A Semigroup Approach to generalized Black-Scholes Type Equations in Incomplete Markets

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Abstract

In this paper we will study an option pricing problem in incomplete markets by an analytic point of view. The incompleteness is generated by the presence of a non-traded asset. The aim of this paper is to use the semigroup theory in order to prove existence and uniqueness of solutions to generalized Black-Scholes type equations that are non-linearly associated with the price of European claims written exclusively on non-traded assets. Then, we derive analytic expressions of the solutions. An approximate representation in terms of a generalized Feynman-Kac type formula is derived in cases where an explicit closed form solution is not available. Numerical examples are also given (see Appendix E) where theoretical approximations and numerical tests reveal a remarkable agreement.

Keywords: Option pricing, indifference price, partial differential equations, (C_0) semigroups, approximate pricing formula, numerical tests.

1. Introduction

As explained by L. E. O. Svensson in [25]: "In the literature of international finance, the existence of income from non-traded assets seems to be the rule rather than the expectation. The existence of non-traded assets could be a result of asset market imperfections, which in turn are caused by the usual reasons: transaction costs,

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moral hazard, legal restrictions, etc. As examples we can think of an individual who cannot trade claims to his future wages for obvious moral hazard reasons; a government which cannot trade claims to future tax receipts; or a country which cannot trade claims to its gross domestic product (GDP) in world capital markets".

Finding solutions to portfolio problems in a continuous time model when there is some income from non-traded assets, can be seen as the problem of pricing and hedging in incomplete markets, where incompleteness is generated just by non-traded assets that prevent the creation of perfectly replicating portfolio. Thus the evaluation based on replication and no arbitrage assumptions is no longer possible and new strategies to price and hedge derivatives that are written on such securities are needed (see, for example, Musiela and Zariphopoulou [19] and references therein). A pricing methodology is based on utility maximization criteria which produce the so called indifference price. "The pricing mechanism is based upon the parity between the maximal utilities, with and without employing the derivative. The residual amount gained from granting the option, which renders the investor impartial towards these two scenarios, is called the indifference price" (see [20, p. 1]). By this approach the price is not determined with respect to the risk neutral measure as in a complete setting, but with respect to an indifference measure:"describing the historical behavior of the non-traded asset. In fact, it refers to which is defined as the closest to the risk neutral one and, at the same time, capable of measuring the unhedgeable risk, by being defined on the filtration of the Brownian motion used for the modelling of the non-traded asset dynamics" (Musiela and Zariphopoulou [19, p.3]). The indifference pricing problem and the underlying utility-optimization problem are well characterized by martingale duality results (see [12]), by stochastic differential equations (see [23]) and by non-linear partial differential equations (see [3]). The utility-based price and the hedging strategy can be described by the solution of a partial differential equation, in analogy to the Black Scholes model [2], but it is more difficult to obtain the explicit solution in specific models.

Musiela and Zariphopoulou in [19] derived the indifference price of a European claim, written exclusively on the non-traded asset, as a non-linear expectation of the derivative's payoff under an appropriate martingale measure. The aim of the present paper is to give a closed form representation of the indifference price by using analytical tools based on (C_0) semigroup theory, which is often useful to study the evolution in time of some problems coming from Mathematical Physics, Mathematical Finance and other applied sciences (see e.g. [10], [13], [14], [5]). A different approach to compute closed form approximate solutions of one-dimensional parabolic equations, with variable coefficients, associated with option pricing problems can be find in [6] (see also references therein).

The paper is organized as follows. After a short discussion in Section 2, in Section 3 we will focus on initial value Cauchy problems associated with operators of the type

$$\mathcal{L}u = \frac{1}{2}(cx + d)^2 u'' + [(\gamma x + \delta) - \theta(cx + d)]u',$$

acting on the space of all real-valued continuous functions in a suitable unbounded real interval having finite limits at the endpoints. Here the domain of $\mathcal{L}$ depends on the coefficients of u', u''. Useful generation results are shown and, further, an analytic expression of the solution to the associated Cauchy problem is derived. In the next Section 4 approximate solutions expressed in terms of a generalized Feynman-Kac type formula are given when explicit closed forms are not available. Some numerical applications and examples are also presented in order to compare the approximate solutions with benchmark formulas in the literature. These applications confirm the accuracy and the fast convergence of the proposed approximation formulas, thus showing that an approximate representation of the indifference price is particularly helpful in cases where exact pricing formulas fail because their coefficients cannot be explicitly computed. Section 5 deals with the conclusions.

2. Analytic Problem

We consider the market environment with two risky assets assumed by Musiela and Zariphopoulou in [19]; the price S of the traded asset is a geometric Brownian motion, i.e. it is the unique solution of the following SDE

$$dS_\tau = \mu S_\tau d\tau + \sigma S_\tau d\widetilde{W}_\tau, \quad 0 \le t < \tau, \tag{1}$$

with initial value $S_t = s > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$. The dynamics of the level Y of the nontraded risky asset is described by a general diffusion process satisfying the SDE

$$dY_\tau = b(Y_\tau, \tau)\, d\tau + a(Y_\tau, \tau)\, dW_\tau, \quad 0 \le t < \tau, \tag{2}$$

with initial value $Y_t = y \in \mathbb{R}$, where the coefficients $b(\cdot, \cdot)$ and $a(\cdot, \cdot)$ satisfy enough regularity for (2) to have a unique (strong) solution. The processes $\{\widetilde{W}_\tau, \tau \ge 0\}$ and $\{W_\tau, \tau \ge 0\}$ are standard one-dimensional Brownian motions defined on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_\tau)_{\tau \ge 0}, \mathbb{P})$, where $\mathcal{F}_\tau$ is the σ algebra generated by $\{\widetilde{W}_u, W_u,\ 0 \le u \le \tau\}$. The Brownian motions are correlated with instantaneous correlation coefficient $\rho \in (-1, 1)$. It is assumed that the derivative to be priced is a European claim written exclusively on the nontraded asset, whose payoff at the maturity $T > t$ is of the form $\varepsilon = G(Y_T)$, being G a bounded function. Moreover,

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trading occurs in the time horizon $[t, T]$ and only between the risky asset with price S and a riskless bond $B = 1$ with maturity T, yielding constant interest rate $0 \leq r < \mu$. Without any loss of generality, we assume that $r = 0$.

In the framework described above, the individual risk preferences are modelled via an exponential utility function

$$U(x) = -e^{-\eta x}, \quad x \in \mathbb{R}, \tag{3}$$

where $\eta > 0$ is the *risk aversion* parameter. Then according to the approach to pricing based on the comparison of maximal expected utility payoffs, Musiela and Zariphopoulou [19] derived the writer's indifference price of a European derivative with payoff $\varepsilon = G(Y_T)$ in the following closed form (see [19, Theorem 3])

$$h(y, t) = \frac{1}{\eta(1 - \rho^2)} \ln \mathbb{E}_{\mathbb{Q}}[e^{\eta(1-\rho^2)G(Y_T)} \mid Y_t = y], \tag{4}$$

for $(y, t) \in \mathbb{R} \times [0, T]$, where the pricing measure $\mathbb{Q}$ is defined as follows

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}\left[exp\left(-\rho \frac{\mu}{\sigma} W_T - \frac{1}{2} \rho^2 \frac{\mu^2}{\sigma^2}\right) I_A\right], \quad A \in \mathcal{F}_T. \tag{5}$$

The above measure is a martingale measure that minimizes the entropy with respect to the probability measure $\mathbb{P}$, i.e. it is the minimum

$$\min_{Q \in \mathcal{M}} \mathbb{E}_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} \ln \frac{dQ}{d\mathbb{P}}\right]$$

in the class $\mathcal{M}$ of all $\mathbb{P}$-absolutely continuous martingale measures Q ([19, Theorem 2]).

In particular, under the measure $\mathbb{Q}$, Y_τ satisfies the following SDE

$$dY_\tau = \left(b(Y_\tau, \tau) - \rho \frac{\mu}{\sigma} a(Y_\tau, \tau)\right) d\tau + a(Y_\tau, \tau) d\overline{W}_\tau, \tag{6}$$

where $\overline{W}_\tau = W_\tau + \rho \frac{\mu}{\sigma} \tau$ is a Brownian motion. Hence, $(Y_\tau)_{\tau \in [0,T]}$ is a diffusion process with the infinitesimal generator

$$\frac{\partial}{\partial \tau} + \frac{1}{2} a^2(y, \tau) \frac{\partial^2}{\partial y^2} + \left(b(y, \tau) - \rho \frac{\mu}{\sigma} a(y, \tau)\right) \frac{\partial}{\partial y}.$$

Remark 2.1. In complete arbitrage-free markets every financial instrument can be replicated. Thus by using self-financing and replicating portfolio strategies, the

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arbitrage-free representation of a contingent claim at a time $0 \leq t < T$ is uniquely determined as the conditional expectation of the discounted payoff function under the unique risk-neutral martingale measure (see, e.g., [24, Section VI.1]). As mentioned in the Introduction, in incomplete arbitrage-free markets financial instruments are not, in general, perfectly replicable. Further, asset pricing will depend on the utility function of investors. Thus a specific martingale measure, which is defined as the closest to the risk-neutral one, must be determined by certain optimality criteria to price a contingent claim. Frittelli in [12] showed that if the minimal entropy martingale measure exists, it is unique and is equivalent to $\mathbb{P}$.

Our aim herein is to give an explicit representation for the indifference pricing function $h(y,t)$ that, by (4), can be written as

$$h(y,t) = \frac{1}{\eta(1-\rho^2)} \ln w(y,t),$$

where

$$w(y,t) = \mathbb{E}_{\mathbb{Q}}[e^{\eta(1-\rho^2)G(Y_T)} \mid Y_t = y]. \tag{7}$$

Under the usual regularity for the Feynman-Kac approach (see, for example, [21, Section 8.2]), $w(y,t)$ solves the Cauchy problem

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{1}{2}a^2(y,t)\frac{\partial^2 w}{\partial y^2} + \left(b(y,t) - \rho\frac{\mu}{\sigma}a(y,t)\right)\frac{\partial w}{\partial y} = 0, & (y,t) \in \mathbb{R} \times [0,T), \\ w(y,T) = e^{\eta(1-\rho^2)G(y)}, & y \in \mathbb{R}, \end{cases} \tag{8}$$

where $y = Y_t$ is a dummy variable for any $t \in [0,T]$. To give an explicit representation of $h(y,t)$, we will prove existence and uniqueness of the solution to problem (8) by a semigroup approach. Observe that, in a perfect correlation between the traded and the non-traded asset, i.e. when $\rho^2 = 1$, and, in addition, the coefficients in (2) are $b(y,t) = \mu y$ and $a(y,t) = \sigma y$, the market becomes complete and the indifference price h reduces to the usual Black-Scholes model [2] (see [20, Theorem 2.3]).

With the variable change $u(y,t) = w(y,T-t)$ the problem (8) can be transformed from a backward to a forward parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}a^2(y,t)\frac{\partial^2 u}{\partial y^2} + \left(b(y,t) - \rho\frac{\mu}{\sigma}a(y,t)\right)\frac{\partial u}{\partial y}, & (y,t) \in \mathbb{R} \times (0,T], \\ u(y,0) = e^{\eta(1-\rho^2)G(y)}, & y \in \mathbb{R}. \end{cases} \tag{9}$$

Once we consider (9), we may do so for $0 \leq t < +\infty$.

The coefficients $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are assumed of the type

$$a(Y_\tau,\tau) := c_\tau Y_\tau + d_\tau, \quad b(Y_\tau,\tau) := \gamma_\tau Y_\tau + \delta_\tau, \tag{10}$$

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where $c_\tau, d_\tau, \gamma_\tau, \delta_\tau$ are suitable functions depending on τ. This assumption is not restrictive from a financial point of view, because it is often possible to reduce $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ to (10) (for more details the reader can refer to [11]).

In the sequel, we shall focus the discussion on the autonomous case, i.e. we assume

$$a(Y_\tau, \tau) \equiv a(Y_\tau) = cY_\tau + d, \quad b(Y_\tau, \tau) \equiv b(Y_\tau) = \gamma Y_\tau + \delta,$$

for any $\tau \geq 0$, with $c, d, \gamma, \delta \in \mathbb{R}$. Under this assumption, obviously, the SDE (6) has a unique strong solution (see, for example, [21, Section 5.2]).

Remark 2.2. We notice that assumption (10) is suggested by Musiela and Zariphopoulou in [20]. Indeed, they indicate as possible candidate for the dynamics of the non-traded asset a class of diffusion processes for which $a(y) = d$, $b(y) = \gamma y + \delta$, with $d, \gamma, \delta \in \mathbb{R}$. We shall examine this particular case in Section 3.

3. Semigroup Approach: generation results and explicit representations

Let $J = (r_1, r_2)$ be a real interval with $-\infty \leq r_1 < r_2 \leq +\infty$ and $C(\bar{J})$ be the space of all real-valued continuous functions in J having finite limits at the endpoints r_1, r_2. We consider the following abstract Cauchy problem

$$(ACP) \qquad \begin{cases} u_t = \mathcal{L}u, & \text{in } J \times (0, +\infty), \\ u(x, 0) = g(x), & x \in J, \end{cases} \tag{11}$$

where $\mathcal{L}$ is a differential operator of the type $\mathcal{L}u = m(x)u'' + q(x)u'$ acting on $C(\bar{J})$, with the maximal domain

$$D_M(\mathcal{L}) = \{u \in C(\bar{J}) \cap C^2(J) \,|\, \mathcal{L}u \in C(\bar{J})\}. \tag{12}$$

For our purposes

$$\mathcal{L}u = \frac{1}{2}a^2(x)u'' + (b(x) - \theta a(x))u',$$

with $a(x) = cx + d$, $b(x) = \gamma x + \delta$, and $c, d, \gamma, \delta, \theta \in \mathbb{R}$ constant parameters. Here $\theta = \rho\mu/\sigma$, where the parameters ρ, μ and σ are defined in Section 2. Thus we can write

$$\mathcal{L}u = \frac{1}{2}(cx + d)^2 u'' + ((\gamma x + \delta) - \theta(cx + d))u'. \tag{13}$$

The initial condition of (11) is

$$g(x) = e^{\eta(1-\rho^2)G(x)}, \quad x \in J, \tag{14}$$

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where $\eta > 0$ is the risk aversion parameter and $G \in C(\bar{J})$.

In order to solve the (ACP) (11), we will start by showing the existence of a (C_0) semigroup generated by $\mathcal{L}$ on $C(\bar{J})$, under the additional assumption $\delta c = d\gamma$. A preliminary result is presented in the following.

Lemma 3.1. *Let us assume that c, d, γ, δ satisfy*

$$(\delta c = d\gamma) \ \ and \ \ (c, d) \neq (0, 0). \tag{15}$$

Then the operator $(\mathcal{L}, D_M(\mathcal{L}))$ has one of the following expressions, for any $u \in D_M(\mathcal{L})$,

$$\mathcal{L}u = \frac{1}{2}c^2 x^2 u'' + k_1 x u', \ \ if \ d = 0, \tag{16}$$

$$\mathcal{L}u = \frac{1}{2}d^2 u'' + k_2 u', \ \ if \ c = 0, \tag{17}$$

$$\mathcal{L}u = \frac{1}{2}d^2 (kx + 1)^2 u'' + k_3 (kx + 1)u', \ \ if \ c \neq 0 \ and \ d \neq 0. \tag{18}$$

Here $k_1 = \gamma - \theta c$, $k_2 = \delta - \theta d$, $k = \frac{d}{c}$, $k_3 = k_2$ if $\gamma \neq 0$ and $k_3 = -\theta d$ if $\gamma = \delta = 0$. Moreover, for $\gamma \neq 0$ and $d \neq 0$ we have $k = \frac{d}{c} = \frac{\gamma}{\delta}$.

Proof. See Appendix B. □

We are now in a position to state our first main result of this section.

Theorem 3.2. *Assume that c, d, γ, δ, $\theta \in \mathbb{R}$ satisfy (15) and the operator $\mathcal{L}$ is defined by (13) with domain $D_M(\mathcal{L})$ given by (12). Then the operator $(\mathcal{L}, D_M(\mathcal{L}))$ generates a positive (C_0) contraction semigroup on $C(\bar{J})$. Here $J = (0, +\infty)$ if $\mathcal{L}$ is of the type (16), $J = \mathbb{R}$ if $\mathcal{L}$ is of the type (17) and $J = \left(-\frac{d}{c}, +\infty\right)$ if $c > 0$ (resp. $J = \left(-\infty, -\frac{d}{c}\right)$ if $c < 0$) if $\mathcal{L}$ is of the type (18).*

Proof. As consequence of Lemma 3.1, the operator $\mathcal{L}$ takes the form (16) or (17) or (18). In the case $\mathcal{L}$ is of the type (16), then, according to [15, Theorem 3.2], the operator $\mathcal{L}$ with domain $D_M(\mathcal{L})$ generates a positive (C_0) contraction semigroup on $C[0, +\infty]$. In the case $\mathcal{L}$ is of the type (17), the assertion follows from [15, Section IV] and $J = \mathbb{R}$.

Finally, let us focus on $\mathcal{L}$ of the form (18). Let us proceed with the change of variable

$$z = \Phi(x) = kx + 1, \tag{19}$$

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where $k = \frac{d}{c}$. If $c > 0$ (analogous arguments work for $c < 0$), then the operator $\mathcal{L}$ acting on $C\left[-\frac{d}{c}, +\infty\right]$ can be transformed in the operator $\widetilde{\mathcal{L}}$ acting on $C[0, +\infty]$, where

$$\widetilde{\mathcal{L}}v = \frac{1}{2}\tilde{k}^2 z^2 v'' + k_3 kzv', \tag{20}$$

with $\tilde{k} = kd$, and k_3 defined as in Lemma 3.1. We observe that $D(\widetilde{\mathcal{L}})$ has the same expression of $D_M(\mathcal{L})$ where J is replaced by $(0, +\infty)$ (see [15, Theorem 3.2]). Consequently, $\widetilde{\mathcal{L}}$ is of the type (16) on $C[0, +\infty]$. Thus, according to the above arguments, $\widetilde{\mathcal{L}}$ generates a positive (C_0) contraction semigroup on $C[0, +\infty]$, where $D_M(\widetilde{\mathcal{L}})$ is obtained easily from $D_M(\mathcal{L})$. Hence, Lemma A.2 and Remark A.3 imply the assertion. $\qquad\square$

Remark 3.3. Direct calculations and known results concerning Feller classification of the endpoints (see [10, Chapter VI, Section 4]) allow to show that $D_M(\mathcal{L})$) coincides with

$$D_W(\mathcal{L}) = \left\{ u \in C(\bar{J}) \cap C^2(J) | \lim_{\substack{x \to r_1 \\ x \to r_2}} \mathcal{L}u(x) = 0 \right\},$$

provided that the assumptions of Theorem 3.2 hold.

The next step is to give an *explicit* representation of the solution to (11), the second main result of this section.

Theorem 3.4. *Under the assumptions of Theorem 3.2, fixed $g \in C(\bar{J})$ and for any $(x, t) \in J \times [0, +\infty)$ the explicit solution to the problem (11) is given by*
Case 1: *If $d = 0$,*

$$u(x, t) = \int_{-\infty}^{+\infty} g\left(e^{\frac{c}{\sqrt{2}}(\psi t + y)}x\right)p(t, y)dy, \tag{21}$$

with $J = (0, +\infty)$ and $\psi = \frac{\sqrt{2}}{c}\gamma - \sqrt{2}\theta - \frac{c}{\sqrt{2}}$.
Case 2: *If $c = \gamma = 0$,*

$$u(x, t) = \int_{-\infty}^{+\infty} g\left(x + \frac{d}{\sqrt{2}}(\omega t + y)\right)p(t, y)dy, \tag{22}$$

with $J = \mathbb{R}$ and $\omega = \frac{\sqrt{2}}{d}(\delta - \theta d)$.
Case 3: *If $d \neq 0$ and $c \neq 0$,*

$$u(x, t) = \int_{-\infty}^{+\infty} g\left(e^{\frac{k}{\sqrt{2}}(\chi t + y)}(kx + 1)\right)p(t, y)dy, \tag{23}$$

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with $J = \left(-\frac{d}{c}, +\infty\right)$ *if* $c > 0$ *(resp.* $J = \left(-\infty, -\frac{d}{c}\right)$ *if* $c < 0$*) and* $\chi = \frac{\sqrt{2}}{d} k_3 - \frac{\tilde{k}}{\sqrt{2}}$.
In all cases

$$p(t, y) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{y^2}{4t}}, \qquad t > 0, \ y \in \mathbb{R}. \tag{24}$$

Proof. **Case 1.** According to Lemma 3.1, if $d = 0$ the operator $\mathcal{L}$ can be written as in (16). Thus define the operator $Gu := \frac{c}{\sqrt{2}} xu'$ with domain

$$D(G) = \{u \in C[0, +\infty] \cap C^1(0, +\infty) | u', xu' \in C[0, +\infty]\}. \tag{25}$$

Hence, the square of G is given by

$$G^2 u = \frac{c}{\sqrt{2}} x \left(\frac{c}{\sqrt{2}} xu'\right)' = \frac{c^2}{2} x^2 u'' + \frac{c^2}{2} xu',$$

with domain

$$\begin{aligned} D(G^2) &= \{u \in D(G) | Gu \in D(G)\} \\ &= \{u \in C[0, +\infty] \cap C^2(0, +\infty) | u', xu', x(xu')' \in C[0, +\infty]\}. \end{aligned}$$

Therefore, for any $u \in D(G^2)$, the operator $\mathcal{L}$ can be written as

$$\mathcal{L}u = G^2 u + \psi Gu, \tag{26}$$

with $\psi = \frac{\sqrt{2}}{c}\gamma - \sqrt{2}\theta - \frac{c}{\sqrt{2}}$. Notice that $(G, D(G))$ generates a (C_0) group on $C[0, +\infty]$ and, according to [15, Section 3], it is a suitable perturbation of $(G^2, D(G^2))$. Hence, by [13, Chapter II, Section 8], $(\mathcal{L}, D(G^2))$ generates the following (C_0) contraction semigroup

$$\begin{aligned} T(t)g(x) &= \int_0^{+\infty} \left(g\left(e^{\frac{c}{\sqrt{2}}(\psi t + y)} x\right) + g\left(e^{\frac{c}{\sqrt{2}}(\psi t - y)} x\right)\right) p(t, y) dy \\ &= \int_{-\infty}^{+\infty} g\left(e^{\frac{c}{\sqrt{2}}(\psi t + y)} x\right) p(t, y) \, dy \end{aligned}$$

for any $t \geq 0$, and therefore the representation (21) follows.
Case 2: If $c = \gamma = 0$, then the operator $\mathcal{L}$ can be written as in (17). Let us define

$$Gu = \frac{d}{\sqrt{2}} u', \quad D(G) = \{u \in C(\bar{\mathbb{R}}) | u' \in C(\mathbb{R})\}. \tag{27}$$

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Then, the square of G is given by

$$G^2 u = \frac{d}{\sqrt{2}}\left(\frac{d}{\sqrt{2}}u'\right)' = \frac{d^2}{2}u'', \tag{28}$$

with domain

$$D(G^2) = \{u \in D(G)|Gu \in D(G)\} = \{u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R})|u', u'' \in C(\overline{\mathbb{R}})\}. \tag{29}$$

Thus, for any $u \in D(G^2)$, $\mathcal{L}$ can be written as

$$\mathcal{L}u = \frac{1}{2}c^2 x^2 u'' + k_1 x u' = G^2 u + \omega G u, \tag{30}$$

with $\omega = \frac{\sqrt{2}}{d}(\delta - \theta d)$. The operator $(G, D(G))$ generates a (C_0) group on $C[0, +\infty]$ and according to [15, Section 4], it is a suitable perturbation of $(G^2, D(G^2))$. Then $(\mathcal{L}, D(G^2))$ generates the following (C_0) contraction semigroup

$$T(t)g(x) = \int_0^{+\infty} \left(g\left(x + \frac{d}{\sqrt{2}}(\omega t + y) \right) + g\left(x + \frac{d}{\sqrt{2}}(\omega t - y) \right) \right) p(t, y)\, dy$$
$$= \int_{-\infty}^{-\infty} g\left(x + \frac{d}{\sqrt{2}}(\omega t + y) \right) p(t, y)\, dy,$$

and therefore the representation (22) follows.

Case 3: If $c \neq 0$, $d \neq 0$ the operator $\mathcal{L}$ can be written as in (18). By the change of variable (19), $\mathcal{L}$ can be transformed into the operator $\widetilde{\mathcal{L}}$ defined in (20), acting on $C[0, +\infty]$. Hence, let us define

$$Gv := \frac{\tilde{k}}{\sqrt{2}}zv', \quad D(G) = \{v \in C[0, +\infty] \cap C^1(0, +\infty)|v', zv' \in C[0, +\infty]\}.$$

Thus, the square of G is given by

$$G^2 v = \frac{\tilde{k}}{\sqrt{2}}z\left(\frac{\tilde{k}}{\sqrt{2}}zv'\right)' = \frac{\tilde{k}^2}{2}z^2 v'' + \frac{\tilde{k}^2}{2}zv',$$

with domain

$$D(G^2) = \{v \in D(G)|Gv \in D(G)\}$$
$$= \{v \in C[0, +\infty] \cap C^2(0, +\infty)|v', zv', z(zv')' \in C[0, +\infty]\}.$$

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Therefore, for any $v \in D(G^2)$, $\widetilde{\mathcal{L}}$ can be written as

$$\widetilde{\mathcal{L}}v = G^2 v + \chi G v, \tag{31}$$

where $\chi = \frac{\sqrt{2}}{d} k_3 - \frac{\tilde{k}}{\sqrt{2}}$. Analogous arguments as for the Case 1 lead to state that $(\mathcal{L}, D(G^2))$ generates the following (C_0) contraction semigroup

$$T(t)g(x) = \int_{-\infty}^{+\infty} g\left(e^{\frac{k}{\sqrt{2}}(\chi t + y)}(kx + 1)\right) p(t, y) \, dy$$

for any $t \geq 0$ and $x \in J = \left(-\frac{d}{c}, +\infty\right)$ if $c > 0$ (resp. $x \in J = \left(-\infty, -\frac{d}{c}\right)$ if $c < 0$), and therefore the representation (23) follows. $\qquad\square$

Now let us prove the existence and uniqueness of the solution to (11) without assuming that (15) holds.

In the next theorem we suppose that $c = 0$, $\gamma \neq 0$. This implies $d > 0$ because the diffusion coefficient $a(x) = cx + d$ in the SDE (6) must be positive.

Theorem 3.5. *Assume $c = 0$, $\gamma \neq 0$, $d > 0$ and $\delta, \theta \in \mathbb{R}$, so that $\mathcal{L}$, with maximal domain $D_M(\mathcal{L})$ defined in (12), takes the form*

$$\mathcal{L}u = \frac{1}{2} d^2 u'' + (\gamma x + (\delta - \theta d)) \, u'. \tag{32}$$

Then, for any $g \in C(\overline{\mathbb{R}})$, there exists a unique solution to the (ACP) (11).

Proof. First of all, we note that the operator $\mathcal{L}$ in (32) can be rewritten as

$$\mathcal{L}u = \frac{1}{2} d^2 u'' + \gamma\left(x + \frac{(\delta - \theta d)}{\gamma}\right) u'. \tag{33}$$

Thus, by the change of variable

$$z = \Phi(x) = x + \frac{(\delta - \theta d)}{\gamma} \tag{34}$$

the operator (33) is transformed in the following operator

$$\widetilde{\mathcal{L}}u = \frac{1}{2} d^2 u'' + \gamma z u'. \tag{35}$$

Hence, to prove the existence and uniqueness of the solution to (ACP), it is sufficient to show that the boundary endpoints $\pm\infty$ are of entrance or natural type (see Theorem A.4).

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We start to study the boundary point $-\infty$. For sake of simplicity, we set $z_0 = -1$ and compute (see Definition A.1)

$$W(z) = \exp\left(-\int_{-1}^{z} \frac{2\gamma s}{d^2}\, ds\right) = \exp\left(-\frac{\gamma}{d^2}(z^2 - 1)\right) = e^{-\alpha}e^{\alpha z^2},$$

with $\alpha = -\frac{\gamma}{d^2} \neq 0$. Then

$$\int_{-\infty}^{-1} W(z)\, dz = \frac{e^{-\alpha}}{2\alpha}\int_{-\infty}^{-1} \frac{2\alpha z e^{\alpha z^2}}{z}\, dz = \frac{e^{-\alpha}}{2\alpha}\int_{-\infty}^{-1} \frac{1}{z}\frac{d}{dz}(e^{\alpha z^2})\, dz.$$

Observe that, by twice integration by parts,

$$\int \frac{1}{z}\frac{d}{dz}(e^{\alpha z^2})\, dz = \frac{1}{z}e^{\alpha z^2} + \frac{1}{2\alpha z^3}e^{\alpha z^2} + \frac{3}{4\alpha^2}\int \frac{1}{z^5}\frac{d}{dz}(e^{\alpha z^2})\, dz,$$

and therefore, as $z \to -\infty$

$$\int \frac{1}{z}\frac{d}{dz}(e^{\alpha z^2})\, dz \simeq \frac{1}{z}e^{\alpha z^2}. \tag{36}$$

Moreover, by noting that in our case $m(z) = d^2/2$, so that $(m(z)W(z))^{-1} = \frac{2e^\alpha}{d^2}e^{-\alpha z^2}$, we compute

$$\int_{-\infty}^{z} (m(s)W(s))^{-1}\, ds = \frac{2e^\alpha}{d^2}\int_{-\infty}^{z} e^{-\alpha s^2}\, ds = -\frac{e^\alpha}{\alpha d^2}\int_{-\infty}^{z} \frac{1}{s}\frac{d}{ds}(e^{-\alpha s^2})\, ds.$$

With similar arguments as before it follows that, as $s \to -\infty$

$$\int \frac{1}{s}\frac{d}{ds}(e^{-\alpha s^2})\, ds \simeq \frac{1}{s}e^{-\alpha s^2}. \tag{37}$$

Assume $\alpha > 0$ (i.e. $\gamma < 0$). Thus by (36) we can conclude that $W \notin L^1(-\infty, z_0)$ and hence, by Lemma A.5, $R \notin L^1(-\infty, z_0)$.
Further, by (37) and Remark A.6, we compute

$$\int_{-\infty}^{-1} Q(z)\, dz = \int_{-\infty}^{-1}\left(\int_{-\infty}^{z} (m(s)W(s))^{-1}\, ds\right)W(z)\, dz$$

$$\simeq -\frac{1}{\alpha d^2}\int_{-\infty}^{-1} \frac{1}{z}e^{-\alpha z^2}e^{\alpha z^2}\, dz = \frac{1}{\alpha d^2}\int_{1}^{+\infty} \frac{1}{z}\, dz = +\infty,$$

and therefore $Q \notin L^1(-\infty, z_0)$.

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Assume now $\alpha < 0$ (i.e. $\gamma > 0$). Thus by (37) we can conclude that $(m(z)W(z))^{-1} \notin L^1(-\infty, z_0)$ and hence, by Lemma A.5, $Q \notin L^1(-\infty, z_0)$.

Further, by (36) and Remark A.6 we compute

$$\int_{-\infty}^{-1} R(z)\, dz = \int_{-\infty}^{-1} \left(\int_{-\infty}^{z} W(s)\, ds \right) (m(z)W(z))^{-1}\, dz$$

$$\simeq \frac{1}{\alpha d^2} \int_{-\infty}^{-1} \frac{1}{z} e^{\alpha z^2} e^{-\alpha z^2}\, dz = -\frac{1}{\alpha d^2} \int_{1}^{+\infty} \frac{1}{z}\, dz = +\infty,$$

and therefore $R \notin L^1(-\infty, z_0)$. We can then conclude that $-\infty$ is of natural type for all $\alpha \neq 0$.

We consider now the endpoint $+\infty$, set $z_0 = 1$ for sake of simplicity. Similar calculations as carried out for $-\infty$ allow to conclude that $+\infty$ is of natural type too, for all $\alpha \neq 0$. By Lemma A.2 the boundary points $\pm\infty$ have the same type of Feller classification with respect to the operator $\mathcal{L}$. This completes the proof. $\qquad\square$

We will proceed to prove a generation result for the solution to the (ACP) (11) when condition (15) is not verified and $c \neq 0$.

Thus the operator $\mathcal{L}$ is defined by

$$\mathcal{L}u = \frac{1}{2}(cx + d)^2 u'' + ((\gamma x + \delta) - \theta(cx + d))u', \tag{38}$$

Fix $c > 0$ (analogous arguments work for $c < 0$) and consider the change of variable $z = \Phi(x) = cx + d$, where Φ is the isomorphism that maps $C\left(-\frac{d}{c}, +\infty\right)$ into $C(0, +\infty)$. Then $\mathcal{L}$ is transformed in the following operator

$$\widetilde{\mathcal{L}}v = \alpha_0 z^2 v'' + (\alpha_1 z + \alpha_2)\, v', \tag{39}$$

where

$$\alpha_0 = \frac{c^2}{2} > 0, \quad \alpha_1 = (\gamma - \theta\, c) \in \mathbb{R}, \quad \alpha_2 = (\delta\, c - \gamma\, d) \neq 0 \tag{40}$$

(if $\alpha_2 = 0$, then condition (15) holds).

We are now in position to prove the third main result of this section.

Theorem 3.6. *Assume $c \neq 0$, d, γ, δ, $\theta \in \mathbb{R}$ and consider the operator $\widetilde{\mathcal{L}}$ defined in* (39) *with the parameters given in* (40). *Denote by*

$$h_1 = -\frac{\alpha_1}{\alpha_0} \in \mathbb{R}, \; h_2 = \frac{\alpha_2}{\alpha_0} \neq 0. \tag{41}$$

Then

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i) for any $h_1 \in \mathbb{R}$ and $h_2 > 0$, the operator $\widetilde{\mathcal{L}}$ with maximal domain

$$D_M(\widetilde{\mathcal{L}}) = \{v \in C[0, +\infty] \cap C^2[0, +\infty)| \, \widetilde{\mathcal{L}}v \in C[0, +\infty]\},$$

generates a positive (C_0) contraction semigroup on $C[0, +\infty]$.

ii) for any $h_1 \in \mathbb{R}$ and $h_2 < 0$, the operator $\widetilde{\mathcal{L}}$ with Wentzell domain

$$D_W(\widetilde{\mathcal{L}}) = \left\{v \in C[0, +\infty] \cap C^2[0, +\infty) \, \bigg| \lim_{\substack{z \to 0 \\ z \to +\infty}} \widetilde{\mathcal{L}}v(z) = 0\right\},$$

generates a positive (C_0) contraction semigroup on $C[0, +\infty]$.

Proof. Consider $J = (0, +\infty)$. According to Theorem A.4, the proof is based on the Feller classification of the endpoints $0, +\infty$. First, we study the boundary point $z = +\infty$. Set $z_0 = 1$ for sake of simplicity. For any $z > 0$, we compute (see Definition A.1)

$$W(z) = \exp\left(-\int_1^z \frac{(\alpha_1 s + \alpha_2)}{\alpha_0 s^2} \, ds\right) = \exp\left(-\frac{1}{\alpha_0}\left[\alpha_1 \ln z - \frac{\alpha_2}{z} + \alpha_2\right]\right)$$

$$= e^{-h_2} z^{h_1} e^{h_2/z},$$

and

$$(m(z)W(z))^{-1} = \frac{e^{h_2}}{\alpha_0} \frac{e^{-h_2/z}}{z^{h_1+2}}.$$

with $h_1 = -\frac{\alpha_1}{\alpha_0} \in \mathbb{R}$ and $h_2 = \frac{\alpha_2}{\alpha_0} \neq 0$.

Assume $h_2 > 0$. Since $e^{h_2/z} > 1$ for all $z > 0$, we obtain

$$\int_z^{+\infty} W(s) \, ds = e^{-h_2} \int_z^{+\infty} s^{h_1} e^{h_2/s} \, ds$$

$$> e^{-h_2} \int_z^{+\infty} s^{h_1} \, ds = \begin{cases} +\infty, & if \ h_1 \geq -1 \\ -\dfrac{e^{-h_2} z^{h_1+1}}{h_1 + 1}, & if \ h_1 < -1. \end{cases} \tag{42}$$

Hence $W \notin L^1(z_0, +\infty)$ if $h_1 \geq -1$ and therefore, by Lemma A.5, $R \notin L^1(z_0, +\infty)$.

If $h_1 < -1$, by Remark A.6 and (42) we have

$$\int_1^{+\infty} R(z)\,dz = \int_1^{+\infty} \left(\int_z^{+\infty} W(s)\,ds \right) (m(z)W(z))^{-1}\,dz$$

$$> -\frac{1}{\alpha_0(h_1+1)} \int_1^{+\infty} z^{h_1+1}\, \frac{e^{-h_2/z}}{z^{h_1+2}}\,dz$$

$$> -\frac{1}{\alpha_0(h_1+1)} \int_1^{+\infty} \frac{1}{z}\,dz = +\infty,$$

where the last inequality follows by observing that $e^{-h_2/z} < 1$ for $z > 1$ Moreover, for any $z > 0$ we compute

$$\int_z^{+\infty} (m(s)W(s))^{-1}\,ds = \frac{e^{h_2}}{\alpha_0} \int_z^{+\infty} \frac{e^{-h_2/s}}{s^{h_1+2}}\,ds$$

$$(e^{-h_2/s} > e^{-h_2/z} \text{ for all } s > z)$$

$$> \frac{e^{h_2}}{\alpha_0} e^{-h_2/z} \int_z^{+\infty} \frac{1}{s^{h_1+2}}\,ds = \begin{cases} +\infty, & if\ h \le -1 \\ \frac{e^{h_2}}{\alpha_0(h_1+1)} \frac{e^{-h_2/z}}{z^{h_1+1}}, & if\ h > -1. \end{cases} \tag{43}$$

Hence $(m\,W)^{-1} \notin L^1(z_0, +\infty)$ if $h_1 \le -1$ and therefore, by Lemma A.5, $Q \notin L^1(z_0, +\infty)$.

If $h_1 > -1$, by Remark A.6 and (43) we have

$$\int_1^{+\infty} Q(z)\,dz = \int_1^{+\infty} \left(\int_z^{+\infty} (m(s)W(s))^{-1}\,ds \right) W(z)\,dz$$

$$> \frac{1}{\alpha_0(h_1+1)} \int_1^{+\infty} \frac{e^{-h_2/z}}{z^{h_1+1}} z^{h_1} e^{h_2/z}\,dz = \frac{1}{\alpha_0(h_1+1)} \int_1^{+\infty} \frac{1}{z}\,dz = +\infty.$$

We have then proved that $+\infty$ is natural for any $h_2 > 0$ and $h_1 \in \mathbb{R}$.

Now, assume $h_2 < 0$. Observe that $e^{h_2/s} > e^{h_2/z}$ for all $s > z > 0$. Thus

$$\int_z^{+\infty} W(s)\,ds = e^{-h_2} \int_z^{+\infty} e^{h_2/s} s^{h_1}\,ds$$

$$> e^{-h_2} e^{h_2/z} \int_z^{+\infty} s^{h_1}\,ds = \begin{cases} +\infty, & if\ h \ge -1 \\ -\frac{e^{-h_2}}{h_1+1} \frac{e^{h_2/z}}{z^{h_1+1}}, & if\ h < -1. \end{cases}$$

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By Lemma A.5, this implies $R \notin L^1(z_0, +\infty)$ if $h_1 \geq -1$. If $h_1 < -1$, with similar calculations as in the case $h_2 > 0$, we obtain that

$$\int_1^{+\infty} R(z)\,dz = +\infty,$$

and therefore, we can conlude that $R \notin L^1(z_0, +\infty)$ for any $h_2 < 0$ and $h_1 \in \mathbb{R}$. Moreover, by observing that $e^{-h_2/z} > 1$ for any $z > 0$, we have

$$\int_z^{+\infty} (m(s)W(s))^{-1}\,ds = \frac{e^{h_2}}{\alpha_0}\int_z^{+\infty} \frac{e^{-h_2/s}}{s^{h_1+2}}\,ds$$

$$> \frac{e^{h_2}}{\alpha_0}\int_z^{+\infty} \frac{1}{s^{h_1+2}}\,ds = \begin{cases} +\infty, & if\ h_1 \leq -1 \\ \frac{e^{h_2}}{\alpha_0(h_1+1)}\frac{1}{z^{h_1+1}}, & if\ h_1 > -1. \end{cases}$$

Then, by Lemma A.5 it follows that $Q \notin L^1(z_0, +\infty)$ when $h \leq -1$.
If $h > -1$, by similar calculations as in the case $h_2 > 0$, we obtain that

$$\int_1^{+\infty} Q(z)\,dz = +\infty.$$

Hence, the endpoint $+\infty$ is natural when $h_2 < 0$ and $h_1 \in \mathbb{R}$. Therefore, we have proved that $+\infty$ is natural for all $h_2 \neq 0$ and $h_1 \in \mathbb{R}$.

We study now the behaviour at the boundary point 0.

First, assume $h_2 > 0$. Without any loss of generality, set $z_0 = 1$. Observe that $e^{h_2/s} > e^{h_2/z}$ for any $0 < s < z$. Thus

$$\int_0^z W(s)\,ds > e^{-h_2}e^{h_2/z}\int_0^z s^{h_1}\,ds = \begin{cases} +\infty, & if\ h_1 \leq -1 \\ \frac{e^{-h_2}}{h_1+1} e^{h_2/z}z^{h_1+1}, & if\ h_1 > -1, \end{cases} \quad (44)$$

which implies $R \notin L^1(0, z_0)$ when $h_1 \leq -1$. If $h_1 > -1$, by (44) we obtain

$$\int_0^1 R(z)\,dz = \int_0^1 \left(\int_1^z W(s)\,ds\right)(m(z)W(z))^{-1}\,dz$$

$$> \frac{1}{\alpha_0(h_1+1)}\int_0^1 \frac{e^{h_2/z}e^{-h_2/z}z^{h_1+1}}{z^{h_1+2}}\,dz = \frac{1}{\alpha_0(h_1+1)}\int_0^1 \frac{1}{z}\,dz = +\infty,$$

and therefore, $R \notin L^1(0, z_0)$ for any $h_2 > 0$ and $h_1 \in \mathbb{R}$. Moreover, for any $z > 0$

$$0 < \int_0^z (m(s)W(s))^{-1}\,ds < +\infty.$$

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Hence $Q \in L^1(0, z_0)$ or $Q \notin L^1(0, z_0)$ that is, the endpoint 0 may be of entrance or natural type for any $h_2 > 0$ and $h_1 \in \mathbb{R}$. Then from Theorem A.4 the assertion $i)$ holds.

Assume now $h_2 < 0$. Since $e^{-h_2/s} > e^{-h_2/z}$ for $0 < s < z$, we compute

$$\int_0^z (m(z)W(z))^{-1} \, dz = \frac{e^{h_2}}{\alpha_0} \int_0^z \frac{e^{-h_2/s}}{s^{h_1+2}} \, ds$$

$$> \frac{e^{h_2}}{\alpha_0} e^{-h_2/z} \int_0^z \frac{1}{s^{h_1+2}} \, ds = \begin{cases} +\infty, & \text{if } h_1 \geq -1 \\ -\frac{e^{h_2}}{\alpha_0(h_1+1)} \frac{e^{-h_2/z}}{z^{h_1+1}}, & \text{if } h_1 < -1, \end{cases}$$

which implies $Q \notin L^1(0, z_0)$ when $h_1 \geq -1$. If $h_1 < -1$

$$\int_0^1 Q(z) \, dz = \int_0^1 \left(\int_0^z (m(s)W(s))^{-1} \, ds \right) W(z) \, dz$$

$$> -\frac{1}{\alpha_0(h_1+1)} \int_0^1 \frac{e^{-h_2/z} z^{h_1} e^{h_2/z}}{z^{h_1+1}} \, dz = -\frac{1}{\alpha_0(h_1+1)} \int_0^1 \frac{1}{z} \, dz = +\infty;$$

and therefore, $Q \notin L^1(r_1, x_0)$ for any $h_2 < 0$ and $h_1 \in \mathbb{R}$. Moreover, for any $z > 0$,

$$0 < \int_0^z W(s) \, ds < +\infty,$$

Hence $R \in L^1(0, z_0)$ or $R \notin L^1(0, z_0)$ that is, 0 may be of exit or natural type when $h_2 < 0$ and $h_1 \in \mathbb{R}$. This implies that both the boundary points 0 and $+\infty$ are not of entrance type for any $h_2 < 0$ and $h_1 \in \mathbb{R}$. Therefore, from Theorem A.4, the assertion $ii)$ holds. $\qquad \square$

Remark 3.7. Consider $J = \left(\frac{-d}{c}, +\infty \right)$ if $c > 0$ (resp. $J = \left(-\infty, \frac{-d}{c} \right)$ if $c < 0$). From Remark A.3 and Theorem 3.6 it follows that

i) for any $h_1 \in \mathbb{R}$ and $h_2 > 0$, the operator $(\mathcal{L}, D_M(\mathcal{L}))$ generates a positive (C_0) contraction semigroup on $C(\bar{J})$;

ii) for any $h_1 \in \mathbb{R}$ and $h_2 < 0$, the operator $(\mathcal{L}, D_W(\mathcal{L}))$ generates a positive (C_0) contraction semigroup on $C(\bar{J})$,

where $\mathcal{L}$ is given in (38) and h_1, h_2 are defined in (41). We recall that the case $h_2 = 0$ corresponds to condition (15). Then, for any $g \in C(\bar{J})$, there exists a unique solution to the (ACP) (11).

Remark 3.8. If $c > 0$, $\alpha_1 < 0 (\Rightarrow h_1 > 0)$, $\alpha_2 > 0 (\Rightarrow h_2 > 0)$, the operator $\widetilde{\mathcal{L}}$ in (39) represents the infinitesimal generator of a diffusion process considered by Brennan and Schwartz [4] and, successively, by Courtadon [7] to model the dynamic behaviour of interest rates for the valuation of default-free bonds and prices of European options written on default-free bonds.

4. Semigroup Approach: the approximate representation

In this section we present the approximate representations of the solutions obtained in the theorems of the Section 3. We will apply the Lie-Trotter-Daletskii product formula [14, Proposition 1].

Theorem 4.1. *Assume $c = 0$, $\gamma \neq 0$, $d > 0$ and $\delta, \theta \in \mathbb{R}$, then the solution to the (ACP) (11) admits the following approximate formula*

$$u(x, t) = \lim_{n \to +\infty} u_n(x, t),$$

uniformly in $x \in \mathbb{R}$ and for t in bounded intervals of $[0, \infty)$. Here $u_n(\cdot, \cdot), n \geq 1$, is a sequence of approximate solutions given by

$$u_n(x, t) = \underbrace{\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty}}_{n\,times} L_0(t, n, \{y_j\}_{1 \leq j \leq n}, x, g) \prod_{j=1}^{n} p\left(\frac{t}{n}, y_j\right) dy_1...dy_n, \qquad (45)$$

where

$$L_0(t, n, \{y_j\}_{1 \leq j \leq n}, x, g) = g\left[\left(x + \frac{(\delta - \theta d)}{\gamma}\right) e^{\gamma t} + \xi \sum_{j=1}^{n} y_j e^{\gamma(n-j)t/n}\right], \qquad (46)$$

with $n = 2^k$, $k \in \mathbb{N}$, $\xi = \frac{d}{\sqrt{2}}$, and $p(t, y)$ is defined in (24).

Proof. Consider the operator $\widetilde{\mathcal{L}}$ defined in (35), obtained from $\mathcal{L}$ by the change of variable (34). Moreover, consider the operator $(G, D(G))$ defined in (27) and the operator

$$G_1 v = \gamma z v', \quad D(G_1) = \{v \in C(\bar{\mathbb{R}}) \cap C^1(\mathbb{R}) | z v' \in C(\bar{\mathbb{R}})\}.$$

The square $(G^2, D(G^2))$ defined in (28), (29) represents the well known heat operator that generates the (C_0) contraction semigroup on $C(\bar{\mathbb{R}})$ defined by

$$U(t)f(z) = \int_{-\infty}^{+\infty} f(z + \xi y) p(t, y) \, dy, \qquad (47)$$

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for $t \geq 0$, $z \in \mathbb{R}$, $f \in C(\overline{\mathbb{R}})$ (see, e.g., [16]), with $\xi = \frac{d}{\sqrt{2}}$ and $p(t, y)$ given in (24). Further, the operator $(G_1, D(G_1))$ generates the (C_0) semigroup of isometries on $C(\overline{\mathbb{R}})$ given by

$$V(t)f(z) = f(ze^{\gamma t}), \tag{48}$$

for $t \geq 0$, $z \in \mathbb{R}$, $f \in C(\overline{\mathbb{R}})$.

Then, by the Lie-Trotter-Daletskii formula we can conclude that the closure of the operator $(\widetilde{\mathcal{L}}, D(G^2) \cap D(G_1))$ generates a (C_0) semigroup $(T(t))_{t \geq 0}$ on $C(\overline{\mathbb{R}})$ defined by

$$T(t)g(z) = \lim_{n \to +\infty} [U(t/n)V(t/n)]^n \, g(z), \tag{49}$$

uniformly in $z \in \mathbb{R}$ and for t in bounded intervals of $[0, +\infty)$. Hence, the solution to the (ACP) (11) is given by

$$u(z, t) = T(t)g(z) = \lim_{n \to +\infty} [U(t/n)V(t/n)]^n \, g(z).$$

Denote $u_n(z, t) = [U(t/n)V(t/n)]^n \, g(z)$. To compute the approximate solutions $u_n(\cdot, \cdot), n \geq 1$, we proceed by steps. The details are given in Appendix C.

Finally, by Lemma A.2 the formula (46) holds. $\qquad \square$

Remark 4.2. If $\gamma < 0$, one can compare the operator $\widetilde{\mathcal{L}}$ in (35) and its corresponding semigroup, with the operator describing the Ornstein-Uhlenbeck process and its corresponding semigroup (see, e.g., [18, Lecture 12])

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g\left(ze^{\gamma t} + \xi \sqrt{\frac{e^{2\gamma t} - 1}{\gamma}} y \right) e^{-y^2/2} dy. \tag{50}$$

Below we will show that the approximate solutions (45) converge numerically to (50).

Theorem 4.3. *Assume $c \neq 0$, d, $\gamma \, \delta$, $\theta \in \mathbb{R}$. Consider $J = \left(\frac{-d}{c}, +\infty \right)$ if $c > 0$ (resp. $J = \left(-\infty, \frac{-d}{c} \right)$ if $c < 0$). Then the unique solution u to (ACP) (11) admits the following approximate formula*

$$u(x, t) = \lim_{n \to +\infty} u_n(x, t),$$

uniformly for $x \in J$ and for t in bounded intervals of $[0, \infty)$. The sequence $u_n(\cdot, \cdot), n \geq 1$, of approximate solutions is given by

$$u_n(x, t) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{n \, times} L(t, n, \{y_j\}_{1 \leq j \leq n}, x, g) \prod_{j=1}^{n} p\left(\frac{t}{n}, y_j \right) dy_1 ... dy_n, \tag{51}$$

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where

$$L(t, n, \{y_j\}_{1 \leq j \leq n}, x, g) =$$

$$g\left(e^{\beta(\sum_{i=1}^{n} y_i + \zeta t)}\left(x + \frac{d}{c}\right) - \frac{d}{c} + \frac{\alpha_2 t}{nc}\sum_{j=1}^{n} e^{\left[\beta\left(\sum_{i=j}^{n} y_i + \left(\frac{n-j+1}{n}\right)\zeta t\right)\right]}\right), \quad (52)$$

for $g \in C(\bar{J})$, with $n = 2^k$, $k \in \mathbb{N}$. The parameters α_1, α_2 are defined in (40), $\zeta = \sqrt{2}\left(\frac{\alpha_1}{c} - \frac{c}{2}\right)$, and $p(t, y)$ is given in (24).

Proof. We start by rewriting the operator $\mathcal{L}$ defined in (38) as follows

$$\begin{aligned}
\mathcal{L}u &= \frac{1}{2}(cx + d)^2 u'' + ((\gamma - \theta c)x + (\delta - \theta d))u' \\
&= \frac{1}{2}(cx + d)^2 u'' + \left(\left(\frac{\gamma}{c} - \theta\right)(cx + d) + \left(\delta - \frac{\gamma d}{c}\right)\right)u' \\
&= \frac{1}{2}(cx + d)^2 u'' + \left(\frac{\alpha_1}{c}(cx + d) + \frac{\alpha_2}{c}\right)u'. \quad (53)
\end{aligned}$$

Let us introduce the operator

$$Gu = \frac{cx + d}{\sqrt{2}}u', \quad D(G) = \{u \in C(\bar{J}) \cap C^1(J) | u', xu' \in C(\bar{J})\}.$$

Then the square of G is given by

$$G^2 u = \frac{cx + d}{\sqrt{2}}\left(\frac{cx + d}{\sqrt{2}}u'\right)' = \frac{1}{2}(cx + d)^2 u'' + \frac{c}{2}(cx + d)u',$$

with domain

$$\begin{aligned}
D(G^2) &= \{u \in D(G) | Gu \in D(G)\} \\
&= \{u \in C(\bar{J}) \cap C^2(J) | u', xu', x(xu')' \in C(\bar{J})\}.
\end{aligned}$$

Hence, the operator $\mathcal{L}$ can be written in the form

$$\mathcal{L}u = P_1 u + P_2 u, \quad (54)$$

where

$$P_1 u = G^2 u + \zeta Gu, \quad D(P_1) = D(G^2),$$

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with $\zeta = \sqrt{2}\left(\frac{\alpha_1}{c} - \frac{c}{2}\right)$, and

$$P_2 u = \frac{\alpha_2}{c} u', \quad D(P_2) = \{u \in C(\bar{J}) \cap C^1(J) | u' \in C(\bar{J})\}.$$

According to [14, Section 5, Lemma 3], $(G, D(G))$ generates a (C_0) group of isometries, $(\mathcal{S}(t))_{t \geq 0}$, on $C(\bar{J})$, given by

$$\mathcal{S}(t)f(x) := e^{tG}f(x) = f\left(e^{\beta t}x + \frac{d}{c}(e^{t\beta} - 1)\right), \tag{55}$$

with $\beta = \frac{c}{\sqrt{2}}$, for any $f \in C(\bar{J})$, $t > 0$, $x \in J$. Indeed, $(G, D(G))$ and $(-G, D(G))$ are respectively generators of the (C_0) semigroups $(\mathcal{S}_+(t))_{t \geq 0}$ and $(\mathcal{S}_-(t))_{t \geq 0}$ on $C(\bar{J})$, where $\mathcal{S}_+(t) = \mathcal{S}(t)$ and $\mathcal{S}_-(t) = \mathcal{S}(-t)$ for $t > 0$. The first result is proved by [14, Section 5, Lemma 3], the second one can be proved analogously.

Thus, $(P_1, D(P_1))$ generates a (C_0) contraction semigroup (see [15, Sections 3-4] and [13, Chapter I, Section 9, and Chapter II, Section 8]) given by

$$U(t)f(x) = \int_0^{+\infty} \mathcal{S}(y)\,\mathcal{S}(\zeta\,t)f(x)p(t, y)dy + \int_0^{+\infty} \mathcal{S}(-y)\,S(\zeta\,t)f(x)p(t, y)\,dy$$

$$= \int_{-\infty}^{+\infty} \mathcal{S}(\zeta\,t + y)f(x)p(t, y\,)dy$$

$$= \int_{-\infty}^{+\infty} f\left[e^{\beta(\zeta\,t+y)}x + \frac{d}{c}(e^{\beta(\zeta\,t+y)} - 1)\right] p(t, y)\,dy, \tag{56}$$

for any $f \in C(\bar{J})$, $t > 0$, $x \in J$, and $p(t, y)$ is defined in (24).

Further, it is well known that $(P_2, D(P_2))$ generates the (C_0) contraction semigroup given by

$$V(t)f(x) = f\left(x + \frac{\alpha_2}{c}t\right), \tag{57}$$

for any $f \in C(\bar{J})$, $t > 0$, $x \in J$. Finally, by the Lie-Trotter-Daletskii product formula the operator $(\mathcal{L}, D(G^2))$ generates a (C_0) semigroup $(T(t))_{t \geq 0}$ on $C(\bar{J})$ defined as

$$T(t)g(x) = \lim_{n \to +\infty} \left[U(t/n)V(t/n)\right]^n g(x), \tag{58}$$

uniformly in $x \in J$ and for t in bounded intervals of $[0, +\infty)$. Hence, the solution to the (ACP) (11) is given by

$$u(x, t) = T(t)g(x) = \lim_{n \to +\infty} \left[U(t/n)V(t/n)\right]^n g(x),$$

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for any $g \in C(\bar{J})$, $t > 0$, $x \in J$. Denote $u_n(x,t) = [U(t/n)V(t/n)]^n g(x)$. Thus $u_n(\cdot,\cdot), n \geq 1$, is a sequence of approximating solutions whose explicit expression (51)-(52) is computed in the Appendix D. $\qquad\square$

Remark 4.4. From [10, Chapter III, Theorem 5.2, formula (5.4)] one deduces that the rate of convergence of the approximate solutions u_n to the unique solution u to the (ACP) (11) in both Theorems (4.1) and (4.3), is of order

$$||u_n(\cdot,t) - u(\cdot,t)|| = O\left(\frac{1}{\sqrt{n}}\right) \quad \text{as} \ \ n \to \infty,$$

for t in bounded intervals of $[0, \infty)$.

5. Conclusions

In this paper, we considered the option pricing problem in incomplete markets studied by M. Musiela and T. Zariphopoulou [20], where they derived the indifference price of European claims expressed as a non-linear function of the conditional expectation of a European derivative's payoff, written exclusively on non-traded assets, under the appropriate pricing measure. This conditional expectation is the classical Feynman-Kac representation of solutions to linear second order parabolic equations. Our goal herein was to determine an explicit or approximate formula for the indifference price. By the (C_0) semigroup theory we proved existence and uniqueness of solutions to generalized Black-Scholes type equations. Further, following the same idea proposed in [14], we derived an approximate representation of these solutions in terms of a generalized Feynman-Kac type formula when an explicit closed form is not available. In the Appendix E we performed two examples of numerical evaluations where the approximate solutions described in Section 4 are compared with benchmark formulas in the literature. The obtained results show a fast convergence and an excellent agreement between our theoretical results and the numerical tests. This confirms that the approximate representation of the indifference price here proposed is particularly helpful in cases where closed form pricing formulas fail because their coefficients cannot be explicitly computed.

Acknowledgments

Appendices

A. Boundary Feller's Classification

Definition A.1. If $J = (r_1, r_2)$ is a real interval, with $-\infty \leq r_1 < r_2 \leq +\infty$, let $\mathcal{L}$ be a second order differential operator of the type $\mathcal{L}u = m(x)u'' + q(x)u'$, where m, q are real continuous functions on J such that $m(x) \geq 0$ for any $x \in J$. Introduce the following functions

$$W(x) = exp\left(-\int_{x_0}^x \frac{q(s)}{m(s)}ds\right),$$

$$Q(x) = \frac{1}{m(x)W(x)} \int_{x_0}^x W(s)ds,$$

$$R(x) = W(x) \int_{x_0}^x \frac{1}{m(s)W(s)}ds,$$

where $x \in J$, x_0 is fixed in J. The boundary point r_2 is said to be

   i) *regular* if $Q \in L^1(x_0, r_2)$ and $R \in L^1(x_0, r_2)$;

   ii) *exit* if $Q \notin L^1(x_0, r_2)$ and $R \in L^1(x_0, r_2)$;

   iii) *entrance* if $Q \in L^1(x_0, r_2)$ and $R \notin L^1(x_0, r_2)$;

   iv) *natural* if $Q \notin L^1(x_0, r_2)$ and $R \notin L^1(x_0, r_2)$.

Analogous definitions can be given for r_1 by considering the interval (r_1, x_0) instead of (x_0, r_2).

Lemma A.2. *Let us consider $J = (r_1, r_2)$ and $\Gamma = (s_1, s_2)$, with $-\infty \leq r_1 < r_2 \leq +\infty$, $-\infty \leq s_1 < s_2 \leq +\infty$. Let $\Phi : J \to \Gamma$ be a bijective function with inverse Φ^{-1}, such that $\Phi \in C^2(J)$, $\Phi'(x) > 0$ and for any $v \in C(\bar{\Gamma})$, define $T_\Phi(v) = v \circ \Phi$. Then $T_\Phi(v) \in C(\bar{J})$. Moreover T_Φ is an invertible bounded linear operator from $C(\bar{\Gamma})$ to $C(\bar{J})$ such that $\|T_\Phi\| \leq 1$ and $(T_\Phi)^{-1} = T_{\Phi^{-1}}$. Further, s_i has the same type of Feller classification of r_i, $i = 1, 2$.*

Remark A.3. Under the assumptions of Lemma A.2, according to [10, Chapter II, Section 2.a], if $(T(t))_{t\geq 0}$ is a (C_0) semigroup on $C(\bar{J})$ having $(A, D(A))$ as generator, then $(S(t))_{t\geq 0}$ given by $S(t) = T_{\Phi^{-1}} \circ T(t) \circ T_{\Phi}$, $t \geq 0$, is a (C_0) semigroup on $C(\bar{\Gamma})$ with generator $(B, D(B))$. Here $B = T_{\Phi^{-1}} \circ A \circ T_{\Phi}$, and $D(B) = \{g \in C(\bar{\Gamma}) | T_{\Phi}(g) \in D(A)\}$. Moreover, if $(T(t))_{t\geq 0}$ is a positive contraction semigroup on $C(\bar{J})$ (i.e.

$$(f \in C(\bar{J}),\ f \geq 0) \Rightarrow (T(t)f \geq 0,\ t \geq 0,\ \|T(t)\| \leq 1),$$

then $(S(t))_{t\geq 0}$ is a positive contraction semigroup on $C(\bar{\Gamma})$.

Theorem A.4. *Let us consider $J = (r_1, r_2)$. The operator $\mathcal{L}$ with maximal domain*

$$D_M(\mathcal{L}) = \{u \in C(\bar{J}) \cap C^2(J) | \mathcal{L}u \in C(\bar{J})\},$$

generates a Feller semigroup on $C(\bar{J})$ if and only if r_1 and r_2 are of entrance or natural type.
In particular, the operator $\mathcal{L}$ with the so called Wentzell domain

$$D_W(\mathcal{L}) = \left\{u \in C(\bar{J}) \cap C^2(J) \,\middle|\, \lim_{\substack{x \to r_1 \\ x \to r_2}} \mathcal{L}u(x) = 0\right\},$$

generates a Feller semigroup on $C(\bar{J})$ if and only if both the endpoints r_1 and r_2 are not of entrance type.

The next lemma will be useful in the sequel to study the behaviour of the boundary points according to the Feller classification.

Lemma A.5. *Fix $x_0 \in J = (r_1, r_2)$, then we have*

   i) $W \notin L^1(x_0, r_2)$ implies $R \notin L^1(x_0, r_2)$;

   ii) $R \in L^1(x_0, r_2)$ implies $W \in L^1(x_0, r_2)$;

   iii) $(mW)^{-1} \notin L^1(x_0, r_2)$ implies $Q \notin L^1(x_0, r_2)$;

   iv) $Q \in L^1(x_0, r_2)$ implies $(mW)^{-1} \in L^1(x_0, r_2)$.

Analogously for r_1.

Remark A.6. By some interchanges between the integration variables, the integrals of Q and R can be written as

$$\int_{x_0}^{r_2} Q(x)\,dx = \int_{x_0}^{r_2} \left(\int_{x_0}^{x} W(s)\,ds \right) (m(x)W(x))^{-1}\,dx$$

$$= \int_{x_0}^{r_2} \left(\int_{x}^{r_2} (m(s)W(s))^{-1}\,ds \right) W(x)\,dx$$

and

$$\int_{x_0}^{r_2} R(x)\,dx = \int_{x_0}^{r_2} \left(\int_{x_0}^{x} (m(s)W(s))^{-1}\,ds \right) W(x)\,dx$$

$$= \int_{x_0}^{r_2} \left(\int_{x}^{r_2} W(s)\,ds \right) (m(x)W(x))^{-1}\,dx$$

for any fixed $x_0 \in (r_1, r_2)$. Analogously for r_1.

B. Proof of Lemma 3.1

We examine the following cases

$$(i)\ d = 0, \qquad (ii)\ d \neq 0.$$

In the case (i), due to (15), we deduce that $c \neq 0$ and $\delta = 0$. Hence, for any $u \in D_M(\mathcal{L})$, we obtain

$$\mathcal{L}u = \frac{1}{2}c^2 x^2 u'' + (\gamma - \theta c)\,x\,u'$$

and hence, $\mathcal{L}$ has the form (16).
In the case (ii), we have to examine the following subcases

$$(ii)_1\ \gamma = c = 0, \qquad (ii)_2\ \gamma = \delta = 0, \qquad (ii)_3\ \gamma \neq 0.$$

In the subcase $(ii)_1$, we obtain

$$\mathcal{L}u = \frac{1}{2}d^2 u'' + (\delta - \theta d)u'.$$

Thus $\mathcal{L}$ is of the form (17).
In the subcase $(ii)_2$, we can assume $c \neq 0$, otherwise we come back to the case $(ii)_1$. Hence

$$\mathcal{L}u = \frac{1}{2}(cx + d)^2 u'' - \theta(cx + d)u' = \frac{1}{2}d^2(kx + 1)^2 u'' - \theta d(kx + 1)u',$$

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with $k = \frac{d}{c}$ and $\mathcal{L}$ is of the type (18).

In the subcase $(ii)_3$, $\delta \neq 0$ and $c \neq 0$, being $\gamma \neq 0$ and $d \neq 0$. Hence

$$\mathcal{L}u = \frac{1}{2}d^2(\frac{d}{c}x + 1)^2 u'' + \left[\delta\left(\frac{\gamma}{\delta}x + 1\right) - \theta d\left(\frac{d}{c}x + 1\right)\right]u'$$
$$= \frac{1}{2}d^2(kx + 1)^2 u'' + (\delta - \theta d)(kx + 1)u',$$

where $k = \frac{d}{c} = \frac{\gamma}{\delta}$. Thus $\mathcal{L}$ is of the type (18).

C. The Case $c = 0$

Remark C.1. It is worth noting that in the case $c = 0$ the non-traded asset level Y is an affine process since both the drift b and the square of the diffusion coefficient a in (2) are time-homogeneous affine functions in $y = Y_t$.

$$\begin{cases} b(y, t) = b(y) = \gamma y + (\delta - \theta d) \\ a^2(y, t) = a^2(y) = d^2, \end{cases}$$

for any γ, δ, θ, $d \in \mathbb{R}$. This condition is equivalent to state that the problem (8) admits an affine term structure (for more details on affine processes and affine term structures the reader can refer, for example, to [8] or [9]) and then, its solution is of the form

$$w(y, t) = e^{B(t,T) - A(t,T)y}, \quad (y, t) \in \mathbb{R} \times [0, T], \tag{59}$$

where A and B are deterministic functions satisfying the following differential equations

$$\frac{\partial A(t, T)}{\partial t} = -\gamma A(t, T), \tag{60}$$

$$\frac{\partial B(t, T)}{\partial t} = -\frac{d^2}{2}A^2(t, T) + A(t, T)(\delta - \theta d), \tag{61}$$

obtained by plugging the partial derivatives of w into the parabolic equation in (8). For fixed T, equations (60) and (61) are uniquely solvable ODEs when the final conditions $A(T, T)$ and $B(T, T)$ are known. The final condition in (8) implies that $A(T, T)$ and $B(T, T)$ must satisfy the following equation

$$e^{B(T,T) - A(T,T)y} = e^{\eta(1-\rho^2)G(Y_T)}, \tag{62}$$

which can be explicitly solved if and only if the payoff function $G(Y_T)$ is a polynomial.

Hence the closed form (59) is a useful explicit representation of the solution to problem (8) provided that the deterministic functions A and B are explicitly known. For this reason, the approximation formula (45) for the case $c = 0$ may be considered a helpful alternative to the Feynman-Kac formula (7). We will show this by the following example.

We consider a European option that conveys the opportunity, but not the obligation, to sell an underlying asset at time $t > 0$ for some fixed price $K > 0$. This is known as a "put" option; the corresponding "call" option to buy the asset may be treated similarly. Assuming that the option is written exclusively on the non-traded asset Y, its payoff at the maturity date $T > t$ is

$$G(Y_T) = (K - Y_T)^+ = \begin{cases} K - Y_T, & if \ K > Y_T, \\ 0, & if \ K \leq Y_T. \end{cases}$$

Thus equation (62) becomes

$$e^{B(T,T) - A(T,T)y} = e^{r(K-y)^+} = \begin{cases} e^{r(K-y)}, & if \ K > y, \\ 1, & if \ K \leq y, \end{cases}$$

with $r = \eta(1 - \rho^2)$ and $Y_T = y \in \mathbb{R}$, and hence

$$\begin{cases} B(T,T) = rK, & A(T,T) = r, & if \ K > y, \\ B(T,T) = 0, & A(T,T) = 0, & if \ K \leq y. \end{cases}$$

In the case $K > y$ (from (7) we deduce that w reduces to the constant function 1 when $K \leq y$), the functions A and B solve the following systems

$$\begin{cases} \dfrac{dA(t,T)}{dt} = -\gamma A(t,T) \\ A(T,T) = r, \end{cases} \tag{63}$$

$$\begin{cases} \dfrac{dB(t,T)}{dt} = (\delta - \theta d)A(t,T) - \dfrac{d^2}{2} A^2(t,T) \\ B(T,T) = rK. \end{cases} \tag{64}$$

Since (63) is a simple linear ODE, for fixed T, we immediately obtain

$$A(t,T) = re^{\gamma(T-t)}.$$

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Plugging this expression into the so called Riccati equation (64) and integrating in the interval $[t, T]$

$$B(t, T) = r(\delta - \theta d) \int_t^T e^{\gamma(T-s)} ds - \frac{r^2 d^2}{2} \int_t^T e^{2\gamma(T-s)} ds,$$

we obtain

$$B(t, T) = -\frac{r(\delta - \theta d)}{\gamma}(1 - e^{\gamma(T-t)}) + \frac{r^2 d^2}{4\gamma}(1 - e^{2\gamma(T-t)}) + rK.$$

In the following we will compute the approximate solution (45)-(46). For any $t > 0$, consider a uniform partition of $[0, t]$ into $n \in \mathbb{N}$ subintervals of length t/n, say $\{t_{j,n}\}_{j=0}^n$ ($0 = t_{0,n} < t_{1,n} < ... < t_{n,n} = t$), where $t_{j,n} = jt/n$, $j = 0, 1, ..., n$. Suppose $n = 2^k$, $k \in \mathbb{N}$, so that each partition is obtained by bisecting the previous one.

We refer to Theorem 4.1 and consider the (C_0) semigroups U and V defined in (47) and (48), respectively. Fix $g \in C(\overline{\mathbb{R}}), t > 0, z \in \mathbb{R}$. The same holds for the approximate solution (51)-(52), computed in Appendix D.

Step 1. Take $n = 1$ ($k = 0$). We have

$$U(t)V(t)g(z) = \int_{-\infty}^{+\infty} g(ze^{\gamma t} + \xi y_1)p(t, y_1)dy_1.$$

Step 2: Take $n = 2$ ($k = 1$). Then

$$[U(t/2)V(t/2)]^2 g(z) = U(t/2)V(t/2)h(z)$$
$$= \int_{-\infty}^{+\infty} h(ze^{\gamma t/2} + \xi y_1)p(t/2, y_1)dy_1$$

where

$$h(z) = U(t/2)V(t/2)g(z) = \int_{-\infty}^{+\infty} g(ze^{\gamma t/2} + \xi y_2)p(t/2, y_2)dy_2.$$

Hence,

$$[U(t/2)V(t/2)]^2 g(z)$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g((ze^{\gamma t/2} + \xi y_1)e^{\gamma t/2} + \xi y_2) \prod_{i=1}^2 p(t/2, y_i) dy_1 dy_2$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g((ze^{\gamma t} + \xi(e^{\gamma t/2} y_1 + y_2)) \prod_{i=1}^2 p(t/2, y_i) dy_1 dy_2.$$

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By induction, we can conclude that, for $n = 2^k$, $k \in \mathbb{N}$,

$$u_n(z, t) := [U(t/n)V(t/n)]^n g(z)$$

$$= \underbrace{\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty}}_{n\,times} L_0(t, n, \{y_j\}_{1 \leq j \leq n}, z, g) \prod_{i=1}^{n} p\left(\frac{t}{n}, y_i\right) dy_1 \ldots dy_n,$$

where

$$L_0(t, n, \{y_j\}_{1 \leq j \leq n}, z, g) = g\left(ze^{\gamma t} + \xi \sum_{j=1}^{n} y_j e^{\gamma(n-j)t/n}\right).$$

D. The Case $c \neq 0$

We refer to Theorem 4.3 and consider the (C_0) semigroups U and V defined in (56) and (57), respectively.

Assume $c > 0$ and consider $J = \left(-\frac{d}{c}, +\infty\right)$. Denote $\beta = c/\sqrt{2}$ and

$$R(t, x) = e^{t\beta}x + \frac{d}{c}(e^{t\beta} - 1). \tag{65}$$

Observe that for any $x_0 \in J$, $R(t, x_0) \in J$. Indeed,

$$R(t, x_0) = \underbrace{e^{t\beta}\left(x_0 + \frac{d}{c}\right)}_{>0} + \left(-\frac{d}{c}\right) > -\frac{d}{c}.$$

To compute the approximate functions $u_n(\cdot, \cdot), n \geq 1$, given in (51) we proceed by steps. Fix $g \in C(\bar{J})$.

As in Appendix C, we consider, for any $t > 0$, a uniform partition of $[0, t]$ into $n \in \mathbb{N}$ subintervals of length t/n, say $\{t_{j,n}\}_{j=0}^{n}$ $(0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = t)$, where $t_{j,n} = jt/n$, $j = 0, 1, \ldots, n$. Suppose $n = 2^k$, $k \in \mathbb{N}$, so that each partition is obtained by bisecting the previous one.

Step 1: Take $n = 1$ $(k = 0)$. Thus

$$U(t)V(t)g(x) = \int_{-\infty}^{+\infty} g(R(y_1 + \zeta t, x + \alpha_2 t/c))p(t, y_1)dy_1,$$

where

$$R(y_1 + \zeta t, x + \alpha_2 t/c) = e^{\beta(y_1 + \zeta t)}\left(x + \frac{\alpha_2}{c}t + \frac{d}{c}\right) - \frac{d}{c}. \tag{66}$$

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Step 2: Take $n = 2$, $(k = 1)$. By applying (66) and replacing t by $t/2$, we get

$$\left[U(t/2)V(t/2)\right]^2 g(x) = U(t/2)V(t/2)\, h(x)$$

$$= \int_{-\infty}^{+\infty} h(R(y_1 + \zeta t/2,\ x + \alpha_2\, t/2\, c))p(t/2, y_1)dy_1,$$

where

$$h(x) = U(t/2)V(t/2)\, g(x) = \int_{-\infty}^{+\infty} g(R(y_2 + \zeta t/2,\ x + \alpha_2\, t/2\, c))p(t/2, y_2)dy_2.$$

Hence,

$$\left[U(t/2)V(t/2)\right]^2 g(x) =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(R(y_2 + \zeta t/2,\ R(y_1 + \zeta t/2,\ x + \alpha_2\, t/2\, c) + \alpha_2\, t/2\, c)) \prod_{i=1}^{2} p(t/2, y_i)dy_1 dy_2,$$

$$\tag{67}$$

where

$$R(y_2 + \zeta t/2,\ R(y_1 + \zeta t/2,\ x + \alpha_2\, t/2\, c) + \alpha_2\, t/2\, c)$$

$$= e^{\beta(y_2 + \zeta t/2)}\left(e^{\beta(y_1 + \zeta t/2)}\left(x + \frac{\alpha_2 t}{2c} + \frac{d}{c}\right) - \frac{d}{c} + \frac{\alpha_2 t}{2c} + \frac{d}{c}\right) - \frac{d}{c}$$

$$= e^{\beta(y_1 + y_2 + \zeta t)}\left(x + \frac{\alpha_2 t}{2c} + \frac{d}{c}\right) + \frac{\alpha_2 t}{2c}e^{\beta(y_2 + \zeta t/2)} - \frac{d}{c}.$$

Step 3: Take $n = 4$, $(k = 2)$. Analogously

$$\left[U(t/4)V(t/4)\right]^4 g(x) = \left[U(t/4)V(t/4)\right]^2 h(x),$$

that is given by (67) replacing $t/2$ by $t/4$ and g by h. Moreover, h is also given by (67) replacing $t/2$ by $t/4$ and y_1, y_2 by y_3, y_4. So, after many calculations, we can

write

$$\left[U(t/4)V(t/4)\right]^4 g(x) =$$

$$\underbrace{\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty}}_{4\,times} g(R(y_4 + \zeta t/4, R(y_3 + \zeta t/4, R(y_2 + \zeta t/4,$$

$$R(y_1 + \zeta t/4,\ x + \alpha_2\,t/4\,c) + \alpha_2\,t/4\,c) + \alpha_2\,t/4\,c) + \alpha_2\,t/4\,c) \prod_{i=1}^{4} p(t/4, y_i)dy_i,$$

with

$$R(y_4 + \zeta t/4,\ R(y_3 + \zeta t/4,\ R(y_2 + \zeta t/4,\ R(y_1 + \zeta t/4,\ x + \alpha_2\,t/4\,c)+$$
$$+ \alpha_2\,t/4\,c) + \alpha_2\,t/4\,c) + \alpha_2\,t/4\,c)$$

$$= e^{\beta(\sum_{i=1}^4 y_i + \zeta t)} \left(x + \frac{d}{c}\right) - \frac{d}{c} + \frac{\alpha_2 t}{4c}\left[e^{\beta(\sum_{i=1}^4 y_i + \zeta t)} + e^{\beta(\sum_{i=2}^4 y_i + 3\zeta t/4)}\right.$$

$$\left. + e^{\beta(\sum_{i=3}^4 y_i + \zeta t/2)} + e^{\beta(y_4 + \zeta t/4)}\right]$$

$$= R(\sum_{i=1}^4 y_i + \zeta t,\ x) + \frac{\alpha_2 t}{4c} \sum_{j=1}^4 e^{\left[\beta\left(\sum_{i=j}^4 y_i + \left(\frac{4-j+1}{4}\right)\zeta t\right)\right]}.$$

By induction, we can conclude that, for $n = 2^k$, $k \in \mathbb{N}$,

$$\left[U(t/n)V(t/n)\right]^n g(x) =$$

$$\underbrace{\int_0^{+\infty} ... \int_0^{+\infty}}_{n\,times} L(t, n, \{y_j\}_{1\leq j \leq n}, x, g) \prod_{j=1}^n p\left(\frac{t}{n}, y_j\right) dy_1...dy_n,$$

where

$$L(t, n, \{y_j\}_{1\leq j \leq n}, x, g) = g\left(R(\sum_{i=1}^n y_i + \zeta t,\ x) + \frac{\alpha_2 t}{nc}\sum_{j=1}^n e^{\left[\beta\left(\sum_{i=j}^n y_i + \left(\frac{n-j+1}{n}\right)\zeta t\right)\right]}\right).$$

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E. Examples of Numerical Evaluations

Now we focus on some numerical applications and examples related to the approximate solutions given in (45) and (51). It is well known that very few exact solution formulas to Cauchy problems associated with financial models are available. Therefore, one can choose among different computational methods: finite difference methods, finite element methods, finite volume methods, spectral methods, etc...(the reader can refer to [1] and references therein), which are in general very slow.

It is worth noting that formulas (45) and (51) seem to be hard to numerically calculate because of the high computational cost for solving a n-dimensional integral when n is large. Some numerical methods have been proposed in literature to faster evaluate a multidimensional integral (see, for example, [22]). However, since the function $p(t, y)$ defined in (24) is the probability density of a normal distribution, $N(0, 2t)$, with mean 0 and variance $2t$, the integration variables $\{y_j\}_{1 \le j \le n}$ are n independent realizations of a normal distribution $N(0, 2t/n)$. Thus, the n-dimensional integrals (45) and (51) are nothing but the conditional expected value of a function of n independent and normally distribuited random variables $Y_j \sim N(0, 2t/n)$, given the initial condition $Y_0 = x$, that can be estimated by a Monte Carlo integration method.

In the following examples we compute the approximate solutions for both cases $c = 0$ and $c \ne 0$, when the payoff G refers to some well known fixed income derivatives. In particular, we consider a put European option with maturity T and strike price K, whose payoff is given by $G(x) = (K - x)^+$ (where $x \equiv Y_T$). Hence, the initial condition of (11), defined in (14), can be written as

$$g(x) = e^{r(K-x)^+}, \tag{68}$$

with $r = \eta(1 - \rho^2)$.

Example E.1. (The case $c = 0$)
As observed at the beginning of this section, the solution to the (ACP) (11) admits the approximate formula (Theorem 3.6)

$$u(x, t) = \lim_{n \to +\infty} \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{n \, times} L_0(t, n, \{y_j\}_{1 \le j \le n}, x, g) \prod_{j=1}^{n} p\left(\frac{t}{n}, y_j\right) dy_1 ... dy_n$$

$$= \lim_{n \to +\infty} \mathbb{E}_{\mathbb{Q}}[L_0(t, n, \{Y_j\}_{1 \le j \le n}, x, g) | Y_0 = x], \tag{69}$$

for any $x \in \mathbb{R}$, $t \ge 0$ and $g \in C(\overline{\mathbb{R}})$, where the function L_0 is defined in (46).

Assuming that the initial condition g is of type (68), L_0 may be written as

$$L_0(t, n, \{y_j\}_{1 \leq j \leq n}, x, g) = \exp\left[r \left(K - \left(x + \frac{(\delta - \theta d)}{\gamma} \right) e^{\gamma t} - \xi \sum_{j=1}^{n} y_j e^{\gamma(n-j)t/n} \right)^+ \right].$$

As shown in Remark C.1, the solution to the (ACP) (11) is also given by the affine type solution formula derived in Appendix C by taking into account the time transformation $t \to (T - t)$. Therefore, the affine type solution (ATS) is as follows

$$u(t, x) = e^{B(t) - A(t)x}, \quad Y_0 = x \quad (70)$$

with

$$\begin{cases} A(t) = re^{\gamma t}, \\ B(t) = \dfrac{r^2 d^2}{4\gamma}(e^{2\gamma t} - 1) - \dfrac{r(\delta - \theta d)}{\gamma}(e^{\gamma t} - 1) + Kr. \end{cases}$$

Further, as observed in Remark 4.2, if the parameter $\gamma < 0$, the solution to (11) can be also expressed in closed form by analogy with the Ornstein-Uhlenbeck semigroup as follows

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[r\left(K - \left(x + \frac{(\delta - \theta d)}{\gamma} \right) e^{\gamma t} - \xi \sqrt{\frac{e^{2\gamma t} - 1}{\gamma}} y \right)^+ \right] e^{-y^2/2} dy, \quad (71)$$

for $t \geq 0$, $z \in \mathbb{R}$. Using basic probability calculations one can easily show that the term $\sum_{j=1}^{n} y_j e^{\gamma(n-j)t/n}$ in (E.1) converges to $\sqrt{\frac{e^{2\gamma t} - 1}{\gamma}} y$ as $n \to +\infty$, where y is a realization of a standard normal distribution $N(0, 1)$.

Figure 1 plots the behaviour of both the function w solving the parabolic problem (8) or, equivalently, (9), and the corresponding indifference price h, defined in (4), for a put European option, under the following suitably chosen parameter values

$$d = 0.2016, \quad \gamma = -0.9593, \quad \delta = 0.3209, \quad \mu = 0.0380, \quad \sigma = 0.0300,$$
$$\rho = 0.8, \quad \eta = 2, \quad Y_0 = 8, \quad K = 12, \quad T = 8.$$

The function w is computed via the approximate formula (69), for $n = 2^k$ with $k = 1, 3, 5$, by applying a Monte Carlo integration method. Indeed, for each n, if we denote by $U_i^{(n)} = (Y_1^{(i)}, ..., Y_n^{(i)})$, $i \geq 1$, a sequence of independent normally distributed random vectors, then by the Strong Law of large Numbers we have that, uniformly with respect to x and for $t > 0$,

$$E_{\mathbb{Q}}[L_0(t, n, \{Y_j\}_{1 \leq j \leq n}, x, g)|Y_0 = x] \simeq \frac{1}{M} \sum_{i=1}^{M} L_0(t, n, U_i^{(n)}, x, g),$$

33

as $M \to \infty$ (we set $M = 1000$). From the Central Limit Theorem we know that the estimation error is of order $O\left(\dfrac{1}{\sqrt{M}}\right)$.

The plots in Figure 1 show an accurate goodness-of-fit to the curves obtained, respectively, by the closed form ATS (70) and the OU type solution (71), when $k = 5$.
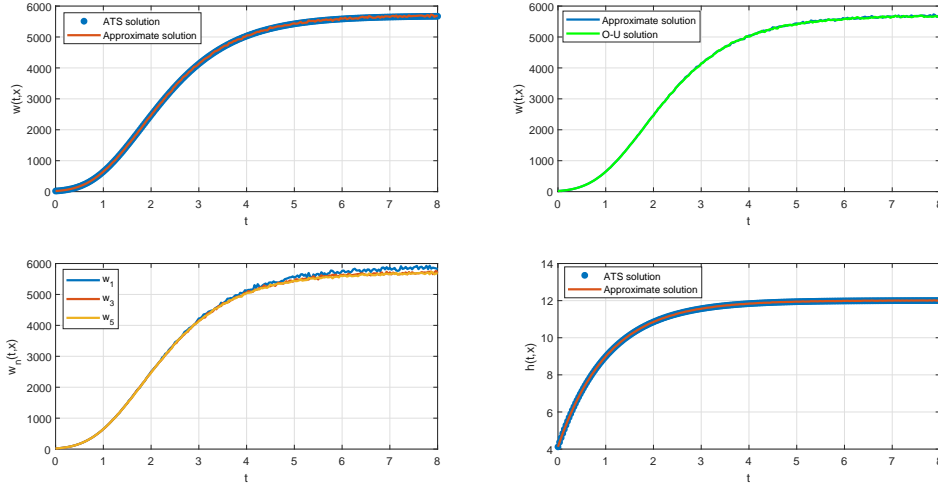

Figure 1: **Top line:** approximate solution w_n $(n = 2^5)$ versus the ATS solution *(left plot)* and the OU type solution *(right plot)*. **Bottom line:** approximate solutions for $n = 2^k$, $k = 1, 3, 5$ *(left plot)*; indifference price h of a put European option corresponding to the function w computed, respectively, by the approximate solution w_n $(n = 2^5)$ and the ATS solution *(right plot)*.

Example E.2. (The case $c \neq 0$)
Let $c > 0$ and consider the interval $J = \left(-\frac{d}{c}, +\infty\right)$. By Theorem 4.3, the solution to the (ACP) (11) admits the approximate formula

$$u(x,t) = \lim_{n \to +\infty} \underbrace{\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty}}_{n\,times} L(t, n, \{y_j\}_{1 \leq j \leq n}, x, g) \prod_{j=1}^{n} p\left(\frac{t}{n}, y_j\right) dy_1 \ldots dy_n$$

$$= \lim_{n \to +\infty} \mathbb{E}_{\mathbb{Q}}[L(t, n, \{Y_j\}_{1 \leq j \leq n}, x, g)|Y_0 = x]. \tag{72}$$

for any $x \in J$, $t \geq 0$ and $g \in C(\bar{J})$. Assuming the initial condition g to be of type

(68), the function L, defined in (52), can be written as

$$
\begin{aligned}
L(t, n, \{y_j\}_{1 \leq j \leq n}, x, g) = \\
\exp\left[r\left(K - e^{\beta(\sum_{i=1}^{n} y_i + \zeta t)}\left(x + \frac{d}{c}\right) + \frac{d}{c} - \frac{\alpha_2 t}{nc} \sum_{j=1}^{n} e^{\left[\beta\left(\sum_{i=j}^{n} y_i + \left(\frac{n-j+1}{n}\right)\zeta t\right)\right]}\right)^+\right].
\end{aligned}
$$

As in Example E.1, Figure 2 shows the behaviour of the function w and of the corresponding indifference price h of a put European option, under the following suitably chosen parameter values

$$
\begin{aligned}
c &= 0.0300, \quad d = -0.0300, \quad \gamma = -0.9593, \quad \delta = 0.3209, \quad \mu = 0.0380, \quad \sigma = 0.0300, \\
\rho &= 0.8, \qquad \eta = 2, \qquad\quad Y_0 = 8, \qquad\quad K = 12, \qquad T = 8.
\end{aligned}
$$

The function w is computed via the approximate formula (72) for $n = 2^k$ with $k = 1, 3, 5, 7, 9$, by applying a Monte Carlo integration method as done previously in Example E.1. In particular, when $k = 9$, an accurate goodness-of-fit to the curve obtained by the classical Feynman-Kac formula (7) is showed.
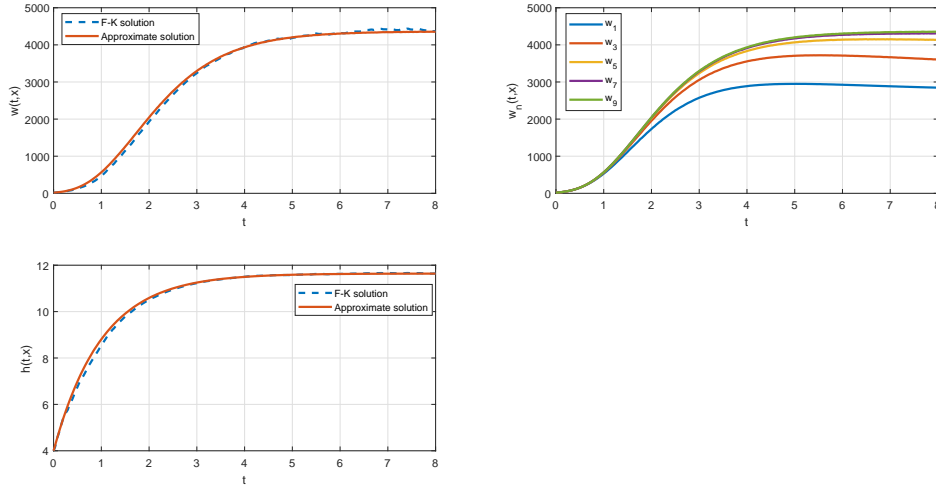

Figure 2: **Top line:** approximate solution w_n ($n = 2^9$) versus the classical Feynman-Kac solution *(left plot)*; approximate solutions for $n = 2^k$, $k = 1, 3, 5, 7, 9$ *(right plot)*. **Bottom line:** indifference price h of a put European option corresponding to the function w computed, respectively, by the approximate solution w_n ($n = 2^9$) and the Feynman-Kac solution.

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