# A shift in the Strauss exponent
# for semilinear wave equations
# with a not effective damping

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## Abstract

In this note we study the global existence of small data solutions to the Cauchy problem for the semilinear wave equation with a *not effective scale-invariant* damping term, namely

$$v_{tt} - \triangle v + \frac{2}{1+t} v_t = |v|^p, \quad v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x),$$

where $p > 1$, $n \geq 2$. We prove blow-up in finite time in the subcritical range $p \in (1, p_2(n)]$ and existence theorems for $p > p_2(n)$, $n = 2, 3$. In this way we find the critical exponent for small data solutions to this problem. Our results lead to the conjecture $p_2(n) = p_0(n + 2)$ for $n \geq 2$, where $p_0(n)$ is the Strauss exponent for the classical semilinear wave equation with power nonlinearity.

*Keywords:* semilinear damped wave equation, not effective damping, small data global existence, Strauss exponent
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## 1. Introduction

In this paper we study the global existence (in time) of small data solutions to

$$\begin{cases} v_{tt} - \triangle v + \dfrac{2}{1+t} v_t = |v|^p, & t > 0, \quad x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \\ v_t(0, x) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

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in space dimensions $n = 2, 3$. The damping term of this model is not effective (see [29]). Nevertheless, there should be an improving influence on the critical exponent $p_{crit}$ in comparison with Strauss exponent $p_0(n)$ for

$$\begin{cases} w_{tt} - \triangle w = |w|^p, & t > 0, \quad x \in \mathbb{R}^n, \\ w(0, x) = w_0(x), & x \in \mathbb{R}^n, \\ w_t(0, x) = w_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{2}$$

here $p_0(n)$ is the positive solution to

$$(n - 1) p^2 - (n + 1) p - 2 = 0.$$

By *critical exponent* $p_{crit}$ for (1) or (2) we mean, that for small initial data in a suitable space, there exist global (in time) solutions for some range of admissible $p > p_{crit}$, and there exist suitable (even small) data such that there exist no global (in time) solutions if $p \in (1, p_{crit}]$.

It has been recently shown that the *critical exponent* for models with effective dissipation, this means, $\mu$ is sufficiently large (see [29]) in the model

$$\begin{cases} v_{tt} - \triangle v + \dfrac{\mu}{1 + t} v_t = |v|^p, & t > 0, \quad x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \\ v_t(0, x) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

is $p_{crit} = 1 + 2/n$ (see Section 2 for details). The exponent $1 + 2/n$ is the same as for the semilinear heat equation, and it is related to the *effectiveness* of the damping, i.e., the property of the damping term to make suitable linear estimates for the wave equation similar to the ones for the corresponding heat equation $\mu v_t - (1 + t)\triangle v = 0$ (in particular, the $L^1 - L^p$ low frequencies estimates). We set

$$p_\infty(n) = 1 + 2/n,$$

where the index $\infty$ means that $\mu$ is sufficiently large.

On the contrary, it seems to be not so easy to show that for small positive values of $\mu$, for example, $\mu = 2$ the *critical exponent* $p_\mu(n)$ is strictly larger than $p_\infty(n)$. Summarizing all these explanations one would expect

$$p_\infty(n) < p_\mu(n) < p_0(n)$$

for $\mu$ in a suitable range. Indeed, this happens because one of the main goals of the paper is to prove $p_\infty(3) < p_2(3) < p_0(3)$.
In this paper, we reach this aim by setting $\mu = 2$ in (3), and showing that

$$p_2(n) := \max\{p_0(n + 2); p_\infty(n)\} = \begin{cases} 3 & \text{if } n = 1, \\ p_0(n + 2) & \text{if } n = 2, 3. \end{cases} \tag{4}$$

We notice that $p_\infty(2) = p_0(4) = 2$ and $p_\infty(3) < p_0(5)$. Hence, for $n = 3$ and $\mu = 2$ in (3), we really feel the influence of the not effective dissipation term.

We prove the following results:

**Theorem 1.** *Assume that $v \in C^2([0, T) \times \mathbb{R}^n)$ is a solution to (1) with (even small) initial data $(v_0, v_1) \in C_c^2(\mathbb{R}^n) \times C_c^1(\mathbb{R}^n)$ such that $v_1, v_0 \geq 0$, and $(v_0, v_1) \not\equiv (0, 0)$. If $p \in (1, p_2(n)]$, then $T < \infty$.*

Being the 1-dimensional existence result already proved in [3], in one space dimension, since the dissipation in (1) is effective, with respect to the linear estimates employed in [3], we prove the existence result for space dimension $n = 2$ and $n = 3$.

**Theorem 2.** *Let $n = 2$ and $p > 2 = p_0(4)$. Let $(\bar{v}_0, \bar{v}_1) \in C_c^2(\mathbb{R}^2) \times C_c^1(\mathbb{R}^2)$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, if $v_0 = \varepsilon\bar{v}_0$ and $v_1 = \varepsilon\bar{v}_1$, then the Cauchy problem (1) admits a unique global (in time) small data solution $v \in C([0, \infty), H^2) \cap C^1([0, \infty), H^1) \cap C^2([0, \infty), L^2)$.*

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**Theorem 3.** *Let $n = 3$ and $p > p_0(5)$. Let $(\bar{v}_0, \bar{v}_1) \in C_c^2(\mathbb{R}^3) \times C_c^1(\mathbb{R}^3)$, be radial. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, if $v_0 = \varepsilon \bar{v}_0$ and $v_1 = \varepsilon \bar{v}_1$, then the Cauchy problem (1) admits a unique global (in time) small data radial solution $v \in C([0, \infty) \times \mathbb{R}^3) \cap C^2([0, \infty) \times (\mathbb{R}^3 \setminus \{0\}))$.*

For the sake of brevity, we use the notation

$$\langle y \rangle = 1 + |y| \text{ for any } y \in \mathbb{R}^n.$$

To prove our results we perform the change of variable $u(t, x) = \langle t \rangle v(t, x)$. So, problem (1) becomes

$$\begin{cases} u_{tt} - \triangle u = \langle t \rangle^{-(p-1)} |u|^p, & t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n \end{cases} \tag{5}$$

with $u_0 = v_0$ and $u_1 = v_0 + v_1$. This means we are dealing with a semilinear wave equation with a time-dependent coefficient in the nonlinearity. For proving Theorem 1 we will extend to this equation the classical blow-up technique due to R.T. Glassey. For Theorem 2 we use Klainerman vector fields. Due to the lack of regularity of the nonlinear term, for $p \in (p_0(5), 2)$, the proof of Theorem 3 requires a different idea and we shall restrict to radial solutions. We will establish an appropriate version of the pointwise estimates for the wave equation.

By the aid of these estimates, in this latter case, we will also find a decay behaviour for the solution to (1) which is the same as for solutions to the $(n + 2)$-dimensional wave-equations. For details, see Theorem 6 and Remark 6. We expect that the technique of the pointwise estimates could be applied to prove the existence for $p > p_0(n + 2)$ with $n \geq 4$. Consequently, the improving influence of the dissipation term on the one hand and the not effective behaviour on the other hand can be expressed by a shift of Strauss exponent $p_0(n)$ to $p_0(n + 2)$ as in the title of this paper (see also Remark 10 and [5]).

## 2. An overview of some existing results

### 2.1. Wave model

For the Cauchy problem (2) it is well-known that the *critical exponent* for the existence of global (in time) small data solutions is $p_0(n)$. More precisely, if $1 < p \leq p_0(n)$, then solutions to (2) blow-up in finite time for a suitable choice of initial data (see [9], [13], [14], [20], [21], [26]), whereas for $p \in (p_0(n), (n + 3)/(n - 1)]$ a unique global (in time) small data solution exists (see [8], [10], [13], [24], [30]). In space dimension $n = 1$, solutions to (2) blow-up in finite time for any $p > 1$, hence, we put $p_0(1) = \infty$ (see [9]).

### 2.2. Scale-invariant damped wave models

Known results on the global existence of small data solutions to (3) can be summarized as follows:

- Non-existence of weak solutions for $\mu > 1$ and $p \leq 1 + 2/n$, provided $\int_{\mathbb{R}^n} (u_0 + (\mu - 1)^{-1} u_1) \, dx > 0$ (see Theorem 1.1 and Example 3.1 in [4]).

- Non-existence of weak solutions for $\mu \in (0, 1]$ and $p \leq 1 + 2/(n - 1 + \mu)$, provided $\int_{\mathbb{R}^n} u_1 \, dx > 0$ (see Theorem 1.4 in [28]).

- According to Theorems 2 and 3 in [3] global (in time) existence of energy solutions for small data if $p > 1 + 2/n$ and

  – if $n = 1$ and $\mu \geq 5/3$,
  – if $n = 2$ and $\mu \geq 3$,
  – for any $n \geq 3$ if $\mu \geq n + 2$.

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*2.3. Useful transformations*

The equation in (3) has many interesting properties. In particular, if $\mu \in \mathbb{R}$, by the change of variables

$$v^{\sharp}(t, x) = \langle t \rangle^{\mu - 1} v(t, x) \tag{6}$$

one sees that $v$ solves (3) if, and only if, $v^{\sharp}(t, x)$ solves

$$\begin{cases} v_{tt}^{\sharp} - \triangle v^{\sharp} + \dfrac{\mu^{\sharp}}{\langle t \rangle} v_t^{\sharp} = \langle t \rangle^{(\mu^{\sharp} - 1)(p-1)} |v^{\sharp}|^p, \\ v^{\sharp}(0, x) = v_0(x), \\ v_t^{\sharp}(0, x) = v_1(x) + (1 - \mu^{\sharp}) v_0(x) \end{cases} \tag{7}$$

with $\mu^{\sharp} = 2 - \mu$.

If $\mu \in (-\infty, 1)$ in (3), by introducing the change of variable $\tilde{v}(t, x) = v(\Lambda(t) - 1, x)$, where

$$\Lambda(t) := \frac{\langle t \rangle^{\ell+1}}{\ell + 1}, \qquad \text{and} \quad \ell = \frac{\mu}{1 - \mu}, \tag{8}$$

the Cauchy problem (3) becomes a Cauchy problem for a semilinear free wave equation with polynomial propagation speed

$$\begin{cases} \tilde{v}_{tt} - \langle t \rangle^{2\ell} \triangle \tilde{v} = \langle t \rangle^{2\ell} |\tilde{v}|^p, \\ \tilde{v}(\bar{t}, x) = v_0(x), \\ \tilde{v}_t(\bar{t}, x) = (1 - \mu)^{-\mu} v_1(x), \end{cases} \tag{9}$$

where $\bar{t} = (1 - \mu)^{-(1-\mu)} - 1$. We notice that:

- $\ell > 0$ if, and only if, $\mu \in (0, 1)$. On the other hand, $\ell \in (-1, 0)$ if $\mu \in (-\infty, 0)$;

- $\bar{t} \in (0, e^{\frac{1}{e}} - 1]$ if $\mu \in (0, 1)$ and $\bar{t} \to 0$ as $\mu \to 0$ and as $\mu \to 1$;

- $\bar{t} \in (-1, 0)$ if $\mu \in (-\infty, 0)$.

Similarly, by virtue of (6), (7) and (8), if $\mu > 1$, the Cauchy problem (3) becomes

$$\begin{cases} \tilde{v}_{tt}^{\sharp} - \langle t \rangle^{2\ell^{\sharp}} \triangle \tilde{v}^{\sharp} = c_{\mu} \langle t \rangle^{2\ell^{\sharp} - (p-1)} |\tilde{v}^{\sharp}|^p, \\ \tilde{v}^{\sharp}(t^{\sharp}, x) = v_0(x), \\ \tilde{v}_t^{\sharp}(t^{\sharp}, x) = (\mu - 1)^{-(2-\mu)} (v_1(x) + (\mu - 1) v_0(x)), \end{cases} \tag{10}$$

where $\ell^{\sharp} = (2 - \mu)/(\mu - 1)$, $t^{\sharp} = (\mu - 1)^{-(\mu - 1)} - 1$, and $c_{\mu} = (\mu - 1)^{(\mu - 1)(p-1)}$.

On the other hand, if $\mu = 1$, by setting $\Lambda(t) = e^t$, the Cauchy problem (3) becomes

$$\begin{cases} \tilde{v}_{tt} - e^{2t} \triangle \tilde{v} = e^{2t} |\tilde{v}|^p, \\ \tilde{v}(0, x) = v_0(x), \\ \tilde{v}_t(0, x) = v_1(x). \end{cases} \tag{11}$$

By means of all these transformations, following the reasoning in Example 4.4 in [4], we can obtain as in [28] the non-existence of global (in time) weak solutions to (3) for $\mu \in (0, 1)$ and

$$p \leq 1 + \frac{2(\ell + 1)}{n(\ell + 1) - 1} = 1 + \frac{2}{n - 1 + \mu}.$$

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*2.4. Blow-up dynamics*

Since in [4, 28] the test function method is employed, the blow-up dynamic remains unknown. However, one can apply an argument similar to those developed in [9], to (9), (10) and (11), obtaining that all the $L^q$ norms of local solutions blow-up in finite time. Indeed, in Example 2a in [25] the author gives sufficient conditions on

$$u_{tt} - a^2(t)\Delta u = m(t)|u|^p,$$

which guarantee that $\lim_{t \to T} \|u(t)\|_q = +\infty$ for any $1 \le q \le +\infty$, where $T$ is the maximal existence time for a smooth solution with nonnegative, compactly supported, initial data. See also [7] for the 1-dimensional case. By means of (9), (10) and (11) from these results one can deduce the blow-up in finite time for (3)

- if $\mu \in (0, 1)$ and $p < 1 + \dfrac{2}{n - 1 + \mu}$,

- if $\mu = 1$ and $p \le p_\infty$,

- if $\mu \in (1, 2]$ and $p < p_\infty$.

We notice that blow-up in finite time is proved for the limit case $p = 1 + 2/(n - [1 - \mu]^+)$ only for $\mu = 1$, while non-existence of weak solutions for $\mu \in (0, 1) \cup (1, 2]$ is also known for $p = 1 + 2/(n - [1 - \mu]^+)$. Up to our knowledge we have no other information from literature about existence or non-existence for (3). In particular, the blow-up dynamic is unknown for $\mu > 2$.

After this discussion, it was natural to ask if the blow-up exponent $p_\infty(n) = 1 + 2/n$ could be improved for some $\mu \in [1, 5/3)$ if $n = 1$, for some $\mu \in [1, 3)$ if $n = 2$, or for some $\mu \in [1, n + 2)$ if $n \ge 3$. On the other hand, one may ask if a counterpart result of global (in time) existence can be proved. Theorems 1, 2, 3 give a positive answer to these questions in the special case $\mu = 2$. These results may give precious hints about the general case of small $\mu$.

*2.5. Space-dependent damping term*

For the sake of completeness, we remark that the case of wave equation with space-dependent damping

$$\begin{cases} v_{tt} - \Delta v + \mu\langle x\rangle^{-\alpha} v_t = |v|^p, & t > 0, \quad x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \\ v_t(0, x) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{12}$$

where $\mu > 0$ and $\alpha \in (0, 1]$, is also particularly difficult when $\alpha = 1$. On the one hand, in [11], the authors proved that the critical exponent for the existence of global (in time) small data solutions is $1 + 2/(n - \alpha)$ if $\alpha \in (0, 1)$. On the other hand, in [12] they proved for $\alpha = 1$ that the estimates for the energy of solutions to the linear model of (12) show a decay rate which depends on $\mu$ for $\mu < n$. This property hints to a $\mu$-depending critical exponent for (12) for small $\mu$.

To complete our overview, we mention that the critical exponent for the wave equation with time-dependent damping $\mu\langle t\rangle^\kappa u_t$ is $1 + 2/n$ if $\kappa \in (-1, 1)$ (see [6, 18, 19]), whereas global existence of small data solutions for $p > 1 + 2/(n - \alpha)$ for the wave equation with damping $\mu\langle x\rangle^{-\alpha}\langle t\rangle^{-\beta}$, if $\alpha, \beta > 0$ and $\alpha + \beta < 1$ has been derived in [27].

## 3. Proof of Theorem 1

Let us remind a lemma on the blow-up dynamics for ordinary differential inequalities with polynomial nonlinearity. This result will play a fundamental role in our approach.

**Lemma 1.** *Let $p > 1$, $q \in \mathbb{R}$. Let $F \in C^2([0, T))$, positive, satisfying*

$$\ddot{F}(t) \ge K_1(t + R)^{-q}(F(t))^p \qquad \text{for any } t \in [T_1, T) \tag{13}$$

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*for some $K_1, R > 0$, and $T_1 \in [0, T)$. If*

$$F(t) \geq K_0(t + R)^a \qquad \text{for any } t \in [T_0, T), \tag{14}$$

*for some $a \geq 1$ satisfying $a > (q - 2)/(p - 1)$, and for some $K_0 > 0$, $T_0 \in [0, T)$, then $T < \infty$.*

*Moreover, let $q \geq p + 1$ in (13). Then there exists a constant $K_0 = K_0(K_1) > 0$ such that, if (14) holds with $a = (q - 2)/(p - 1)$ for some $T_0 \in [0, T)$, then $T < \infty$.*

Proof. The case $a > (q-2)/(p-1)$ corresponds to Lemma 4 in [21]. Let $a = (q-2)/(p-1)$. Following Lemma 2.1 in [26], our problem reduces to find $K_0$ such that (14) holds and the function $G(s) = (T_0 + 1)^{-a} F((T_0 + 1)s + 1)$ blows up. One has

$$\ddot{G}(s) \geq K_1 \langle s \rangle^{-q} (G(s))^p,$$
$$G(s) \geq \tilde{K}_0 \langle s \rangle^{\frac{q-2}{p-1}},$$

respectively, with $\tilde{K}_0 = K_0 \min\{1; (1 + R)/(1 + T_0)\}$. It follows with a possibly larger constant $K_0$ that $\dot{G}$ is positive, so that from $\ddot{G}(s) \geq K_1 \tilde{K}_0^{p-1} \langle s \rangle^{-2} (G(s))$ one has $G(s) \geq \langle s \rangle^{\tilde{K}_0^{p-1} K_2}$. For large $K_0$, the exponent $a := \tilde{K}_0^{p-1} K_1$ satisfies $a > (q - 2)/(p - 1)$, and we may conclude the proof. These ideas are contained in [9]. $\square$

Transforming problem (1) into (5) Theorem 1 follows as a consequence of the next proposition. Here we follow [26], taking into account of the time-dependence of the nonlinear term.

**Proposition 1.** *Let $f \in C^2(\mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$ be nonnegative, compactly supported, such that $f + g \not\equiv 0$. Assume that $u \in C^2([0, T) \times \mathbb{R}^n)$ is the maximal (with respect to the time interval) solution to*

$$\begin{cases} u_{tt} - \triangle u = \langle t \rangle^{-(p-1)} |u|^p, \\ u(0, x) = f(x), \\ u_t(0, x) = g(x). \end{cases} \tag{15}$$

*If $p \leq p_2(n)$, with $p_2(n)$ as in (4), then $T < \infty$.*

Let $R > 0$ be such that $\operatorname{supp} f, \operatorname{supp} g \subset B(R)$. Therefore, $\operatorname{supp} u(t, \cdot) \subset B(R + t)$. Without loss of generality we assume $R = 1$. Let us define

$$F(t) := \int_{\mathbb{R}^n} u(t, x) \, dx.$$

Thanks to the finite speed of propagation of $u$ and by Hölder's inequality

$$\ddot{F}(t) = \langle t \rangle^{-(p-1)} \int_{\mathbb{R}^n} |u(t, x)|^p \, dx = \langle t \rangle^{-(p-1)} \int_{B(\langle t \rangle)} |u(t, x)|^p \, dx \gtrsim \langle t \rangle^{-(n+1)(p-1)} |F(t)|^p. \tag{16}$$

In order to apply Lemma 1 we need to establish that $F(t)$ is positive. For this reason we consider the functions

$$\phi_1(x) := \int_{S^{n-1}} e^{x \cdot \omega} \, d\omega, \qquad \psi_1(t, x) := \phi_1(x) e^{-t}$$

and

$$F_1(t) := \int_{\mathbb{R}^n} u(t, x) \psi_1(t, x) \, dx.$$

It follows that

$$\ddot{F}(t) \gtrsim \langle t \rangle^{-(p-1)} |F_1(t)|^p \left( \int_{B(1+t)} (\psi_1(t, x))^{\frac{p}{p-1}} \, dx \right)^{-(p-1)}.$$

Let us estimate the last integral. Recalling that $\psi_1(t, x) = e^{-t} \phi_1(x)$ we see that

$$\int_{B(K)} (\psi_1(t, x))^{\frac{p}{p-1}} \, dx \leq C(K, A, p) \langle t \rangle^{-A}$$

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for any fixed $K < 1 + t$ and $A > 0$. By using

$$\phi_1(x) \lesssim |x|^{-\frac{n-1}{2}} e^{|x|} \quad \text{as } |x| \to \infty$$

(see [2], pages 184,185), we get for large $t$ and $K$ the estimate

$$\int_{B(1+t)\setminus B(K)} (\psi_1(t,x))^{\frac{p}{p-1}} dx \lesssim \int_K^{t+1} \langle \rho \rangle^{n-1-\frac{(n-1)p}{2(p-1)}} e^{\frac{p}{p-1}(\rho-t)} d\rho.$$

Putting

$$\alpha := n - 1 - (n-1)p/(2(p-1)),$$

we have

$$\int_K^{t+1} \langle \rho \rangle^\alpha e^{\frac{p}{p-1}(\rho-t)} d\rho \leq \frac{p-1}{p} e^{\frac{p}{p-1}} (2+t)^\alpha - \frac{\alpha(p-1)}{p} \int_K^{t+1} e^{\frac{p}{p-1}(\rho-t)} \langle \rho \rangle^{\alpha-1} d\rho.$$

If $\alpha \geq 0$, i.e. $p \geq 2$, then we may immediately conclude

$$\int_K^{t+1} \langle \rho \rangle^\alpha e^{\frac{p}{p-1}(\rho-t)} d\rho \lesssim \langle t \rangle^\alpha. \tag{17}$$

The same estimate holds if $\alpha < 0$, i.e. $p \in (1,2)$, since we may write

$$\left(1 + \frac{\alpha(p-1)}{p(1+K)}\right) \int_K^{t+1} \langle \rho \rangle^\alpha e^{\frac{p}{p-1}(\rho-t)} d\rho \lesssim \langle t \rangle^\alpha$$

and for large $K$ and $t$ we turn to (17). As a conclusion

$$\ddot{F}(t) \gtrsim \langle t \rangle^{-n(p-1)+(n-1)\frac{p}{2}} |F_1(t)|^p.$$

To estimate $|F_1(t)|^p$ the sign of the nonlinear term comes into play. More precisely, the following result holds for any smooth solution to $u_{tt} - \Delta u = G(t,x,u)$ with positive $G$.

**Lemma 2.** [Lemma 2.2 in [26]] *There exists a positive constant $t_0$ such that it holds*

$$F_1(t) \gtrsim \frac{1}{2}(1 - e^{-2t}) \int_{\mathbb{R}^n} (f(x) + g(x))\phi_1(x)\,dx + e^{-2t} \int_{\mathbb{R}^n} f(x)\phi_1(x)\,dx \tag{18}$$

*for $t \geq t_0$.*

In particular, due to our assumption on $f$ and $g$, it holds $F_1(t) > c > 0$. Therefore, we proved

$$\ddot{F}(t) \gtrsim \langle t \rangle^{-n(p-1)+(n-1)\frac{p}{2}} = \langle t \rangle^{-\frac{n+1}{2}p+n}. \tag{19}$$

Integrating twice we obtain

$$F(t) \gtrsim \langle t \rangle^{\max\{-\frac{n+1}{2}p+n+2,1\}} + t\dot{F}(0) + F(0) \gtrsim \langle t \rangle^{\max\{-\frac{n+1}{2}p+n+2,1\}}, \tag{20}$$

since $\dot{F}(0) \geq 0$ and $F(0) \geq 0$.

### 3.1. The subcritical case

Recalling (16) we may apply the first part of Lemma 1 once we have one of the following conditions:

$$-\frac{n+1}{2}p + n + 2 > \frac{(n+1)(p-1)-2}{p-1}, \tag{21}$$

$$1 > \frac{(n+1)(p-1)-2}{p-1}. \tag{22}$$

Condition (21) corresponds to $p < p_0(n+2)$, whereas condition (22) corresponds to $p < p_\infty(n)$, hence, we derived $p < \max\{p_0(n+2), p_\infty(n)\}$.

### 3.2. Critical case in 1d

First, let $n = 1$ and $p = 3$. By (16) it follows (13) with $q = 4$. Setting (20) into (16) leads to

$$\ddot{F}(t) \gtrsim \langle t \rangle^{-4} F(t)^3 \gtrsim \langle t \rangle^{-1}.$$

Integrating twice implies $F(t) \gtrsim \langle t \rangle \ln\langle t \rangle$. Therefore, for any $K_0 > 0$ there exists $T_0 > 0$ such that (14) holds with $a = 1$. The proof follows from Lemma 1.

### 3.3. Critical case in 2d

By (16) and (20) we have again $\ddot{F}(t) \gtrsim \langle t \rangle^{-1}$. Consequently, the conclusion follows.

### 3.4. Critical case in n dimensions with $n \geq 3$

We notice that $p_2(n) = p_0(n + 2) < 2$. Due to the lack of $C^2$ regularity of solutions we shall prove a blow-up behaviour for the spherical mean of $u$, that is, for

$$\tilde{u}(t, r) = \frac{1}{\omega_n} \int_{|\omega|=1} u(t, r\omega) \, dS_\omega.$$

This mean satisfies the differential inequality (see [14])

$$\tilde{u}_{tt} - \Delta \tilde{u} \geq \langle t \rangle^{-(p-1)} |\tilde{u}|^p.$$

We can assume that $u$ is radial. Following [26] we consider the Radon transform of $u$ on the hyper-planes orthogonal to a fixed $\omega \in \mathbb{R}^n$:

$$Ru(t, \rho) := \int_{x \cdot w = \rho} u(t, x) \, dS_x,$$

where $dS_x$ is the Lebesgue measure of $\{x : x \cdot w = \rho\}$. One can see that $Ru$ is independent of $w$ and that

$$Ru(t, \rho) = c_n \int_{|\rho|}^{\infty} u(t, r) \, (r^2 - \rho^2)^{\frac{n-3}{2}} r \, dr. \tag{23}$$

We will assume that $\rho \geq 0$. Since $Ru$ satisfies

$$\partial_t^2 Ru - \partial_\rho^2 Ru = \langle t \rangle^{-(p-1)} R(|u|^p)$$

and $f \geq 0$, $g \geq 0$, it follows

$$Ru(t, \rho) \geq \frac{1}{2} \int_0^t \langle s \rangle^{-(p-1)} \int_{\rho-(t-s)}^{\rho+(t-s)} R(|u|^p)(s, \rho_1) \, d\rho_1 \, ds.$$

Since $\operatorname{supp} R(|u|^p)(s, \cdot) \subset B(s + 1)$, following [26] we may estimate

$$Ru(t, \rho) \geq \frac{1}{2} \int_0^{\frac{t-\rho-1}{2}} \langle s \rangle^{-(p-1)} \int_{\mathbb{R}} R(|u|^p)(s, \rho_1) \, d\rho_1 \, ds$$

$$= \frac{1}{2} \int_0^{\frac{t-\rho-1}{2}} \langle s \rangle^{-(p-1)} \int_{\mathbb{R}^n} |u(s, y)|^p \, dy \, ds = \frac{1}{2} \int_0^{\frac{t-\rho-1}{2}} \ddot{F}(s) \, ds.$$

Recalling (19) we get

$$Ru(t, \rho) \geq \frac{1}{2} \int_0^{\frac{t-\rho-1}{2}} \langle s \rangle^{-\frac{n+1}{2} p+n} \, ds.$$

Since $p = p_2(n) \leq 2$ it holds

$$Ru(t, \rho) \gtrsim (1 + t - \rho)^{-\frac{n+1}{2} p+n+1}. \tag{24}$$

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Coming back to (23) and recalling that $\text{supp}\, u(t, \cdot) \subset B(1 + t)$, since $r + \rho \leq 2r$ in the integral, we may estimate

$$Ru(t, \rho) = c_n \int_\rho^{1+t} u(t, r)\, r(r + \rho)^{\frac{n-3}{2}} (r - \rho)^{\frac{n-3}{2}}\, dr \leq c_n 2^{\frac{[n-3]^+}{2}} \int_\rho^{1+t} u(t, r)\, r^{\frac{n-1}{2}} (r - \rho)^{\frac{n-3}{2}}\, dr. \tag{25}$$

The operator $T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, which is defined by

$$Tf(\tau) := \frac{1}{|1 + t - \tau|^{\frac{n-1}{2}}} \int_\tau^{1+t} f(r)\, |r - \tau|^{\frac{n-3}{2}}\, dr \qquad \text{for any } \tau \in \mathbb{R},$$

is bounded. Therefore, if we put

$$f(r) = \begin{cases} |u(t, r)|\, r^{\frac{n-1}{p}} & \text{if } r \geq 0, \\ 0 & \text{if } r \leq 0, \end{cases}$$

so that $f(r)^p = |u(t, r)|^p\, r^{n-1}$ for $r \geq 0$, then we get

$$\int_0^{1+t} \left( \frac{1}{|1 + t - \rho|^{\frac{n-1}{2}}} \int_\rho^{1+t} |u(t, r)|\, r^{\frac{n-1}{p}} (r - \rho)^{\frac{n-3}{2}}\, dr \right)^p d\rho \lesssim \int_0^\infty |u(t, r)|^p\, r^{n-1}\, dr = C \int_{\mathbb{R}^n} |u(t, x)|^p\, dx.$$

Due to $p \leq 2$ and $r \geq \rho$ it holds $r^{\frac{n-1}{p}} \geq r^{\frac{n-1}{2}} \rho^{(n-1)\left(\frac{1}{p} - \frac{1}{2}\right)}$, so that, by (25), we conclude

$$\int_0^{1+t} \frac{(Ru(t, \rho))^p}{|1 + t - \rho|^{\frac{n-1}{2} p}} \rho^{(n-1)-(n-1)\frac{p}{2}}\, d\rho \lesssim \int_{\mathbb{R}^n} |u(t, x)|^p\, dx.$$

Thanks to (24) we get

$$\int_{\mathbb{R}^n} |u(t, x)|^p\, dx \gtrsim \int_0^{1+t} (1 + t - \rho)^{-\frac{n+1}{2} p^2 + \frac{n+3}{2} p} \rho^{(n-1)-(n-1)\frac{p}{2}}\, d\rho.$$

Recalling that $p = p_2(n)$ we may use

$$\frac{n+1}{2} p^2 - \frac{n+3}{2} p = 1,$$

and obtain

$$\int_{\mathbb{R}^n} |u(t, x)|^p\, dx \gtrsim \int_0^{1+t} (1 + t - \rho)^{-1} \rho^{(n-1)-(n-1)\frac{p}{2}}\, d\rho \gtrsim \langle t \rangle^{(n-1)-(n-1)\frac{p}{2}} \ln\langle t \rangle.$$

Thus,

$$\ddot{F}(t) \gtrsim \langle t \rangle^{-(p-1)} \int_{\mathbb{R}^n} |u|^p\, dx \gtrsim \langle t \rangle^{(n-1)-(n+1)\frac{p}{2}+1} \ln\langle t \rangle,$$

hence,

$$F(t) \gtrsim \langle t \rangle^{(n+1)-(n-1)\frac{p}{2}} \ln\langle t \rangle.$$

Similarly to the case $n = 1$ the end of the proof follows by Lemma 1.

## 4. Proof of Theorem 2

Let $p, q \in [1, \infty]$. As in [17] we define

$$\|f\|_{(p,q)} := \left\| f(r\omega)\, r^{\frac{n-1}{p}} \right\|_{L_r^p([0,\infty), L_\omega^q(S^{n-1}))}.$$

It holds $\|f\|_{(p,p)} = \|f\|_{L^p}$ and Hölder's inequality

$$\|f_1 f_2\|_{(p,q)} \lesssim \|f_1\|_{(p_1,q_1)} \|f_2\|_{(p_2,q_2)} \tag{26}$$

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is valid if

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \leq 1.$$

Moreover, since $S^{n-1}$ is compact, it holds

$$\|f\|_{(p,q_1)} \lesssim \|f\|_{(p,q_2)} \quad \text{for any } q_2 \geq q_1. \tag{27}$$

Let $i, j = 1, 2$ with $i \neq j$. We introduce the vector fields

$$\Gamma = (D, L_0, L_j, \Omega_{ij}), \qquad D = (\partial_t, \partial_j), \quad L_0 = \langle t \rangle \partial_t + x \cdot \nabla,$$
$$L_j = \langle t \rangle \partial_j + x_j \partial_t, \quad \Omega_{ij} = x_i \partial_j - x_j \partial_i,$$

and the norms

$$\|f\|_{\Gamma,s,(p,q)} = \sum_{|\alpha| \leq s} \|\Gamma^\alpha f\|_{(p,q)},$$
$$\|f\|_{\Gamma,s,p} = \|f\|_{\Gamma,s,(p,p)},$$
$$\|f\|_{\Gamma,s,\infty} = \sum_{|\alpha| \leq s} \|\Gamma^\alpha f\|_\infty.$$

To a given $\alpha$ the following relations hold with suitable $a_\beta$ and $b_\beta$:

$$[\Box, \Gamma^\alpha] = \sum_{|\beta| \leq |\alpha|-1} a_\beta \, \Gamma^\beta \Box, \tag{28}$$

$$[D, \Gamma^\alpha] = \sum_{|\beta| \leq |\alpha|-1} b_\beta \, \Gamma^\beta D. \tag{29}$$

By using arguments from [16] one has the following Sobolev-type inequalities in these generalized Sobolev spaces:

$$\|w(t, \cdot)\|_\infty \lesssim \langle t \rangle^{-\frac{n-1}{2}} \|w(t, \cdot)\|_{\Gamma,s,2} \quad \text{if } s > n/2, \tag{30}$$

$$\|w(t, \cdot)\|_\infty \lesssim \langle t \rangle^{1-\frac{n-1}{2}} \left( \|w(t, \cdot)\|_{\Gamma,s,2} + \|Dw(t, \cdot)\|_{\Gamma,s,2} \right) \quad \text{if } s + 1 > n/2, \tag{31}$$

$$\|w(t, \cdot)\|_q \lesssim \langle t \rangle^{-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)} \|w(t, \cdot)\|_{\Gamma,s,2} \quad \text{if } 2 \leq q < \infty, \quad \frac{1}{q} \geq \frac{1}{2} - \frac{s}{n} \geq 0, \tag{32}$$

for any $t > 0$ and any $w(t, \cdot)$ such that the right-hand sides are well-defined. The previous statements can be found in [30].

In these spaces energy estimates are given by

$$\|Du\|_{\Gamma,s,2} \lesssim \|\nabla u_0\|_{\Gamma,s,2} + \|u_1\|_{\Gamma,s,2} + \int_0^t \|f(\tau, x)\|_{\Gamma,s,2} \, d\tau, \qquad s \in \mathbb{N} \tag{33}$$

for solutions to the Cauchy problem for the inhomogeneous wave equation

$$\begin{cases} u_{tt} - \triangle u = f(t, x), & t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{34}$$

Indeed, we may combine (28), (29) with the classical energy estimate

$$\|Du\|_2 \lesssim \|\nabla u_0\|_2 + \|u_1\|_{L^2} + \int_0^t \|f(\tau, x)\|_{L^2} \, d\tau.$$

It is also necessary to estimate $\|u\|_{\Gamma,s,2}$. Here the space dimension $n = 2$ comes into play.

**Lemma 3.** *Let $n = 2$, and $u$ be the solution to (34). Then, for any $\epsilon > 0$ there exists a constant $\delta(\epsilon) > 0$, satisfying $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, such that*

$$\|u(t, \cdot)\|_{\Gamma, s, 2} \lesssim \|u_0\|_{\Gamma, s, 2} + t^\delta \|u_1\|_{\Gamma, s, (1+\epsilon, 2)} + \int_0^t (t - \tau)^\delta \|f(\tau, \cdot)\|_{\Gamma, s, (1+\epsilon, 2)} \, d\tau.$$

PROOF. Due to (28) it suffices to consider the case $s = 0$. First, let $f \equiv 0$. Following [17], by using the change of variables $x = ty$, we may estimate

$$\|u(t, \cdot)\|_2 \lesssim \|u_0\|_2 + t^2 \|G\|_{H^{-1}(\mathbb{R}_y^2)}, \qquad \text{where } G(y) = u_1(ty).$$

Recalling that

$$\|G\|_{H^{-1}} = \sup_{v \in H^1, v \neq 0} \frac{|\int_{\mathbb{R}^2} G(y) v(y) \, dy|}{\|v\|_{H^1}},$$

by virtue of (26)-(27) and Sobolev embeddings it holds

$$\|Gv\|_{L^1} \lesssim \|G\|_{(q, 2)} \|v\|_{(q', 2)} \lesssim \|G\|_{(q, 2)} \|v\|_{q'} \lesssim \|G\|_{(q, 2)} \|v\|_{H^1},$$

where $q = 1 + \epsilon$ with some $\epsilon \in (0, 1)$. Since

$$\|G\|_{(q, 2)} \lesssim t^{-\frac{n}{q}} \|u_1\|_{(q, 2)},$$

summarizing, we proved that

$$\|u(t, \cdot)\|_2 \lesssim \|u_0\|_2 + t^{2(1 - \frac{1}{1+\epsilon})} \|u_1\|_{(1+\epsilon, 2)}.$$

The case $f \not\equiv 0$ follows by Duhamel's principle.

Now we come back to the semilinear problem. For any $T > 0$, we introduce the space $X(T)$ with norm

$$\|u\|_{X(T)} := \sup_{t \in [0, T]} (\langle t \rangle^{-\delta} \|u\|_{\Gamma, 1, 2} + \|Du\|_{\Gamma, 1, 2}),$$

where $\delta$ is given by Lemma 3. For any $w \in X(T)$ let $u = S[w]$ be the solution to

$$u_{tt} - \triangle u = \langle t \rangle^{-(p-1)} |w|^p, \qquad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x)$$

with compactly supported data. Thanks to Lemma 3 for $s = 1$ we may estimate

$$\|u(t, \cdot)\|_{\Gamma, 1, 2} \lesssim \|u_0\|_{\Gamma, 1, 2} + t^\delta \|u_1\|_{\Gamma, 1, (1+\epsilon, 2)} + \int_0^t (t - \tau)^\delta \left\| \langle \tau \rangle^{-(p-1)} |w(\tau, \cdot)|^p \right\|_{\Gamma, 1, (1+\epsilon, 2)} \, d\tau.$$

Since

$$[\partial_\tau, \langle \tau \rangle^\alpha] = \alpha \frac{1}{\langle \tau \rangle} \langle \tau \rangle^\alpha, \qquad [L_0, \langle \tau \rangle^\alpha] = \alpha \langle \tau \rangle^\alpha, \qquad [L_j, \langle \tau \rangle^\alpha] = \alpha \frac{x_j}{\langle \tau \rangle} \langle \tau \rangle^\alpha,$$

thanks to the finite speed of propagation, i.e. $|x| \lesssim \langle t \rangle$ on $\operatorname{supp} u$, we get

$$\|u(t, \cdot)\|_{\Gamma, 1, 2} \lesssim \|u_0\|_{\Gamma, 1, 2} + t^\delta \|u_1\|_{\Gamma, 1, (1+\epsilon, 2)} + \int_0^t (t - \tau)^\delta \langle \tau \rangle^{-(p-1)} \left\| |w(\tau, \cdot)|^p \right\|_{\Gamma, 1, (1+\epsilon, 2)} \, d\tau.$$

Now we may estimate

$$\left\| |w(\tau, \cdot)|^p \right\|_{\Gamma, 1, (1+\epsilon, 2)} \lesssim \||w|^{p-1}\|_{(\bar{q}, \infty)} \|w(\tau, \cdot)\|_{\Gamma, 1, 2},$$

where $\bar{q}(\epsilon) \in (2, \infty)$ is given by

$$\frac{1}{1 + \epsilon} = \frac{1}{2} + \frac{1}{\bar{q}}.$$

Let $\gamma(\epsilon):=2/\bar{q} = (1-\epsilon)/(1+\epsilon)$. Since $p > 2$, it holds $\gamma + 1 < p$. Then

$$\||w(\tau,\cdot)|^{p-1}\|_{(\bar{q},\infty)} \lesssim \|w(\tau,\cdot)\|_{\infty}^{p-1-\gamma} \|w(\tau,\cdot)\|_{(2,\infty)}^{\gamma}.$$

Applying Sobolev embeddings on the unit sphere $S^1$ leads to

$$\|w(\tau,\cdot)\|_{(2,\infty)} \lesssim \|w(\tau,\cdot)\|_{H^1} \leq \|w(\tau,\cdot)\|_2 + \|Dw(\tau,\cdot)\|_2.$$

Thanks to (31) we have

$$\|w(\tau,\cdot)\|_{\infty} \lesssim \langle\tau\rangle^{\frac{1}{2}} (\|w(\tau,\cdot)\|_{\Gamma,1,2} + \|Dw(\tau,\cdot)\|_{\Gamma,1,2}).$$

Therefore, taking into account that $w \in X(T)$ we conclude

$$\int_0^t (t-\tau)^{\delta} \langle\tau\rangle^{-(p-1)} \||w(\tau,\cdot)|^p\|_{\Gamma,1,(1+\epsilon,2)} \, d\tau \lesssim \|w\|_{X(T)}^p \int_0^t (t-\tau)^{\delta} \langle\tau\rangle^{-(p-1)+p\delta+\frac{p-(1+\gamma)}{2}} \, d\tau.$$

Since $\delta(\epsilon) \to 0$ and $\gamma(\epsilon) \to 1$ as $\epsilon \to 0$, for any $p > 2$, one can find a sufficiently small $\epsilon$ such that

$$-(p-1) + p\delta + \frac{p-(1+\gamma)}{2} < -1.$$

To estimate $\|Du\|_{\Gamma,1,2}$ we apply (33). Now

$$\||w(\tau,\cdot)|^p\|_{\Gamma,1,2} \lesssim \||w(\tau,\cdot)|^{p-1}\|_{2+\epsilon_1} \|w(\tau,\cdot)\|_{\Gamma,1,q_1}$$

for some $\epsilon_1 > 0$, where $q_1(\epsilon_1)$ is such that

$$\frac{1}{2} = \frac{1}{2+\epsilon_1} + \frac{1}{q_1}.$$

Sobolev embeddings yield

$$\|w(\tau,\cdot)\|_{\Gamma,1,q_1} \lesssim \|w(\tau,\cdot)\|_{\Gamma,1,2} + \|Dw(\tau,\cdot)\|_{\Gamma,1,2}.$$

On the other hand, since $p > 2$, we have

$$\||w(\tau,\cdot)|^{p-1}\|_{2+\epsilon_1} \leq \|w(\tau,\cdot)\|_{(2+\epsilon_1)(p-1)}^{p-1} \lesssim \langle\tau\rangle^{-\left(\frac{1}{2}-\frac{1}{(2+\epsilon_1)(p-1)}\right)(p-1)} \|w(\tau,\cdot)\|_{\Gamma,1,2}^{p-1} \leq \|w(\tau,\cdot)\|_{\Gamma,1,2}^{p-1}.$$

In turn, this gives

$$\int_0^t \langle\tau\rangle^{-(p-1)} \||w(\tau,\cdot)|^p\|_{\Gamma,1,2} \, d\tau \lesssim \|w\|_{X(T)}^p \int_0^t \langle\tau\rangle^{-(p-1)+p\delta} \, d\tau.$$

Since $p > 2$, it is sufficient to fix $\epsilon$ such that $\delta(\epsilon)$ satisfies $p(1-\delta) > 2$.
Summarizing, we proved that

$$\|u(t,\cdot)\|_{\Gamma,1,2} \lesssim \|u_0\|_{\Gamma,1,2} + \langle t\rangle^{\delta} \|u_1\|_{\Gamma,1,(1+\epsilon,2)} + \langle t\rangle^{\delta} \|w\|_{X(T)}^p,$$

$$\|Du(t,\cdot)\|_{\Gamma,1,2} \lesssim \|\nabla u_0\|_{\Gamma,1,2} + \|u_1\|_{\Gamma,1,2} + \|w\|_{X(T)}^p.$$

Recalling that initial data are compactly supported, we derive

$$\|u\|_{X(T)} \lesssim (\|u_0\|_{\Gamma,1,2} + \|Du_0\|_{\Gamma,1,2} + \|u_1\|_{\Gamma,1,(1+\epsilon,2)} + \|u_1\|_{\Gamma,1,2}) + \|w\|_{X(T)}^p.$$

By a standard argument, this estimate guarantees that with small data the operator $S[w]$ has a unique fixed point, that is the required solution.

## 5. Proof of Theorem 3

**Remark 1.** *In the statement of Theorem 3 we may relax the assumptions of compact support of the initial data. More precisely, we will prove that for any $p > p_0(5)$ and for any $\kappa \geq (3 - p)/(p - 1)$ if $p < 2$, or $\kappa > 1$ if $p \geq 2$, there exists $\varepsilon_0 > 0$ such that if $(v_0, v_1) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ are radial, namely, $v_0 = v_0(|x|)$, $v_1 = v_1(|x|)$, and*

$$\langle r \rangle^{\kappa+1} (|v_0(r) + v_1(r)| + \langle r \rangle |v_0'(r) + v_1'(r)|) + \langle r \rangle^\kappa (|v_0(r)| + \langle r \rangle |v_0'(r)| + \langle r \rangle^2 |v_0''(r)|) < \varepsilon, \tag{35}$$

*for some $\varepsilon < \varepsilon_0$, then (1) admits a (radial) global (in time), solution $v \in C([0, \infty) \times \mathbb{R}^3) \cap C^2([0, \infty) \times (\mathbb{R}^3 \setminus \{0\}))$. We set $r = |x|$ in (35).*

**Remark 2.** *We may also replace the nonlinear term $|u|^p$ in (1) by $f(u)$, where $f \in C^1$ is an even function satisfying $|f^{(h)}(u)| \lesssim |u|^{p-h}$ for $h = 0, 1$. In particular, it holds*

$$f(0) = 0, \qquad |f(u) - f(v)| \lesssim |u - v|(|u|^{p-1} + |v|^{p-1}). \tag{36}$$

To fulfill our objective we apply to (5) a technique introduced by Asakura [1] and developed in different works, in particular in [15]. For the sake of simplicity, let $v_0 = 0$ and let $g := u_1 = v_1$. Then condition (35) becomes

$$|g^{(h)}(r)| \leq \varepsilon \langle r \rangle^{-(\kappa+1+h)} \text{ for } h = 0, 1. \tag{37}$$

We extend $g$ to negative values of $r$ by defining $g(r) := g(-r)$ for any $r < 0$. Then, by symmetry, we rewrite (5) as

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{2}{r} u_r = \langle t \rangle^{-(p-1)} |u|^p, & t > 0, \quad r \in \mathbb{R}, \\ u(0, r) = 0, \quad u_t(0, r) = g(r), & r \in \mathbb{R}. \end{cases} \tag{38}$$

**Definition 1.** *We say that $u(t, |x|) = u(t, r)$ is a radial global (in time) solution to (38) if $u \in C([0, \infty) \times \mathbb{R})$, $ru \in C^1([0, \infty) \times \mathbb{R})$, $r^2 u \in C^2([0, \infty) \times \mathbb{R})$ and*

$$\begin{cases} r^2 u_{tt} - (r^2 u_{rr} + 2r u_r) = r^2 \langle t \rangle^{-(p-1)} |u|^p, & t > 0, \quad r \in \mathbb{R}, \\ u(0, r) = 0, \quad u_t(0, r) = g(r), & r \in \mathbb{R}. \end{cases}$$

**Remark 3.** *Any solution to (38) in the sense of Definition 1 gives a $C([0, \infty) \times \mathbb{R}^3) \cap C^2([0, \infty) \times (\mathbb{R}^3 \setminus \{0\}))$ solution to (5) and in turn (1).*

### 5.1. The linear equation

**Definition 2.** *Let us consider*

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{2}{r} u_r = 0, & t > 0, \quad r \in \mathbb{R}, \\ u(0, r) = 0, \quad u_t(0, r) = g(r), & r \in \mathbb{R}. \end{cases} \tag{39}$$

*We say that $u \in C([0, \infty) \times \mathbb{R})$ is a solution to (39) if $ru \in C^1([0, \infty) \times \mathbb{R})$, $r^2 u \in C^2([0, \infty) \times \mathbb{R})$ and*

$$\begin{cases} r^2 u_{tt} - (r^2 u_{rr} + 2r u_r) = 0, & t > 0, \quad r \in \mathbb{R}, \\ u(0, r) = 0, \quad u_t(0, r) = g(r), & r \in \mathbb{R}. \end{cases} \tag{40}$$

We see that $r^2 u \in C^2([0, \infty) \times \mathbb{R})$, $ru \in C^1([0, \infty) \times \mathbb{R})$, and $u \in C([0, \infty) \times \mathbb{R})$ give sufficient regularity for solutions to the equation in (40). Indeed, we have $r^2 u_{rr} + 2r u_r = \partial_{rr}(r^2 u) - 2u - 2r u_r \in C([0, \infty) \times \mathbb{R})$. Hence, $r^2 u_{tt} \in C([0, \infty) \times \mathbb{R})$, too. According to Definition 2 the function

$$u^{\text{lin}}(t, r) = \int_{-1}^1 H_g(t + r\sigma) \, d\sigma \text{ with } H_g(\rho) := \frac{\rho g(\rho)}{2}$$

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is the solution to (39). This result can be found in [2], but we rewrite the computation for completeness. Indeed, for any $H = H(\rho)$, $H \in C^1$, we put

$$v(t, r) := \frac{1}{r} \int_{t-r}^{t+r} H(\rho) \, d\rho = \int_{-1}^{1} H(t + r\sigma) \, d\sigma. \tag{41}$$

For any $r \neq 0$ it holds

$$v_t = \int_{-1}^{1} H'(t + r\sigma) \, d\sigma = \frac{1}{r} \left( H(t + r) - H(t - r) \right), \tag{42}$$

$$v_{tt} = \frac{1}{r} \left( H'(t + r) - H'(t - r) \right), \tag{43}$$

$$v_r = \int_{-1}^{1} \sigma H'(t + r\sigma) \, d\sigma = \frac{1}{r} \left( H(t + r) + H(t - r) \right) - \frac{1}{r} v, \tag{44}$$

$$v_{rr} = -\frac{1}{r^2} \left( H(t + r) + H(t - r) \right) + \frac{1}{r} \left( H'(t + r) - H'(t - r) \right) + \frac{1}{r^2} v - \frac{1}{r} v_r$$
$$= \frac{1}{r} \left( H'(t + r) - H'(t - r) \right) - \frac{2}{r} v_r. \tag{45}$$

In particular, $v$ solves the equation in (40) for any $r \neq 0$. Moreover, $r^2 v \in C^2([0, \infty) \times \mathbb{R})$, and $v$ solves the equation in (40) for any $r \in \mathbb{R}$, as one may immediately check by multiplying (42)-(44) by $r$ and (43)-(45) by $r^2$. We remark that $v(0, r) = 0$ if $H$ is odd. In this latter case, $rv_t(0, r) = 2H(r)$. In particular, this proves that $u^{\text{lin}}$ solves (39).

For our convenience we also compute

$$\partial_r (rv) = v + rv_r = H(t + r) + H(t - r), \tag{46}$$

$$\partial_r^2 (r^2 v) = \partial_r (rv) + r \partial_r^2 (rv) = H(t + r) + H(t - r) + r(H'(t + r) - H'(t - r)). \tag{47}$$

For any fixed $\kappa > 1$, we introduce the Banach space

$$X_\kappa := \{ u \in C([0, \infty), \mathbb{R}), \ u \text{ is even in } r : \ \partial_r (ru) \in C([0, \infty), \mathbb{R}), \ \|u\|_{X_\kappa} < \infty \}$$

with the norm

$$\|u\|_{X_\kappa} := \{ \langle t + |r| \rangle \langle t - |r| \rangle^{\kappa-1} u \} + \{ \langle r \rangle^{-1} \langle t + |r| \rangle \langle t - |r| \rangle^{\kappa-1} \partial_r (ru) \}.$$

**Theorem 4.** *Suppose that (37) holds for some $\kappa > 1$. Then*

$$\|u^{\text{lin}}\|_{X_\kappa} \leq C\varepsilon$$

*for a suitable constant $C > 0$.*

PROOF. We notice that
$$|H_g^{(h)}(\rho)| \leq \varepsilon \langle \rho \rangle^{-\kappa-h} \quad \text{for} \ \ h = 0, 1.$$

Thanks to (46) we immediately derive

$$|\partial_r (ru^{\text{lin}})| = \left| H_g(t + r) + H_g(t - r) \right| \lesssim \varepsilon \langle t - |r| \rangle^{-\kappa}.$$

We distinguish two cases.
If $t \geq 2|r|$, then $\langle t \pm |r| \rangle \simeq \langle t \rangle$ and we get

$$|\partial_r (ru^{\text{lin}})| \lesssim \varepsilon \langle t - |r| \rangle^{-(\kappa-1)} \langle t + |r| \rangle^{-1} \lesssim \varepsilon \langle t - |r| \rangle^{-(\kappa-1)} \langle t + |r| \rangle^{-1} \langle r \rangle,$$

where in the last inequality we used the trivial estimate $1 \leq \langle r \rangle$.
If $t \leq 2|r|$, then $\langle t + |r| \rangle \leq 3 \langle r \rangle$. Therefore,

$$|\partial_r (ru^{\text{lin}})| \lesssim \varepsilon \langle t - |r| \rangle^{-\kappa} \langle t + |r| \rangle^{-1} \langle r \rangle \lesssim \varepsilon \langle t - |r| \rangle^{-(\kappa-1)} \langle t + |r| \rangle^{-1} \langle r \rangle,$$

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where in the last inequality we use the trivial estimate $\langle t - |r| \rangle^{-1} \le 1$.

In order to estimate $\left\| \langle t + |r| \rangle \langle t - |r| \rangle^{\kappa-1} u^{\text{lin}} \right\|_{L^\infty_t L^\infty_r}$ we observe that

$$|u^{\text{lin}}(t, r)| \lesssim \frac{\varepsilon}{r} \int_{t-r}^{t+r} \langle \rho \rangle^{-\kappa} \, d\rho = \frac{1}{|r|} C \varepsilon \int_{t-|r|}^{t+|r|} \langle \rho \rangle^{-\kappa} \, d\rho.$$

If $t \ge 2|r|$, then $\langle t \pm |r| \rangle \simeq \langle t \rangle$. Hence,

$$|u^{\text{lin}}(t, r)| \simeq \varepsilon \langle t - |r| \rangle^{-(\kappa-1)} \langle t + |r| \rangle^{-1}.$$

If $t \le 2|r|$, then we also distinguish two cases. If $|r| \le 1$, then $\langle t + |r| \rangle \langle t - |r| \rangle^{\kappa-1} \simeq 1$ and it is sufficient to estimate

$$|u^{\text{lin}}(t, r)| \le \int_{-1}^{1} H_g(t + r\sigma) \, d\sigma \le C.$$

On the other hand, if $t \le 2|r|$ and $|r| \ge 1$, then $\langle t + |r| \rangle \le 3\langle r \rangle$ and $|r| \simeq \langle r \rangle$. Therefore,

$$|u^{\text{lin}}(t, r)| \lesssim \frac{1}{|r|} \varepsilon \int_{t-|r|}^{t+|r|} \langle \rho \rangle^{-\kappa} \, d\rho \simeq \frac{1}{\langle r \rangle} \varepsilon \langle t - |r| \rangle^{-(\kappa-1)}$$
$$\lesssim \varepsilon \langle t + |r| \rangle^{-1} \langle t - |r| \rangle^{-(\kappa-1)},$$

thanks to $\kappa > 1$. This concludes the proof that $\|u^{\text{lin}}\|_{X_\kappa} \le C\varepsilon$.

### 5.2. Duhamel's principle and basic nonlinear estimates

For any $u \in X_\kappa$ let

$$Lu(t, r) := \int_0^t \langle s \rangle^{-(p-1)} \int_{-1}^{1} H_u[s](t - s + r\sigma) \, d\sigma \, ds = \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} H_u[s](\rho) \, d\rho \, ds,$$

where

$$H_u[s](\rho) := \frac{\rho \, f(u(s, \rho))}{2} . \tag{48}$$

We denote by $H_u[s]'(\rho)$ the derivative of $H_u[s](\rho)$ with respect to $\rho$, considering $s$ as a parameter.

Let us consider $f(u(s, \rho))$ and $\rho \partial_\rho f(u(s, \rho))$. If $u \in X_\kappa$, recalling that $r u_r = \partial_r(ru) - u$, then we may estimate

$$|f(u(s, \rho))| \lesssim \|u\|_{X_\kappa}^p \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)},$$
$$\langle \rho \rangle^{-1} |\rho \, \partial_\rho f(u(s, \rho))| \lesssim \|u\|_{X_\kappa}^p \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)},$$

Having in mind (48), it follows, in particular, that

$$|H_u[s](\rho)| + |H_u[s]'(\rho)| \lesssim \|u\|_{X_\kappa}^p \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)} \langle \rho \rangle. \tag{49}$$

**Proposition 2.** *Let $u \in X_\kappa$ be even with respect to $r$. Then $Lu \in X_\kappa$ and $r^2 Lu \in C^2([0, \infty) \times \mathbb{R})$. Moreover, $Lu$ is even with respect to $r$ and satisfies*

$$r^2 \left( \partial_t^2 - \partial_r^2 \right) Lu - 2r \partial_r Lu = \langle t \rangle^{-(p-1)} r^2 f(u), \quad t > 0, \quad r \in \mathbb{R} \tag{50}$$

*with initial data $g \equiv 0$.*

PROOF. From the continuity of $H_u[s](\rho)$ (which follows from $u \in X_\kappa \subset C$), it follows that $Lu \in X_\kappa$, i.e. $Lu, \partial_r(rLu) \in C$. Being $u$ even with respect to $r$, and $f$ even in $u$, we get that $H_u[s]$ is odd for any $s$. It follows that $Lu$ is even.

We notice that

$$\partial_t Lu = \int_0^t \langle s \rangle^{-(p-1)} \int_{-1}^1 \partial_t H_u[s](t - s + r\sigma)\, d\sigma\, ds + \langle t \rangle^{-(p-1)} \int_{-1}^1 H_u[t](r\sigma)\, d\sigma$$

$$= \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \left( H_u[s](t - s + r) - H_u[s](t - s - r) \right) ds,$$

$$\partial_t^2 Lu = \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \left( H_u[s]'(t - s + r) - H_u[s]'(t - s - r) \right) ds + \frac{1}{r} \langle t \rangle^{-(p-1)} \left( H_u[t](r) - H_u[t](-r) \right)$$

$$= \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \left( H_u[s]'(t - s + r) - H_u[s]'(t - s - r) \right) ds + \langle t \rangle^{-(p-1)} f(u(t, r)).$$

In particular, we gain $\partial_t^2 Lu \in C$. Recalling (45), we see that $Lu$ solves (50) and we get the continuity of the $r$-derivatives for $r^2 Lu$.

In order to prove global (in time) existence trough contraction mapping principle we shall prove the following statement.

**Theorem 5.** *Let $p > p_0(5)$ and let*

$$\frac{3 - p}{p - 1} \le \kappa \le 2(p - 1) \quad \text{if } p \in (p_0(5), 2), \text{ or } 1 < \kappa \le 2(p - 1) \text{ if } p \ge 2. \tag{51}$$

*If $u \in X_\kappa$, then*

$$\|Lu\|_{X_\kappa} \lesssim \|u\|_{X_\kappa}^p; \tag{52}$$

$$\|Lu - Lv\|_{X_\kappa} \lesssim \|u - v\|_{X_\kappa} (\|u\|_{X_\kappa}^{p-1} + \|u\|_{X_\kappa}^{p-1}). \tag{53}$$

Recalling the definition of the involved norm, for proving (52) it suffices to show

$$|Lu(t, r)| \lesssim \langle t + |r| \rangle^{-1} \langle t - |r| \rangle^{-(\kappa-1)} \|u\|_{X_\kappa}^p, \tag{54}$$

$$|\partial_r (rLu)(t, r)| \lesssim \langle t + |r| \rangle^{-1} \langle t - |r| \rangle^{-(\kappa-1)} \langle r \rangle \|u\|_{X_\kappa}^p. \tag{55}$$

Since $Lu$ is even in $r$, it suffices to deal with $r > 0$. Proceeding as in (41), from (49) we have

$$|Lu(t, r)| \lesssim \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} |H_u[s](\rho)|\, d\rho\, ds$$

$$\lesssim \frac{1}{r} \|u\|_{X_\kappa}^p \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)} \langle \rho \rangle\, d\rho\, ds.$$

By using (46) we get

$$|\partial_r (rLu)(t, r)| \le \int_0^t \langle s \rangle^{-(p-1)} \left| H_u[s](t - s + r) + H_u[s](t - s - r) \right| ds$$

$$\lesssim \|u\|_{X_\kappa}^p \sum_{\pm} \int_0^t \langle s \rangle^{-(p-1)} \langle s + |t - s \pm r| \rangle^{-p} \langle s - |t - s \pm r| \rangle^{-p(\kappa-1)} \langle t - s \pm r \rangle ds.$$

Consequently, our aim reduces to estimate the quantities

$$I_0(t, r) := \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)} \langle \rho \rangle\, d\rho\, ds,$$

$$I_{1,\pm}(t, r) := \int_0^t \langle s \rangle^{-(p-1)} \langle s + |t - s \pm r| \rangle^{-p} \langle s - |t - s \pm r| \rangle^{-p(\kappa-1)} \langle t - s \pm r \rangle ds.$$

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Similarly, to prove (53) it suffices to show

$$|Lu(t,r) - Lv(t,r)| \lesssim \langle t + |r| \rangle^{-1} \langle t - |r| \rangle^{-(\kappa-1)} \|u - v\|_{X_\kappa} (\|u\|_{X_\kappa}^{p-1} + \|v\|_{X_\kappa}^{p-1}), \tag{56}$$

$$|\partial_r(rLu)(t,r) - \partial_r(rLv)(t,r)| \lesssim \langle t + |r| \rangle^{-1} \langle t - |r| \rangle^{-(\kappa-1)} \langle r \rangle \|u - v\|_{X_\kappa} (\|u\|_{X_\kappa}^{p-1} + \|v\|_{X_\kappa}^{p-1}). \tag{57}$$

We have

$$|Lu(t,r) - Lv(t,r)| \lesssim \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} \left| H_u[s](\rho) - H_v[s](\rho) \right| d\rho \, ds.$$

Moreover, from (36) and (48), it follows that

$$\left| H_u[s](\rho) - H_v[s](\rho) \right| \lesssim |\rho| |u(s,\rho) - v(s,\rho)| (|u(s,\rho)|^{p-1} + |v(s,\rho)|^{p-1}).$$

As a conclusion

$$|Lu(t,r) - Lv(t,r)| \lesssim \|u - v\|_{X_\kappa} (\|u\|_{X_\kappa}^{p-1} + \|v\|_{X_\kappa}^{p-1}) I_0(t,r).$$

Similarly, we get

$$|\partial_r(rLu(t,r) - rLv(t,r))| \lesssim \|u - v\|_{X_\kappa} (\|u\|_{X_\kappa}^{p-1} + \|v\|_{X_\kappa}^{p-1}) \sum_{\pm} I_{1,\pm}(t,r).$$

If $t \leq r$, then we may simplify our approach, thanks to the following observation.

**Remark 4.** *If $t \leq r$, then it holds*

$$Lu(t,r) = \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \int_{r-(t-s)}^{r+(t-s)} H_u[s](\rho) \, d\rho \, ds.$$

*Indeed,*

$$\int_{(t-s)-r}^{r-(t-s)} H_u[s](\rho) \, d\rho = 0,$$

*being $H_u[s]$, defined in (48) odd, thanks to the assumption that $f(u)$ is even with respect to u, and thanks to the fact that u is even with respect to r. Therefore, we may replace $I_0(t,r)$ by*

$$I_0'(t,r) := \int_0^t \langle s \rangle^{-(p-1)} \int_{r-(t-s)}^{r+(t-s)} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)} \langle \rho \rangle \, d\rho \, ds.$$

The estimates for $I_0$, $I_{1,\pm}$ and $I_0'$ are based on the following lemma.

**Lemma 4.** *Let $p > p_0(5)$ and let*

$$\frac{3-p}{p-1} \leq \kappa \leq 2(p-1) \qquad \text{if } p \in (p_0(5), 2), \tag{58}$$

$$1 < \kappa \leq 2 \qquad \text{if } p = 2, \tag{59}$$

$$\frac{1}{p-1} \leq \kappa \leq 2(p-1) \qquad \text{if } p > 2. \tag{60}$$

*Then*

$$I(\xi) = \int_{-\xi}^{\xi} \langle \eta + \xi \rangle \langle \eta - \xi \rangle^{-(p-1)} \langle \eta \rangle^{-p(\kappa-1)} d\eta \lesssim \langle \xi \rangle^{-(\kappa-p)}. \tag{61}$$

**Remark 5.** *The interval (58), i.e. $(3-p)(p-1)^{-1} \leq \kappa \leq 2(p-1)$, is nonempty if, and only if, $p > p_0(5)$. If $p > 2$, then the interval (60), i.e. $(p-1)^{-1} \leq \kappa \leq 2(p-1)$ is nonempty for any $p > 2$.*
*We observe that this latter range contains the range $(1, 2(p-1)]$ required in the assumption (51).*

PROOF. We split $I(\xi)$ in $I_1(\xi) = \int_{-\xi/2}^{\xi/2} \dots d\eta$ and $I_2(\xi)$ as the remaining integral.

Let $\eta \in [0, \xi/2]$. Then we have $\langle \xi \rangle \simeq \langle \eta + \xi \rangle \simeq \langle \xi - \eta \rangle$. Hence,

$$I_1(\xi) \lesssim \langle \xi \rangle^{2-p} \int_0^{\xi/2} \langle \eta \rangle^{-p(\kappa-1)} \, d\eta.$$

We get $I_1(\xi) \lesssim \langle \xi \rangle^{-(\kappa-p)}$ if

$$\kappa < 1 + \frac{1}{p} \quad \text{and} \quad -3 + p\kappa \geq \kappa - p,$$

$$\kappa = 1 + \frac{1}{p} \quad \text{and} \quad -2 + p > \kappa - p,$$

$$\kappa > 1 + \frac{1}{p} \quad \text{and} \quad -2 + p \geq \kappa - p.$$

The first condition corresponds to the interval

$$\Big[ \frac{3-p}{p-1}, 1 + \frac{1}{p} \Big),$$

which is nonempty for any $p > p_0(5)$. The second condition holds for any $p > p_0(5)$, therefore $\kappa = 1 + 1/p$ is admissible. The third condition corresponds to the interval

$$\Big( 1 + \frac{1}{p}, 2(p-1) \Big],$$

which is nonempty for any $p > p_0(5)$. Gluing together the above three intervals we obtain the admissible range in (58), i.e.,

$$\Big[ \frac{3-p}{p-1}, 2(p-1) \Big].$$

Now, let $\eta \in [\xi/2, \xi]$. We have $\langle \eta \rangle \simeq \langle \xi \rangle \simeq \langle \xi + \eta \rangle$. It follows that

$$I_2(\xi) \simeq \langle \xi \rangle^{1-p(\kappa-1)} \int_{\xi/2}^{\xi} \langle \eta - \xi \rangle^{-(p-1)} \, d\eta + \langle \xi \rangle^{-(p-1)-p(\kappa-1)} \int_{\xi/2}^{\xi} \langle \eta - \xi \rangle \, d\eta = I_{2,1}(\xi) + I_{2,2}(\xi).$$

For any $p > 1$ we have

$$I_{2,2}(\xi) \lesssim \langle \xi \rangle^{-(p-1)-p(\kappa-1)+2},$$

in particular, $I_{2,2} \leq \langle \xi \rangle^{-(\kappa-p)}$ for any

$$\kappa \geq \frac{3-p}{p-1}. \tag{62}$$

The estimate of $I_{2,1}$ depends on the range of $p$:

$$I_{2,1}(\xi) \lesssim \langle \xi \rangle^{1-p(\kappa-1)} \text{ if } p > 2,$$
$$I_{2,1}(\xi) \lesssim \langle \xi \rangle^{1-p(\kappa-1)} \ln\langle \xi \rangle \text{ if } p = 2,$$
$$I_{2,1}(\xi) \lesssim \langle \xi \rangle^{1-p(\kappa-1)-(p-1)+1} \text{ if } p < 2.$$

For $p < 2$ the assumption $\kappa \geq \frac{3-p}{p-1}$ gives directly $I_{2,1}(\xi) \leq \langle \xi \rangle^{-(\kappa-p)}$. For $p = 2$, we get $1 - p(\kappa - 1) < p - \kappa$ if and only if $\kappa > 1$. For $p > 2$, we get $1 - p(\kappa - 1) \leq p - \kappa$ if and only if $\kappa \geq \frac{1}{p-1}$.

Therefore, combining the lower bound on $\kappa$ obtained for $I_{2,1}$ with the upper bound for $\kappa$ derived for $I_1$, we obtain (59) if $p = 2$ and (60) if $p > 2$.

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**Proposition 3.** *Let $p > p_0(5)$ and $\kappa$ be as in (51). It holds*

$$I_0(t,r) \lesssim \begin{cases} r\,\langle t+r\rangle^{-\kappa} & \text{if } t \geq 2r \quad \text{or } r \leq 1, \\ \langle t-r\rangle^{-(\kappa-1)} & \text{if } r \leq t \leq 2r \text{ and } r \geq 1. \end{cases}$$

*Moreover,*

$$I_0'(t,r) \lesssim \langle t-r\rangle^{-(\kappa-1)} \qquad \text{if } t \leq r \text{ and } r \geq 1.$$

*In particular, the estimates (54) and (56) hold.*

Proof. First, let us estimate $I_0$. Being $|t - s - r| < t - s + r$ we have

$$I_0(t,r) \leq 2 \int_0^t \langle s\rangle^{-(p-1)} \int_{\max\{0,t-s-r\}}^{t-s+r} \langle s+\rho\rangle^{-p}\,\langle s-\rho\rangle^{-p(\kappa-1)}\langle\rho\rangle\,d\rho\,ds.$$

Now we use the change of variables $\xi = s + \rho$, $\eta = \rho - s$. Since $\rho \geq 0$, we have $|\eta| \leq \xi$. Moreover, $\xi = s + \rho \leq s + (t - s - r) = t + r$ and $\xi \geq s + \max\{0, t - s - r\} \geq (t-r)_+$. Finally, we arrive at

$$I_0(t,r) \lesssim \int_{(t-r)_+}^{t+r} \langle\xi\rangle^{-p} \int_{-\xi}^{\xi} \langle\eta+\xi\rangle\langle\eta-\xi\rangle^{-(p-1)}\langle\eta\rangle^{-p(\kappa-1)}\,d\eta\,d\xi = \int_{(t-r)_+}^{t+r}\langle\xi\rangle^{-p} I(\xi)\,d\xi \tag{63}$$

with $I(\xi)$ as in Lemma 4. From Lemma 4 we conclude

$$I_0(t,r) \lesssim \int_{(t-r)_+}^{t+r} \langle\xi\rangle^{-\kappa}d\xi. \tag{64}$$

In the following we shall use different ideas in different zones of the $(t,r)$ plane.

*5.2.1. The zone $t \geq 2r$*

Here, we have in $[(t-r), (t+r)]$ the equivalence $\langle\xi\rangle \simeq \langle t+r\rangle$, therefore, $I_0(t,r) \lesssim r\langle t+r\rangle^{-\kappa}$.

*5.2.2. The zone $r \leq 1$ and $t \leq 2r$*

In this zone it holds $\langle t+r\rangle \simeq 1$. It is enough to show $I_0(t,r) \lesssim r$, which follows from (64) being $\kappa \geq 0$.

*5.2.3. The zone $r \geq 1$ and $r \leq t \leq 2r$*

If $r \leq t \leq 2r$, then from (64) we derive

$$I_0(t,r) \lesssim \int_{t-r}^{t+r} \langle\xi\rangle^{-\kappa}d\xi \lesssim \langle t-r\rangle^{-(\kappa-1)},$$

where we used $\kappa > 1$.

Now, let us estimate $I_0'$ for $r \geq 1$ and $t \leq r$. Applying the same change of variables to

$$I_0'(t,r) = \int_0^t \langle s\rangle^{-(p-1)} \int_{r-(t-s)}^{r+(t-s)} \langle s+\rho\rangle^{-p}\,\langle s-\rho\rangle^{-p(\kappa-1)}\langle\rho\rangle\,d\rho\,ds$$

we obtain

$$I_0'(t,r) \lesssim \int_{r-t}^{r+t} \langle\xi\rangle^{-p} \int_{r-t}^{\xi} \langle\eta+\xi\rangle\langle\eta-\xi\rangle^{-(p-1)}\langle\eta\rangle^{-p(\kappa-1)}\,d\eta\,d\xi.$$

Moreover, $[r-t, \xi] \subset [-\xi, \xi]$. From Lemma 4 we have

$$I_0(t,r) \lesssim \int_{r-t}^{r+t} \langle\xi\rangle^{-p} I(\xi)\,d\xi \lesssim \int_{r-t}^{r+t} \langle\xi\rangle^{-\kappa}\,d\xi \lesssim \langle t-r\rangle^{1-\kappa},$$

where we used again $\kappa > 1$. Finally, we prove (54). If $t \geq 2r$ or $r \leq 1$, then from $\langle t+r\rangle \geq \langle t-r\rangle$ it follows

$$|Lu(t,r)| \lesssim \langle t+r\rangle^{-\kappa}\|u\|_{X_\kappa}^p \lesssim \langle t+r\rangle\langle t-r\rangle^{-(\kappa-1)}.$$

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For $r \geq 1$ and $t \leq 2r$ we have

$$|Lu(t,r)| \lesssim \langle r \rangle^{-1} \langle t-r \rangle^{-(\kappa-1)} \|u\|_{X_\kappa}^p \simeq \langle t+r \rangle^{-1} \langle t-r \rangle^{-(\kappa-1)} \|u\|_{X_\kappa}^p.$$

The same arguments lead to (56).

**Proposition 4.** *Let $p > p_0(5)$ and $\kappa$ be as in (51). Then we have*

$$I_{1,-}(t,r) \lesssim \begin{cases} \langle t-r \rangle^{-\kappa} & \text{if } t \geq 2r, \\ \langle t-r \rangle^{-(\kappa-1)} & \text{if } t \leq 2r, \end{cases}$$

*and $I_{1,+} \lesssim \langle t+r \rangle^{-\kappa}$. In particular, the estimates (55) and (57) hold.*

Proof. We start with the estimate of $I_{1,-}$.

*5.2.4. The zone $t \geq 2r$*

Since $t+r \simeq t-r$ and if $s \in [t-r,t]$, then

$$s + |t-s-r| \simeq t-r.$$

Conversely, if $s \in [0, t-r]$, then

$$s + |t-s-r| = s+t-s-r = t-r.$$

Therefore,

$$I_{1,-} \lesssim \langle t-r \rangle^{-p} \int_0^t \langle s \rangle^{-(p-1)} \langle s - |t-s-r| \rangle^{-p(\kappa-1)} \langle t-s-r \rangle ds = \langle t-r \rangle^{-p}(Q_- + Q_+),$$

where

$$Q_- = \int_0^{t-r} \langle s \rangle^{-(p-1)} \langle 2s-t+r \rangle^{-p(\kappa-1)} \langle t-s-r \rangle ds,$$

$$Q_+ = \langle t-r \rangle^{-p(\kappa-1)} \int_{t-r}^t \langle s \rangle^{-(p-1)} \langle t-s-r \rangle ds.$$

So, we may directly estimate

$$Q_+ \leq \langle t-r \rangle^{-p(\kappa-1)-(p-1)} \int_{t-r}^t \langle t-s-r \rangle d\tau = \langle t-r \rangle^{-p(\kappa-1)-(p-1)} \int_{-r}^0 \langle \rho \rangle d\rho$$
$$\lesssim \langle t-r \rangle^{-p(\kappa-1)-(p-1)+2}.$$

Being $\kappa \geq \dfrac{3-p}{p-1}$ we have the required estimate $Q_+ \lesssim \langle t-r \rangle^{p-\kappa}$.

In order to estimate $Q_-$ we use Lemma 4. By the change of variables $\eta = \dfrac{t-r}{2} - s$, we have

$$Q_- \lesssim \int_{-\frac{t-r}{2}}^{\frac{t-r}{2}} \left\langle \eta + \frac{t-r}{2} \right\rangle^{-(p-1)} \langle \eta \rangle^{-p(\kappa-1)} \left\langle \eta - \frac{t-r}{2} \right\rangle d\eta = I\left(\frac{t-r}{2}\right) \lesssim \langle t-r \rangle^{p-\kappa}. \tag{65}$$

Together with the estimate of $Q_+$ this gives $I_{1,-} \lesssim \langle t-r \rangle^{-\kappa}$.

*5.2.5. The zone $t \le 2r$*

We write $I_{1,-} = \tilde{Q}_+ + \tilde{Q}_-$, where

$$
\begin{aligned}
\tilde{Q}_- &= \int_0^{(t-r)_+} \langle s \rangle^{-(p-1)} \langle t-r \rangle^{-p} \langle 2s-t+r \rangle^{-p(\kappa-1)} \langle t-s-r \rangle ds \\
&= \langle t-r \rangle^{-p} \, Q_-, \\
\tilde{Q}_+ &= \int_{(t-r)_+}^t \langle s \rangle^{-(p-1)} \langle 2s-t+r \rangle^{-p} \langle t-r \rangle^{-p(\kappa-1)} \langle t-s-r \rangle ds \\
&= \langle t-r \rangle^{-p(\kappa-1)} \int_{(t-r)_+}^t \langle s \rangle^{-(p-1)} \langle 2s-t+r \rangle^{-p} \langle t-s-r \rangle ds = \langle t-r \rangle^{-p(\kappa-1)} Q_+^{\sharp}.
\end{aligned}
$$

Since the estimate (65) holds for any $t \ge r$ we may directly conclude $\tilde{Q}_- \lesssim \langle t-r \rangle^{-\kappa}$. Since $p > 1$, in order to gain $\tilde{Q}_+ \lesssim \langle t-r \rangle^{-(\kappa-1)}$, it suffices to estimate $Q_+^{\sharp}$ by a constant.

Since $2s - (t-r) \ge s - (t-r)$ we have

$$
\begin{aligned}
Q_+^{\sharp} &\lesssim \int_0^\infty \langle s \rangle^{-(p-1)} \langle s-(t-r) \rangle^{-(p-1)} ds \\
&\lesssim \int_0^{(t-r)/2} \langle s \rangle^{-2(p-1)} ds + \int_{(t-r)/2}^\infty \langle t-s-r \rangle^{-2(p-1)} ds \le 2 \int_0^{+\infty} \langle s \rangle^{-2(p-1)} ds.
\end{aligned}
$$

This quantity is finite taking into consideration $2(p-1) > 2(p_0(5)-1) > 1$.

The estimate for $I_{1,+}$ is easier to obtain. Indeed

$$
I_{1,+} = \langle t+r \rangle^{-p} \int_0^t \langle s \rangle^{-(p-1)} \langle 2s-t-r \rangle^{-p(\kappa-1)} \langle t-s+r \rangle ds,
$$

due to $t + r - s \ge 0$. After the change of variables $\eta = \dfrac{t+r}{2} - s$ we are in position to apply Lemma 4 and conclude

$$
I_{1,+} \lesssim \langle t+r \rangle^{-p} \int_{-\frac{t+r}{2}}^{\frac{t+r}{2}} \left\langle \eta + \frac{t+r}{2} \right\rangle^{-(p-1)} \langle \eta \rangle^{-p(\kappa-1)} \left\langle \eta - \frac{t+r}{2} \right\rangle d\eta = \langle t+r \rangle^{-p} I\left(\frac{t+r}{2}\right) \lesssim \langle t+r \rangle^{-\kappa}.
$$

Now, we can gain (56), and similarly (57). If $t \ge 2r$, then we use $\langle t+r \rangle \simeq \langle t-r \rangle$ and $\langle r \rangle \ge 1$ to conclude

$$
|\partial_r (rLu)(t,r)| \lesssim \|u\|_{X_\kappa}^p \sum_{\pm} I_{1,\pm} \lesssim \|u\|_{X_\kappa}^p \langle t-r \rangle^{-\kappa} \lesssim \langle r \rangle \langle t+r \rangle^{-1} \langle t-r \rangle^{-\kappa-1} \|u\|_{X_\kappa}^p.
$$

If $t \le 2r$, then $\langle r \rangle \simeq \langle t+r \rangle$. Hence,

$$
|\partial_r (rLu)(t,r)| \lesssim \|u\|_{X_\kappa}^p \sum_{\pm} I_{1,\pm} \lesssim \|u\|_{X_\kappa}^p \langle t-r \rangle^{-(\kappa-1)} \lesssim \langle r \rangle \langle t+r \rangle^{-1} \langle t-r \rangle^{-\kappa-1} \|u\|_{X_\kappa}^p.
$$

*5.3. Existence theorem*

**Theorem 6.** *Let $p > p_0(5)$ and $\kappa$ as in (51). There exists a constant $\varepsilon_0 > 0$ such that if (37) holds with $\varepsilon < \varepsilon_0$, then the Cauchy problem (38) admits a unique global (in time) small data solution $u(t,r)$ in the sense of Definition 1. In particular, $u \in X_\kappa$ and the following decay estimate holds:*

$$
|u(t,r)| + |\partial_r u(t,r)| \lesssim \langle t+|r| \rangle^{-1} \langle t-|r| \rangle^{-(\kappa-1)}. \tag{66}
$$

Proof. Let us define the sequence

$$
u_0 = u^{\mathrm{lin}}, \quad u_{n+1} = u^{\mathrm{lin}} + Lu_n.
$$

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By using Theorems 4 and 5 we get

$$\|u_{n+1}\|_{X_\kappa} \le \|u^{\text{lin}}\|_{X_\kappa} + C_1\|u_n\|_{X_\kappa}^p \le C_0\varepsilon + C_1\|u_n\|_{X_\kappa}^p,$$

$$\|u_{n+1} - u_n\|_{X_\kappa} \le C_2\|u_n - u_{n-1}\|_{X_\kappa}\left(\|u_n\|_{X_\kappa}^{p-1} + \|u_{n-1}\|_{X_\kappa}^{p-1}\right)$$

with suitable constant $C_0, C_1, C_2 > 0$. For $\varepsilon_0 < (2C_0 C_1^{1/(p-1)})^{-1}$, we find via induction argument

$$\|u_n\|_{X_\kappa} \le 2\|u^{\text{lin}}\|_{X_\kappa} \le 2C_0\varepsilon_0.$$

In turn, for $\varepsilon_0 < (2^{p+1}C_2 C_0^{p-1})$, this gives

$$\|u_{n+1} - u_n\|_{X_\kappa} \le 2^{-n}\|u_1 - u_0\|_{X_\kappa}.$$

We can conclude that $\{u_n\}_n$ is a Cauchy sequence. It converges in $X_\kappa$ to the solution to $u = u^{\text{lin}} + Lu$. According to Proposition 2 this solution is the required one. The decay estimates follow from the definition of $X_\kappa$.

**Remark 6.** *From the decay estimate* (66) *we may derive an estimate for the solution to the scale-invariant damping Cauchy problem* (1). *Coming back, by the inverse Liouville transformation, we find*

$$|v(t, |x|)| \le \langle t\rangle^{-1}\langle t + |x|\rangle^{-1}\langle t - |x|\rangle^{-(\kappa-1)}.$$

*The worst situation is close to the light cone, where we only have*

$$|v(t, |x|)| \le \langle t\rangle^{-2}.$$

*The decay behavior* $\langle t\rangle^{-2}$ *in the 3-dimensional case can be interpreted as* $\langle t\rangle^{-\frac{(n+2)-1}{2}}$: *the same decay for the wave equation in dimension* $n + 2$. *This motivates the shift of 2-dimensions of the critical exponent* $p_0(n) \to p_0(n + 2)$. *The same effect can be observed in* [5] *for others space dimensions.*

## 6. Expectations for $\mu \ne 2$

The same type of transformation we used in the treatment of (1) allows us to transform the Cauchy problem with scale-invariant mass and dissipation

$$\begin{cases} v_{tt} - \triangle v + \dfrac{\mu}{1+t}v_t + \dfrac{m}{4(1+t)^2}v = |v|^p, & t > 0, \quad x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \\ v_t(0, x) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{67}$$

where $\mu > 0$ and $m \in \mathbb{R}$, to

$$\begin{cases} u_{tt} - \triangle u + \dfrac{\mu(2-\mu)+m}{4(1+t)^2}u = \langle t\rangle^{-\frac{\mu}{2}(p-1)}|u|^p, & t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{68}$$

where we set $u(t, x) = \langle t\rangle^{\frac{\mu}{2}}v(t, x)$, $u_0 = v_0$ and $u_1 = v_1 + (\mu/2)v_0$.

In particular, in the special case $m = (\mu - 2)\mu$, the equation in (68) becomes a wave equation with the nonlinearity $\langle t\rangle^{-\frac{\mu}{2}(p-1)}|u|^p$. We may directly follow the proof of Theorem 1 to obtain a nonexistence result for this special problem.

**Theorem 7.** *Assume that* $v \in C^2([0, T) \times \mathbb{R}^n)$ *is a solution to* (67) *with* $m = (\mu - 2)\mu$ *and initial data* $(v_0, v_1) \in C_c^2(\mathbb{R}^n) \times C_c^1(\mathbb{R}^n)$ *such that* $v_1, v_0 \ge 0$, *and* $(v_0, v_1) \not\equiv (0, 0)$. *If* $p \in (1, \tilde{p}_\mu(n)]$, *then* $T < \infty$, *where*

$$\tilde{p}_\mu(n) = \max\{p_\infty(n - 1 + \mu/2); p_0(n + \mu)\}.$$

**Remark 7.** *Let us compute $\tilde{p}_\mu(n)$. The critical exponent $p_0(n+\mu)$ may be written as*

$$p_0(n+\mu) = \frac{q + \sqrt{q^2 + 4(q-1)}}{2}, \qquad \text{where} \quad q = q(n+\mu) = 1 + \frac{2}{n-1+\mu} = p_\infty(n-1+\mu).$$

*To determine $\tilde{p}_\mu(n)$ we remark that $p_0(n+\mu) \geq r := p_\infty(n-1+\mu/2)$ if and only if*

$$\sqrt{q^2 + 4(q-1)} \geq 2r - q.$$

*Being $q < r$ for any $\mu > 0$ we may take the squared powers:*

$$q^2 + 4(q-1) \geq 4r^2 - 4rq + q^2,$$

*that is, $q - 1 \geq r(r-q)$, explicitly,*

$$\frac{2}{n-1+\mu} \geq \frac{n+1+\mu/2}{n-1+\mu/2}\Big(\frac{2}{n-1+\mu/2} - \frac{2}{n-1+\mu}\Big) = \frac{\mu(n+1+\mu/2)}{(n-1+\mu/2)^2(n-1+\mu)},$$

*that is,*

$$\mu(n-3) + 2(n-1)^2 \geq 0.$$

*It follows that $\tilde{p}_\mu(n) = p_0(n+\mu)$ for any $n \geq 3$, $\tilde{p}_\mu(1) = p_\infty(\mu/2)$, and*

$$\tilde{p}_\mu(2) = \begin{cases} p_\infty(1+\mu/2) & \text{if } \mu \geq 2, \\ p_0(2+\mu) & \text{if } \mu \in [0,2]. \end{cases}$$

The statements of Theorem 7 are consistent with known results for the classical semilinear wave equation (i.e. $\mu = 0$) and with Theorem 1 (i.e. $\mu = 2$), which are the only two cases in which $m = 0$.

PROOF (PROOF OF THEOREM 7). We only sketch the differences to the proof of Theorem 1. For the sake of brevity we only consider the subcritical case. It is clear that we obtain

$$\ddot{F}(t) \gtrsim \langle t \rangle^{-(n+\mu/2)(p-1)} |F(t)|^p,$$
$$\ddot{F}(t) \gtrsim \langle t \rangle^{-(n-1+\mu/2)(p-1)+(n-1)\frac{p}{2}} |F_1(t)|^p.$$

By virtue of Lemma 2 we derive again $F_1(t) > 0$, and integrating twice the estimate for $\ddot{F}(t)$, we derive $F(t) \gtrsim \langle t \rangle^a$, where

$$a = \max\Big\{-\frac{n-1+2\mu}{2}p + n + 2, 1\Big\}.$$

Setting $q = (n+\mu/2)(p-1)$ we immediately obtain the blow-up in finite time if $1 > (q-2)/(p-1)$, i.e. $p < p_\infty(n-1+\mu/2)$, or if

$$-\frac{n-1+2\mu}{2}p + n + 2 > \frac{q-2}{p-1} = n + \frac{\mu}{2} - \frac{2}{p-1},$$

i.e. $p < p_0(n+\mu)$.

We may conjecture that global existence of small data solutions holds for some range of $p > \tilde{p}_\mu(n)$. For this reason we propose as critical exponent $p_{crit}$ the value $\tilde{p}_\mu(n)$.

**Remark 8.** *Let $\mu \in (0,2)$ in (3). We may expect that the critical exponent $p_\mu(n)$ is not larger than $\tilde{p}_\mu(n)$ due to the fact that the model in (67) with $m = (\mu-2)\mu$ has an additional negative mass term with respect to the model in (3). Moreover, we know that the critical exponent has to be not smaller than $p_\infty(n-(1-\mu)_+)$. Therefore, we expect that*

$$p_\infty(n-(1-\mu)_+) \leq p_\mu(n) \leq \tilde{p}_\mu(n).$$

If $n \geq 2$, then we may replace $\tilde{p}_\mu(n) = p_0(n + \mu)$, whereas if $n = 1$, we replace $\tilde{p}_\mu(1) = p_\infty(\mu/2)$, i.e. we expect that $p_\mu(1) \in [1 + 2/\mu, 1 + 4/\mu]$. Indeed, if $n = 1$ our considerations are restricted to $\mu \in (0, 5/3)$, since we already know that the critical exponent is $3$ for $\mu \geq 5/3$.

On the other hand, if $n \geq 3$ and $\mu \in (2, n + 2)$ in (3), then we may expect that the critical exponent $p_\mu(n)$ is not smaller than $\tilde{p}_\mu(n)$, due to the fact that the model in (67) with $m = (\mu - 2)\mu$ has an additional positive mass term with respect to the model in (3). Moreover, we know that the critical exponent may not be smaller than $p_\infty(n)$. Therefore, we expect that

$$\max\{p_0(n + \mu); p_\infty(n)\} \leq p_\mu(n) \leq p_0(n + 2).$$

## 7. Concluding remarks and open problems

**Remark 9.** *In the statement of Theorem 2 we may weaken the assumption on the data from $(\bar{v}_0, \bar{v}_1) \in C_c^2 \times C_c^1$ to $(\bar{v}_0, \bar{v}_1) \in H^2 \times H^1$, compactly supported.*

**Remark 10.** *In the paper [5] the first two authors deal with the odd dimensional cases $n \geq 5$. They prove the global existence of small data solutions to (1) for some admissible range of $p \in (p_0(n+2), p_1)$. This yields together with the statement from Theorem 1, that $p_0(n + 2)$ is the critical exponent for (1) in odd space dimensions $n \geq 5$, too. It remains to analyze the case of even $n \geq 4$. But the authors expect the shift $p_0(n) \rightarrow p_0(n + 2)$ in all space dimensions $n \geq 4$, too.*

**Remark 11.** *Prof. H.Takamura (Future University Hakodate, Japan) has given the authors the following hint: He is able to prove the estimate of the lifespan of solutions of semilinear wave equations with the scaling invariant damping term of (1). He will use the method from [22] for the sub-critical case and the method from [23] for the critical case. This is possible because we apply in Section 3 the usual functional method to semilinear wave equations without damping but with time dependent coefficient in the semilinear term. It just shifts the critical exponent $p_0(n)$ to $p_0(n + 2)$. The following life-span estimates can be proved for $n \geq 2$:*

$$T(\varepsilon) \leq \exp(C\varepsilon^{-p(p-1)}) \quad \text{for } p = p_0(n + 2),$$
$$T(\varepsilon) \leq C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+2)}} \quad \text{for } 1 < p < p_0(n + 2).$$

*In the second life-span estimate the parameter $\gamma(p, n + 2)$ is defined by*

$$\gamma(p, n + 2) = 2 + (n + 3)p - (n + 1)p^2.$$

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