ON COMPLETE INTERSECTIONS CONTAINING A LINEAR SUBSPACE

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Abstract. Consider the Fano scheme $F_k(Y)$ parameterizing k–dimensional linear subspaces contained in a complete intersection $Y \subset \mathbb{P}^m$ of multi–degree $\underline{d} = (d_1, \ldots, d_s)$. It is known that, if $t := \sum_{i=1}^{s} \binom{d_i+k}{k} - (k+1)(m-k) \leqslant 0$ and $\Pi_{i=1}^{s} d_i > 2$, for Y a general complete intersection as above, then $F_k(Y)$ has dimension $-t$. In this paper we consider the case $t > 0$. Then the locus $W_{\underline{d},k}$ of all complete intersections as above containing a k–dimensional linear subspace is irreducible and turns out to have codimension t in the parameter space of all complete intersections with the given multi–degree. Moreover, we prove that for general $[Y] \in W_{\underline{d},k}$ the scheme $F_k(Y)$ is zero–dimensional of length one. This implies that $W_{\underline{d},k}$ is rational.

1. Introduction

In this paper we will be concerned with the *Fano scheme* $F_k(Y)$, parameterizing k–dimensional linear subspaces contained in a subvariety $Y \subset \mathbb{P}^m$, when Y is a complete intersection of multi–degree $\underline{d} = (d_1, \ldots, d_s)$, with $1 \leqslant s \leqslant m-2$. We will assume that Y is neither a linear subspace nor a quadric, cases to be considered as trivial. Thus we will constantly assume that $\Pi_{i=1}^{s} d_i > 2$.

Let $S_{\underline{d}} := \bigoplus_{i=1}^{s} H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i))$, and consider its Zariski open subset $S_{\underline{d}}^* := \bigoplus_{i=1}^{s} \left(H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i)) \setminus \{0\}\right)$. For any $u := (g_1, \ldots, g_s) \in S_{\underline{d}}^*$, let $Y_u := V(g_1, \ldots, g_s) \subset \mathbb{P}^m$ denote the closed subscheme defined by the vanishing of the polynomials $g_1, \ldots, g_s$. When $u \in S_{\underline{d}}^*$ is general, Y_u is a smooth, irreducible variety of dimension $m - s \geqslant 2$, so that $S_{\underline{d}}^*$ contains an open dense subset parametrizing s–tuples u such that Y_u is a smooth complete intersection. For any integer $k \geqslant 1$, we define the locus

$$W_{\underline{d},k} := \left\{ u \in S_{\underline{d}}^* \,\middle|\, F_k(Y_u) \neq \emptyset \right\} \subseteq S_{\underline{d}}^*$$

and set

$$t(m,k,\underline{d}) := \sum_{i=1}^{s} \binom{d_i+k}{k} - (k+1)(m-k).$$

If no confusion arises, we will simply denote $t(m,k,\underline{d})$ by t.

First of all, consider the case $t \leqslant 0$. This is the most studied case in the literature, and it is now well understood (cf. e.g. [2, 3, 6, 7]). In particular, the following holds.

Result 1. *Let m, k, s and $\underline{d} = (d_1, \ldots, d_s)$ be such that $\Pi_{i=1}^{s} d_i > 2$ and $t \leqslant 0$. Then:*
(a) $W_{\underline{d},k} = S_{\underline{d}}^*$;
(b) for general $u \in S_{\underline{d}}^*$, $F_k(Y_u)$ is smooth, of dimension $\dim(F_k(Y_u)) = -t$ *and it is irreducible when* $\dim(F_k(Y_u)) \geqslant 1$.

The proof of this result can be found e.g. in [2, Prop.2.1, Cor.2.2, Thm. 4.1], for the complex case, and in [3, Thm. 2.1, (b) & (c)], for any algebraically closed field. In addition, in [3, Thm. 4.3] the authors compute $\deg(F_k(Y_u))$ under the Plücker embedding $F_k(Y_u) \subset \mathbb{G}(k,m) \hookrightarrow \mathbb{P}^N$, with $N = \binom{m+1}{k+1} - 1$. Their formulas extend to any $k \geqslant 1$ enumerative formulas by Libgober in [4], who computed $\deg(F_1(Y_u))$ when $t(m,1,\underline{d}) = 0$.

On the other hand, we are interested in the case $t > 0$, where the known results can be summarized as follows.

Result 2. *Let m, k, s and $\underline{d} = (d_1, \ldots, d_s)$ be such that $\Pi_{i=1}^{s} d_i > 2$ and $t > 0$. Then:*
(a) $W_{\underline{d},k} \subsetneq S_{\underline{d}}^*$.
(b) $W_{\underline{d},k}$ contains points u for which $Y_u \subset \mathbb{P}^m$ is a smooth complete intersection of dimension $m-s$ if and only if $s \leqslant m - 2k$.
(c) For $s \leqslant m-2k$, set $H_{\underline{d},k} := \left\{ u \in W_{\underline{d},k} \,\middle|\, Y_u \subset \mathbb{P}^m \text{ is smooth, of dimension } m-s \right\}$. If $d_i \geqslant 2$ for any $1 \leqslant i \leqslant s$, then $H_{\underline{d},k}$ is irreducible, unirational and $\operatorname{codim}_{S_{\underline{d}}^*}(H_{\underline{d},k}) = t$. Moreover, for general $u \in H_{\underline{d},k}$, $F_k(Y_u)$ is a zero–dimensional scheme.

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The proof of Result 2 (a) is contained in [3, Thm. 2.1 (a)], whereas that of assertions (b) and (c) is contained in [5, Cor. 1.2, Rem. 3.4]; both proofs therein hold for any algebraically closed field.

The main result of this paper, which improves on Result 2, is the following.

Theorem 1.1. *Let m, k, s and $\underline{d} = (d_1, \dots, d_s)$ be such that $\Pi_{i=1}^s d_i > 2$ and $t > 0$. Then $W_{\underline{d},k} \subsetneq S^*_{\underline{d}}$ is non–empty, irreducible and rational, with $\operatorname{codim}_{S^*_{\underline{d}}}(W_{\underline{d},k}) = t$. Furthermore, for a general point $u \in W_{\underline{d},k}$, the variety $Y_u \subset \mathbb{P}^m$ is a complete intersection of dimension $m-s$ whose Fano scheme $F_k(Y_u)$ is a zero–dimensional scheme of length one. Moreover, Y_u has singular locus of dimension $\max\{-1, 2k+s-m-1\}$ along its unique k–dimensional linear subspace (in particular Y_u is smooth if and only if $m-s \geqslant 2k$).*

The proof of this theorem is contained in Section 2 and it extends [1, Prop. 2.3] to arbitrary $k \geqslant 1$. Theorem 1.1 improves, via different and easier methods, Miyazaki's results in [5, Cor. 1.2], showing that for general $u \in W_{\underline{d},k}$ one has $\deg(F_k(Y_u)) = 1$, which implies the rationality of $W_{\underline{d},k}$. Moreover we also get rid of Miyazaki's hypothesis $m-s \geqslant 2k$.

2. The proof

This section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\mathbb{G} := \mathbb{G}(k,m)$ be the Grassmannian of k-linear subspaces in $\mathbb{P}^m$ and consider the incidence correspondence

$$J := \left\{ ([\Pi], u) \in \mathbb{G} \times S^*_{\underline{d}} \,\middle|\, \Pi \subset Y_u \right\} \subset \mathbb{G} \times S^*_{\underline{d}}$$

with the two projections

$$\mathbb{G} \xleftarrow{\pi_1} J \xrightarrow{\pi_2} S^*_{\underline{d}}.$$

The map $\pi_1 \colon J \to \mathbb{G}$ is surjective and, for any $[\Pi] \in \mathbb{G}$, one has $\pi_1^{-1}([\Pi]) = \bigoplus_{i=1}^s \left(H^0\left(\mathcal{I}_{\Pi/\mathbb{P}^m}(d_i)\right) \setminus \{0\}\right)$, where $\mathcal{I}_{\Pi/\mathbb{P}^m}$ denotes the ideal sheaf of Π in $\mathbb{P}^m$.

Thus J is irreducible with $\dim(J) = \dim(\mathbb{G}) + \dim(\pi_1^{-1}([\Pi])) = (k+1)(m-k) + \sum_{i=1}^s h^0\left(\mathcal{I}_{\Pi/\mathbb{P}^m}(d_i)\right)$. From the exact sequence

$$0 \to \bigoplus_{i=1}^s \mathcal{I}_{\Pi/\mathbb{P}^m}(d_i) \to \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^m}(d_i) \to \bigoplus_{i=1}^s \mathcal{O}_{\Pi}(d_i) \cong \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^k}(d_i) \to 0, \tag{2.1}$$

one gets

$$\dim(J) = (k+1)(m-k) + \sum_{i=1}^s \binom{d_i+m}{m} - \sum_{i=1}^s \binom{d_i+k}{k} = \dim(S^*_{\underline{d}}) - t. \tag{2.2}$$

The next step recovers [5, Cor. 1.2] via different and easier methods, and we also get rid of the hypothesis $m-s \geqslant 2k$ present there. We essentially adapt the argument in [2, Proof of Prop. 2.1], used for the case $t \leqslant 0$.

Step 1. *The map $\pi_2 \colon J \to S^*_{\underline{d}}$ is generically finite onto its image $W_{\underline{d},k}$, which is therefore irreducible and unirational. Moreover $\operatorname{codim}_{S^*_{\underline{d}}}(W_{\underline{d},k}) = t$.*
For general $u \in W_{\underline{d},k}$, $F_k(Y_u)$ is a zero–dimensional scheme and Y_u has singular locus of dimension $\max\{-1, 2k+s-m-1\}$ along any of the k–dimensional linear subspaces in $F_k(Y_u)$.

Proof of Step 1. One has $W_{\underline{d},k} = \pi_2(J)$, hence $W_{\underline{d},k} \subsetneq S^*_{\underline{d}}$ is irreducible and unirational, because J is rational, being an open dense subset of a vector bundle over $\mathbb{G}$. Once one shows that $\pi_2 \colon J \to W_{\underline{d},k}$ is generically finite, one deduces that $\operatorname{codim}_{S^*_{\underline{d}}}(W_{\underline{d},k}) = t$ from (2.2). Therefore, we focus on proving that π_2 is generically finite, i.e. that if $u \in W_{\underline{d},k}$ is a general point, then $\dim(\pi_2^{-1}(u)) = 0$.

Let $[\Pi] \in \mathbb{G}$ and choose $[y_0, y_1, \dots, y_m]$ homogeneous coordinates in $\mathbb{P}^m$ such that the ideal of Π is $I_\Pi := (y_{k+1}, \dots, y_m)$. For general $([\Pi], u) \in \pi_1^{-1}([\Pi]) \subset J$, with $u = (g_1, \dots, g_s) \in W_{\underline{d},k}$, we can write

$$g_i = \sum_{h=k+1}^m y_h \, p_i^{(h)} + r_i, \; 1 \leqslant i \leqslant s,$$

with

$$r_i \in (I^2_\Pi)_{d_i} \;\text{ whereas }\; p_i^{(h)} = \sum_{|\underline{\mu}| = d_i - 1} c^{(h)}_{i,\underline{\mu}} \; \underline{y}^{\underline{\mu}} \in \mathbb{C}[y_0, y_1, \dots, y_k]_{d_i-1}, \quad 1 \leqslant i \leqslant s, \; k+1 \leqslant h \leqslant m, \tag{2.3}$$

where $(I^2_\Pi)_{d_i}$ is the homogenous component of degree d_i of the ideal I^2_Π, $\underline{\mu} := (\mu_0, \cdots, \mu_k) \in \mathbb{Z}^{k+1}_{\geqslant 0}$, $|\underline{\mu}| := \sum_{r=0}^k \mu_r$, and $\underline{y}^{\underline{\mu}} := y_0^{\mu_0} y_1^{\mu_1} \cdots y_k^{\mu_k}$. By the generality assumption on u, the polynomials $p_i^{(h)}$ and r_i are general.

The Jacobian matrix $(\frac{\partial g_i}{\partial y_j})_{1\leqslant i\leqslant s;0\leqslant j\leqslant m}$ computed along Π takes the block form

$$M = (\mathbf{0} \quad \mathbf{P}) \quad \text{where} \quad \mathbf{P} := (p_i^{(h)})_{1\leqslant i\leqslant s;k+1\leqslant h\leqslant m}$$

where the $\mathbf{0}$–block has size $s\times(k+1)$ and $\mathbf{P}$ has size $s\times(m-k)$, where $m-k\geqslant s$ because of course $\dim(Y_u)=m-s\geqslant k$. By the generality of the polynomials $p_i^{(h)}$, the locus of Π where $\mathrm{rk}(M)<s$, which coincides with the singular locus of Y_u along Π, has dimension $\max\{-1, 2k+s-m-1\}$ and, by Bertini's theorem, it coincides with the singular locus of Y_u.

Next we consider the following exact sequence of normal sheaves

$$0 \to N_{\Pi/Y_u} \to N_{\Pi/\mathbb{P}^m} \cong \mathcal{O}_{\mathbb{P}^k}(1)^{\oplus(m-k)} \to N_{Y_u/\mathbb{P}^m}\big|_{\Pi} \cong \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^k}(d_i) \tag{2.4}$$

(see [8, Lemma 70.5.6]; when $2k\leqslant m-s$, namely when Y_u is smooth along Π, the map on the right-side in (2.4) is more precisely surjective). Any $\xi\in H^0(\Pi, N_{\Pi/\mathbb{P}^m})$ can be identified with a collection of $m-k$ linear forms on $\Pi\cong\mathbb{P}^k$

$$\varphi_h^{\xi}(\underline{y}) := a_{h,0}y_0 + a_{h,1}y_1 + \cdots + a_{h,k}y_k,\ k+1\leqslant h\leqslant m,$$

whose coefficients fill up the $(m-k)\times(k+1)$ matrix

$$A_\xi := (a_{h,j}),\ k+1\leqslant h\leqslant m,\ 0\leqslant j\leqslant k;$$

by abusing notation, one may identify ξ with A_ξ.

Thus the map $H^0\left(\Pi, N_{\Pi/\mathbb{P}^m}\right) \xrightarrow{\sigma} H^0\left(\Pi, N_{Y_u/\mathbb{P}^m}\big|_{\Pi}\right)$, arising from (2.4) is given by (cf. e.g. [2, formula (4)])

$$A_\xi \xrightarrow{\sigma} \left(\sum_{0\leqslant j\leqslant k<h\leqslant m} a_{h,j}y_jp_i^{(h)}\right)_{1\leqslant i\leqslant s}. \tag{2.5}$$

Notice that the assumption $t>0$ reads as

$$(k+1)(m-k) = h^0\left(\Pi, N_{\Pi/\mathbb{P}^m}\right) < h^0\left(\Pi, N_{Y_u/\mathbb{P}^m}\big|_{\Pi}\right) = \sum_{i=1}^{s}\binom{d_i+k}{k}.$$

Claim 2.1. *The map $H^0\left(\Pi, N_{\Pi/\mathbb{P}^m}\right) \xrightarrow{\sigma} H^0\left(\Pi, N_{Y_u/\mathbb{P}^m}\big|_{\Pi}\right)$ is injective, equivalently $h^0(N_{\Pi/Y_u})=0$. In particular, for a general point $u\in W_{\underline{d},k}$, the Fano scheme $F_k(Y_u)$ contains $\{[\Pi]\}$ as a zero–dimensional integral component.*

Proof of Claim 2.1. Using (2.3), the polynomials on the right–hand–side of (2.5) read as

$$\sum_{h=k+1}^{m}\sum_{j=0}^{k} a_{h,j}y_j\left(\sum_{|\underline{\mu}|=d_i-1} c_{i,\underline{\mu}}^{(h)}\ \underline{y}^{\underline{\mu}}\right),\ \ 1\leqslant i\leqslant s.$$

Ordering the previous polynomial expressions via the standard lexicographical monomial order on the canonical basis $\{\underline{y}^{\underline{\mu}}\}$ of $\mathbb{C}[y_0,y_1,\ldots,y_k]_{d_i} = H^0(\mathcal{O}_{\mathbb{P}^k}(d_i))$, $1\leqslant i\leqslant s$, the injectivity of the map σ is equivalent for the homogeneous linear system

$$\sum_{0\leqslant j\leqslant k<h\leqslant m} c_{i,\underline{\nu}-\underline{e}_j}^{(h)}\ a_{h,j} = 0,\ 1\leqslant i\leqslant s, \tag{2.6}$$

to have only the trivial solution, where $\underline{\nu}:=(\nu_0,\nu_1,\ldots,\nu_k)\in\mathbb{Z}_{\geqslant 0}^{k+1}$ is such that $|\underline{\nu}|=d_i$, $\underline{e}_j$ is the $(j+1)$–th vertex of the standard $(k+1)$-simplex in $\mathbb{Z}_{\geqslant 0}^{k+1}\setminus\{\underline{0}\}$, and $c_{i,\underline{\nu}-\underline{e}_j}^{(h)}=0$ when $\underline{\nu}-\underline{e}_j\notin\mathbb{Z}_{\geqslant 0}^{k+1}$ (this last condition stands for "$\underline{\nu}-\underline{e}_j$ *improper*" as formulated in [2, p. 29]). The linear system (2.6) consists of $\sum_{i=1}^{s}\binom{d_i+k}{k}$ equations in the $(k+1)(m-k)$ indeterminates $a_{h,j}$, with coefficients $c_{i,\underline{\mu}}^{(h)}$, $0\leqslant j\leqslant k<h\leqslant m$.

Let $C:=(c_{i,\underline{\nu}-\underline{e}_j}^{(h)})$ be the coefficient matrix of (2.6); one is reduced to show that, for general choices of the entries $c_{i,\underline{\nu}-\underline{e}_j}^{(h)}$, the matrix C has maximal rank $(k+1)(m-k)$. This can be done arguing as in [2, p. 29]. Namely, row–indices of C are determined by the standard lexicographical monomial order on the canonical basis of $\bigoplus_{i=1}^{s}\mathbb{C}[y_0,y_1,\ldots,y_k]_{d_i}$, whereas column–indices of C are determined by the standard lexicographic order on the set of indices (h,j). If one considers the square sub–matrix $\widehat{C}$ of C formed by the first $(k+1)(m-k)$ rows and by all the columns of C, then $\det(\widehat{C})$ is a non–zero polynomial in the indeterminates $c_{i,\underline{\mu}}^{(h)}$. Indeed, take the lexicographic order on the set of indices

$$(h,\ i,\ \underline{\mu}),\ \text{ where }\ k+1\leqslant h\leqslant m,\ |\underline{\mu}|=d_i-1,\ 1\leqslant i\leqslant s,$$

and order the monomials appearing in the expression of $\det(\widehat{C})$ according to the following rule: the monomials m_1 and m_2 are such that $m_1 > m_2$ if, considering the smallest index $(h,\ i,\ \underline{\mu})$ for which $c_{i,\underline{\mu}}^{(h)}$ occurs in the monomial m_1 with exponent p_1 and in the monomial m_2 with exponent $p_2 \neq p_1$, one has $p_1 > p_2$. The greatest monomial (in the monomial ordering described above) appearing in $\det(\widehat{C})$ has coefficient $\pm\, 1$, since in each column the choice of the $c_{i,\underline{\mu}}^{(h)}$ entering in this monomial is uniquely determined. By maximality of such monomial, it follows that $\det(\widehat{C}) \neq 0$, which shows that C has maximal rank $(k+1)(m-k)$, i.e. the map σ is injective.

The injectivity of σ and (2.4) yield $h^0(N_{\Pi/Y_u}) = 0$. Since $H^0(N_{\Pi/Y_u})$ is the tangent space to $F_k(Y_u)$ at its point $[\Pi]$, one deduces that $\{[\Pi]\}$ is a zero–dimensional, reduced component of $F_k(Y_u)$, as claimed. □

Finally, by monodromy arguments, the irreducibility of J and Claim 2.1 ensure that for general $u \in W_{\underline{d},k}$, the Fano scheme $F_k(Y_u)$ is zero-dimensional and reduced, i.e. $\pi_2\colon J \to W_{\underline{d},k}$ is generically finite, and that Y_u has a singular locus of dimension $\max\{-1, 2k+s-m-1\}$ along any of the k–dimensional linear subspaces in $F_k(Y_u)$. This completes the proof of Step 1. □

To conclude the proof of Theorem 1.1, we need the following numerical result.

Step 2. *For* $0\leqslant h\leqslant k-1$ *integers, consider the integer*

$$\delta_h(m,k,\underline{d}) := \sum_{i=1}^{s}\binom{d_i+k}{k} - \sum_{i=1}^{s}\binom{d_i+h}{h} - (k-h)(m+h+1-k).$$

If $\delta_h(m,k,\underline{d}) \leqslant 0$*, then*

$$t(m,k,\underline{d}) \leqslant 0.$$

Proof of Step 2. In order to ease notation, we set $\delta_h := \delta_h(m,k,\underline{d})$. Therefore, the condition $\delta_h \leqslant 0$ implies $m\geqslant\frac{1}{k-h}\left[\sum_{i=1}^{s}\binom{d_i+k}{k} - \binom{d_i+h}{h}\right] - (h+1-k)$. Plugging the previous inequality in the expression of t, one has

$$t\leqslant -\sum_{i=1}^{s}\left[\frac{h+1}{k-h}\binom{d_i+k}{k} - \frac{k+1}{k-h}\binom{d_i+h}{h}\right] + (k+1)(h+1). \tag{2.7}$$

Set $D(x) := \frac{h+1}{k-h}\binom{x+k}{k} - \frac{k+1}{k-h}\binom{x+h}{h}$. Thus, (2.7) reads

$$t\leqslant -\sum_{i=1}^{s} D(d_i) + (k+1)(h+1). \tag{2.8}$$

The assumption $0\leqslant h\leqslant k-1$ gives

$$D(d_i) = \frac{(h+1)(d_i+1)\cdots(d_i+h)}{k!(k-h)}\Big((d_i+h+1)\cdots(d_i+k) - (k+1)k\cdots(h+2)\Big),\ 1\leqslant i\leqslant s.$$

The polynomial $D(x)$ vanishes for $x=1$, which is its only positive root. Notice that

$$D(2) = \frac{h+1}{k-h}\binom{k+2}{k} - \frac{k+1}{k-h}\binom{h+2}{h} = \frac{(h+1)(k+1)}{2} > 0.$$

In particular, $D(x)$ is increasing and positive for $x>1$, so from (2.8) it follows that

$$t\leqslant -\sum_{i=1}^{s} D(d_i) + (k+1)(h+1)\leqslant -\,s\,D(2) + (k+1)(h+1) = (k+1)(h+1)\left(1-\frac{s}{2}\right).$$

Therefore, when $s\geqslant 2$, we have $t\leqslant 0$ and we are done in this case.

If $s=1$, set $d := d_1$. In this case (2.8) is $t\leqslant -D(d) + (k+1)(h+1)$, where again $D(d)$ is increasing and positive for $d>1$. When $s=1$, we have $d\geqslant 3$ by assumption. Thus, one computes

$$D(3) = (k+1)(h+1)\frac{k+h+5}{6}$$

and so, for any $d\geqslant 3$, one has

$$t\leqslant -D(d) + (k+1)(h+1)\leqslant -D(3) + (k+1)(h+1) = (k+1)(h+1)\frac{1-k-h}{6}.$$

Being $0\leqslant h\leqslant k-1$, one deduces that $t\leqslant 0$, completing the proof of Step 2. □

The final step of the proof of Theorem 1.1 is the following.

Step 3. *For general $u \in W_{\underline{d},k}$, the zero–dimensional Fano scheme $F_k(Y_u)$ has length one. In particular, the map $\pi_2\colon J \to W_{\underline{d},k}$ is birational and $W_{\underline{d},k}$ is rational.*

Proof of Step 3. Let us consider the (locally closed) incidence correspondence

$$I := \left\{ ([\Pi_1], [\Pi_2], u) \in \mathbb{G} \times \mathbb{G} \times S^*_{\underline{d}} \middle| \Pi_1 \neq \Pi_2,\ \Pi_i \subset Y_u,\ 1 \leqslant i \leqslant 2 \right\} \subset \mathbb{G} \times \mathbb{G} \times S^*_{\underline{d}}.$$

If I is not empty, let $\varphi\colon I \to J$ be the map defined by

$$\varphi\left(([\Pi_1], [\Pi_2], u)\right) = ([\Pi_1], u).$$

We need to prove that φ is not dominant. To do this, consider the (locally closed) subset

$$I_h := \left\{ ([\Pi_1], [\Pi_2], u) \in I \mid \Pi_1 \cap \Pi_2 \cong \mathbb{P}^h \right\},\ \text{where } -1 \leqslant h \leqslant k-1$$

(we set $\mathbb{P}^{-1} = \emptyset$, i.e. the case $h = -1$ occurs when Π_1 and Π_2 are skew). Clearly, one has $I = \bigsqcup_{h=-1}^{k-1} I_h$. Setting $\varphi_h := \varphi_{|I_h}$, it is sufficient to prove that φ_h is not dominant, for any $-1 \leqslant h \leqslant k-1$.

So, let h be such that I_h is not empty, and let T_h be an irreducible component of I_h. Of course, if $\dim(T_h) < \dim(J)$, the restriction $\varphi_{h|T_h}\colon T_h \to J$ is not dominant. On the other hand, suppose that $\dim(T_h) > \dim(J)$. For any such a component, the map $\varphi_{h|T_h}$ cannot be dominant, otherwise the composition $T_h \xrightarrow{\varphi_{h|T_h}} J \xrightarrow{\pi_2} W_{\underline{d},k}$ would be dominant, as π_2 is, which would imply that the general fiber of π_2 is positive dimensional, contradicting Step 1.

Therefore, it remains to investigate the case $\dim(T_h) = \dim(J)$. We estimate the dimension of T_h as follows. Consider

$$\mathbb{G}^2_h := \left\{ ([\Pi_1], [\Pi_2]) \in \mathbb{G} \times \mathbb{G} \mid \Pi_1 \cap \Pi_2 \cong \mathbb{P}^h \right\} \subset \mathbb{G} \times \mathbb{G},$$

which is locally closed in $\mathbb{G} \times \mathbb{G}$. The projection

$$\widehat{\pi}_1\colon \mathbb{G}^2_h \to \mathbb{G},\quad ([\Pi_1], [\Pi_2]) \longmapsto [\Pi_1]$$

is surjective onto $\mathbb{G}$ and any $\widehat{\pi}_1$–fiber is irreducible, of dimension equal to $\dim\left(\mathbb{G}(h,k) \times \mathbb{G}(k-h-1, m-h-1)\right) = (h+1)(k-h) + (k-h)(m-k)$. Thus

$$\dim \mathbb{G}^2_h = (k+1)(m-k) + (h+1)(k-h) + (k-h)(m-k).$$

One has the projection

$$\psi_h\colon T_h \longrightarrow \mathbb{G}^2_h,\quad ([\Pi_1], [\Pi_2], u) \longmapsto ([\Pi_1], [\Pi_1]),$$

which is surjective, because the projective group acts transitively on $\mathbb{G}^2_h$. Hence $\dim(T_h) = \dim(\mathbb{G}^2_h) + \dim(\mathfrak{F}_{\mathfrak{h}})$, where $\mathfrak{F}_{\mathfrak{h}} := \bigoplus_{i=1}^s \left(H^0\left(\mathcal{I}_{\Pi_1 \cup \Pi_2/\mathbb{P}^m}(d_i)\right) \setminus \{0\} \right)$ is the general fiber of $\psi_{h|T_h}$ and where $\mathcal{I}_{\Pi_1 \cup \Pi_2/\mathbb{P}^m}$ denotes the ideal sheaf of $\Pi_1 \cup \Pi_2$ in $\mathbb{P}^m$.

Claim 2.2. *For every positive integer d one has*

$$h^0(\mathcal{I}_{\Pi_1 \cup \Pi_2/\mathbb{P}^m}(d)) = \dim(S_d) - 2\binom{d+k}{k} + \binom{d+h}{h}.$$

Proof of Claim 2.2. We have

$$h^0(\mathcal{I}_{\Pi_1/\mathbb{P}^m}(d)) = \dim(S_d) - \binom{d+k}{k}. \tag{2.9}$$

Consider the linear system Σ cut out on Π_2 by $|\mathcal{I}_{\Pi_1/\mathbb{P}^m}(d)|$. We claim that Σ is the complete linear system of hypersurfaces of degree d of Π_2 containing $\Pi := \Pi_1 \cap \Pi_2$. Indeed Σ contains all hypersurfaces consisting of a hyperplane through Π plus a hypersurface of degree $d-1$ of Π_2, which proves our claim. In the light of this fact, and arguing as in (2.1) and (2.2), we deduce that

$$h^0(\mathcal{I}_{\Pi_1 \cup \Pi_2/\mathbb{P}^m}(d)) = h^0(\mathcal{I}_{\Pi_1/\mathbb{P}^m}(d)) - (\dim(\Sigma) + 1) = h^0(\mathcal{I}_{\Pi_1/\mathbb{P}^m}(d)) - \left(\binom{d+k}{k} - \binom{d+h}{h}\right),$$

which, by (2.9), yields the assertion. □

By Claim 2.2 we have

$$\dim(\mathfrak{F}_{\mathfrak{h}}) = \dim(S^*_{\underline{d}}) - 2\sum_{i=1}^s \binom{d_i+k}{k} + \sum_{i=1}^s \binom{d_i+h}{h}.$$

Hence

$$\begin{aligned}\dim(T_h) &= \dim(\mathfrak{F}_\mathfrak{h}) + \dim(\mathbb{G}^2_h) = \\ &= \dim(S^*_{\underline{d}}) - 2\sum_{i=1}^{s}\binom{d_i+k}{k} + \sum_{i=1}^{s}\binom{d_i+h}{h} + \\ &+ (k+1)(m-k) + (h+1)(k-h) + (k-h)(m-k) = \\ &= \dim(J) - \sum_{i=1}^{s}\binom{d_i+k}{k} + \sum_{i=1}^{s}\binom{d_i+h}{h} + (k-h)(m+h+1-k) = \\ &= \dim(J) - \delta_h.\end{aligned} \tag{2.10}$$

Since $\dim(T_h) = \dim(J)$, (2.10) implies $\delta_h = 0$. When $0\leqslant h\leqslant k-1$, Step 2 gives $t\leqslant 0$, contrary to our assumption. When $h = -1$, one has $0 = \delta_{-1} = t$, again against our assumptions.

Since no component $T_h \subset I_h$ can dominate J, the map $\varphi\colon I \to J$ is not dominant. We conclude therefore that the map $\pi_2\colon J \to W_{\underline{d},k}$ is birational, completing the proof of Step 3. □

Steps 1–3 prove Theorem 1.1. □

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