) we obtain

$$\|(-\Delta)^{\frac{n}{2}} \eta_k\|_{L^p(B_R)} \leq C(p, R) \quad \text{for } 1 \leq p < \bar{p},$$

and

$$\|(-\Delta)^{\frac{n}{2}} \eta_k\|_{L^\infty(K)} \leq C(K) \quad \text{for every } K \Subset \mathbb{R}^n \setminus \{0\}.$$

Moreover, differentiating in (30) as in Lemma 4.3 (notice that the part $v^* x_1$ does not play a role in $\Delta \eta_k$), we obtain

$$\int_{B_R} |\Delta \eta_k(x)| dx \leq CR^{n-2}, \quad \int_{\mathbb{R}^n \setminus B_R} \frac{|\Delta \eta_k(x)|}{|x|^{2n-2}} dx \leq \frac{C}{R^n}. \tag{33}$$

We also have $\eta_k(\bar{x}_k) = 0$, where $\bar{x}_k := \frac{x_k}{r_k}$ satisfies $|\bar{x}_k| = O(1)$. Then by Proposition A.3, up to a subsequence, $\eta_k \to \eta$ in $C^0_{\text{loc}}(\mathbb{R}^n) \cap C^{n-1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ and $\Delta \eta_k \to \Delta \eta$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ for some η.

Then, with the same argument of Lemma 4.1 we obtain that η satisfies the integral equation

$$\eta(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1 + |y|}{|x - y|} \right) |y|^{n\alpha} e^{n\eta(y)} dy + c. \tag{34}$$

In particular, differentiating (34) we obtain that for every $R > 0$

$$\int_{B_R} |\nabla \eta(x)| dx \leq CR^{n-1}.$$

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Using (30) one obtains for every $R > 0$

$$\int_{B_R} |\nabla \eta_k(x) - t_k r_k v_k^* e_1| dx \le t_k \int_{B_R} \int_{\mathbb{R}^n} \frac{r_k}{|x_k + r_k x - y|} |y|^{n\alpha} e^{-n|y|^2} e^{n(v_k(y) + c_{v_k})} dy dx$$
$$\le CR^{n-1}.$$

Therefore, for every $R > 0$

$$\lim_{k\to\infty} t_k r_k v_k^* |B_R| \le \lim_{k\to\infty} \int_{B_R} \left(|\nabla \eta_k(x) - t_k r_k v_k^* e_1| + |\nabla \eta_k(x)| \right) dx \le CR^{n-1}.$$

This shows that

$$\lim_{k\to\infty} t_k r_k v_k^* = 0.$$

Since $t_k \to t_\infty \ne 0$, we must have

$$\lim_{k\to\infty} r_k v_k^* = 0.$$

This is a contradiction since from $\liminf_{k\to\infty} t_k > 0$ we infer $v_k^* \ge \frac{1}{C} e^{\frac{2}{1+\alpha} w_k(x_k)}$, hence

$$r_k v_k^* \ge \frac{1}{C} e^{\frac{1}{1+\alpha} w_k(x_k)} \to \infty.$$

Case 2 Up to a subsequence $|x_k|^{1+\alpha} e^{w_k(x_k)} \to \infty$.

We set

$$\eta_k(x) = w_k(x_k + r_k x) - w_k(x_k),$$

where

$$r_k := |x_k|^{-\alpha} e^{-w_k(x_k)}.$$

Notice that by (31) for every $R > 0$ we have $\eta_k(x) \le \log 2$ on B_R for $k \ge k_0(R)$. Moreover $r_k = o(|x_k|)$ and we compute

$$(-\Delta)^{\frac{n}{2}} \eta_k(x) = e^{-n|x_0|^2} (1 + o(1)) \left| \frac{x_k}{|x_k|} + \frac{r_k}{|x_k|} x \right|^{n\alpha} e^{n\eta_k} = (c_0 + o(1)) e^{n\eta_k},$$

on compact sets, where $x_k \to x_0$ and $c_0 := e^{-n|x_0|^2}$. Then, similarly to Case 1, we obtain $\eta_k \to \eta$ where η is a normal solution to

$$(-\Delta)^{\frac{n}{2}} \eta = c_0 e^{n\eta} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^{n\eta} dx < \infty,$$

that is, η is a spherical, a contradiction as $\Lambda < \Lambda_1$. $\qquad\square$

Lemma 5.3. *There exists $C > 0$ such that for every $(t, v) \in (0, 1] \times X$ such that $v = tT(v)$ one has $v^* \le C$ and $c_v \le C$.*

Proof. Take $(t, v) \in (0, 1] \times X$ be such that $v = tT(v)$ and let w be as in Lemma 5.2. For $|x| \leq 1$ we obtain from (30)

$$v(x) - tv^*x_1 = \frac{1}{\gamma_n} \left(\int_{|y|<2} + \int_{|y|>2} \right) \log \left(\frac{1 + |y|}{|x - y|} \right) |y|^{n\alpha} e^{-n|y|^2} e^{nw(y)} dy$$

$$= O(1) \int_{|y|<2} \log \left(\frac{1 + |y|}{|x - y|} \right) |y|^{n\alpha} dy + O(1) \int_{|y|>2} |y|^{n\alpha} e^{-n|y|^2} e^{nw(y)} dy$$

$$= O(1), \tag{35}$$

where the last equality follows from (29), while in the second inequality we have used that $w \leq C$ on B_2 thanks to Lemma 5.2, and the estimate $\log \left(\frac{1+|y|}{|x-y|} \right) = O(1)$ for $(x, y) \in B_1 \times B_2^c$. Therefore,

$$v(x) + c_v = c_v + tv^*x_1 + O(1) \quad \text{on } B_1.$$

This and (29) imply that $v + c_v \leq C$ on B_1, which is equivalent to $v^* \leq C$. In particular, from (35), we have $v = O(1)$ in B_1, and (29) yields $c_v \leq C$. $\square$

Lemma 5.4. *For every $\varepsilon > 0$ there exists $R > 0$ such that the following holds: Given $(t, v) \in (0, 1] \times X$ with $v = tT(v)$ one has*

$$\int_{|x|>R} e^{-n|x|^2} |x|^{n\alpha} e^{nw(x)} dx < \varepsilon,$$

where $w := v + c_v + \frac{1}{n} \log t$.

Proof. We use a Pohozaev type identity for the integral equation

$$\tilde{w}(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log \left(\frac{1}{|x - y|} \right) e^{-n|y|^2 - tv^*y_1} |y|^{n\alpha} e^{n\tilde{w}(y)} dy + nc_v + \frac{1}{n} \log t,$$

where $\tilde{w} := w + tv^*x_1$. Since $(t, v) \in (0, 1] \times X$, we can apply Proposition A.1 to get

$$t\Lambda(t\Lambda - \Lambda_1) = c \int_{\mathbb{R}^n} y \cdot \nabla \left(e^{-n|y|^2 - tv^*y_1} |y|^{n\alpha} \right) e^{n\tilde{w}(y)} dy$$

$$= cn\alpha \int_{\mathbb{R}^n} |y|^{n\alpha} e^{-n|y|^2} e^{nw(y)} dy - c \int_{\mathbb{R}^n} (2n|y|^2 + tv^*y_1) e^{-n|y|^2} |y|^{n\alpha} e^{nw(y)} dy$$

$$= cn\alpha t\Lambda - c \left(\int_{B_{R_0}} + \int_{B_{R_0}^c} \right) (2n|y|^2 + tv^*y_1) e^{-n|y|^2} |y|^{n\alpha} e^{nw(y)} dy$$

$$=: cn\alpha t\Lambda - c(I_1 + I_2),$$

where $R_0 > 0$ is such that $2n|y|^2 + tv^*y_1 \geq |y|^2$ on $B_{R_0}^c$. We observe that $|I_1| \leq C(R_0, \Lambda, n)$,

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thanks to Lemma 5.2 and the estimate $v^* \leq C$ of Lemma 5.3. Therefore, for $R > R_0$ we obtain

$$
\begin{aligned}
\int_{B_R^c} e^{-n|y|^2} |y|^{n\alpha} e^{nw(y)} dy &\leq \frac{1}{R^2} \int_{B_R^c} |y|^2 e^{-n|y|^2} |y|^{n\alpha} e^{nw(y)} dy \\
&\leq \frac{1}{R^2} \int_{B_R^c} (2n|y|^2 + tv^* y_1) e^{-n|y|^2} |y|^{n\alpha} e^{nw(y)} dy \\
&\leq \frac{1}{R^2} \int_{B_{R_0}^c} (2n|y|^2 + tv^* y_1) e^{-n|y|^2} |y|^{n\alpha} e^{nw(y)} dy \\
&\leq \frac{1}{R^2} C(R_0, \Lambda, n).
\end{aligned}
$$

This proves the lemma. □

Proof of Proposition 5.1. Since $v^* \leq C$, thanks to Lemma 5.3, it is sufficient to show that $\tilde{v} := v - tv^* x_1$ is bounded in X (we want to show that $|\tilde{v}(x)| \leq C \log(2 + |x|)$). We have

$$
\tilde{v}(x) := \int_{\mathbb{R}^n} \log \left(\frac{1 + |y|}{|x - y|} \right) K(y) e^{n\tilde{v}(y)} dy, \quad K(y) := \frac{t_k}{\gamma_n} |y|^{n\alpha} e^{-n|y|^2} e^{nc_v + ntv^* y_1}. \tag{36}
$$

As $v^* \leq C$ and $c_v \leq C$, there exists $R > 0$ such that

$$
K(y) \leq e^{-2|y|^2} \quad \text{on } B_R^c. \tag{37}
$$

By Lemma 5.4 we can also assume that

$$
\int_{B_R^c} K e^{n\tilde{v}} dy \leq \frac{1}{4}.
$$

Then, as in Lemma 3.4 one obtains

$$
\tilde{v}(x) \leq \left(-t\Lambda + \frac{1}{4} \right) \log |x| + \int_{B_1(x)} \log \left(\frac{1}{|x - y|} \right) K(y) e^{n\tilde{v}(y)} dy, \quad |x| \geq R. \tag{38}
$$

In the spirit of Lemma 3.5 we get

$$
\int_{B_1(x_0)} e^{2n\tilde{v}(z)} dz \leq C |x_0|^{\frac{1}{4}}, \quad |x_0| \geq R + 2.
$$

By (37) and Hölder inequality, from (38)

$$
\tilde{v}(x) \leq \frac{1}{4} \log |x| + C, \quad |x| \geq R + 2.
$$

Therefore,

$$
|\tilde{v}(x)| \leq C \int_{\mathbb{R}^n} \left| \log \left(\frac{1 + |y|}{|x - y|} \right) \right| e^{-|y|^2} dy \leq C \log(2 + |x|),
$$

thanks to Lemma 5.2. □

A Appendix

A.1 A Pohozaev-type result

Proposition A.1. *Assume that $K \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) \cap L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p > 1$, $n \geq 2$. Let η be a solution to the integral equation*

$$\eta(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1 + |y|}{|x - y|}\right) K(y) e^{n\eta(y)} dy + c \tag{39}$$

for some $c \in \mathbb{R}$, with $Ke^{n\eta} \in L^1(\mathbb{R}^n)$ and $(\nabla K \cdot x) e^{n\eta} \in L^1(\mathbb{R}^n)$. If there exists $R_0, \varepsilon > 0$ such that

$$|K(x)|e^{n\eta(x)} \leq \frac{1}{|x|^{n+\varepsilon}} \quad \text{for } |x| \geq R_0, \tag{40}$$

then, denoting $\Lambda := \int_{\mathbb{R}^n} K(x)e^{n\eta(x)} dx$, we have

$$\frac{\Lambda}{\gamma_n}(\Lambda - 2\gamma_n) = \frac{2}{n}\int_{\mathbb{R}^n}(x \cdot \nabla K(x))\, e^{n\eta(x)} dx, \tag{41}$$

Proof. Noticing that $\eta \in C^0(\mathbb{R}^n) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, in the spirit of [29, Theorem 1.1], for any $R > 0$ we can multiply $\nabla\eta$ by $x \cdot \nabla\eta$ and integrate on B_R using the divergence theorem to get

$$\int_{B_R} Ke^{n\eta} dx + \frac{1}{n}\int_{B_R}(x \cdot \nabla K(x))\, e^{n\eta} dx - \frac{R}{n}\int_{\partial B_R} Ke^{n\eta} d\sigma(x)$$
$$= \frac{1}{2\gamma_n}\int_{B_R}\int_{B_R^c} \frac{(x - y) \cdot (x + y)}{|x - y|^2} K(y)e^{n\eta(y)}K(x)e^{n\eta(x)} dy dx$$
$$+ \frac{1}{2\gamma_n}\int_{\mathbb{R}^n} K(y)e^{n\eta(y)} dy \int_{B_R} K(x)e^{n\eta(x)} dx.$$

It follows from (40) that the boundary term and the double integral term go to 0 as $R \to \infty$. Therefore, taking $R \to \infty$, we obtain (41).

$\square$

We are interested in the following special cases of the above proposition.

Proposition A.2. *Given $n \geq 2$, $\alpha > -1$, $\mu \geq 0$ let η be a solution to (39) with $c \in \mathbb{R}$, $K := |\cdot|^{n\alpha}e^{-\mu|\cdot|^2}$ and*

$$\Lambda := \int_{\mathbb{R}^n} K(x)e^{n\eta(x)} dx < +\infty.$$

Then, $\Lambda \leq \Lambda_1(1 + \alpha)$ and the equality holds if and only if $\mu = 0$.

Proof. First, we claim that (40) holds. If $\mu = 0$, then η is a normal solution to $(-\Delta)^{\frac{n}{2}}\eta = |x|^{n\alpha}e^{n\eta}$ and the classification part of Theorem 1.1 implies that

$$\lim_{|x|\to\infty} \frac{\eta(x)}{\log|x|} = -\beta, \quad \beta := \frac{\Lambda}{\gamma_n}. \tag{42}$$

Moreover, arguing as in Lemma 3.1 we get

$$v(x) \geq -\beta \log |x| + c, \quad |x| \geq 1, \tag{43}$$

and from $Ke^{n\eta} \in L^1$ we find that $\beta > 1 + \alpha$. Then (40) follows at once form (42). If $\mu > 0$, we get (42) from Remark 3.2 and the function $Ke^{n\eta}$ decays exponentially at infinity, so that (40) trivially holds. Observe now that the condition $(\nabla K \cdot x)e^{n\eta} \in L^1(\mathbb{R}^n)$ is satisfied, since $\nabla K \cdot x = n\alpha K$ for $\mu = 0$, and since $(\nabla K \cdot x)e^{n\eta}$ decays exponentially for $\mu > 0$ thanks to (42). Then, we can apply Proposition A.1 to get

$$\frac{\Lambda}{\gamma_n}(\Lambda - 2\gamma_n) = \frac{2}{n}\int_{\mathbb{R}^n}\left(x \cdot \nabla\left(|x|^{n\alpha}e^{-\mu|x|^2}\right)\right)e^{n\eta(x)}dx$$

$$= 2\alpha\Lambda - \frac{4\mu}{n}\int_{\mathbb{R}^n}|x|^{n\alpha+2}e^{-\mu|x|^2}e^{n\eta(x)}dx$$

$$\leq 2\alpha\Lambda,$$

with equality holding iff $\mu = 0$. Since $\Lambda_1 = 2\gamma_n$, the proof is complete. $\square$

A.2 A Liouville-type theorem

Lemma A.1. *Let $\alpha \in (-1,0)$ and u be a measurable function such that $|\cdot|^{n\alpha}e^{nu} \in L^1(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$ we have*

$$\fint_{B_R(x)} u^+ \, dy \to 0$$

as $R \to \infty$.

Proof. Fix $x \in \mathbb{R}^n$. From $nu^+ \leq e^{nu}$, multiplying and dividing by $|y|^{n\alpha}$ we get

$$n\fint_{B_R(x)} u^+ \, dy \leq \fint_{B_R(x)} e^{nu} \, dy$$

$$\leq \frac{C(R+|x|)^{-n\alpha}}{R^n}\int_{B_R(x)}|y|^{n\alpha}e^{nu} \, dy$$

$$\leq \frac{C(R+|x|)^{-n\alpha}}{R^n},$$

where we used that for $y \in B_R(x)$ we have $|y| \leq R + |x|$ and that $\int_{\mathbb{R}^n}|x|^{n\alpha}e^{nu} \, dx < \infty$. The claim follows letting $R \to \infty$, since $\alpha \in (-1,0)$. $\square$

Theorem A.1. *Let $\alpha > -1$, $m \geq 1$ and consider $h \colon \mathbb{R}^n \to \mathbb{R}$ with $(-\Delta)^{\frac{m}{2}}h = 0$ in the sense of Definition 1.1. If m is even, assume further that $h(x) \leq u + C\log(1+|x|) + C$ for any $x \in \mathbb{R}^n$, with $\int_{\mathbb{R}^n}|x|^{n\alpha}e^{nu} \, dx < \infty$. Then h is a polynomial of degree at most $m - 1$.*

Proof. If $m \geq 1$ is even, the proof is almost identical to the one of Theorem 6 in [23], the only difference being the estimate of the term $\fint_{B_R(x)} u^+ \, dy \to 0$ for $\alpha \in (-1,0)$, which is true thanks to Lemma A.1. In the case $m \geq 1$ odd, notice that $h \in L_{\frac{m}{2}}(\mathbb{R}^n)$ by Definition 1.1. This implies that h is a polynomial of degree at most $m - 1$ (see e.g. the proof of [18, Lemma 2.4]). $\square$

A.3 A non-local elliptic estimate

Proposition A.3. *Assume $n \geq 3$. Let $(u_k) \subset L_{\frac{n}{2}}(\mathbb{R}^n)$ be a sequence of solutions to*

$$(-\Delta)^{\frac{n}{2}} u_k = f_k \quad in \ \mathbb{R}^n$$

for some $f_k \in L^1(\mathbb{R}^n)$ satisfying

$$\|f_k\|_{L^1(\mathbb{R}^n)} \leq C, \quad \|f_k\|_{L^p(B_R)} \leq C, \quad \|f_k\|_{L^\infty(A)} \leq C, \tag{44}$$

for some $p > 1$, $R > 0$ and an open set $A \Subset \mathbb{R}^n \setminus \{0\}$. Assume further that

$$u_k \leq u_k(0) = 0 \ in \ B_R \quad and \quad \int_{B_R} |\Delta u_k| dx \leq C.$$

Then the sequence (u_k) is bounded $C_{\text{loc}}^{0,\sigma_1}(B_R)$ and in $C_{\text{loc}}^{n-1,\sigma_2}(A)$ for some $\sigma_1 = \sigma_1(p) \in (0,1)$ and for every $\sigma_2 \in (0,1)$.

Proof. We set

$$v_k(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) f_k(y) dy, \quad p_k := u_k - v_k. \tag{45}$$

Then by [18, Lemma 2.4] we have that p_k is a polynomial of degree at most $n-1$. It follows from the assumptions on f_k and from (45) that (v_k) is bounded in $C_{\text{loc}}^{0,\sigma_1}(B_R)$ and in $C_{\text{loc}}^{n-1,\sigma_2}(A)$. Therefore, p_k satisfies

$$\sup_{B_R} p_k \leq C(R), \quad |p_k(0)| \leq C, \quad \int_{B_{\frac{R}{2}}} |\Delta p_k| dx \leq C.$$

Hence, (p_k) is bounded in $C_{\text{loc}}^{\ell}(\mathbb{R}^n)$ for every $\ell \in \mathbb{N}$. $\qquad\square$

A.4 Some integral estimates

Lemma A.2. *There exists $C > 0$ such that for every $\rho \in (0,1]$ we have for any $x, \xi \in \mathbb{R}^n$*

i)

$$\int_{B_\rho(x)} \int_{B_\rho(x)} \left| \log\left(\frac{|z-\xi|}{|y-\xi|}\right) \right| dz dy \leq C\rho^{2n}.$$

ii) If $|x - \xi| \geq 2\sqrt{\rho}$ then

$$\int_{B_\rho(x)} \int_{B_\rho(x)} \left| \log\left(\frac{|z-\xi|}{|y-\xi|}\right) \right| dz dy \leq C\rho^{2n+\frac{1}{2}}.$$

iii)

$$\int_{B_\rho(x)} \int_{B_\rho(x)} \left| \log\left(\frac{|z-\xi|}{|y-\xi|}\right) \right| dz dy = o(1)\rho^{2n}, \quad o(1) \xrightarrow{|x-\xi|\to\infty} 0.$$

Proof. Under the rescaling

$$y' = \frac{y - x}{\rho} \qquad z' = \frac{z - x}{\rho} \qquad \xi' := \frac{\xi - x}{\rho},$$

i) will be equivalent to showing

$$\int_{B_1(0)} \int_{B_1(0)} \left| \log \left(\frac{|z' - \xi'|}{|y' - \xi'|} \right) \right| dz' dy' \leq C,$$

independently on x, ξ'.

If $|\xi'| \geq 2$, then it suffices to apply the triangular inequality to get $\left| \log \left(\frac{|z' - \xi'|}{|y' - \xi'|} \right) \right| \leq \log 3$.

On the other hand, for $|\xi'| \leq 2$ one has

$$\int_{B_1(0)} \int_{B_1(0)} \left| \log \left(\frac{|z' - \xi'|}{|y' - \xi'|} \right) \right| dz' dy' \leq \int_{B_3(\xi')} \int_{B_3(\xi')} \left| \log \left(\frac{|z' - \xi'|}{|y' - \xi'|} \right) \right| dz' dy'$$

$$= \int_{B_3(0)} \int_{B_3(0)} \left| \log \left(\frac{|z''|}{|y''|} \right) \right| dz'' dy''$$

$$\leq 2|B_3(0)| \int_{B_3(0)} |\log |z''|| dz'''$$

$$\leq C.$$

To prove *ii)* first notice that for $|x - \xi| \geq 2\sqrt{\rho}$ one has

$$\frac{|z - \xi|}{|y - \xi|} \leq \frac{|y - \xi| + |z - y|}{|y - \xi|} \leq 1 + 2\sqrt{\rho}.$$

Exchanging the role of y and z we find the same estimate for $\frac{|y - \xi|}{|z - \xi|}$. Therefore

$$\left| \log \left(\frac{|z - \xi|}{|y - \xi|} \right) \right| \leq \log(1 + 2\sqrt{\rho}) \leq 2\sqrt{\rho}.$$

The proof follows immediately.

As for the proof of *iii)* it suffices to notice that as $|x - \xi| \to \infty$ we have

$$\frac{|z - \xi|}{|y - \xi|} \to 1 \quad \text{uniformly for } y, z \in B_\rho(x) \subset B_1(x).$$

$\square$

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