An Optimal Control Problem in Coefficients for a Strongly Degenerate Parabolic Equation with Interior Degeneracy

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Abstract. We deal with an optimal control problem in coefficients, for a strongly degenerate diffusion equation with interior degeneracy which is due to the nonnegative diffusion coefficient vanishing with some rate at an interior point of a multi-dimensional space domain. The optimal control is searched in the class of functions having essentially bounded partial derivatives. The existence of the state system and of the optimal control is proved in a functional framework constructed on weighted spaces. The conditions of optimality are expressed in a generalized form only, due to the insufficient regularity of the state. By an approximating control process, explicit approximating optimality conditions are deduced and a representation theorem allows to express the approximating optimal control as the solution to the eikonal equation. Under certain hypotheses, further properties of the approximating optimal control can be proved, including uniqueness in some situations. An algorithm for the construction of the optimal control and a numerical example in a square domain are provided.

Keywords Optimal control, Optimality conditions, Degenerate diffusion equations, Interior degeneracy, Coefficient identification, Eikonal equation

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1 Introduction

In engineering and applied sciences several processes are often described by mathematical models, whose parameters should be determined in such a way that the system can reach a certain objective. Therefore, an optimal control procedure must be implemented in order to construct or approximate these parameters. A first example arises in material science, where the design of a composite material having a certain thermal diffusivity is done in relation with an optimal control problem for the heat equation (see [1], [2]). Problems in meteorology (see [3]), pollutant propagation in fluids, flows in porous media (see [4], [5]), population dynamics (see [6]), biology (see [7], [8]), structural

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engineering (see [9]) or many other (see [10], [11], [12], [13], [7], [14], [15], [16], [17], [18], [2]) involve models with parabolic diffusion equations and require the determination of the diffusion coefficient generating a certain dynamics of the solution. A model with a vanishing diffusion coefficient governing e.g., diffusion of a substance in water, soil or air (see e.g., [3]), heat flow in a material (see e.g., [2]), or diffusion of a population in a habitat (see e.g., [13], [6]) raises interest since a particular local behavior may induce long distance effects. In a degenerate situation, there are different problems classified with respect to the rate of degeneracy, which can be weak or strong. Here, we are interested in a related control problem, provided that the diffusion coefficient is non-negative and vanishes with a certain rate in an interior point of a suitable domain. The one-dimensional case was treated in [19], for the strongly degenerate parabolic equation in divergence form, and in [20] for the strongly degenerate nondivergence case. Now, we deal with the multi-dimensional case for the strongly degenerate situation in divergence form. Here, we will have to face some additional difficulties with respect to the one-dimensional case.

2 Statement of the problem

Let us denote by Ω an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, having a connected boundary $\Gamma := \partial\Omega$, of class C^1. Let $T > 0$ and denote $Q :=]0, T[\times\Omega$ and $\Sigma :=]0, T[\times\Gamma$. We consider the following problem

$$\frac{\partial y}{\partial t} - \nabla \cdot (u(x)\nabla y) = f \text{ in } Q, \tag{1}$$

$$y(0, x) = y_0(x) \text{ in } \Omega, \tag{2}$$

$$y = 0 \text{ on } \Sigma. \tag{3}$$

Our aim is to to control the dynamics of the solution y by the means of the diffusion coefficient u, supposed to be a nonnegative function, vanishing at an interior point of Ω. Moreover, we shall look for a function $u \in W^{1,\infty}(\Omega)$ such that $\frac{1}{u}$ is not integrable, case classified as strongly degenerate in [21].

An example is indicated by the function $u(x) = |x - x_0|_N^k$, $k \geq N$, where $|\cdot|_N$ is the Euclidean norm (see Th. 1.1.11 in [22], p. 9).

The objective is to force the solution to approach either a certain spatial mean M_T at a final time T, a mean value M_Q over Q, or both. This suggests to study the problem

$$\text{Minimize } \left\{ \frac{\lambda_1}{2} \left(\int_\Omega y^u(T, x)dx - M_T \right)^2 + \frac{\lambda_2}{2} \left(\int_Q y^u(t, x)dxdt - M_Q \right)^2 \right\} \tag{P}$$

subject to (1)-(3), for all u in a set U, which will be further specified. Here, M_T, M_Q, λ_1, λ_2, are nonnegative real numbers, and there exists at least one $i \in \{1, 2\}$ such that $\lambda_i > 0$. The notation y^u indicates the solution to (1)-(3) corresponding to u. The different choices of the constants λ_i, $i = 1, 2$ are used to induce a higher or lower importance to the terms in the functional, as required by the problem.

Control problems with the optimal control in $W^{1,\infty}(\Omega)$ were treated in the non-degenerate one-dimensional case in [12], [11] and in the two-dimensional case in [9], the latter in connection with an elliptic equation modeling the deformation of a thick membrane under the action of a force. The particularities of the problems considered in these works allowed the computation of exact expressions for u.

Here we study a degenerate case in N-dimensions, related to a vanishing u with a certain rate inside the domain and approach the above nonlinear minimization problem as an optimal control problem in coefficients. We prove the existence for the state system in Section 3, then the existence of a solution to the nonlinear control problem (P) in Section 4, and deduce the optimality conditions. Due to the strong restrictions imposed for u, the last ones remain in a generalized form. In order to get a clearer characterization of the structure of the optimal control we introduce in Section 5 an approximating problem (P_ε) involving a nondegenerate state system and prove that a solution to (P) can be obtained as the uniform limit in Ω of a sequence of solutions to this approximating problem (P_ε). The optimality conditions for (P_ε) are determined in a more explicit form. The study of the problem in N dimensions requires some more results than in the one dimensional case, namely a density result in weighted spaces (Lemma 3.1), representation theorems of the optimal control (Propositions 5.1-5.3) extended in dimension N, and a new optimal control algorithm for its construction in a square domain (Section 6).

3 Existence for the state system

We begin with some notation and definitions. Let us consider $x_0 \in \Omega$ and u such that

$$u \in W^{1,\infty}(\Omega),\ u(x_0) = 0,\ u > 0 \text{ on } \overline{\Omega}\diagdown\{x_0\},\ \frac{1}{u} \notin L^1(\Omega). \tag{4}$$

We consider the space $L^2(\Omega)$, the standard Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$ and define the weighted spaces

$$H_u^1(\Omega) = \{y \in L^2(\Omega);\ \sqrt{u}\nabla y \in (L^2(\Omega))^N,\ y = 0 \text{ on } \Gamma\},$$
$$H_u^2(\Omega) = \{y \in H_u^1(\Omega);\ u\nabla y \in (H^1(\Omega))^N\}.$$

For convenience, and where no confusion can arise we shall not write the function arguments in the integrands. It is obvious that $H_u^1(\Omega)$ is a Hilbert space with the norm

$$\|y\|_{H_u^1(\Omega)} = \left(\|y\|_{L^2(\Omega)}^2 + \left\|\sqrt{u}\nabla y\right\|_{L^2(\Omega)}^2\right)^{1/2}. \tag{5}$$

Moreover, $H_u^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (H_u^1(\Omega))'$, where $(H_u^1(\Omega))'$ is the dual of $H_u^1(\Omega)$ and "$\hookrightarrow$" means a continuous and dense embedding. For simplicity, sometimes we denote

$$H = L^2(\Omega),\ V_u = H_u^1(\Omega),\ V_u' = (H_u^1(\Omega))'.$$

Lemma 3.1. *Let $N \geq 2$, $x_0 \in \Omega$, $u \in W^{1,\infty}(\Omega)$, $u(x_0) = 0$, $u > 0$ on $\overline{\Omega}\diagdown\{x_0\}$. Assume in addition that u has the local property*

$$u(x) \leq L\,|x - x_0|_N^k,\ \textit{for}\ \ |x - x_0|_N^k \leq \delta, \tag{6}$$

for some $\delta > 0$, small, where $k \geq N$ and $L > 0$. Then, the space $H_0^1(\Omega)$ is dense in $H_u^1(\Omega)$.

Proof. Let $\rho \in C^\infty(\mathbb{R}^+)$ be such that

$$\rho(r) = \begin{cases} 1 \text{ for } r \geq 2 \\ 0 \text{ for } r \leq 1 \end{cases}$$

and $\rho'(r) \leq C$ for $r \in [1, 2]$. Also, the notation C stands further for several positive constants.

Now, let $y \in H_u^1(\Omega)$ and set

$$y_\varepsilon(x) = y(x)\rho\left(\frac{|x - x_0|_N}{\sqrt{\varepsilon}}\right) \quad \text{for } 0 < \varepsilon < \frac{\delta^2}{4}, \ x \in \Omega. \tag{7}$$

Then,

$$y_\varepsilon(x) = \begin{cases} y(x) \text{ for } |x - x_0|_N \geq 2\sqrt{\varepsilon} \\ 0 \quad\quad \text{for } |x - x_0|_N \leq \sqrt{\varepsilon}. \end{cases} \tag{8}$$

It is obvious that $y_\varepsilon \in L^2(\Omega)$, $y_\varepsilon \to y$ a.e. in Ω, as $\varepsilon \to 0$ and $\|y_\varepsilon\|_H \leq \|y\|_H$, hence, by the Lebesgue dominated convergence theorem we get

$$y_\varepsilon \to y \text{ strongly in } L^2(\Omega), \text{ as } \varepsilon \to 0.$$

For $x \neq x_0$ we compute

$$\nabla y_\varepsilon = \nabla y \rho\left(\frac{|x - x_0|_N}{\sqrt{\varepsilon}}\right) + y\frac{x - x_0}{|x - x_0|_N}\frac{1}{\sqrt{\varepsilon}}\rho'\left(\frac{|x - x_0|_N}{\sqrt{\varepsilon}}\right)$$

and note that $\nabla y_\varepsilon \in (L^2(\Omega))^N$ (because ρ vanishes on the subset $|x - x_0|_N \leq \sqrt{\varepsilon}$). Hence $y_\varepsilon \in H_0^1(\Omega)$. Next

$$\sqrt{u}\nabla y_\varepsilon = \sqrt{u}\nabla y \rho\left(\frac{|x - x_0|_N}{\sqrt{\varepsilon}}\right) + y\frac{x - x_0}{|x - x_0|_N}\frac{\sqrt{u}}{\sqrt{\varepsilon}}\rho'\left(\frac{|x - x_0|_N}{\sqrt{\varepsilon}}\right). \tag{9}$$

Since ρ' vanishes for $|x - x_0|_N \geq 2\sqrt{\varepsilon}$ we have

$$\frac{\sqrt{u}}{\sqrt{\varepsilon}}\rho'\left(\frac{|x - x_0|_N}{\sqrt{\varepsilon}}\right) \leq C\frac{\varepsilon^{k/2}}{\sqrt{\varepsilon}} \to 0 \text{ as } \varepsilon \to 0$$

(because $u(x) \leq L\,|x - x_0|_N^k \leq 2^k L\varepsilon^{k/2}$, for $|x - x_0|_N \leq 2\sqrt{\varepsilon} \leq \delta$ and $k \geq 2$). Then, $\sqrt{u}\nabla y_\varepsilon \to \sqrt{u}\nabla y$ a.e. in Ω, $\|\sqrt{u}\nabla y_\varepsilon\|_{L^2(\Omega)} \leq \|\sqrt{u}\nabla y\|_{L^2(\Omega)}$, hence

$$\sqrt{u}\nabla y_\varepsilon \to \sqrt{u}\nabla y \text{ strongly in } (L^2(\Omega))^N.$$

This implies that $y_\varepsilon \to y$ in $H_u^1(\Omega)$, as claimed. □

It is clear that u having the properties in Lemma 3.1 obeys the condition $\frac{1}{u} \notin L^1(\Omega)$, since $\frac{1}{u} \geq \frac{1}{L|x-x_0|_N^k}$ which is not integrable for $k \geq N$ in the ball $|x - x_0|_N^k \leq \delta$ (see again [22], p. 9).

Let us introduce the linear operator

$$A : H^1_u(\Omega) \to (H^1_u(\Omega))'$$

by

$$\langle Az, \psi\rangle_{V_u', V_u} = \int_\Omega u\nabla z \cdot \nabla\psi dx, \text{ for any } \psi \in H^1_u(\Omega), \tag{10}$$

and the Cauchy problem

$$\begin{aligned} \frac{dy}{dt}(t) + Ay(t) &= f(t), \text{ a.e. } t \in (0,T) \\ y(0) &= y_0. \end{aligned} \tag{11}$$

Definition 3.1. The operator A is called *strongly degenerate* if there exists $x_0 \in \Omega$ and u with the properties listed in (4).

Definition 3.2. Let $y_0 \in L^2(\Omega)$, $f \in L^2(0,T;(H^1_u(\Omega))')$. We call a *solution* to (1)-(3) with A strongly degenerate, a function

$$y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_u(\Omega)) \cap W^{1,2}([0,T];(H^1_u(\Omega))'), \tag{12}$$

which satisfies the equation

$$\int_0^T \left\langle \frac{dy}{dt}(t), \psi(t)\right\rangle_{V_u', V_u} dt + \int_Q u\nabla y \cdot \nabla\psi dxdt = \int_0^T \langle f(t), \psi(t)\rangle_{V_u', V_u} dt, \tag{13}$$

for any $\psi \in L^2(0,T;H^1_u(\Omega))$, and the initial condition $y(0) = y_0$.

We remark that this is a solution to (1)-(3) in the sense of distributions.

By C we shall denote several positive constants.

Theorem 3.1. *If $y_0 \in L^2(\Omega)$, $f \in L^2(0,T;(H^1_u(\Omega))')$, then* (11) *has a unique solution* (12) *satisfying the estimate*

$$\sup_{t\in[0,T]} \|y(t)\|_H^2 + \int_0^T \|y(t)\|_{V_u}^2 dt \leq C(\|y_0\|_H^2 + \|f\|^2_{L^2(0,T;V_u')}). \tag{14}$$

If, in addition $y_0 \in H^1_u(\Omega)$ and $f \in L^2(Q)$, then

$$y \in W^{1,2}([0,T];L^2(\Omega)) \cap L^2(0,T;H^2_u(\Omega)) \cap L^\infty(0,T;H^1_u(\Omega)) \tag{15}$$

and it satisfies

$$\begin{aligned} &\sup_{t\in[0,T]} \|y(t)\|_{V_u}^2 + \int_0^T \left(\left\| \frac{dy}{dt}(t)\right\|_H^2 + \|\nabla\cdot(u\nabla y(t))\|_H^2 \right) dt \\ \leq\ & C\left(\|y_0\|_{V_u}^2 + \|f\|^2_{L^2(Q)}\right). \end{aligned} \tag{16}$$

If $y_0 \geq 0$ a.e. in Ω and $f \geq 0$ a.e. in Q, *then $y(t) \geq 0$ a.e. in Ω for all $t \in [0,T]$.*

Proof. We immediately observe that the linear operator A is continuous and monotone

$$\|Az\|_{V_u'} = \sup_{\psi \in V_u, \|\psi\|_{V_u} \leq 1} \left| \langle Az, \psi \rangle_{V_u', V_u} \right| \leq \|z\|_{V_u}, \tag{17}$$

$$\langle Az, z \rangle_{V_u', V_u} \geq 0, \tag{18}$$

and has the property

$$\langle Az, z \rangle_{V_u', V_u} = \int_\Omega u \left| \nabla z \right|_N^2 dx = \|z\|_{V_u}^2 - \|z\|_H^2. \tag{19}$$

Let $y_0 \in L^2(\Omega)$, $f \in L^2(0,T;(H_u^1(\Omega))')$. Then the Cauchy problem has a unique solution belonging to the spaces indicated in (12) (see e.g. [23], p. 162).

The estimate (14) follows by setting in (13) $\psi = y$ and performing a few computations involving the Gronwall's lemma. If $y_0 \in H_u^1(\Omega)$ and $f \in L^2(Q)$, the results follow by Th. 4.2.1 in [22], p. 190. Then, we multiply the first equation in (11) by $\frac{dy}{dt}$ and integrate with respect to time, obtaining

$$\int_0^t \int_\Omega \left(\frac{dy}{ds} \right)^2 dxds + \frac{1}{2} \int_\Omega u \left| \nabla y \right|_N^2 dx = \frac{1}{2} \int_\Omega u \left| \nabla y_0 \right|_N^2 dx + \int_0^t \int_\Omega f \frac{dy}{ds} dxds,$$

whence we deduce

$$\left\| \frac{dy}{dt} \right\|_{L^2(0,T;H)}^2 + \left\| \sqrt{u} \nabla y(t) \right\|_H^2 \leq C \left(\left\| \sqrt{u} \nabla y_0 \right\|_H^2 + \|f\|_{L^2(Q)}^2 \right), \text{ for any } t \in [0,T].$$

By (1) we get that $\nabla \cdot (u \nabla y) \in L^2(Q)$ and so we obtain (16), as claimed.

Finally, the nonnegativity of the solution follows by testing (11) by the negative part $y^-(t)$ and integrating over $(0,t)$. We mention that $g \geq 0$ when $g \in (H_u^1(\Omega))'$ means that $\langle g, \psi \rangle_{V_u', V_u} \geq 0$ for any $\psi \in V_u$. We get, by Stampacchia Lemma

$$\frac{1}{2} \left\| y^-(t) \right\|_H^2 + \int_0^t \left\| \sqrt{u} \nabla y^-(s) \right\|_H^2 ds = -\int_0^t \left\langle f(s), y^-(s) \right\rangle_{V_u', V_u} ds + \frac{1}{2} \left\| y_0^- \right\|_H^2.$$

Since $y_0^- = 0$, f and y^- are nonnegative, it follows that $y^-(t) = 0$ and so $y(t) \geq 0$ a.e. in Ω, for all $t \in [0,T]$. □

For a later use we immediately introduce an approximating nondegenerate Cauchy problem and show that it approximates in some sense (11), as $\varepsilon \to 0$.

Thus, let ε be positive, consider $u_\varepsilon \in W^{1,\infty}(\Omega)$, $u_\varepsilon > 0$, and the problem

$$\begin{aligned} \frac{\partial y}{\partial t} - \nabla \cdot (u_\varepsilon \nabla y) &= f \text{ in } Q, \\ y(0) &= y_0 \text{ in } \Omega, \\ y &= 0 \text{ on } \Sigma. \end{aligned} \tag{20}$$

We introduce the Cauchy problem

$$\begin{aligned} \frac{dy}{dt}(t) + A_\varepsilon y(t) &= f(t), \text{ a.e. } t \in (0,T) \\ y(0) &= y_0, \end{aligned} \tag{21}$$

where $A_\varepsilon : V = H_0^1(\Omega) \to V' = H^{-1}(\Omega)$ by

$$\langle A_\varepsilon z, \psi \rangle_{V',V} = \int_\Omega u_\varepsilon \nabla z \cdot \nabla \psi dx, \text{ for any } \psi \in H_0^1(\Omega). \tag{22}$$

Also, we note that the solution to (21) defined as in Definition 3.2 (with the replacements $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ instead of $H_u^1(\Omega)$ and $(H_u^1(\Omega))'$) satisfies (20) in the sense of distributions. For each $\varepsilon > 0$, the operator A_ε is nondegenerate and if $y_0 \in L^2(\Omega)$, $f \in L^2(0,T;(H_u^1(\Omega))')$, it is immediately seen that (21) has, again by the Lions theorem, a unique solution

$$y_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap W^{1,2}([0,T];H^{-1}(\Omega)) \tag{23}$$

with the estimate

$$\sup_{t\in[0,T]} \|y_\varepsilon(t)\|_H^2 + \int_0^T \|\sqrt{u_\varepsilon}\nabla y_\varepsilon(t)\|_H^2 \, dt \le C(\|y_0\|_H^2 + \|f\|_{L^2(0,T;V_u')}^2). \tag{24}$$

If $y_0 \in H_u^1(\Omega)$, then the solution is more regular

$$y_\varepsilon \in W^{1,2}([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$$

and satisfies

$$\begin{aligned} &\sup_{t\in[0,T]} \|y_\varepsilon(t)\|_V^2 + \left\| \frac{dy_\varepsilon}{dt} \right\|_{L^2(0,T;H)}^2 + \int_0^T \|\nabla \cdot (u_\varepsilon \nabla y_\varepsilon(t))\|_H^2 \\ \le\ & C(\|\sqrt{u_\varepsilon}\nabla y_0\|_H^2 + \|f\|_{L^2(Q)}^2). \end{aligned} \tag{25}$$

Theorem 3.2. *Let $y_0 \in L^2(\Omega)$, $f \in L^2(Q)$, $u_\varepsilon \in W^{1,\infty}(\Omega)$, $u_\varepsilon(x) > 0$ for all $x \in \overline{\Omega}$, such that $u_\varepsilon \to u$ weak* in $L^\infty(\Omega)$ and $\nabla u_\varepsilon \to \nabla u$ weak* in $(L^\infty(\Omega))^N$. Let y_ε be the solution to* (21) *corresponding to u_ε. Then, as $\varepsilon \to 0$,*

$$y_\varepsilon \to y \text{ weakly in } L^2(0,T;H_u^1(\Omega)) \cap W^{1,2}([0,T];(H_u^1(\Omega))'), \tag{26}$$

$$y_\varepsilon(t) \to y(t) \text{ weakly in } L^2(\Omega) \text{ for all } t \in [0,T] \tag{27}$$

and y is the solution to (1)-(3) *corresponding to u. If $y_0 \in H_0^1(\Omega)$, then y belongs to the spaces indicated in* (15). *If y_0 and f are nonnegative then $y(t) \ge 0$ a.e. on Ω, for any $t \in [0,T]$.*

Proof. Let $y_0 \in L^2(\Omega)$. The solution y_ε satisfies (24) and we deduce the boundedness of the following sequences: $(y_\varepsilon)_\varepsilon$ in $L^\infty(0,T;H)$, $(\sqrt{u_\varepsilon}\nabla y_\varepsilon)_\varepsilon$ in $(L^2(0,T;H))^N$. On a subsequence (denoted still by $_\varepsilon$) we get

$$y_\varepsilon \to y \text{ weakly in } L^2(Q), \tag{28}$$

$$\sqrt{u_\varepsilon}\nabla y_\varepsilon \to \xi \text{ weakly in } (L^2(Q))^N. \tag{29}$$

We have to show that $\xi = \sqrt{u}\nabla y$ a.e. on Q. We denote $\xi_\varepsilon^i = \sqrt{u_\varepsilon}\frac{\partial y_\varepsilon}{\partial x_i}$, $i = 1,...,N$.

Let δ be positive, arbitrary and define the open subset $\Omega_\delta := \{x \in \Omega; u(x) > \delta\}$. Then $\xi^i_\varepsilon \to \xi^i$ weakly in $L^2(0,T;L^2(\Omega_\delta))$ too. Moreover, by the hypotheses it follows that $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$, as $\varepsilon \to 0$, and we have

$$\frac{\partial y_\varepsilon}{\partial x_i} = \frac{1}{\sqrt{u_\varepsilon}}\xi^i_\varepsilon \to \frac{1}{\sqrt{u}}\xi^i \text{ weakly in } L^2(0,T;L^2(\Omega_\delta)).$$

On the other hand, by (28) we deduce that $\frac{\partial y_\varepsilon}{\partial x_i} \to \frac{\partial y}{\partial x_i}$ in the sense of distributions. We conclude that $\xi^i = \sqrt{u}\frac{\partial y}{\partial x_i}$ a.e. on $(0,T)\times\Omega_\delta$ and since δ is arbitrary we finally get $\xi^i = \sqrt{u}\frac{\partial y}{\partial x_i}$ a.e. on Q. It follows that $y \in L^2(0,T;H^1_u(\Omega))$ and

$$y_\varepsilon \to y \text{ weakly in } L^2(0,T;H^1_u(\Omega)), \text{ as } \varepsilon \to 0.$$

By the definition (22) we have that $(A_\varepsilon y_\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;V')$ and consequently, by (21), $\left(\frac{dy_\varepsilon}{dt}\right)_\varepsilon$ is bounded in the same space. So, for any $\psi \in H^1_0(\Omega)$ we have that

$$\int_0^T \langle A_\varepsilon y_\varepsilon(t), \psi\rangle_{V',V}\, dt \to \int_Q u\nabla y\cdot\nabla\psi dx dt$$

and

$$\frac{dy_\varepsilon}{dt} \to \frac{dy}{dt} \text{ weakly in } L^2(0,T;V'),$$

hence $y \in C([0,T];L^2(\Omega))$.

Then, passing to the limit as $\varepsilon \to 0$ in the weak form of (21)

$$\int_0^T \left\langle \frac{dy_\varepsilon}{dt}(t), \psi\right\rangle_{V',V} dt + \int_Q u_\varepsilon \nabla y_\varepsilon\cdot\nabla\psi dx dt = \int_0^T\int_\Omega f\psi dx dt$$

we get

$$\int_0^T \left\langle \frac{dy}{dt}(t), \psi\right\rangle_{V',V} dt + \int_Q u \nabla y\cdot\nabla\psi dx dt = \int_0^T\int_\Omega f\psi dx dt$$

for any $\psi \in L^2(0,T;V)$. Since by Lemma 3.1, $V = H^1_0(\Omega)$ is dense in $H^1_u(\Omega)$ this relation takes place also for any $\psi \in H^1_u(\Omega)$ and so we proved the convergence of the solution to (21) to the solution to (11).

Still by (24) we have on a subsequence that $y_\varepsilon(t)$ converges weakly in $L^2(\Omega)$ and we shall prove that

$$y_\varepsilon(t) \to y(t) \text{ weakly in } L^2(\Omega), \text{ for all } t \in [0,T]. \tag{30}$$

Since y_ε is the solution to (21) we get that

$$\begin{aligned} y_\varepsilon(t) &= y_0 + \int_0^t (-A_\varepsilon y_\varepsilon(s) + f(s))ds \qquad (31)\\ &\to y_0 + \int_0^t (-Ay(s) + f(s))ds, \text{ weakly in } (H^1_u(\Omega))', \text{ for all } t \in [0,T]. \end{aligned}$$

From here we get that

$$\int_\Omega y_\varepsilon(t)\phi_0 dx = \int_\Omega y_0\phi_0 dx - \int_0^t\int_\Omega u_\varepsilon \nabla y_\varepsilon(s)\cdot\nabla\phi_0 + \int_0^t\int_\Omega f\phi_0 dxds \tag{32}$$

for any $\phi_0 \in H_0^1(\Omega)$ and $t\in[0,T]$. Passing to the limit we obtain

$$l(t) = \lim_{\varepsilon\to 0}\int_\Omega y_\varepsilon(t)\phi_0 dx = \int_\Omega y_0\phi_0 dx - \int_0^t\int_\Omega u\nabla y(s)\cdot\nabla\phi_0 + \int_0^t\int_\Omega f\phi_0 dxds.$$

We multiply this relation by $\phi_1\in L^2(0,T)$ and integrate over $(0,T)$, getting

$$\begin{aligned}&\int_0^T \phi_1(t)l(t)dt \\ = \ &\int_0^T\left(\int_\Omega y_0\phi_0 dx - \int_0^t\int_\Omega u\nabla y(s)\cdot\nabla\phi_0 + \int_0^t\int_\Omega f\phi_0 dxds\right)\phi_1(t)dt.\end{aligned} \tag{33}$$

We multiply (31) by $\phi_1(t)\phi_0(x)$ and integrate over $(0,T)\times\Omega$,

$$\begin{aligned}&\int_Q \phi_0\phi_1 y_\varepsilon dxdt \\ = \ &\int_0^T\left(\int_\Omega y_0\phi_0 dx - \int_0^t\int_\Omega u_\varepsilon\nabla y_\varepsilon(s)\cdot\nabla\phi_0 + \int_0^t\int_\Omega f\phi_0 dxds\right)\phi_1(t)dt.\end{aligned}$$

By the weak convergence $y_\varepsilon\to y$ in $L^2(Q)$ we get that

$$\begin{aligned}&\int_Q \phi_0\phi_1 y dxdt \\ = \ &\int_0^T\left(\int_\Omega y_0\phi_0 dx - \int_0^t\int_\Omega u\nabla y(s)\cdot\nabla\phi_0 + \int_0^t\int_\Omega f\phi_0 dxds\right)\phi_1(t)dt.\end{aligned} \tag{34}$$

Comparing (33) and (34) we deduce that

$$\int_0^T \phi_1(t)l(t)dt = \int_Q \phi_0\phi_1 y dxdt, \text{ for any } \phi_0\in H_0^1(\Omega), \phi_1\in L^2(0,T).$$

Using again the density proved in Lemma 3.1 we get

$$l(t) = \lim_{n\to\infty}\int_\Omega y_n(t)\phi_0 dx = \int_\Omega y(t)\phi_0 dx, \text{ for any } \phi_0\in H_u^1(\Omega), \ t\in[0,T],$$

i.e., $y_n(t)\to y(t)$ weakly in $(H_u^1(\Omega))'$. Since $y_\varepsilon(t)$ converges weakly also in $L^2(\Omega)$ it follows that this limit is $y(t)$ and

$$y_0 = y_\varepsilon(0)\to y(0),\ y_\varepsilon(T)\to y(T) \text{ weakly in } L^2(\Omega), \text{ as } \varepsilon\to 0. \tag{35}$$

If y_0 and f are nonnegative then $y_\varepsilon(t)\geq 0$ a.e. on Ω, for any $t\in[0,T]$, by Theorem 3.1 and this property is preserved by passing to the limit.

If $y_0 \in H_0^1(\Omega)$, the approximating solution satisfies (25). Then, on a subsequence $(\varepsilon \to 0)$ we have

$$\frac{dy_\varepsilon}{dt} \to \frac{dy}{dt} \text{ weakly in } L^2(Q),$$

$$\sqrt{u_\varepsilon}\nabla y_\varepsilon \to \sqrt{u}\nabla y \text{ weak* in } (L^\infty(0,T;L^2(\Omega)))^N.$$

By (20) we get that $\nabla \cdot (u_\varepsilon \nabla y)$ is bounded in $L^2(Q)$ and so

$$\nabla \cdot (u_\varepsilon \nabla y) \to \nabla \cdot (u \nabla y) \text{ weakly in } L^2(Q).$$

This implies that $u\nabla y \in (L^2(0,T;H^1(\Omega)))^N$ and so y belongs to the spaces indicated in (15). □

Lemma 3.2. *Let $u \in W^{1,\infty}(\Omega)$, $y_0 \in L^2(\Omega)$, $f \in L^2(Q)$ and let $y \in H_u^1(\Omega)$ be the solution to problem* (11) *corresponding to y_0. Then*

$$-\int_0^T \langle \nabla \cdot (u\nabla y(t)), z(t)\rangle_{V_u', V_u} dt = \int_Q u\nabla y \cdot \nabla z dx dt, \text{ for any } z \in H_u^1(\Omega). \tag{36}$$

Proof. Let us consider the approximating problem (21) corresponding to $u_\varepsilon(x) = u(x) + \varepsilon$. Then $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$ and we get the solution y_ε with the regularity (24). For any $z \in H_0^1(\Omega)$ we have

$$-\int_0^T \langle \nabla \cdot (u_\varepsilon \nabla y_\varepsilon(t)), z(t)\rangle_{V',V} dt = \int_Q u_\varepsilon \nabla y_\varepsilon \cdot \nabla z dx dt.$$

On the basis of Theorem 3.2 we pass to the limit and get (36). Next, we see that (36) is still preserved if $z \in H_u^1(\Omega)$, by the density result of Lemma 3.1. □

Remark 3.1. Let us observe that the operator defined by (10) is the operator associated to the sesquilinear form $a(u,v)$ defined as in the right hand side of (10) with domain V_u. Since, according to [24], Definitions 1.4-1.5, a is densely defined, accretive, continuous, closed and symmetric, then by [24], Propositions 1.24-1.51, we can deduce that the operator $-A$ generates a (C_0) contractive analytic semigroup on $L^2(\Omega)$. Note that the results of this Section were obtained in the context of real Hilbert spaces, but the conclusions remain valid also in the complex case. Hence one can also interpret the results of this Section as the N-dimensional extensions of the results in [19], but obtained without using the general framework given in [21].

4 Optimal control existence and generalized optimality conditions

In this section we will face the minimization problem (P) by focusing on the existence of a solution and on the necessary conditions it must satisfy.

In the sequel we assume the following hypotheses:

$$\begin{aligned} x_0 &\in \Omega,\ \rho \in [0,\infty[,\ u_m,\ u_M \in C(\overline{\Omega}), \\ 0 &< u_m(x) < u_M(x) \text{ for } x \in \overline{\Omega}\backslash\{x_0\}\ , u_m(x_0) = u_M(x_0) = 0, \\ \text{there exists } \alpha &\in C(\overline{\Omega}), \alpha \geq 1, \text{ such that } u_M(x) \leq \alpha(x) u_m(x) \text{ for } x \in \overline{\Omega}, \\ u_m(x) &\leq L\,|x - x_0|_N^k \text{ for } |x - x_0|_N^k \leq \delta,\ \delta > 0,\ k \geq N, \\ |\nabla u_M(x)|_N &< \rho,\ |\nabla u_m(x)|_N < \rho. \end{aligned} \tag{37}$$

Immediately it follows that

$$\int_\Omega \frac{1}{u_M(x)} dx = +\infty. \tag{38}$$

We denote

$$J(u) = \frac{\lambda_1}{2}\left(\int_\Omega y^u(T,x)dx - M_T\right)^2 + \frac{\lambda_2}{2}\left(\int_Q y^u dxdt - M_Q\right)^2 \tag{39}$$

and introduce the minimization problem

$$\text{Minimize } J(u) \text{ for all } u \in U, \tag{P}$$

subject to (1)-(3), where

$$\begin{aligned} U &= \{u \in W^{1,\infty}(\Omega);\ u_m(x) \leq u(x) \leq u_M(x), \\ u &= u_\Gamma \text{ a.e. on } \Gamma,\ |\nabla u(x)|_N \leq \rho \text{ a.e. } x \in \Omega\}, \end{aligned} \tag{40}$$

where u_Γ and ρ are known data. Then, by (37), for all $u \in U$ we have

$$0 < u_m|_\Gamma \leq u_\Gamma \leq u_M|_\Gamma\,. \tag{41}$$

Conditions (37) and $u \in U$ ensure that the operator A is strongly degenerate, because they imply that $u(x_0) = 0$, $u > 0$ in $\overline{\Omega}\backslash\{x_0\}$, and (38) establishes that $\frac{1}{u} \notin L^1(\Omega)$.

Moreover, the assumption $u_M(x) \leq \alpha(x)u_m(x)$ implies that if $u, v \in U$ then $\frac{v}{u}(x) \leq \|\alpha\|_{L^\infty(\Omega)}$ for $x \in \overline{\Omega}\backslash\{x_0\}$ and

$$H^1_u(\Omega) = H^1_v(\Omega) \text{ for any } u, v \in U. \tag{42}$$

Indeed, if $y \in H^1_u(\Omega)$ then $y \in L^2(\Omega)$, $\sqrt{u}\nabla y \in (L^2(\Omega))^N$, $y = 0$ on Γ. Let $v \in U$. By a simple calculation

$$\int_\Omega v\,|\nabla y|_N^2\,dx \leq \|\alpha\|_{L^\infty(\Omega)} \left\|\sqrt{u}\nabla y\right\|_H^2 \tag{43}$$

and so $y \in H^1_v(\Omega)$. Analogously, we get $H^1_v(\Omega) \subset H^1_u(\Omega)$.

Theorem 4.1. *Let $y_0 \in L^2(\Omega)$, $f \in L^2(Q)$. Then, (P) has at least one solution u with the corresponding state*

$$y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_u(\Omega)) \cap W^{1,2}([0,T]; (H^1_u(\Omega))').$$

If, in addition, $y_0 \in H^1_u(\Omega)$, then the state y is regular, as given by (15)*. For $y_0 \geq 0$ and $f \geq 0$ the state is nonnegative, $y(t) \geq 0$ for all $t \in [0,T]$.*

Proof. Under the first specified hypotheses, problem (1)-(3) has a unique (nonnegative) solution given by Theorem 3.1. Then, $J(u) \geq 0$, its infimum exists and it is nonnegative. Let us denote it by d.

For not overloading the notations we shall drop the superscript ^{u}.

We consider a minimizing sequence $(u_n)_{n\geq 1}$, $u_n \in U$ which satisfies

$$d \leq J(u_n) \leq d + \frac{1}{n} \tag{44}$$

where the corresponding state y_n is the solution to (1)-(3) (equivalently to (11)) with u replaced by u_n. By Theorem 3.1, first part, the solution y_n exists for each n, it is unique and satisfies

$$\sup_{t\in[0,T]} \|y_n(t)\|^2_H + \int_0^T \|\sqrt{u_n}\nabla y_n(t)\|^2_H\, dt \leq C\left(\|y_0\|^2_{V_u} + \|f\|^2_{L^2(0,T;V_u')}\right), \tag{45}$$

with C a positive constant independent of n, by (14).

Since $u_n \in U$, we deduce that there exists a subsequence (denoted still by the subscript $_n$) such that

$$u_n \to u \text{ weak* in } L^\infty(\Omega),\ \text{ as } n \to \infty,$$

$$\nabla u_n \to \nabla u \text{ weak* in } (L^\infty(\Omega))^N,\ \text{ as } n \to \infty,$$

$$y_n \to y \text{ weak* in } L^\infty(0,T;L^2(\Omega)),\ \text{ as } n \to \infty,$$

$$y_n(T) \to \zeta \text{ weakly in } L^2(\Omega),\ \text{ as } n \to \infty.$$

By (45), $(\sqrt{u_n}\nabla y_n)_n$ is bounded in $(L^2(Q))^N$ and there exists $\xi \in (L^2(Q))^N$ such that, on a subsequence (denoted still by the subscript $_n$)

$$\xi_n := \sqrt{u_n}\nabla y_n \to \xi \text{ weakly in } (L^2(Q))^N,\ \text{ as } n \to \infty.$$

The previous first two convergences for u_n imply that

$$u_n \to u \text{ uniformly on } \overline{\Omega},\ \text{ as } n \to \infty.$$

Since U is closed, then $u(x) \in [u_m(x), u_M(x)]$ which implies by (37) and (38) that $u(x) > 0$ on $\overline{\Omega}\backslash\{x_0\}$, $u(x_0) = 0$ and $\frac{1}{u} \notin L^1(\Omega)$, so that $u \in U$ and the corresponding operator A is strongly degenerate.

By a similar argument as developed in Theorem 3.2 it follows that

$$y_n \to y \text{ weakly in } L^2(0,T;H^1_u(\Omega)),\ \text{ as } n \to \infty.$$

Next, by the definition of A, see (10), we have that

$$Ay_n \to Ay \text{ weakly in } L^2(0,T;(H^1_u(\Omega))'),\ \text{ as } n \to \infty$$

and so by the state system

$$\frac{dy_n}{dt} \to \frac{dy}{dt} \text{ weakly in } L^2(0,T;(H^1_u(\Omega))'), \text{ as } n \to \infty.$$

Now, y_n satisfies (13)

$$\int_0^T \left\langle \frac{dy_n}{dt}(t), \psi(t) \right\rangle_{V_u', V_u} dt + \int_Q u_n \nabla y_n \cdot \nabla \psi dxdt = \int_Q f\psi dxdt,$$

for any $\psi \in L^2(0,T;H^1_u(\Omega))$ and passing to the limit as $n \to \infty$ we get that y satisfies (13), too. Moreover, we prove, as in Theorem 3.2, that

$$y_n(t) \to y(t) \text{ weakly in } L^2(\Omega) \text{ for all } t \in [0,T]$$

implying that $y_n(0) \to y(0)$ and $y_n(T) \to y(T) = \zeta$ weakly in $L^2(\Omega)$. All these assertions prove that y is the solution to (1)-(3) corresponding to u.

If $y_0 \in H^1_u(\Omega)$, then the solution y previously obtained has the regularity as in (15), according to Theorem 3.1.

Finally, we pass to the limit in (44) as $n \to \infty$, on the basis of the weakly lower semicontinuity of each convex term in $J(u_n)$, and get that $d = J(u)$. □

Optimality conditions. Next we are interested in determining the necessary conditions for problem (P).

Proposition 4.1. *Assume hypotheses of* Theorem 4.1 *and let* (u^*, y^*) *be a solution to* (P). *Then* u^* *satisfies the necessary condition*

$$\int_Q (u^* - u)(-\nabla y^* \cdot \nabla p) dxdt \geq 0 \tag{46}$$

for all $u \in U$, *where* p *is the solution to*

$$\frac{\partial p}{\partial t} + \nabla \cdot (u^* \nabla p) = \lambda_2 \left(\int_Q y^* dxdt - M_Q \right) \quad \text{in } Q, \tag{47}$$

$$p(T,x) = -\lambda_1 \left(\int_\Omega y^*(T,x) dx - M_T \right) \quad \text{in } \Omega, \tag{48}$$

$$p = 0 \quad \text{on } \Sigma. \tag{49}$$

Proof. Let (u^*, y^*) be a solution to (P), with the regularity (16), let $\lambda \in]0,1[$, $u \in U$ and denote

$$u^\lambda(x) = u^*(x) + \lambda v(x),$$

where

$$v(x) = u(x) - u^*(x), \ u \in U. \tag{50}$$

It is obvious that $v \in W^{1,\infty}(\Omega)$, $v(x_0) = 0$, $v = 0$ on Γ and $\frac{1}{v} \notin L^1(\Omega)$. We introduce the system

$$\frac{\partial Y}{\partial t} - \nabla \cdot (u^* \nabla Y) = \nabla \cdot (v \nabla y^*) \text{ in } Q, \tag{51}$$

$$Y(0,x) = 0 \text{ in } \Omega, \tag{52}$$

$$Y = 0 \text{ on } \Sigma. \tag{53}$$

We note that

$$\int_Q v \left|\nabla y^*\right|_N^2 dxdt \leq (\|\alpha\|_{L^\infty(\Omega)} + 1) \int_0^T \left\|\sqrt{u^*}\nabla y^*(t)\right\|_H^2 dt.$$

Problem (51)-(53) can be written as

$$\begin{aligned} \frac{dY}{dt}(t) + AY(t) &= f_1(t) \text{ a.e. } t \in (0,T), \\ Y(0) &= 0 \end{aligned} \tag{54}$$

where $f_1(t) \in V'_{u^*}$ and it is defined by $f_1(t)(\psi) = \int_\Omega v \nabla y^*(t) \cdot \nabla \psi dx$, a.e. $t \in (0,T)$, for any $\psi \in H^1_{u^*}(\Omega)$.

This problem has a unique solution

$$Y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_{u^*}(\Omega)) \cap W^{1,2}([0,T]; H^1_{u^*}(\Omega))') \tag{55}$$

which follows by Theorem 3.1.

Moreover, denoting by $y^\lambda(t,x)$ the solution to (1)-(3) corresponding to $u^\lambda(x)$, one can prove that actually

$$Y(t,x) = \lim_{\lambda \to 0} \frac{y^\lambda(t,x) - y^*(t,x)}{\lambda},$$

so that (51)-(53) is the system of first order variations. For simplicity we denote

$$I_T = \int_\Omega y^*(T,x)dx - M_T, \; I_Q = \int_Q y^* dxdt - M_Q. \tag{56}$$

We introduce the dual system (47)-(49), make the transformation $t \to T - t$ and, since the right-hand side in (47) is in $L^\infty(Q)$, we deduce that the transformed system has a unique regular solution given by Theorem 3.1, second part

$$p \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H^1_{u^*}(\Omega) \cap W^{1,2}([0,T]; L^2(\Omega)).$$

Now, we write that (u^*, y^*) is a solution to (P), that is

$$J(u^*) \leq J(u), \text{ for all } u \in U,$$

and, in particular, for $u = u^\lambda$. After some algebra, taking into account that v vanishes on the boundary, we get

$$\lambda_1 I_T \int_\Omega Y(T,x)dx + \lambda_2 I_Q \int_Q Y dxdt \geq 0. \tag{57}$$

We multiply (51) by $p(t)$ and integrate over $(0,T)$. After some calculations, applying Lemma 3.2 and recalling that $v=0$ on the boundary, we get

$$\int_Q -\left(p_t + \nabla\cdot(u^*\nabla p)\right) Y dxdt + \int_\Omega p(T,x)Y(T,x)dx = -\int_Q v\nabla y^*\cdot\nabla p dxdt.$$

This yields, by (47)-(49),

$$\lambda_1 I_T \int_\Omega Y(T,x)dx + \lambda_2 I_Q \int_Q Y dxdt = \int_Q v\nabla y^*\cdot\nabla p dxdt. \tag{58}$$

Comparing with (57) it follows that

$$\int_Q v\nabla y^*\cdot\nabla p dxdt \geq 0 \tag{59}$$

with $v=u-u^*$, for all $u\in U$. We note that this integral makes sense since

$$\int_Q v\nabla y^*\cdot\nabla p dxdt \leq (\|\alpha\|_{L^\infty(\Omega)}+1)\int_0^T \left\|\sqrt{u^*}\nabla y^*(t)\right\|_H \left\|\sqrt{u^*}\nabla p(t)\right\|_H dt.$$

Then, (59) implies (46), as claimed. □

5 Approximating problem

As we can see, (46) cannot provide any more information about u^*. However, the characterization of the structure of u^* can be established by deducing an approximating form of it. To this end, for each $\varepsilon>0$, we introduce an approximating problem (P_ε) involving a nondegenerate state equation. The approximating optimality conditions may be written more explicitly due to the better regularity of the approximating state and dual variable. Then we show that (P_ε) tends in some sense to (P) and so the sequence u^*_ε for which we prove a representation theorem can be used to approach u^*.

We introduce the problem

$$\text{Minimize } J(u) \text{ for all } u\in U_\varepsilon, \tag{P_ε}$$

subject to the state system (1)-(3), where

$$\begin{aligned} U_\varepsilon &= \{u\in W^{1,\infty}(\Omega);\ u_m(x)+\varepsilon \leq u(x)\leq u_M(x)+2\varepsilon, \\ u &= u_\Gamma^\varepsilon \text{ on } \Gamma,\ \ |\nabla u(x)|_N \leq \rho \text{ a.e. } x\in\Omega\}, \end{aligned} \tag{60}$$

with

$$u_m|_\Gamma + \varepsilon \leq u_\Gamma^\varepsilon \leq u_M|_\Gamma + 2\varepsilon, \text{ for all } x\in\Gamma.$$

The hypotheses made in (37), (38) remain the same, with the modification in the previous boundary condition and with $\varepsilon\leq u(x_0)\leq 2\varepsilon$.

For all $u\in U_\varepsilon$, $u(x)\geq u_m(x)+\varepsilon\geq\varepsilon$, and then system (1)-(3) with $u\in U_\varepsilon$ is nondegenerate. In fact, it becomes (20) and has a unique solution according to (23) and (24).

Obviously, the control problem (P_ε) has at least a solution $(u_\varepsilon, y_\varepsilon)$, with $u_\varepsilon \in U_\varepsilon$ and y_ε being the solution to (20) corresponding to u_ε.

Next we prove the convergence result of (P_ε) to (P) as $\varepsilon \to 0$.

Theorem 5.1. *Let $y_0 \in L^2(\Omega)$, $f \in L^2(Q)$. Let $(u_\varepsilon^*, y_\varepsilon^*)_{\varepsilon>0}$ be a sequence of solutions to (P_ε). Then, on a subsequence, as $\varepsilon \to 0$ we have,*

$$u_\varepsilon^* \to u^* \text{ weak* in } L^\infty(\Omega), \tag{61}$$

$$\nabla u_\varepsilon^* \to \nabla u^* \text{ weak* in } (L^\infty(\Omega))^N, \tag{62}$$

$$y_\varepsilon^* \to y^* \text{ weakly in } L^2(0,T;H^1_{u^*}(\Omega)) \cap W^{1,2}([0,T];(H^1_{u^*}(\Omega))'), \tag{63}$$

$$y_\varepsilon^*(T) \to y^*(T) \text{ weakly in } L^2(\Omega). \tag{64}$$

Moreover, y^* is the solution to (1)-(3) *corresponding to u^* and (u^*, y^*) is a solution to* (P).

Proof. Let $(u_\varepsilon^*, y_\varepsilon^*)$ be a solution to (P_ε), i.e.,

$$J(u_\varepsilon^*) \leq J(u_\varepsilon), \text{ for all } u_\varepsilon \in U_\varepsilon,$$

that is

$$\begin{aligned} &\frac{\lambda_1}{2}\left(\int_\Omega y_\varepsilon^*(T,x)dx - M_T\right)^2 + \frac{\lambda_2}{2}\left(\int_Q y_\varepsilon^* dxdt - M_Q\right)^2 \\ \leq\ &\frac{\lambda_1}{2}\left(\int_\Omega y_\varepsilon(T,x)dx - M_T\right)^2 + \frac{\lambda_2}{2}\left(\int_Q y_\varepsilon dxdt - M_Q\right)^2 \end{aligned} \tag{65}$$

for all $u_\varepsilon \in U_\varepsilon$.

Under our hypotheses it follows that problem (1)-(3) where u is replaced by $u_\varepsilon \in U_\varepsilon$ has a unique solution $y_\varepsilon \in C([0,T];L^2(\Omega)) \cap W^{1,2}([0,T];H^{-1}(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$. The sequence $(u_\varepsilon)_\varepsilon \subset U_\varepsilon$ is bounded and has a bounded gradient, so that one can extract a subsequence such that $u_\varepsilon \to u$ weak* in $L^\infty(\Omega)$, $\nabla u_\varepsilon \to \nabla u$ weak* in $(L^\infty(\Omega))^N$ and $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$. Obviously, we also get that $u \in U$. Consequently, the state $(y_\varepsilon)_\varepsilon$ converges on a subsequence to y the solution to (1)-(3) corresponding to u, as established by (26)-(27) in Theorem 3.2. It turns out that the right-hand side in (65) is bounded independently of ε. Similarly we argue for the pair $(u_\varepsilon^*, y_\varepsilon^*)$ getting (61)-(64).

Passing to the limit in (65) we get

$$\begin{aligned} &\frac{\lambda_1}{2}\left(\int_\Omega y^*(T,x)dx - M_T\right)^2 + \frac{\lambda_2}{2}\left(\int_Q y^* dxdt - M_Q\right)^2 \\ &\qquad \leq \liminf_{\varepsilon\to 0}\left(\frac{\lambda_1}{2}\left(\int_\Omega y_\varepsilon^*(T,x)dx - M_T\right)^2 + \frac{\lambda_2}{2}\left(\int_Q y_\varepsilon^* dxdt - M_Q\right)^2\right) = L_{\inf}. \end{aligned}$$

On the other hand

$$\begin{aligned} L_{\inf} &\leq \limsup_{\varepsilon\to 0}\left(\frac{\lambda_1}{2}\left(\int_\Omega y_\varepsilon(T,x)dx - M_T\right)^2 + \frac{\lambda_2}{2}\left(\int_Q y_\varepsilon dxdt - M_Q\right)^2\right) \\ &\leq \frac{\lambda_1}{2}\left(\int_\Omega y(T,x)dx - M_T\right)^2 + \frac{\lambda_2}{2}\left(\int_Q y dxdt - M_Q\right)^2, \end{aligned}$$

for all $u \in U$. This implies that (u^*, y^*) is a solution to (P). $\square$

Approximating optimality conditions. In the following we will deal with the approximating optimality conditions.

Let K be a closed convex subset of a Banach space X having the dual X'. We recall that the indicator function of K is

$$I_K(\xi) = \begin{cases} 0, & \text{if } \xi \in K, \\ +\infty, & \text{if } \xi \notin K, \end{cases}$$

and the subdifferential of I_K coincides with the normal cone to K at ξ (see [11], p. 4)

$$\partial I_K(\xi) = N_K(\xi) = \{\xi^* \in X';\ \ \langle \xi^*, \xi \rangle_{X',X} \geq 0\}.$$

Let $(u_\varepsilon^*, y_\varepsilon^*)$ be a solution to (P_ε). The system in variations, the dual system and the optimality conditions for (P_ε) are similarly obtained as those for (P). Namely, we introduce

$$\frac{\partial p_\varepsilon}{\partial t} + \nabla \cdot (u_\varepsilon^* \nabla p_\varepsilon) = \lambda_2 \left(\int_Q y_\varepsilon^* dxdt - M_Q \right) \text{ in } Q, \tag{66}$$

$$p_\varepsilon(T, x) = -\lambda_1 \left(\int_\Omega y_\varepsilon^*(T, x) dx - M_T \right) \text{ in } \Omega, \tag{67}$$

$$p_\varepsilon = 0 \text{ on } \Sigma. \tag{68}$$

By the transformation $t \to T - t$, we easily see that the approximating dual system (which is nondegenerate) has a unique solution in the same spaces as indicated in (23)

$$p_\varepsilon \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap W^{1,2}([0,T]; H^{-1}(\Omega)), \tag{69}$$

and satisfies the estimate deduced by (24)

$$\begin{aligned} &\sup_{t \in [0,T]} \|p_\varepsilon(t)\|_H^2 + \int_0^T \|\sqrt{u_\varepsilon} \nabla p_\varepsilon(t)\|_H^2 \, dt \qquad (70) \\ \leq\ & C \left\{ \lambda_1^2 \left(\int_\Omega y_\varepsilon^*(T,x) dx - M_T \right)^2 \text{meas}^2(\Omega) \right. \\ & \qquad \left. + \lambda_2^2 \left(\int_Q y_\varepsilon^* dxdt - M_Q \right)^2 \text{meas}^2(Q) \right\}. \end{aligned}$$

Let us denote $N_{U_\varepsilon}(u_\varepsilon^*) \subset L^1(\Omega)$ the normal cone to $U_\varepsilon \subset L^\infty(\Omega)$, in the duality pair $(L^\infty(\Omega), L^1(\Omega))$, that is (see e.g. [11], p. 13 and p. 232])

$$N_{U_\varepsilon}(u_\varepsilon^*) = \{w \in L^1(\Omega);\ \int_\Omega (u_\varepsilon^* - u_\varepsilon) w dx \geq 0, \text{ for all } u_\varepsilon \in U_\varepsilon\}.$$

Proposition 5.1. *Let $(u_\varepsilon^*, y_\varepsilon^*)$ be an optimal pair in (P_ε). Then, the approximating optimality condition reads*

$$\Phi_\varepsilon(x) = -\int_0^T \nabla y_\varepsilon^*(t,x) \cdot \nabla p_\varepsilon(t,x) dt, \tag{71}$$

where p_ε is the solution to (66)-(68). *Moreover, $\Phi_\varepsilon \in N_{U_\varepsilon}(u_\varepsilon^*)$ if and only if it has the representation*

$$\Phi_\varepsilon(x) = -\nabla \cdot \theta(x) + \mu(x) \textit{ in } \mathcal{D}'(\Omega), \tag{72}$$

where $\mu \in L^1(\Omega)$ and $\theta \in (L^1(\Omega))^N$ satisfy the system

$$\begin{cases} \mu(x) \leq 0 \textit{ in } \{x \in \Omega;\ u_\varepsilon^*(x) = u_m(x) + \varepsilon\} \\ \mu(x) = 0 \textit{ in } \{x \in \Omega;\ u_\varepsilon^*(x) \in (u_m(x) + \varepsilon, u_M(x) + 2\varepsilon)\} \\ \mu(x) \geq 0 \textit{ in } \{x \in \Omega;\ u_\varepsilon^*(x) = u_M(x) + 2\varepsilon\}, \end{cases} \tag{73}$$

and

$$\theta(x) = \begin{cases} 0 & \textit{a.e. in } \{x \in \Omega; |\nabla u_\varepsilon^*(x)|_N < \rho\} \\ \nu(x)\nabla u_\varepsilon^*(x) & \textit{a.e. in } \{x \in \Omega; |\nabla u_\varepsilon^*(x)|_N = \rho\} \end{cases} \tag{74}$$

with $\nu \in L^1(\Omega)$, $\nu \geq 1$ a.e. $x \in \Omega$.

Proof. The computations for the optimality condition are led in the same way as in Proposition 4.1 and it reads

$$\int_Q (u_\varepsilon^* - u_\varepsilon)\nabla y_\varepsilon^* \cdot \nabla p_\varepsilon dxdt \leq 0 \tag{75}$$

for all $u_\varepsilon \in U_\varepsilon$. Since y_ε^* and p_ε are in $L^2(0,T;H_0^1(\Omega))$, we can write (75) in the form

$$\int_\Omega (u_\varepsilon^* - u_\varepsilon)(x) \int_0^T \nabla y_\varepsilon^*(t,x) \cdot \nabla p_\varepsilon(t,x) dtdx \leq 0, \text{ for all } u_\varepsilon \in U_\varepsilon, \tag{76}$$

and observe that $\Phi_\varepsilon \in L^1(\Omega)$. In (72), $(\nabla \cdot)$ is the divergence operator and $\nabla \cdot \theta$ is considered in the sense of distributions. However, since Φ_ε and μ are in $L^1(\Omega)$ it follows that $\theta \in (W^{1,1}(\Omega))^N$.

Let $w \in N_{U_\varepsilon}(u_\varepsilon^*)$. We show that

$$w = -\nabla \cdot \theta(x) + \mu(x) \text{ a.e. } x \in \Omega, \tag{77}$$

with θ and μ previously defined. Let us note that U_ε can be written $U_\varepsilon = U_{1\varepsilon} + U_{2\varepsilon}$, where

$$U_{1\varepsilon} = \{v \in W^{1,\infty}(\Omega);\ \ |\nabla v(x)|_N \leq \rho \text{ a.e. } x \in \Omega,\ \ v|_\Gamma = u_\Gamma^\varepsilon\},$$

$$U_{2\varepsilon} = \{v \in L^\infty(\Omega);\ u_m(x) + \varepsilon \leq v(x) \leq u_M(x) + 2\varepsilon \text{ a.e. } x \in \Omega\}.$$

Observing that $U_{1\varepsilon} \cap$ int $U_{2\varepsilon} \neq \varnothing$, it follows that $\partial(I_1 + I_2) = \partial I_1 + \partial I_2$ (see [11], p. 7), where I_i is the indicator function of $U_{i\varepsilon}$ $(i = 1,2)$ and ∂I_i denotes its subdifferential. Since $N_{U_\varepsilon}(u_\varepsilon^*) = \partial I_{U_\varepsilon}(u_\varepsilon^*)$ it follows that $w \in$ is given by

$$w = \xi + \mu,\ \xi \in \partial I_1(u_\varepsilon^*),\ \mu \in \partial I_2(u_\varepsilon^*). \tag{78}$$

It is obvious that $\mu(x) \in \partial I_2(u_\varepsilon^*) = N_{U_{2\varepsilon}}(u_\varepsilon^*)$ if and only if $\mu(x)$ satisfies (73).

Let $\xi \in \partial I_1(u_\varepsilon^*)$ and $\gamma \in (W^{1,1}(\Omega))^N$ such that $\xi = -\nabla \cdot \gamma$ a.e. in Ω. Then, $\xi = -\nabla \cdot \gamma \in \partial I_1(u_\varepsilon^*)$ and this implies that

$$\gamma \in \partial I_F(\nabla u_\varepsilon^*), \tag{79}$$

where I_F is the indicator function of the set

$$F = \{\zeta \in (L^\infty(\Omega))^N;\ \zeta = \nabla v \text{ a.e. in } \Omega,\ v \in U_{1\varepsilon}\}.$$

Indeed, for all $\zeta = \nabla v \in F$, with $v \in U_{1\varepsilon}$, we have

$$\int_\Omega \gamma \cdot (\nabla u_\varepsilon^* - \nabla v)dx = -\int_\Omega (u_\varepsilon^* - v)\nabla \cdot \gamma dx = \int_\Omega \xi(u_\varepsilon^* - v)dx \geq 0.$$

The set F can be decomposed as $F_1 \cap F_2$ where

$$F_1 = \{\zeta \in (L^\infty(\Omega))^N;\ \zeta = \nabla v \text{ a.e. in } \Omega,\ v \in W^{1,\infty}(\Omega),\ \ v|_\Gamma = u_\Gamma^\varepsilon\},$$

$$F_2 = \{\zeta \in (L^\infty(\Omega))^N;\ \ |\zeta(x)|_N \leq \rho \text{ a.e. } x \in \Omega\}.$$

We may assume that there exists $w_0 \in W^{1,\infty}(\overline{\Omega})$ such that $|w_0| < \rho,\ \ w|_\Gamma = u_\Gamma^\varepsilon$. We note that $\nabla w_0 \in F_1 \cap$ int F_2 and so $\partial I_F = \partial I_{F_1} + \partial I_{F_2}$ (see again [11], p. 7). Therefore, $\gamma \in \partial I_F(\nabla u_\varepsilon^*)$ can be written $\gamma = \gamma_1 + \gamma_2$ with $\gamma_i \in \partial I_{F_i}(\nabla u_\varepsilon^*)$. The subdifferential ∂I_F is an application from $(L^\infty(\Omega))^N$ to $((L^\infty(\Omega))')^N$ (with $(L^\infty(\Omega))'$ the dual of $L^\infty(\Omega)$) and so, γ_i can be seen as an element belonging to $((L^\infty(\Omega))')^N$. It is represented as the sum of a continuous part $\gamma_{ia} \in (L^1(\Omega))^N$ and a singular part γ_{is} (see [11], p. 15). Then, $\gamma_{2a} \in \partial I_{F_2}(\nabla u_\varepsilon^*)$ a.e. $x \in \Omega$ and so it reads

$$\gamma_{2a}(x) = \begin{cases} 0 & \text{a.e. in } \{x \in \Omega; |\nabla u_\varepsilon^*(x)|_N < \rho\} \\ \nu(x)\nabla u_\varepsilon^*(x) & \text{a.e. in } \{x \in \Omega; |\nabla u_\varepsilon^*(x)|_N = \rho\}, \end{cases} \tag{80}$$

where $\nu \in L^1(\Omega)$, $\nu \geq 1$ a.e. in Ω (see [11], p. 13). Next, $\gamma_{1a} \in \partial I_{F_1}(\nabla u_\varepsilon^*)$ and so

$$\int_\Omega \gamma_{1a}(\nabla u_\varepsilon^* - \nabla v)dx = -\int_\Omega (u_\varepsilon^* - v)\nabla \cdot \gamma_{1a} dx \geq 0, \text{ for any } v \in F_1.$$

In particular setting $v := u_\varepsilon^* + l\phi$ with $\phi \in C_0^\infty(\Omega)$ and $l > 0$, we get

$$\int_\Omega \phi \nabla \cdot \gamma_{1a} dx \geq 0 \text{ for any } \phi \in C_0^\infty(\Omega).$$

Setting $v := u_\varepsilon^* - l\phi$ we deduce the inverse inequality and so it follows that

$$\int_\Omega \phi \nabla \cdot \gamma_{1a} dx = 0 \text{ for any } \phi \in C_0^\infty(\Omega),$$

which implies that $\nabla \cdot \gamma_{1a}(x) = 0$.

In conclusion, $\gamma = \gamma_{1a} + \gamma_{2a}$, where $\nabla \cdot \gamma_{1a} = 0$ a.e. in Ω and $\gamma_{2a} := \theta$ satisfies (80), so that we have obtained $\xi = -\nabla \cdot \gamma = -\nabla \cdot \theta$. We note also that

$$-\nabla \cdot \theta \in \partial I_1(u_\varepsilon^*) \text{ iff } \theta \in \partial I_F(\nabla u_\varepsilon^*). \tag{81}$$

Hence we get (77) and thus relation (72) follows because $\Phi_\varepsilon(x) \in N_{U_\varepsilon}(u_\varepsilon^*)$.

Conversely, if w is given by (71)-(74) then

$$\begin{aligned}
&\int_\Omega w(x)(u_\varepsilon^* - u_\varepsilon)dx = \int_\Omega \mu(x)(u_\varepsilon^* - u_\varepsilon)dx - \int_\Omega (u_\varepsilon^* - u_\varepsilon)\nabla \cdot \theta(x)dx \\
= &\int_\Omega \mu(x)(u_\varepsilon^* - u_\varepsilon)dx + \int_\Omega \theta(x) \cdot (\nabla u_\varepsilon^* - \nabla u_\varepsilon)dx \geq 0,
\end{aligned}$$

for any $u_\varepsilon \in U_\varepsilon$, by (73) for the first term and (81) for the last term. This means that $w \in N_{U_\varepsilon}(u_\varepsilon^*)$. □

On the subset $\{x \in \Omega; \theta(x) \neq 0\}$ the optimal control is given by the eikonal equation

$$|\nabla u_\varepsilon^*(x)|_N = \rho. \tag{82}$$

Weak solutions for this class of equations (with Dirichlet boundary conditions) are studied e.g., in [25] and [26].

Next, we provide a representation of the control u_ε^* in relation with the sign of Φ_ε.

Let us denote

$$U_M^\varepsilon = \{x \in \Omega; u_\varepsilon^*(x) < u_M(x) + 2\varepsilon\},\ U_m^\varepsilon = \{x \in \Omega; u_\varepsilon^*(x) > u_m(x) + \varepsilon\}$$

and

$$U_+^\varepsilon = \{x \in \Omega;\ \Phi_\varepsilon(x) > 0\},\ U_-^\varepsilon = \{x \in \Omega;\ \Phi_\varepsilon(x) < 0\}.$$

Proposition 5.2. *Assume that the set $D_+^\varepsilon = U_+^\varepsilon \cap U_M^\varepsilon$ is open, connected and has a smooth boundary. Then, u_ε^* is given by*

$$\begin{aligned}
u_\varepsilon^* = \sup \big\{z;\ z \in W^{1,\infty}(\Omega),\ |\nabla z(x)|_N \leq \rho \text{ a.e. } x \in D_+^\varepsilon, \\
z(x) \leq u_\varepsilon^*(x), \text{ for any } x \in \partial D_+^\varepsilon \big\}
\end{aligned} \tag{83}$$

$$|\nabla u_\varepsilon^*(x)|_N = \rho \text{ a.e. } x \in D_+^\varepsilon. \tag{84}$$

Proof. By Proposition 5.1 we have (72)-(74). On D_+^ε we have

$$\mu = \Phi_\varepsilon + \nabla \cdot \theta > \nabla \cdot \theta$$

because $\Phi_\varepsilon > 0$ on U_+^ε. We note that on the subset U_M^ε the function $\mu \leq 0$ and so it follows that $\nabla \cdot \theta < 0$ on D_+^ε. Then we prove that $F_+^\varepsilon = \varnothing$, where F_+^ε is defined as the interior of the subset $\{x \in D_+^\varepsilon; \theta(x) = 0\}$. Indeed, if $F_+^\varepsilon \neq \varnothing$, then there exists a ball $B \subset F_+^\varepsilon$, meas$(B) \neq 0$, such that

$$\int_B \varphi \nabla \cdot \theta dx = -\int_B \theta \cdot \nabla \varphi dx = 0, \text{ for any } \varphi \in C_0^\infty(B).$$

On the other hand,

$$\int_{D_+^\varepsilon} \varphi \nabla \cdot \theta dx < 0, \text{ for any } \varphi \in C_0^\infty(D_+^\varepsilon),\ \varphi \geq 0 \text{ on } D_+^\varepsilon,\ \varphi > 0 \text{ on } B.$$

This contradicts the previous equality. Then, by (82) we get (84).

Then we prove that u_ε^* is the maximum element of the set

$$E_+^\varepsilon = \{z \in W^{1,\infty}(\Omega);\ |\nabla z(x)|_N \leq \rho \text{ a.e. } x \in D_+^\varepsilon,\ z(x) \leq u_\varepsilon^* \ \forall x \in \partial D_+^\varepsilon\}.$$

To this end, let $z \in E_+^\varepsilon$, and use (72) to get on the right-hand that

$$\begin{aligned} & \int_{D_+^\varepsilon} (u_\varepsilon^*(x) - z(x))^- \nabla \cdot \theta(x) dx \\ = & \int_{D_+^\varepsilon} \mu(x)(u_\varepsilon^*(x) - z(x))^- dx - \int_{D_+^\varepsilon} \Phi_\varepsilon(x)(u_\varepsilon^*(x) - z(x))^- dx \leq 0. \end{aligned} \tag{85}$$

Since $\nabla \cdot \theta < 0$ on D_+^ε, we necessarily obtain that $(u_\varepsilon^*(x) - z(x))^- = 0$, meaning that $z(x) \leq u_\varepsilon^*(x)$ for any $z \in E_+^\varepsilon$ and so u_ε^* turns out to be the maximal element, as claimed. □

Remark 5.1. According to Proposition 5.2, p. 137 in [26], it follows that u_ε^* given by (83) is a viscosity solution, and also a weak solution for the eikonal equation (84). As the largest element of the set E_+^ε, it is unique.

As regards the assumption that D_+^ε is open we mention that this is the case when for instance Φ_ε is continuous. This can follow by the regularity results induced by more regular data (see [27]). Then, U_+^ε and consequently D_+^ε are open and D_+^ε can be written as a union of connected subsets, formula (83) being valid on any such subset.

We also specify that in the one-dimensional case there exist situations where the result of Proposition 5.3 (and also of Proposition 5.5 below) can be proved without any other hypotheses because the sign of Φ_ε is constant in Ω (see e.g., [12]). Such a situation was treated in [19] for mixed homogeneous Dirichlet and Neumann boundary conditions for the cost functional with $\lambda_2 = 0$.

Analogously, one can state the following result:

Proposition 5.3. *Assume that the set $D_-^\varepsilon = U_-^\varepsilon \cap U_m^\varepsilon$ is open, connected and has a smooth boundary. Then, u_ε^* is given by (84) and*

$$u_\varepsilon^* = \inf\left\{z;\ z \in W^{1,\infty}(\Omega),\ |\nabla z(x)|_N \leq \rho \ \textit{a.e. } x \in D_-^\varepsilon, \\ z(x) \geq u_\varepsilon^* \ \textit{for any } x \in \partial D_-^\varepsilon\right\}. \tag{86}$$

$$|\nabla u_\varepsilon^*(x)|_N = \rho \ \textit{a.e. } x \in D_-^\varepsilon. \tag{87}$$

In this case we get $\nabla \cdot \theta > 0$ on D_-^ε, and we prove that u_ε^* is the minimum element of the set

$$E_-^\varepsilon = \{z \in W^{1,\infty}(\Omega);\ |\nabla z(x)|_N \leq \rho \text{ a.e. } x \in D_-^\varepsilon,\ z(x) \geq u_\varepsilon^* \ \forall x \in \partial D_-^\varepsilon\}$$

and it is unique. It follows that $-u_\varepsilon^*$ is the viscosity solution to the eikonal equation (see [26]).

Remark 5.2. As seen, Theorem 5.1 allows the possibility of the computation of the optimal control u^* in (P) as the limit (61)-(62) which implies that $u^*_\varepsilon \to u^*$ uniformly in $\overline{\Omega}$. Moreover, we can retrieve the optimality conditions but (46) by passing directly to the limit in the approximating corresponding equations.

We recall that by Theorem 4.1 the approximating state $y^*_\varepsilon \to y$ weakly in $L^2(Q)$, $y^*_\varepsilon(T) \to y^*(T)$ weakly in $L^2(\Omega)$ and so it follows that the right-hand side in (70) is bounded independently on ε. Proceeding further as in the proof of Theorem 3.5 we obtain that

$$\begin{aligned} p_\varepsilon &\to p \textit{ weakly in } L^2(0,T;H^1_u(\Omega)) \cap W^{1,2}([0,T];(H^1_u(\Omega))'), \\ p_\varepsilon(T) &\to p(T) \text{ weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0, \end{aligned} \tag{88}$$

and p is the solution to the dual system (47)-(49).

Since we do not have estimates for ∇y^*_ε and ∇p_ε we cannot pass to the limit in (71), but we can do it in (83). We denote

$$D_+ = \bigcap_{\varepsilon \in (0,1)} D^\varepsilon_+$$

and assume that this is not empty. On each D^ε_+ we have (83). We define the subset

$$\{z \in W^{1,\infty}(\Omega);\ \ |\nabla z(x)|_N \le \rho \text{ a.e. } x \in D_+,\ z(x) \le u^*_\varepsilon\ \forall x \in \partial D_+\},$$

and pass to the limit as $\varepsilon \to 0$, getting

$$E_+ = \{z \in W^{1,\infty}(\Omega);\ \ |\nabla z(x)|_N \le \rho \text{ a.e. } x \in D_+,\ z(x) \le u^*\ \forall x \in \partial D_+\}.$$

Next, (85) follows exactly on D_+ and so we have $u^* = \sup E_+$.

We mention that in [12], [19], [20] exact expressions for the optimal control have been obtained in the one-dimensional case and in [9] in the two-dimensional case using the results in [26].

6 Construction of the optimal control in a particular case

In this section we describe an algorithm for constructing the controller u^*_ε in the square $\Omega = [0,L] \times [0,L]$, under the hypotheses of Proposition 5.2, where $\Phi_\varepsilon(x) > 0$ a.e. $x \in \Omega$. We also recall that $u_m(x) + \varepsilon < u^\varepsilon_\Gamma(x) < u_M(x) + 2\varepsilon$ for all $x \in \Gamma$.

We denote $x = (x_1, x_2)$ and $x_0 = (x^1_0, x^2_0)$ and describe the construction of u^*_ε.

Let us fix $x_1 = 0$ and let x_2 vary in $[0,L]$. In each section $x_2 = c$ (constant) we consider the curve γ_ε obtained by intersecting the surface $z = u_M(x) + 2\varepsilon$ with the plane $x_2 = c$. This curve has the equation $z = u_M(x_1, c) + 2\varepsilon$, for $x_1 \in [0,L]$. In the plane of this curve we have the point $z^\varepsilon_0 = u^\varepsilon_\Gamma(0,c)$ and write the equation of the line passing by z^ε_0 and having the slope ρ, i.e., $z = z^\varepsilon_0 + \rho x_1$. We call it line z_+. Assume that this intersects the curve γ_ε in the point $(x^{int}_{0,\varepsilon}, c)$, where $x^{int}_{0,\varepsilon}$ is the smallest solution to the equation

$$z^\varepsilon_0 + \rho x_1 = u_M(x_1, c) + 2\varepsilon.$$

Similarly, we consider the point $z_L^\varepsilon = u_\Gamma^\varepsilon(L,c)$ and write the equation of the line passing by z_L^ε and having the slope $-\rho$, i.e., $z = z_L^\varepsilon - \rho(x_1 - L)$. We call it line z_-. This intersects the curve γ_ε in the point $(x_{L,\varepsilon}^{int}, c)$, where $x_{L,\varepsilon}^{int}$ is the largest solution to the equation

$$z_L^\varepsilon - \rho(x_1 - L) = u_M(x_1, c) + 2\varepsilon.$$

These points of intersection exists under some compatibility conditions expressed by relations between the parameters ρ, u_m, u_M, u_Γ. We assume that they are fulfilled for some ρ large enough.

As $x_2 = c$ varies, the points $x_{0,\varepsilon}^{int}$, $x_{L,\varepsilon}^{int}$ determine a curve γ_ε^{int} in the plane x_1Ox_2 and this delimits the domains in which the expression of u_ε^* changes. It reads as

$$u_\varepsilon^*(x) = \begin{cases} u_{1\varepsilon}^*(x), & \text{for } (x_1,x_2) \in [0, x_{0,\varepsilon}^{int}] \times [0,L] \\ u_M(x) + 2\varepsilon, & \text{for } (x_1,x_2) \in [x_{0,\varepsilon}^{int}, x_{L,\varepsilon}^{int}] \times [0,L] \\ u_{2\varepsilon}^*(x), & \text{for } (x_1,x_2) \in [x_{L,\varepsilon}^{int}, L] \times [0,L] \end{cases} \tag{89}$$

where $z = u_{1\varepsilon}^*(x)$ is the surface generated by the segment $[z_0^\varepsilon, x_{0,\varepsilon}^{int}]$ and $u_{2\varepsilon}^*(x)$ is the surface generated by the segment $[z_L^\varepsilon, x_{L,\varepsilon}^{int}]$ as c varies.

Following Remark 5.2, since $u_\varepsilon^* \to \widehat{u^*}$ uniformly in x we get by (89) that

$$\widehat{u^*}(x) = \begin{cases} \widehat{u_1^*}(x), & \text{for } (x_1,x_2) \in [0, x_0^{int}] \times [0,L] \\ u_M(x), & \text{for } (x_1,x_2) \in [x_0^{int}, x_L^{int}] \times [0,L] \\ \widehat{u_2^*}(x), & \text{for } (x_1,x_2) \in [x_L^{int}, L] \times [0,L] \end{cases} \tag{90}$$

where the point x_0^{int} is the intersection of the line of slope ρ starting by $z_0 = u_\Gamma(0,c)$ with the curve $z = u_M(x_1,c)$ and the point x_L^{int} is the intersection of the line of slope $-\rho$ starting by $z_L = u_\Gamma(L,c)$ with the curve $z = u_M(x_1,c)$. The surfaces $z = \widehat{u_1^*}(x)$ and $z = \widehat{u_2^*}(x)$ are generated by the segments $[z_0, x_0^{int}]$ and $[z_L, x_L^{int}]$, respectively, as c varies.

Now, let us fix $x_2 = 0$ and let $x_1 = c$ vary in $[0,L]$. By a similar procedure we finally construct the points $\widetilde{z_0} = u_\Gamma(c,0)$, $\widetilde{z_L} = u_\Gamma(c,L)$, y_0^{int}, y_L^{int} (the analogues of x_0^{int}, x_L^{int}) and the surfaces $\widetilde{u_1^*}$ and $\widetilde{u_2^*}$ generated by $[\widetilde{z_0}, y_0^{int}]$ and $[z_L, y_L^{int}]$, as c varies. All these lead to the construction of the function

$$\widetilde{u^*}(x) = \begin{cases} \widetilde{u_1^*}(x), & \text{for } (x_1,x_2) \in [0,L] \times [0, y_0^{int}] \\ u_M(x), & \text{for } (x_1,x_2) \in [0,L] \times [y_0^{int}, y_L^{int}] \\ \widetilde{u_2^*}(x), & \text{for } (x_1,x_2) \in [0,L] \times [y_L^{int}, L]. \end{cases} \tag{91}$$

The final surface u^* is represented by the reunion of the surfaces $\widehat{u^*}$ and $\widetilde{u^*}$.

Finally, we present in Fig. 1 the graphic of the optimal control u^* given by the union of the surfaces described by (90)-(91), computed with Comsol Multiphysics v. 3.5a (FLN License 1025226), in the square domain $\Omega = (0,1) \times (0,1)$, for the following data

$$\begin{aligned} x_0 &= (x_0^1, x_0^2) = (0.5, 0.5),\ u_\Gamma(x) = 0.6,\ \rho = 10, \\ u_M(x) &= \alpha_M \left| x - x_0 \right|^2,\ u_m(x) = \alpha_m \left| x - x_0 \right|^2,\ \alpha_M = 3,\ \alpha_m = 0.5. \end{aligned}$$

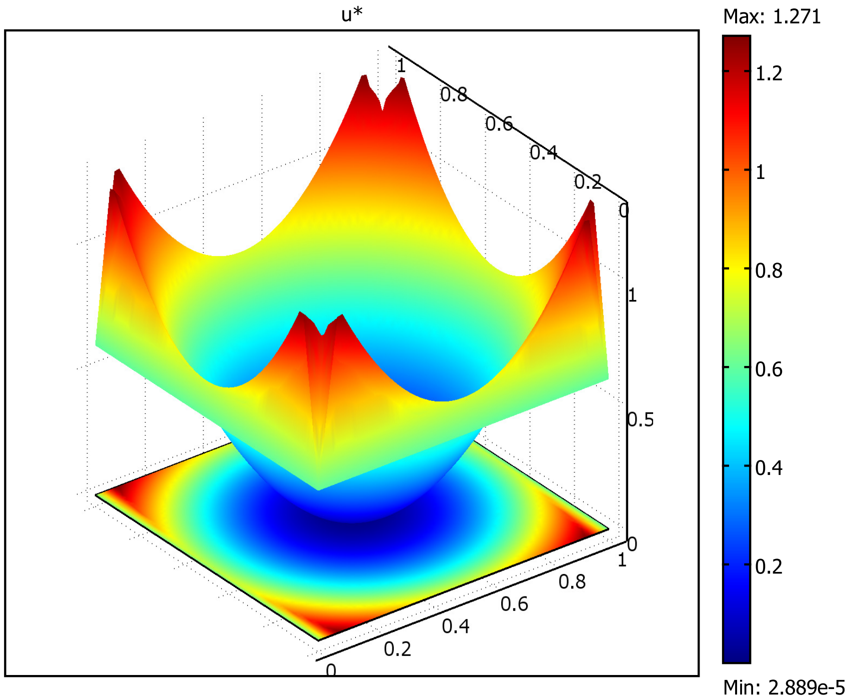

Fig. 1. Optimal control u^* given by (90)-(91)

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