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Article

Advances in Financial Leasing Mechanism Designs

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Abstract: Financial leasing contracts usually include specific clauses for early termination of the agreement in the event that the lessee defaults. The lessor might, therefore, not immediately ask for termination of the financial lease agreement in order to take advantage of the increment in fees accrued due to the default interest. The analysis presented in this work aims to find the conditions determining the optimal time to rescind the contract, when the arrears' interest rate is higher than the outstanding fees' interest rate established in the contract and when the return of alternative investments is deterministic. In this paper, we study the optimal time for advanced termination of a leasing contract to maximize the wealth of the lessor against the natural expiry of the contract. We prove that the optimal time can be found through the intersection of the instantaneous force of interest of a lessor's credit at a certain date and the instantaneous force of interest of the opportunity cost of capital, or at the initial date or maturity date. To this end, we provide a detailed list of cases in which it is appropriate to terminate the contract or not. The problem is formulated in the case where the opportunity cost follows deterministic and semi-deterministic dynamics.

Keywords: leasing; rescinding contracts; economic decision analysis

MSC: 91B05

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1. Introduction

A leasing contract is an agreement between a lessor and a lessee, allowing the latter the right to use a property owned or managed by the former for a stated period of time (see, for instance, ref. [1] or [2]). In a basic leasing contract, the lessor transfers the property asset to the lessee at a specified time; this occurs on the condition that the lessee pays both the agreed-upon stated fees and the redemption price.

The most typical obligation a lessee owes to the lessor is to meet the negotiated conditions stated in the lease contract and to pay what is due; the lessee is also held responsible for taking care of the asset and for conducting regular maintenance, although this clause may vary significantly, depending on the jurisdiction involved. The lessee is also required to return the property at the end of the lease in the initial condition. The lessee is expected to strictly abide by all the lease terms and conditions; if not, the lease is said to be "broken"and the asset is returned to the lessor. The agreement does not provide the lessee with any ownership rights; nevertheless, the lessor may grant certain allowances to modify, change, or otherwise adapt the property to suit the lessee's needs. During the lease period, the lessee is considered accountable for the state of the property.

Most of the existing literature addresses asset leasing contracts. Just to name a few, in 1978, ref. [3] derived a simple formula for lease valuation and studied the impact on their analyses of different tax rates and of different amounts of debt that may be displaced by the lease; ref. [4] examined the firm's lease in the case of a borrowing problem; ref. [5] gave closed-form solutions to price-secured bank loans and financial leases subject to default risk; ref. [6] studied the effect of a leasing agreement when defined as the difference between the potential costs of the lessee enterprise as the ratio of the effect of the leasing contract and the potential costs of the lessee; ref. [7] examined leasing contracts to try to bridge the existing data gaps but focused on features other than those in our paper; and ref. [8] implemented a model of equilibrium determination of lease rates in the event the lessee fails to pay.

Grenadier [8], in one of their earlier works, provided a unified framework for the determination of the equilibrium credit spread on leases subject to default risk. His model can be applied to a wide variety of real-world leasing structures, such as security deposits, required up-front prepayments, embedded lease options, leases indexed to use, and lease credit insurance contracts. However, this paper differs from Grenadier's, as it does not address default risk but, rather, it assesses the chance for an alternative investment given by the available asset, in conformity with European policy provisions (in compliance with the European Union's regulation, in the case that the lessee seriously breaches their contract, the lessor is entitled to obtain the expired unpaid fees plus the interest at the defaulted interest rate, as well as the fees not yet expired and the redemption price discounted at the leasing rate) in the case of the lessee's insolvency. Throughout this paper, we assume that the lessee stops paying fees at a certain date prior to the end of the original lease term. The lessor, following this date, assesses their economic interest to either keep or terminate the lease contract in light of the opportunity cost evaluation.

Herein, the leasing agreement is assumed to indicate a contract's stated interest rate, an arrears interest rate, and an interest rate to be applied to unpaid fees. In leasing contracts [9], the lessor retains ownership of the asset until the contract's expiry date, but in the case of a breach of contract by the lessee, the owner may resort to rescission.

The common features of lease contracts are the object of our investigation, with a focus on the contract rescission on the part of the lessor, pursuant to the lessee's insolvency. In that event, it is within the lessor's rights to be granted the expired fees capitalized in terms of simple interest accumulation at an arrears interest rate predetermined in the contract. In the lease under appraisal, at the occurrence of the lessee's failure to pay the debt [10,11], the leased asset can be sold to clear the unpaid debt; nevertheless, the sale of the asset may not suffice to pay off the debt. This lets us know the percentage on the sum the lessor expects.

Circumstances may exist such that the lessor prefers not to rescind the contract and bide their time. Let us theorize that the lessor may avail themselves of the credit obtained from the contract dissolution to make alternative investments. The purpose of this paper is to find the conditions determining the optimal time to rescind the contract at issue, when the arrears interest rate is higher than the interest rate of the outstanding fees established in the contract and the return of alternative investments is deterministic [12,13]. It could be useful not only for an academician but for financial corporate practitioners in particular.

The lessor is always fully aware of the possibility of exercising an early termination; therefore, he is offered two alternatives: either retain the contract or terminate it.

In this work, we intend to describe the use of financial regulations to capitalize long-overdue fees, which entails interest on defaulted payments; hence, we want to consider the lessee's ability to ensure solvency. Specifically, the paper is structured as follows: Section 2 illustrates the model by which the lessor breaks the contract and the lessee pays the amount due back. In Section 2, we argue for a random variable that is independent of time and represents the credit share that the lessee pays in the case that the lessor decides to rescind the contract at any time. Here, we set a semi-deterministic case in which we analyze the real hypothesis in which the lessee is unable to fully return the amount due to the lessor when he decides to rescind the leasing contract. In Section 3, we propose a numerical example to find the optimal time of termination of the contract. Lastly, in Section 4, some comments conclude the paper.

2. The Model

In this section, we introduce the model. The lessor gives an asset to the lessee. The lessee pays regular fees and the redemption price R at maturity T.

Let us consider a leasing contract in which, from a certain date forward, the lessee no longer pays the fees. By contract, the lessor, in case of default of the lessee, can ask to terminate the contract and invest the proceeds in alternative investments that are obviously subsequent to the termination of the leasing contract. The lessor (see [14]) may decide not to terminate the contract immediately but to wait by effectively exploiting the overdue default interest and any better conditions relating to alternative investments. As a consequence, the lessor evaluates whether or not to rescind the contract at date z by comparing it with the capital cost of an alternative investment. In order to fulfill this objective, we now introduce the model; we assume that the fees have been continuously paid and therefore that the fees due in the interval $[t, t+dt]$ are $\gamma(t)dt$.

Without loss of generality, we can assume that the lessee has been insolvent since the beginning, at time 0. Later, we see that assuming immediate insolvency does not change the generality of our conclusions.

A financial leasing contract provides that the default interest relating to the overdue fees is paid according to the simple capitalization regime, while the installments not yet expired are determined under the compound capitalization regime on the date on which the existing contract is to be terminated. In general, leasing contracts provide that overdue and unpaid fees must be added to interest on arrears. To avoid anatocism, many laws require that default interest must be calculated with the law of simple interest. Upon termination of the contract, the lessor is entitled to the amount of fees expired and not paid until the date of termination and fees that have yet to expire, only in principal line and the price agreed for the exercise of the final purchase option. To calculate the principal portion of the fees to maturity, the fees to expire under compound capitalization at the contractually provided leasing rate are discounted. The credit at time z for the leasing company, in the case that the lessee is insolvent, is the sum of two terms; the first one is the amount of the expired fees and accrued in simple interest at the arrears interest rate m, predetermined in the contract. The second term is the outstanding fees and the redemption price R discounted by a lower force of interest δ, also contractually established. Hence, the lessor's credit in the case of insolvency is:

$$C(z) := \int_0^z \gamma(s)(1 + m(z-s))ds + \int_z^T \gamma(s)e^{-\delta(s-z)}ds + Re^{-\delta(T-z)}.$$

Let us assume that the lessor can use the credit $C(z)$ obtained from the anticipatory breach of contract in alternative investments. Let r indicate the instantaneous force of interest of the opportunity cost of capital for the lessor.

The optimal time in which the lessor decides to terminate the contract is the solution to an optimal problem. By definition of optimal time, terminating the contract sooner or later leads to a worse result than final wealth.

At this end, the lessor chooses an optimal time $z \in [0, T]$ to rescind the contract in order to maximize the amount accrued at time T obtained by closing the contract at a time z and by investing the credit $C(z)$ in the alternative investment for the residual time $T - z$. Let us assume throughout the paper that $m > \delta$ and r and γ are continuous functions.

From the continuity of γ, the credit C admits a derivative, so we can define its instantaneous force of interest:

$$\lambda(z) := \frac{C'(z)}{C(z)}.$$

The amount that the lessor accrues at time T from an anticipatory breach of contract at time z is

$$M(z,T) := C(z)e^{\int_z^T r(s)ds} = C(0)\exp\left(\int_0^z \lambda(s)ds + \int_z^T r(s)ds\right). \quad (1)$$

We show some conditions whereby it is convenient to either rescind or maintain the contract before the natural expiration date. As usually happens in leasing contracts, we assume that the default interest rate is higher than the interest rate of the outstanding fees ($m > \delta$). Moreover, we assume that r is deterministic.

We look for the optimal time needed to close the contract and ask for a settlement of the amount due. In the following theorem, we prove that the optimal time can be found through the intersection of the instantaneous force of interest of C, λ and the instantaneous force of interest of opportunity cost of capital r or at the initial date 0 or maturity date T. In the latter case, the question remains as to whether it is more convenient to end the contract immediately or not.

Derivatives' properties help us to identify the local extreme.

Theorem 1. *Let us call $A = \{z \in [0,T] \| \rho(z) = \lambda(z)\}$ the set of the intersection points between ρ and λ. If $m > \delta$ and ρ is deterministic, then*

$$\arg \max_{z\in[0,T]} M(z,T) = \arg \max_{z\in A\cup\{0;T\}} M(z,T).$$

Proof. Inasmuch as γ and ρ are continuous functions, then M admits a derivative with respect to z. For a differentiable function defined in the closed interval $[0,T]$, the global maxima can be found between the stationary points of M or boundary points of $[0,T]$. From (1), the derivative of M is

$$\frac{\partial M(z,T)}{\partial z} = M(z,T)[\lambda(z) - \rho(z)].$$

The initial credit $C(0)$ for the leasing company is strictly positive, and therefore also $M(z,T) > 0$. It follows that the stationary points of M are the solutions of the equation $\lambda(z) - \rho(z) = 0$.

It follows that, in line with Fermat's theorem, the global maxima, in the closed interval $[0,T]$, can be found between the stationary points of M or boundary points of $[0,T]$. $\square$

Result should be demonstrated that will be useful later.

The following lemma fixed a delta condition.

Lemma 1. *For $0 < \delta \leq m$*

$$\frac{-\delta + \sqrt{\delta(2m-\delta)}}{\delta m} \leq \frac{2m - \delta - \sqrt{(m-\delta)^2 + m^2}}{\delta m}. \tag{2}$$

Proof. By simple calculation, (2) is equivalent to

$$(2m\delta - \delta^2)(\delta^2 - 2m\delta + 2m^2) \leq m^4.$$

Let $x = \frac{\delta}{m} \leq 1$. We note that x is positive. We rewrite the previous inequality as

$$(2x - x^2)(x^2 - 2x + 2) - 1 \leq 0.$$

The function

$$f(x) = (2x - x^2)(x^2 - 2x + 2) - 1$$

is negative for $x = 0$, vanishes for $x = 1$ and $f'(x) = 4(1-x)^3$. Hence, for $x \in [0,1]$, we have the result that $f(x) \leq 0$. $\square$

The following theorem illustrates the possible behavior of the force of interest, λ, if the fee intensity, γ, is constant. We prove the existence of local extreme for λ under some conditions of T and the redemption price, R.

Theorem 2. *Suppose that γ is constant. There exists an instant $z^* \in [0, T]$ such that the force of interest of C, λ, is an increasing function on $[0, z^*]$. Moreover, if T is large enough, i.e.,*

$$T > \frac{2m - \delta + \sqrt{(m-\delta)^2 + m^2}}{\delta m} \tag{3}$$

and if the redemption price is sufficiently high, that is,

$$R > \gamma \, \frac{m(m+\delta)T^2 - 2(m-\delta)T}{\delta^2 m T^2 - 2\delta(2m-\delta)T + 2(m-\delta)} \tag{4}$$

then, there exists $z^{**} \in [z^*, T]$ so that the force of interest λ decreases on $[z^*, z^{**}]$ and increases on $[z^{**}, T]$. If condition (4) does not hold and $T > z^*$, then the force of interest λ decreases along $[z^*, T]$.

Proof. To detect possible monotonic properties of λ, we study the sign of its derivative:

$$\lambda'(z) = \frac{C''(z)C(z) - [C'(z)]^2}{C^2(z)}. \tag{5}$$

The first two derivatives of C are:

$$C'(z) = \gamma(mz + 1) + (\delta R - \gamma)e^{-\delta(T-z)}$$

$$C''(z) = \gamma m + \delta(\delta R - \gamma)e^{-\delta(T-z)}.$$

Let us define the sign of the numerator of (5)

$$\varphi(z) := C''(z)C(z) - \left[C'(z)\right]^2 = -\frac{\gamma^2}{2\delta}\left(P(z) + \alpha Q(z)e^{-\delta(T-z)}\right)$$

where

$$\alpha = 1 - \frac{\delta R}{\gamma}$$

$$P(z) = \delta m^2 z^2 + 2m\delta z - 2(m-\delta)$$

$$Q(z) = \delta^2 m z^2 - 2\delta(2m-\delta)z + 2(m-\delta).$$

Note that the sign of λ' and φ is the same. We have that

$$\varphi(z) > 0 \iff \alpha e^{-\delta(T-z)} Q(z) < -P(z)$$

hence,

$$\varphi(z) > 0 \iff \begin{cases} \alpha e^{-\delta(T-z)} < G(z) & \text{if } Q(z) > 0 \\ \alpha e^{-\delta(T-z)} > G(z) & \text{if } Q(z) < 0 \\ 0 < -P(z) & \text{if } Q(z) = 0 \end{cases} \tag{6}$$

where $G(z) = -\frac{P(z)}{Q(z)}$.

The roots of a quadratic polynomial are real numbers, if its determinant is positive. The discriminants of the quadratic polynomials P and Q are, respectively,

$$\Delta_P = 4m^2\delta(2m - \delta)$$

$$\Delta_Q = 4\delta^2[(m-\delta)^2 + m^2]$$

which are both positive, for $m > \delta$. Applying Descartes's rule of signs, Q has two positive roots

$$q_i = \frac{2m - \delta \pm \sqrt{(m-\delta)^2 + m^2}}{\delta m} \qquad i = 1, 2 \tag{7}$$

and P has only one positive root

$$p_1 = \frac{-\delta + \sqrt{\delta(2m-\delta)}}{\delta m}. \tag{8}$$

It is easy to show (see Lemma 1) that $p_1 \leq q_1 < q_2$, for $m > \delta$. From (6),

$$\varphi(z) > 0 \iff \begin{cases} \alpha e^{-\delta(T-z)} < G(z) & \text{if } z \in [0, q_1[\cup]q_2, +\infty] \\ \alpha e^{-\delta(T-z)} > G(z) & \text{if } z \in [q_1, q_2] \\ P(z) < 0 & \text{if } z = q_1 \text{ or } z = q_2 \end{cases}$$

Note that $P(q_i) > 0$, $i = 1, 2$, and hence, $\varphi(q_i) < 0$.

The function $G(z)$ is positive for $z \in [0, p_1[\cup]q_1, q_2]$ and it is negative for $z \in [p_1, q_1[\cup]q_2, +\infty]$. From the derivative of G:

$$G'(z) = \frac{4\delta}{Q^2(z)}\left(\delta m^3 z^2 - m(m^2 - \delta^2)z + (\delta - m)^2\right)$$

we obtain its critical points:

$$g_1^* = \frac{m-\delta}{m^2} \quad \text{and} \quad g_2^* = \frac{m-\delta}{m\delta}$$

$$G(g_1^*) = m\frac{2m-\delta}{(m-\delta)^2 + m^2} \quad \text{and} \quad G(g_2^*) = \frac{m}{\delta}$$

It is easy to prove that $g_1^* < p_1 < q_1 < g_2^* < q_2$, and g_1^* is a local maximum, and g_2^* is a local minimum.

We examine the following three cases $(a), (b), (c)$ constructed according to the possible relative positions of z.

(a) Let us assume that $0 \leq z \leq q_1$.

We observe that if $\alpha \geq 0$, $\alpha e^{-\delta(T-z)}$ increases on $[0, q_1]$ and G increases on $[0, g_1^*]$. We have

$$\forall z \in [0, g_1^*], \quad \alpha e^{-\delta(T-z)} \leq \alpha \leq 1 = G(0) \leq G(z).$$

Because

$$\lim_{z \to q_1^-} G(z) = -\infty$$

and G decreases on $[g_1^*, q_1]$; there exists a unique $z^* \in [g_1^*, p_1]$ such that $G(z^*) = e^{-\delta(T-z^*)}$.

If $\alpha < 0$, let us assume that $g(z) = \alpha e^{-\delta(T-z)} Q(z)$.

The derivative of g is

$$g'(z) = \alpha\delta e^{-\delta(T-z)}(\delta^2 m^2 z^2 + 2\delta(m^2 - 2m + \delta)z - 2m)$$

hence, function g is an increasing function on $[0, q_1]$; moreover, $g(0) < 0$ and $g(0) = 0$. Because $-P(z)$ is a decreasing function on $[0, q_1]$, $P(0) < 0$ and $P(q_1) > 0$, there exists only one $z^* \in [p_1, q_1]$ such that $g(z^*) = -P(z^*)$; that is, $G(z^*) = e^{-\delta(T-z^*)}$.

From (6) and from

$$\varphi(0) = \frac{\gamma^2(m-\delta)}{\delta}\left(1 - \alpha e^{-\delta T}\right) > 0,$$

we see that

$$\begin{array}{ll} \varphi(z) > 0 & \text{if } z \in [0, z^*] \\ \varphi(z) < 0 & \text{if } z \in [z^*, q_1]. \end{array}$$

(b) Let us assume that $q_1 < z \leq q_2$.

In this case, we have

$$\alpha e^{-\delta(T-z)} \leq \alpha \leq 1 < \frac{m}{\delta} = G(g_2^*) = \min_{z \in [q_1, q_2]} G(z) \leq G(z);$$

hence, from (6), for $z \in [q_1, q_2]$, we see that $\varphi(z) < 0$.

(c) Let us assume that $z > q_2$.
For $z > q_2$ and $\alpha \geq 0$, we have that

$$G(z) < 0 \leq \alpha e^{-\delta(T-z)};$$

hence, from (6), $\varphi(z) < 0$ for $z > q_2$.
For $z > q_2$ and $\alpha < 0$, the function $\alpha e^{-\delta(T-z)}$ is decreasing and the function $G(z)$ is increasing. Moreover,

$$\lim_{z \to q_2^+} G(z) = -\infty.$$

Hence, if $T > q_2$ and $G(T) > \alpha$, then there exists $z^{**} \in [q_2, T]$ such that

$$\begin{array}{ll} G(z) < \alpha e^{-\delta(T-z)} & \text{if } z \in [q_2, z^{**}] \\ G(z) > \alpha e^{-\delta(T-z)} & \text{if } z \in [z^{**}, T]. \end{array}$$

It is easy to show that $G(T) > \alpha$ is equivalent to condition (4). From (6)

$$\begin{array}{ll} \varphi(z) < 0 & \text{if } z \in [q_2, z^{**}] \\ \varphi(z) > 0 & \text{if } z \in [z^{**}, T]. \end{array}$$

If $T > q_2$ and $G(T) \leq \alpha$, then for $z \in [q_2, T]$, we see that $G(z) < \alpha e^{-\delta(T-z)}$ and from (6)

$$\varphi(z) < 0 \quad \text{if } z \in [q_2, T].$$

□

Figure 1 shows the possible behavior of λ. If conditions (3) and (4) are satisfied, then λ has a local maximum z^* and a local minimum z^{**} (see Figure 1b); otherwise, λ has a local maximum z^* only (see Figure 1a).

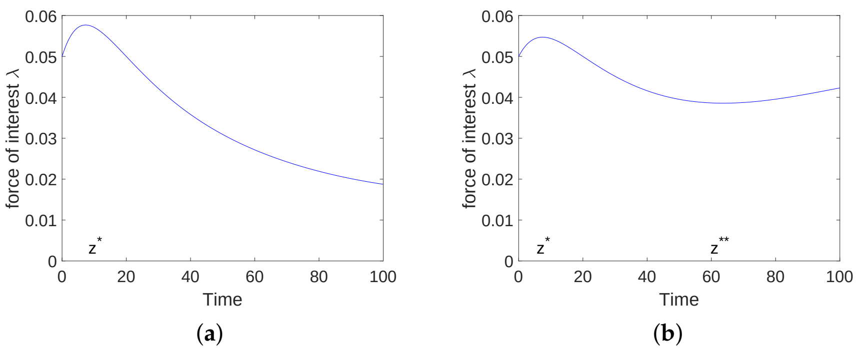

Figure 1. The figure shows the two behaviors of λ for small ($R = 20$ (**a**)) and high ($R = 1000$ (**b**)) redemption prices. The other parameters are $\gamma = 0.5$, $T = 100$, $\delta = 0.05$, and $m = 0.1$.

An immediate consequence of the previous theorem is the following result, showing that the global minima and maxima of λ can be found among the boundary of $[0, T]$ and the local extrema of λ.

Theorem 3. *If conditions (3) and (4) are both verified, then*

$$\min(\delta, \lambda(z^{**})) = \min_{z \in [0,T]} \lambda(z) \qquad \max(\lambda(z^*), \lambda(T)) = \max_{z \in [0,T]} \lambda(z);$$

otherwise,

$$\min(\delta, \lambda(T)) = \min_{z\in[0,T]} \lambda(z) \qquad \lambda(z^*) = \max_{z\in[0,T]} \lambda(z)$$

Remarks:
We observe that $\lambda(0) = \delta$.

1. If $\delta\frac{R}{\gamma} \le 1$, then condition (4) is not verified. Therefore, condition (4) is never verified if the redemption price R is zero or δ is small. In this case, the only possible behavior for λ is shown in Figure 1a.
   From the proof of the Theorem (2), we also obtain some estimates of z^*.
2. If $\delta\frac{R}{\gamma} \le 1$ and $T > p_1$, where p_1 is defined by condition (8), then
$$\frac{m-\delta}{m^2} < z^* < p_1.$$
3. If $\delta\frac{R}{\gamma} > 1$ and $T > q_1$, where q_1 is defined by condition (7), then
$$p_1 < z^* < q_1.$$

To find the optimal time z to rescind the contract, the lessor must solve the equation

$$\lambda(z) = r(z). \tag{9}$$

In the case where the force of interest, r, is a linear function

$$r(z) = az + b$$

(including constant function, i.e., $a = 0$), then Equation (9) can be studied easily. Let us define

$$r_{\min} := \min_{z\in[0,T]} r(z) = \min(r(0), r(T)) \quad r_{\max} := \max_{z\in[0,T]} r(z) = \max(r(0), r(T)).$$

We have the following cases:

(a) $r_{\min} \ge \max_{z\in[0,T]} \lambda(z)$.
In this case, it is immediately convenient to rescind the contract. By applying Theorem (3), if conditions (3) and (4) are both verified, we have that $r_{\min} \ge \max(\lambda(z^*), \lambda(T))$; otherwise $r_{\min} \ge \lambda(z^*)$.

(b) $r_{\max} \le \min_{z\in[0,T]} \lambda(z)$.
In this case, it is not convenient to rescind. By applying Theorem (3), if conditions (3) and (4) are both verified, we have $r_{\max} \le \min(\delta, \lambda(z^{**}))$, otherwise $r_{\max} \le \min(\delta, \lambda(T))$.

(c) $\min_{z\in[0,T]} \lambda(z) < r_{\max}$ or $r_{\min} < \max_{z\in[0,T]} \lambda(z)$.
In this case, if conditions (3) and (4) are both verified, the equation has one, two or three solutions; otherwise, Equation (9) has one or two solutions.

Semi-Deterministic Case

Let us hypothesize that the lessee is unable to fully return the amount due to the lessor at time z, if the lessor decides to rescind the leasing contract at this time. Let us presume that k is the random share of credit that the lessor recovers in this case. We suppose that k does not depend on time z.

In order to make optimal decision to minimize risk, the lessor is believed to maximize the expected utility of the amount accrued, which the lessee will pay at time T. We assume that the lessor's utility function u is sufficiently smooth. Therefore, the optimization problem to be solved is:

$$\tilde{z} := \arg\max_{z\in[0,T]} G(z) = \mathrm{E}u(kM(z,T)) = \int_0^1 u(xM(z,T))d\mathrm{F}(x)$$

where E denotes the expectation operator and F is the cumulative distribution function of k.

It is trivial to show that the lessor's choice does not depend on the utility function u nor on the probability distribution of the random variable k. In fact, the signs of $G'(z,T)$ and $M'(z,T)$ are the same; hence, G and M are the same global maximum points.

3. Numerical Simulation

The intersections of the λ curve with the r curve must be determined. Once the intersection points are found, we calculate the function $M(z,T)$ at those points and at the extremes of the range $[0,T]$. The largest value gives the optimal time of termination of the contract.

In the numerical example, assume that r is constant and that the parameters are

$$\gamma = 1000 \quad m = 0.10 \quad \delta = 0.7 \quad T = 20 \quad R = 500$$

With $r = 0.072$, the intersection of the line in black with the curve of λ (see Figure 2) numerically gives two points of intersection $z_1 = 0.8570$ and $z_2 = 7.0968$, to which they correspond: $M(z_1,T)$ = 45,910.29 and $M(z_2,T)$ = 46,371.84. Taking into account that $M(0,T)$ = 45,947.33 and $M(T,T)$ = 40,500, the largest final amount is then obtained for $z = z_2$.

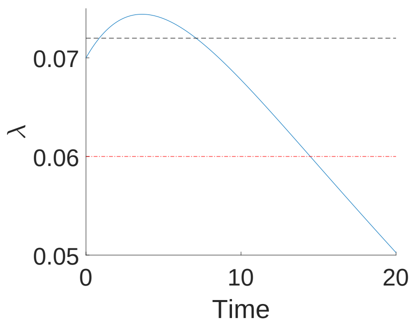

Figure 2. The figure shows the λ curve and the flat r flat curve. The black line corresponds to $r = 0.072$ and the red one to $r = 0.06$.

With $r = 0.06$, the intersection of the red line with the curve of λ (see Figure 2) numerically gives only one point of intersection $z_1 = 14.4682$, to which $M(z_1,T)$ = 41,614.63 corresponds. Taking into account that $M(0,T)$ = 36,143.45 and $M(T,T)$ = 40,500, the largest final amount is then obtained for $z = z_1$.

It is noted that in both cases, terminating the contract immediately or not terminating it at all gives a lower amount than the optimal resolution.

4. Conclusions

The purpose of this paper was to illustrate the convenience of terminating a leasing contract in advance should the lessee stop paying the leasing fees. Under these conditions, the lessor may bide their time in the hopes that the lessee should resume regular payments. Consequently, the lessor may opt not to demand the immediate termination of the contract, but the unpaid accrued fees the lessee is expected to pay will entail the application of default interest, which will be higher than the interest rate agreed. Therefore, the lessor may profit from deferring their contract termination request; in the event of contract termination, the asset will be sold and the proceeds of sale will be used to compensate the lessee for the past due unpaid fees increased at a default interest rate along with the residual value of the asset. The residual value of the asset was calculated by discounting, at the lease rate, the outstanding fees and the (agreed) price for the exercise of the final purchase option. We estimated the lessee's convenience of rescinding the contract, calculating at its natural expiry the wealth obtained through the capitalization using an adequate term structure of interest rates of the amount obtained from the rescission. We determined the optimal contract rescinding time in the case of a deterministic evolution.

The limitation of this model is that, in many cases, it could be ideal rather than real, so an idea for future investigations is the possibility to include into the analysis stochastic process on optimal stopping time and on the real ability of the lessee to fully return the amount due to the lessor at a certain time.

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