

## Research Article

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# Sign-Changing Solutions for the One-Dimensional Non-Local sinh-Poisson Equation

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**Abstract:** We study the existence of sign-changing solutions for a non-local version of the sinh-Poisson equation on a bounded one-dimensional interval  $I$ , under Dirichlet conditions in the exterior of  $I$ . This model is strictly related to the mathematical description of galvanic corrosion phenomena for simple electrochemical systems. By means of the finite-dimensional Lyapunov–Schmidt reduction method, we construct bubbling families of solutions developing an arbitrarily prescribed number sign-alternating peaks. With a careful analysis of the limit profile of the solutions, we also show that the number of nodal regions coincides with the number of blow-up points.

**Keywords:** Fractional Laplacian, Exponential Non-Linearities, Non-Local, Corrosion Modelling, Lyapunov–Schmidt Reduction, One-Dimension, Sign-Changing

**MSC 2010:** 35R11, 35J61, 35B44, 35B38, 35B40

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## 1 Introduction

In this work we consider the non-local sinh-Poisson equation given by

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = \lambda(e^u - e^{-u}) & \text{in } I := (-1, 1), \\ u \equiv 0 & \text{on } \mathbb{R} \setminus I, \end{cases} \quad (1.1)$$

with  $\lambda \in \mathbb{R}^+$ . This equation is related to mathematical models for the description of galvanic corrosion of a planar electrochemical system consisting of an electrolyte solution and an adjoining metal surface. If the electrolyte is confined in a domain  $\Omega \subseteq \mathbb{R}^2$ , the electrolytic voltage potential satisfies the non-linear boundary value problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \lambda(e^{\beta v} - e^{-(1-\beta)v}) + g & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where  $\lambda$  and  $\beta$  are constants depending on the constituents of the system and  $g$  models an externally imposed current. We refer to [13, 36] for the mathematical derivation of this model, which is due to Butler and Volmer.

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If one takes  $\beta = \frac{1}{2}$  and  $g = 0$ , the problem becomes equivalent to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \lambda(e^v - e^{-v}) & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Problems (1.2)–(1.3), and corresponding ones in higher dimension, have been studied by several authors in recent years (see e.g. [10, 21, 28, 29, 36]). Among several generalizations, we mention that Alessandrini and Sincich [1, 2] have considered problems of the form

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \subseteq \mathbb{R}^2, \\ \frac{\partial v}{\partial \nu} = \lambda(e^v - e^{-v}) & \text{on } \Gamma_1, \\ \frac{\partial v}{\partial \nu} = g & \text{on } \Gamma_2, \\ v = 0 & \text{on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \end{cases}$$

where  $\Gamma_1$  and  $\Gamma_2$  are two open, disjoint portions of  $\partial\Omega$ . Here  $\Gamma_1$  represents the corroded part of  $\partial\Omega$ , which is not accessible to direct inspection,  $\Gamma_2$  is the portion of  $\partial\Omega$  where current density can be directly measured, and the remaining part of  $\partial\Omega$  is assumed to be grounded. In this work we want to consider the strictly related problem in which  $\Omega = \mathbb{R}_+^2$  is the upper half plane,  $\Gamma_1$  is a segment and  $\Gamma_2 = \emptyset$ . Namely, we have

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \lambda(e^v - e^{-v}) & \text{on } I \times \{0\}, \\ v = 0 & \text{on } (\mathbb{R} \setminus I) \times \{0\}. \end{cases} \quad (1.4)$$

There is a strict connection between problems (1.4) and (1.1). Indeed, the Poisson harmonic extension of any solution to (1.1) solves (1.4). Viceversa, if  $v$  solves (1.4) and has finite Dirichlet energy, then the boundary trace  $u = v(\cdot, 0)$  is a solution of (1.1).

Equation (1.1) can also be considered as a one-dimensional version of the planar sinh-Poisson problem

$$\begin{cases} -\Delta u = \lambda(e^u - e^{-u}) & \text{in } \Omega \subseteq \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

which arises in the statistical mechanics approach proposed by Onsager [27] and Joyce and Montgomery [20, 26] to the description of two-dimensional turbulent Euler flows with null total vorticity (we refer to [7, 8, 23, 27] for a physical discussion of this problem).

In recent years, there has been a great interest in the construction of sign-changing solutions for problems (1.3) and (1.5). When  $\Omega$  is the unit disk of  $\mathbb{R}^2$ , explicit families of solutions to (1.3) were exhibited by Bryan and Vogelius in [5]. As  $\lambda \rightarrow 0$ , such solutions develop an even number of sign-alternating peaks concentrating in separate points of  $\partial\Omega$ . In [10], Dávila, Del Pino, Musso and Wei proved that solutions of (1.3) with an analogous behavior exist on arbitrary bounded domains with smooth boundary. In fact, for any even  $k \in \mathbb{N}$ , they constructed two independent branches of solutions developing  $k$  sign-alternating peaks on  $\partial\Omega$ . In this result,  $k$  must be even to ensure the existence of sign-alternating peaks configurations on  $\partial\Omega$ , whose connected components are closed curves. We also refer to [21, 25] for a-priori analysis of blowing-up solutions to (1.3).

Concerning problem (1.5), the existence of sign-changing solutions developing exactly two peaks with different sign has been obtained by Bartolucci and Pistoia in [3]. More generally, they proved that if  $\xi_1, \dots, \xi_k \in \Omega$  correspond to a stable critical point of a generalized version of the  $k$ -point Kirchhoff–Routh path function, then there is a solution  $u_\lambda$  of (1.5) such that

$$\lambda(e^u - e^{-u}) \rightarrow 8\pi \sum_{i=1}^k (-1)^i \delta_{\xi_i}$$

as  $\lambda \rightarrow 0$  in the sense of measures, where  $\delta_{\xi_i}$  denotes the Dirac delta at  $\xi_i, i = 1, \dots, k$ . The existence of stable critical points of the generalized Kirchhoff–Routh function for  $k \geq 3$  was studied by Bartsch, Pistoia and Weth in [4]. They give existence results on any domain for  $k = 3, 4$ , and on axially symmetric domains for arbitrary  $k \geq 1$ . In the latter case, the points  $\xi_1, \dots, \xi_k$  are located on the symmetry axis of  $\Omega$  and the corresponding solution of (1.5) develops sign-alternating peaks at  $\xi_1, \dots, \xi_k$ .

Inspired by these results, the main purpose of this paper is to discuss the existence, for small values of  $\lambda$ , of a branch solutions of (1.1) with an arbitrarily prescribed number of nodal regions. Specifically, for any  $k \in \mathbb{N}$ , we will construct a branch of solutions with exactly  $k$  sign-alternating peaks in the interval  $I = (-1, 1)$ .

**Theorem 1.1.** *For any  $k \in \mathbb{N}$ , there exist  $\lambda_0 > 0$  and a family  $(u_\lambda)$ , defined for  $\lambda \in (0, \lambda_0)$ , of weak solutions to (1.1) with exactly  $k$  nodal regions. Moreover, for any sequence  $(\lambda_n)_{n \in \mathbb{N}} \subset (0, \lambda_0)$  with  $\lambda_n \rightarrow 0$  as  $n \rightarrow +\infty$ , there exist  $\xi_1, \dots, \xi_k \in I$  with  $\xi_1 < \xi_2 < \dots < \xi_k$  such that (along a subsequence) we have:*

- $u_{\lambda_n}$  blows-up at  $\xi_i$  as  $n \rightarrow \infty$  for  $i = 1, \dots, k$ . Namely, for any  $\varepsilon > 0$ , we have

$$\sup_{(\xi_i - \varepsilon, \xi_i + \varepsilon)} (-1)^{i-1} u_{\lambda_n} \rightarrow +\infty \quad \text{as } n \rightarrow +\infty. \tag{1.6}$$

- $u_{\lambda_n} \rightarrow 2\pi \sum_{i=1}^k (-1)^{i-1} G_{\xi_i}$ , in  $C_{loc}^\infty(I \setminus \{\xi_1, \dots, \xi_k\}) \cap C_{loc}^{0, \frac{1}{2}}(\bar{I} \setminus \{\xi_1, \dots, \xi_k\})$  as  $n \rightarrow +\infty$ , where  $G_{\xi_i}$  is the Green function for  $(-\Delta)^{\frac{1}{2}}$  with singularity at  $\xi_i$  (explicitly given by (2.3)).

The solutions provided in Theorem 1.1 can be considered as the analogue of the ones constructed in [4] for problem (1.5) but, working in dimension 1, we are able to obtain a stronger result and to show that the number of nodal regions coincides with the number of peaks, which does not seem to be known for the solutions in [4], except for the case  $k = 2$  (see [3]).

Our solutions are also strictly related to the solutions of (1.3) with an even number of peaks constructed in [10]. However, here we do not need to impose the evenness of  $k$  as the interaction between the first and the last peak is weaker than the interaction with intermediate peaks (unless  $k = 2$ ). It should be noted that if  $u$  solves (1.1), then  $-u$  is a solution as well. Thus, Theorem 1.1 provides two distinct branches of solutions. But, due to the lack of topology of  $I$ , we cannot expect the existence of other independent branches of solutions with separate peaks as in [10]. Nevertheless, we strongly believe it would be possible to find a different branch of solutions with  $k$  nodal regions which develops a tower of peaks at the origin. Solutions of this kind were constructed for problem (1.5) by Grossi and Pistoia in [17] (see also [30]).

Differently from the approach in [10], we will not rely on the connection between (1.1) and the extended problem (1.4). Instead, the proof of Theorem 1.1 will be based on the Lyapunov–Schmidt finite-dimensional reduction method, which has successfully been used to find solutions to (1.5) and other similar problems (see e.g. [3, 12, 15]). Here, we will apply this technique on the fractional-order Sobolev space  $X_0^{1/2}(I)$ , which is defined as the space of all the functions in  $H^{\frac{1}{2}}(\mathbb{R})$  which vanish identically outside  $I$ . A Hilbert structure on  $X_0^{1/2}(I)$  is determined by the scalar product

$$\langle u, v \rangle := \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} u (-\Delta)^{\frac{1}{4}} v \, dx, \quad u, v \in X_0^{1/2}(I), \tag{1.7}$$

with the corresponding norm given by

$$\|u\| := \|u\|_{X_0^{1/2}} = \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})} \equiv \|(-\Delta)^{\frac{1}{4}} u\|_2, \quad u \in X_0^{1/2}(I). \tag{1.8}$$

In order to prove Theorem 1.1, we proceed as follows. For  $\delta > 0, \xi \in \mathbb{R}$ , let us consider the one-dimensional bubble

$$U_{\delta, \xi}(x) := \log\left(\frac{2\delta}{\delta^2 + |x - \xi|^2}\right), \tag{1.9}$$

which solves the fractional-order Liouville equation [9, 18]

$$(-\Delta)^{\frac{1}{2}} U_{\delta, \xi} = e^{U_{\delta, \xi}} \quad \text{in } \mathbb{R}. \tag{1.10}$$

Note that  $U_{\delta, \xi}$  blows-up at  $\xi$  as  $\delta \rightarrow 0^+$ , namely  $\lim_{\delta \rightarrow 0^+} U_{\delta, \xi}(\xi) = +\infty$  and  $\lim_{\delta \rightarrow 0^+} U_{\delta, \xi}(x) = -\infty, x \in \mathbb{R} \setminus \{\xi\}$ .

For any  $\delta > 0$ ,  $\xi \in \mathbb{R}$ , let  $PU_{\delta,\xi}$  be the projection of  $U_{\delta,\xi}$  on  $X_0^{1/2}(I)$ , that is

$$PU_{\delta,\xi} := (-\Delta)^{-\frac{1}{2}} e^{U_{\delta,\xi}},$$

where  $(-\Delta)^{-\frac{1}{2}}$  represents the inverse of the half Laplacian  $(-\Delta)^{\frac{1}{2}}$ . Intuitively, for suitable choices of  $\delta$  and  $\xi$  we will use  $PU_{\delta,\xi}$  to model each peak developed by solutions of (1.1) as  $\lambda \rightarrow 0$ . Indeed, Theorem 1.1 can be deduced by the following result:

**Theorem 1.2.** *For any  $k \in \mathbb{N}$ , we can find  $\lambda_0 > 0$  such that, for any  $\lambda \in (0, \lambda_0)$ , there exist  $\delta_i = \delta_i(\lambda) \in (0, +\infty)$ ,  $\xi_i = \xi_i(\lambda) \in I$ ,  $i = 1, \dots, k$ , and  $\varphi_\lambda \in X_0^{1/2}(I) \cap L^\infty(I)$  such that:*

- $\sum_{i=1}^k (-1)^{i-1} PU_{\delta_i,\xi_i} + \varphi_\lambda$  is a solution of (1.1) for any  $\lambda \in (0, \lambda_0)$ .
- $\lim_{\lambda \rightarrow 0} \delta_i(\lambda) = 0$  for any  $i = 1, \dots, k$ .
- There exists a small  $\eta_0 \in (0, \frac{2}{k+1})$  (not depending on  $\lambda$ ) such that  $-1 + \eta_0 \leq \xi_1 < \dots < \xi_k \leq 1 - \eta_0$ , and  $\min_{1 \leq i \leq k-1} \xi_{i+1} - \xi_i \geq \eta_0$  for any  $\lambda \in (0, \lambda_0)$ .
- $\|\varphi_\lambda\| + \|\varphi_\lambda\|_{L^\infty(I)} \rightarrow 0$  as  $\lambda \rightarrow 0$ .

This work is organized as follows. In Section 2, we introduce the notation and state some preliminary results. For the reader’s convenience, we also include, in Section 2.1, an outline of the proof of Theorem 1.2. The technical aspects of the proof are discussed in Sections 3, 4, 5 and 6, as we will detail in Section 2.1. Finally, the conclusion of the proof of Theorem 1.2 and the proof of Theorem 1.1 are given in Section 7.

## 2 Notation and Preliminary Results

We start by recalling the definition and the main properties of the fractional Laplacian operator. For a given function  $u$  in the Schwartz space  $\mathcal{S}$  of rapidly decreasing functions (see e.g. [35]), we can define

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)(\xi)), \quad s \in (0, 1),$$

where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote respectively the Fourier and the inverse Fourier transform operators. In fact, this definition makes sense when  $|\xi|^{2s} \mathcal{F}(u) \in L^2(\mathbb{R})$ . More generally, if  $u$  belongs to the space

$$L_s(\mathbb{R}) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1 + |x|^{1+2s}} dx < +\infty \right\},$$

it is possible to define  $(-\Delta)^s$  as the tempered distribution

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}} u(-\Delta)^s \varphi dx, \quad \varphi \in \mathcal{S}.$$

For  $s \in (0, 1)$ , let  $H^s(\mathbb{R})$  denote the fractional-order Bessel potential space

$$H^s(\mathbb{R}) := \{ u \in L^2(\mathbb{R}) : |\xi|^s \mathcal{F}(u) \in L^2(\mathbb{R}) \}.$$

This space can be equivalently defined as the space of  $L^2$  functions for which the Gagliardo seminorm is finite (see e.g. [14, Section 3]). Similarly,  $(-\Delta)^s$  can be characterized in terms of singular integrals as

$$(-\Delta)^s u(x) = c_{1,s} \text{P. V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy, \quad c_{1,s} = \frac{-2^{2s} \Gamma(\frac{1}{2} + s)}{\pi^{\frac{1}{2}} \Gamma(-s)}.$$

Throughout the paper, we will always denote  $I := (-1, 1) \subseteq \mathbb{R}$  and we will consider the space

$$X_0^{1/2}(I) = \{ u \in H^{\frac{1}{2}}(\mathbb{R}) : u \equiv 0 \text{ in } \mathbb{R} \setminus I \},$$

which is a Hilbert space with respect to the scalar product given in (1.7). The corresponding norm will be denoted as in (1.8).

Given  $p \geq 1$  and a function  $f \in L^p(I)$ , we say that  $u$  is a weak solution of the problem

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = f & \text{in } I, \\ u = 0 & \text{in } \mathbb{R} \setminus I, \end{cases} \tag{2.1}$$

if  $u \in X_0^{1/2}(I)$  and it satisfies

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \varphi (-\Delta)^{\frac{1}{4}} u \, dx = \int_{\mathbb{R}} f \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(I).$$

Let also  $(-\Delta)^{-\frac{1}{2}}$  represent the inverse of  $(-\Delta)^{\frac{1}{2}}$ . For any given  $p \in (1, +\infty)$ , the restriction of  $(-\Delta)^{-\frac{1}{2}}$  to  $L^p(I)$  is defined by

$$(-\Delta)^{-\frac{1}{2}} : L^p(I) \rightarrow X_0^{1/2}(I), \quad f \mapsto u \text{ solution to (2.1).}$$

This operator coincides with the adjoint of the inclusion operator  $i_{p'} : X_0^{1/2}(I) \rightarrow L^{p'}(I)$ , where  $p' = \frac{p}{p-1}$ . In particular, for any  $p \in (1, +\infty)$ , there exists a constant  $C(p)$  such that

$$\|(-\Delta)^{-\frac{1}{2}} f\| \leq C(p) \|f\|_{L^p(I)} \quad \text{for all } f \in L^p(I). \tag{2.2}$$

The operator  $(-\Delta)^{-\frac{1}{2}}$  can also be defined via an explicit representation formula. Throughout the paper, for any  $\xi \in I$  we will denote by  $G_\xi$  the Green function for  $(-\Delta)^{\frac{1}{2}}$  on  $I$  with singularity at  $\xi$ , which is given explicitly (see [6]) by the formula

$$G_\xi(x) := \begin{cases} \frac{1}{\pi} \log\left(\frac{1-\xi x + \sqrt{(1-\xi^2)(1-x^2)}}{|x-\xi|}\right), & x \in I, \\ 0, & x \in \mathbb{R} \setminus I. \end{cases} \tag{2.3}$$

We shall often use the notation  $G(\xi, x)$  in place of  $G_\xi(x)$ . For any  $f \in L^p(I)$ , we have the representation formula

$$(-\Delta)^{-\frac{1}{2}} f(x) = \int_I G_x(y) f(y) \, dy, \quad x \in I.$$

We will also denote by  $H(\xi, x)$  the regular part of  $G_\xi$ , namely

$$H(\xi, x) := G_\xi(x) - \frac{1}{\pi} \log \frac{1}{|x - \xi|}.$$

### 2.1 Outline of the Proof of Theorem 1.2: The Lyapunov–Schmidt Reduction Method

For a given  $k \in \mathbb{N}$ , we fix a small  $\eta > 0$  with  $0 < \eta < \frac{2}{k+1}$  and we define

$$\mathcal{P}_{k,\eta} := \{\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) : 1 + \xi_1 > \eta, \xi_k < 1 - \eta \text{ and } \xi_{i+1} - \xi_i > \eta, i = 1, \dots, k - 1\}.$$

For  $\mathbf{a} = (a_1, \dots, a_k) \in \{-1, 1\}^k$ ,  $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k) \in (0, 1)^k$ , and  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \mathcal{P}_{k,\eta}$ , we denote

$$\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}} := \sum_{i=1}^k a_i P U_{\delta_i, \xi_i}. \tag{2.4}$$

For the proof of Theorems 1.1 and 1.2 we could fix  $a_i = (-1)^{i-1}$ ,  $1 \leq i \leq k$ . But, since many of the estimates given throughout the paper hold true for a generic choice of the coefficients  $a_i$ , we will only fix them when it is necessary.

Our goal is to find solutions for (1.1) of the form

$$u = \omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}} + \varphi,$$

where  $\varphi \in X_0^{1/2}(I) \cap L^\infty(I)$  is small with respect to both the  $X_0^{1/2}(I)$  and the  $L^\infty(I)$  norm. Throughout the paper we will denote  $f_\lambda(u) := \lambda(e^u - e^{-u})$ . Then in terms of  $\varphi$  equation (1.1) reads as

$$\begin{aligned} (-\Delta)^{\frac{1}{2}} \varphi &= f_\lambda(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}} + \varphi) - (-\Delta)^{\frac{1}{2}} \omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}} \\ &= \underbrace{f_\lambda(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}) - (-\Delta)^{\frac{1}{2}} \omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}}_{=: E} + \underbrace{f'_\lambda(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}) \varphi + f_\lambda(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}} + \varphi) - f_\lambda(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}) - f'_\lambda(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}) \varphi}_{=: N(\varphi)}, \end{aligned}$$

that is

$$(-\Delta)^{\frac{1}{2}}\varphi - f_{\lambda}'(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}})\varphi = E + N(\varphi).$$

It is convenient to rewrite this equation as

$$L\varphi = (-\Delta)^{-\frac{1}{2}}E + (-\Delta)^{-\frac{1}{2}}N(\varphi), \quad \text{where } L := \text{Id} - (-\Delta)^{-\frac{1}{2}}f_{\lambda}'(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}). \tag{2.5}$$

For simplicity, here and in the rest of the paper, we will not specify the dependence of  $E$ ,  $N$  and  $L$  on  $\lambda$ ,  $\mathbf{a}$ ,  $\boldsymbol{\delta}$ , and  $\boldsymbol{\xi}$ .

We will prove that there exists a  $k$ -dimensional subspace  $K_{\boldsymbol{\delta},\boldsymbol{\xi}}$  of  $X_0^{1/2}(I)$  such that  $L$  is invertible on  $K_{\boldsymbol{\delta},\boldsymbol{\xi}}^{\perp}$ . The space  $K_{\boldsymbol{\delta},\boldsymbol{\xi}}$  is spanned by the functions  $PZ_{1,i} := (-\Delta)^{-\frac{1}{2}}e^{U_{\delta_i,\xi_i}}Z_{1,i}$ ,  $i = 1, \dots, k$ , where  $Z_{1,i}$  is the unique solution which vanishes at infinity of the linearization of equation (1.10) around  $U_{\delta_i,\xi_i}$  (see Section 4). Let  $\pi : X_0^{1/2}(I) \mapsto K_{\boldsymbol{\delta},\boldsymbol{\xi}}$  and  $\pi^{\perp} : X_0^{1/2}(I) \mapsto K_{\boldsymbol{\delta},\boldsymbol{\xi}}^{\perp}$  be the projections of  $X_0^{1/2}(I)$  into  $K_{\boldsymbol{\delta},\boldsymbol{\xi}}$  and  $K_{\boldsymbol{\delta},\boldsymbol{\xi}}^{\perp}$ , respectively. Since  $X_0^{1/2}(I) = K_{\boldsymbol{\delta},\boldsymbol{\xi}} \oplus K_{\boldsymbol{\delta},\boldsymbol{\xi}}^{\perp}$ , equation (2.5) is equivalent to the following couple of non-linear problems:

$$\pi^{\perp}L\varphi = \pi^{\perp}(-\Delta)^{-\frac{1}{2}}E + \pi^{\perp}(-\Delta)^{-\frac{1}{2}}N(\varphi), \tag{2.6}$$

$$\pi L\varphi = \pi(-\Delta)^{-\frac{1}{2}}E + \pi(-\Delta)^{-\frac{1}{2}}N(\varphi). \tag{2.7}$$

Exploiting the invertibility of  $\pi^{\perp}L$  on  $K_{\boldsymbol{\delta},\boldsymbol{\xi}}$ , one formulates equation (2.6) in terms of a fixed point problem for  $\varphi$ . Such a problem can be solved if the error term  $E$  has small  $L^p(I)$  norm for some  $p \in (1, 2)$ , the non-linear term  $N(\varphi)$  decays faster than  $\|\varphi\|$ , and the operator norm of  $(\pi^{\perp}L)^{-1}$  can be controlled in terms of  $\lambda$ . We will show that for any choice of  $\eta \in (0, \frac{2}{k+1})$ ,  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \mathcal{P}_{k,\eta}$  and  $\lambda$  small enough, these conditions are satisfied by a suitable choice of  $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k)$  depending on  $\lambda$  and  $\boldsymbol{\xi}$ . Specifically, there exists  $\boldsymbol{\delta} = \boldsymbol{\delta}_{\lambda,\boldsymbol{\xi}} \in (0, 1)^k$  and  $\varphi = \varphi_{\lambda,\boldsymbol{\xi}} \in X_0^{1/2}(I)$  such that (2.6) holds. In other words, there exist coefficients  $c_i = c_i(\lambda, \boldsymbol{\xi})$ ,  $i = 1, \dots, k$ , which depend continuously on  $\boldsymbol{\xi}$  such that

$$L\varphi_{\lambda,\boldsymbol{\xi}} = (-\Delta)^{-\frac{1}{2}}E + (-\Delta)^{-\frac{1}{2}}N(\varphi_{\lambda,\boldsymbol{\xi}}) + \sum_{j=1}^k c_j PZ_{1,j}.$$

Then  $\varphi_{\lambda,\boldsymbol{\xi}}$  solves equation (2.7) if and only if

$$c_i(\lambda, \boldsymbol{\xi}) = 0, \quad i = 1, \dots, k. \tag{2.8}$$

The proof of Theorem 1.2 can be concluded by proving that for any small  $\lambda$ , there exist  $\xi_1, \dots, \xi_k$  depending on  $\lambda$  solving the finite-dimensional system (2.8).

The rest of this paper is organized as follows. In Section 3 we choose the parameters  $\delta_1, \dots, \delta_k$  and we provide point-wise and  $L^p$  estimates on the error terms  $E$  and  $N$ . Section 4 contains the precise definition of  $K_{\boldsymbol{\delta},\boldsymbol{\xi}}$  and the analysis of the invertibility properties of  $\pi^{\perp}L$ . The fix point argument which allows to solve (2.6) is explained in Section 5, while system (2.8) is studied in Section 6. Finally, we complete the proof of Theorems 1.1 and 1.2 in Section 7.

For the proof described above it is important to point out that all the estimates in Sections 3–6 will be uniform with respect to the choice of  $\boldsymbol{\xi} \in \mathcal{P}_{k,\eta}$  and of small values of  $\lambda$  and  $\boldsymbol{\delta}$ . For this reason, given two quantities  $\Theta_1, \Theta_2$  depending on  $\lambda, \boldsymbol{\delta}, \boldsymbol{\xi}$  and  $\eta$  (and eventually other parameters), it is convenient to write  $\Theta_1 = O(\Theta_2)$  to indicate  $|\Theta_1| \leq C\Theta_2$ , for some constant  $C > 0$  that does not depend on  $\boldsymbol{\xi}, \boldsymbol{\delta}$  and  $\lambda$  (but may depend on  $\eta$  and the other parameters, unless otherwise specified). This notation will be used several times throughout the paper.

### 3 Choice of the Concentration Parameters and Estimates of the Error Terms

Let  $\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}$  be as in (2.4). In order to perform the perturbation argument explained before, we need to be sure that  $\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}$  is a good approximate solution to (1.1). This means we need to estimate the error term

$$E = f_{\lambda}(\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}}) - (-\Delta)^{\frac{1}{2}}\omega_{\mathbf{a},\boldsymbol{\delta},\boldsymbol{\xi}},$$

as defined in Section 2. As a first step, we need the following lemma.

**Lemma 3.1.** For any  $\eta > 0$ , there exists a constant  $C_\eta$  such that

$$|PU_{\delta,\xi} - U_{\delta,\xi} + \log(2\delta) - 2\pi H(\xi, \cdot)| \leq C_\eta \delta^2,$$

for any  $\delta \in (0, 1)$ ,  $\xi \in (-1 + \eta, 1 - \eta)$ . In particular, we have that

$$PU_{\delta,\xi} = \begin{cases} U_{\delta,\xi} - \log(2\delta) + 2\pi H(\xi, \xi) + O(|\cdot - \xi|) + O(\delta^2) & \text{uniformly in } (\xi - \frac{\eta}{2}, \xi + \frac{\eta}{2}), \\ 2\pi G_\xi + O(\delta^2) & \text{uniformly in } I \setminus (\xi - \frac{\eta}{2}, \xi + \frac{\eta}{2}), \end{cases}$$

independently of the choice of  $\xi \in (-1 + \eta, 1 - \eta)$  and  $\delta \in (0, 1)$ .

*Proof.* Let  $u_{\delta,\xi} := PU_{\delta,\xi} - U_{\delta,\xi} + \log(2\delta) - 2\pi H(\xi, \cdot)$ . First, we observe that

$$(-\Delta)^{\frac{1}{2}} u_{\delta,\xi} = 0 \quad \text{in } I,$$

since  $(-\Delta)^{\frac{1}{2}} H(\xi, \cdot) = 0$  on  $I$  and  $(-\Delta)^{\frac{1}{2}} PU_{\delta,\xi} = (-\Delta)^{\frac{1}{2}} U_{\delta,\xi}$  in  $I$  (by the definition of  $PU_{\delta,\xi}$ ). Next, we study the values of  $u_{\delta,\xi}$  in  $\mathbb{R} \setminus I$ . Here, since  $PU_{\delta,\xi} \in X_0^{1/2}(I)$ , we have  $PU_{\delta,\xi} = 0$ . Then, recalling the expression of  $U_{\delta,\xi}$  given in (1.9), and noting that  $2\pi H(\xi, x) = 2 \log|x - \xi|$  in  $\mathbb{R} \setminus I$ , we find that

$$u_{\delta,\xi}(x) = -U_{\delta,\xi}(x) + \log(2\delta) - 2 \log|x - \xi| = \log(\delta^2 + |x - \xi|^2) - 2 \log|x - \xi|.$$

Since  $x \in \mathbb{R} \setminus I$  and  $\xi \in (-1 + \eta, 1 - \eta)$ , we get  $|x - \xi| \geq \eta$ . Then we can find  $C_\eta$  s.t.  $|u_{\delta,\xi}| \leq C_\eta \delta^2$  in  $\mathbb{R} \setminus I$ . By the Maximum Principle (see [33, Lemma 6]) we have the desired result.  $\square$

Next, we shall fix  $\delta_1, \dots, \delta_k$  in order to make the error term  $E$  small near each of the points  $\xi_1, \dots, \xi_k$ . Note that, for any  $1 \leq i \leq k$ , in the interval  $(\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})$  we have the uniform expansion

$$\begin{aligned} E &= \lambda e^{\sum_{j=1}^k a_j PU_{\delta_j, \xi_j}} - \lambda e^{-\sum_{j=1}^k a_j PU_{\delta_j, \xi_j}} - \sum_{j=1}^k a_j e^{U_{\delta_j, \xi_j}} \\ &= \frac{\lambda}{(2\delta_i)^{a_i}} e^{a_i U_{\delta_i, \xi_i} + 2\pi a_i H(\xi_i, \xi_i) + 2\pi \sum_{j \neq i} a_j G_{\xi_j} + \sum_{i=1}^k O(\delta_i^2) + O(|\cdot - \xi_i|)} \\ &\quad - \lambda (2\delta_i)^{a_i} e^{-a_i U_{\delta_i, \xi_i} - 2\pi a_i H(\xi_i, \xi_i) - 2\pi \sum_{j \neq i} a_j G_{\xi_j} + \sum_{i=1}^k O(\delta_i^2) + O(|\cdot - \xi_i|)} - a_i e^{U_{\delta_i, \xi_i}} - \sum_{j \neq i} a_j e^{U_{\delta_j, \xi_j}} \\ &= \frac{\lambda a_i}{2\delta_i} e^{U_{\delta_i, \xi_i}} e^{2\pi H(\xi_i, \xi_i) + 2\pi a_i \sum_{j \neq i} a_j G_{\xi_j} + O(|\delta|^2) + O(|\cdot - \xi_i|)} + O(\lambda \delta_i e^{-U_{\delta_i, \xi_i}}) - a_i e^{U_{\delta_i, \xi_i}} - \sum_{j \neq i} a_j e^{U_{\delta_j, \xi_j}}, \end{aligned} \tag{3.1}$$

where we have used Lemma 3.1 and  $a_i \in \{-1, 1\}$ ,  $i = 1, \dots, k$ . Moreover, we have that

$$\delta_i e^{-U_{\delta_i, \xi_i}(x)} = \frac{\delta_i^2 + |x - \xi_i|^2}{2} = O(\delta_i^2) + O(|x - \xi_i|^2) \tag{3.2}$$

and, for  $j \neq i$ , that

$$e^{U_{\delta_j, \xi_j}(x)} = \frac{2\delta_j}{\delta_j^2 + |x - \xi_j|^2} = \frac{2\delta_j}{|x - \xi_j|^2} + O(\delta_j^3) = \frac{2\delta_j}{|\xi_i - \xi_j|^2} + O(\delta_j|x - \xi_i|) + O(\delta_j^3) = O(\delta_j).$$

For  $i \in \{1, \dots, k\}$ , let us consider the functions

$$F_i(\xi) := 2\pi H(\xi_i, \xi_i) + 2\pi a_i \sum_{j \neq i} a_j G_{\xi_j}(\xi_i), \tag{3.3}$$

so that estimate (3.1) rewrites as

$$E = a_i e^{U_{\delta_i, \xi_i}} \left( \frac{\lambda}{2\delta_i} e^{F_i(\xi) + O(|\delta|^2) + O(|\cdot - \xi_i|)} - 1 \right) + O(\lambda) + \sum_{j \neq i} O(\delta_j). \tag{3.4}$$

In order to make the main term of the above expansion small, we choose

$$\delta_i = \delta_i(\lambda, \xi) := \frac{\lambda}{2} e^{F_i(\xi)} = \frac{\lambda}{2} e^{2\pi H(\xi_i, \xi_i) + 2\pi a_i \sum_{j \neq i} a_j G_{\xi_j}(\xi_i)}, \quad i = 1, \dots, k. \tag{3.5}$$

With this choice, we get the following integral estimate on  $E$ .

**Lemma 3.2.** Let  $\delta = (\delta_1, \dots, \delta_k)$  be as in (3.5). For any  $p \in (1, \infty)$  one has

$$\|E\|_{L^p(I)} = O(\lambda^{\frac{1}{p}}),$$

uniformly with respect to the choice of  $\lambda \in (0, 1)$  and  $\xi = (\xi_1, \dots, \xi_k) \in \mathcal{P}_{k,\eta}$ .

*Proof.* Thanks to (3.5), we have  $\delta_i = O(\lambda)$ ,  $i = 1, \dots, k$  uniformly for  $\xi \in \mathcal{P}_{k,\eta}$ . Then (3.4) and (3.5) yield

$$E(x) = a_i e^{U_{\delta_i, \xi_i}} (O(\lambda^2) + O(|x - \xi_i|)) + O(\lambda) = O(e^{U_{\delta_i, \xi_i}} |x - \xi_i|) + O(\lambda),$$

uniformly in  $(\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})$ ,  $i = 1, \dots, k$ , where the last equality follows by  $e^{U_{\delta_i, \xi_i}} \leq \frac{2}{\delta_i} = O(\lambda^{-1})$ . Moreover, using Lemma 3.1, we get

$$E = \lambda e^{2\pi \sum_{j=1}^k a_j G_{\xi_j} + O(\lambda^2)} - \lambda e^{-2\pi \sum_{j=1}^k a_j G_{\xi_j} + O(\lambda^2)} - \sum_{j=1}^k a_j e^{U_{\delta_j, \xi_j}} = O(\lambda),$$

uniformly in  $I \setminus \bigcup_{i=1}^k (\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})$ . Using these estimates, we can assert that

$$\begin{aligned} \|E\|_{L^p(I)}^p &= \sum_{i=1}^k \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} (O(e^{U_{\delta_i, \xi_i}} |x - \xi_i|) + O(\lambda))^p dx + \int_{I \setminus \bigcup_{i=1}^k (\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})} O(\lambda)^p dx \\ &= \sum_{i=1}^k \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} (O(e^{U_{\delta_i, \xi_i}} |x - \xi_i|))^p dx + O(\lambda)^p. \end{aligned}$$

Since  $p > 1$ , with the change of variable  $y = \frac{|x - \xi_i|}{\delta_i}$ , we find

$$\int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} (e^{U_{\delta_i, \xi_i}} |x - \xi_i|)^p dx = \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} \left( \frac{2\delta_i |x - \xi_i|}{\delta_i^2 + |x - \xi_i|^2} \right)^p dx = \delta_i \int_{-\frac{\eta}{2\delta_i}}^{\frac{\eta}{2\delta_i}} \left( \frac{2y}{1 + y^2} \right)^p dy = O(\delta_i) = O(\lambda)$$

for  $i = 1, \dots, k$ . We can so conclude that

$$\|E\|_{L^p(I)}^p = O(\lambda) + O(\lambda^p) = O(\lambda) \quad \text{for all } p > 1. \quad \square$$

**Remark 3.3.** Using the change of variable of the proof above, one can easily verify that, for any  $p, q \geq 0$ , the following useful estimate holds as  $\delta \rightarrow 0$ , uniformly with respect to  $\xi \in \mathbb{R}$ :

$$\int_{\xi - \frac{\eta}{2}}^{\xi + \frac{\eta}{2}} e^{p U_{\delta, \xi}} |x - \xi|^q dx = \begin{cases} O(\delta^{q-p+1}) & \text{if } 2p - q > 1, \\ O(\delta^p |\log \delta|) & \text{if } 2p - q = 1, \\ O(\delta^p) & \text{if } 2p - q < 1. \end{cases} \quad (3.6)$$

**Remark 3.4.** For  $p = 1$ , the argument of Lemma 3.2 gives

$$\|E\|_{L^1(I)} = O(\lambda |\log \lambda|).$$

### 3.1 Estimates on the Non-Linear Error Term

In this subsection we look for estimates on the non-linear error term

$$N(\varphi) = f_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \varphi) - f_\lambda(\omega_{\mathbf{a}, \delta, \xi}) - f'_\lambda(\omega_{\mathbf{a}, \delta, \xi})\varphi,$$

as defined in Section 2. The following lemma shows that  $N$  depends quadratically on  $\varphi$ .

**Lemma 3.5.** Let  $\delta = (\delta_1, \dots, \delta_k)$  be as in (3.5). For any  $p \geq 1$  and  $s > p$ , there exists a constant  $C_{p,s,\eta} > 0$ , depending only on  $p, s$  and  $\eta$ , such that

$$\|N(\varphi_1) - N(\varphi_2)\|_{L^p(I)} \leq C_{p,s,\eta} \lambda^{\frac{1}{s}-1} \|\varphi_1 - \varphi_2\| (\|\varphi_1\| + \|\varphi_2\|)$$

for any  $\lambda \in (0, 1)$ ,  $\xi \in \mathcal{P}_{k,\eta}$  and  $\varphi_1, \varphi_2 \in X_0^{1/2}(I)$  satisfying  $\|\varphi_1\|, \|\varphi_2\| \leq 1$ .

*Proof.* First of all, we observe that for any  $x \in I$ , there exist  $t_1 = t_1(x), t_2 = t_2(x) \in [0, 1]$  such that

$$\begin{aligned} N(\varphi_1) - N(\varphi_2) &= f_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \varphi_1) - f_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \varphi_2) - f'_\lambda(\omega_{\mathbf{a}, \delta, \xi})\varphi_1 + f'_\lambda(\omega_{\mathbf{a}, \delta, \xi})\varphi_2 \\ &= f'_\lambda(\omega_{\mathbf{a}, \delta, \xi} + t_1\varphi_1 + (1 - t_1)\varphi_2)(\varphi_1 - \varphi_2) - f'_\lambda(\omega_{\mathbf{a}, \delta, \xi})(\varphi_1 - \varphi_2) \\ &= f''_\lambda(\omega_{\mathbf{a}, \delta, \xi} + t_2t_1\varphi_1 + t_2(1 - t_1)\varphi_2)(\varphi_1 - \varphi_2)(t_1\varphi_1 + (1 - t_1)\varphi_2). \end{aligned}$$

Denoting  $\tau_1 = t_1t_2 \in [0, 1]$  and  $\tau_2 = t_2(1 - t_1) \in [0, 1]$ , we get that

$$\begin{aligned} |N(\varphi_1) - N(\varphi_2)| &\leq |f''_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \tau_1\varphi_1 + \tau_2\varphi_2)| |\varphi_1 - \varphi_2| |t_1\varphi_1 + (1 - t_1)\varphi_2| \\ &\leq |f''_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \tau_1\varphi_1 + \tau_2\varphi_2)| |\varphi_1 - \varphi_2| (|\varphi_1| + |\varphi_2|). \end{aligned}$$

Noting that  $f''_\lambda = f_\lambda$  and that  $|f_\lambda(t)| \leq 2\lambda e^{|t|}$ , we get

$$|N(\varphi_1) - N(\varphi_2)| \leq 2\lambda e^{|\omega_{\mathbf{a}, \delta, \xi}| + |\varphi_1| + |\varphi_2|} |\varphi_1 - \varphi_2| (|\varphi_1| + |\varphi_2|) \leq 2\lambda e^{|\omega_{\mathbf{a}, \delta, \xi}| + \varphi_3} |\varphi_1 - \varphi_2| \varphi_3, \tag{3.7}$$

where  $\varphi_3 = |\varphi_1| + |\varphi_2|$ . Additionally, for any choice of  $s > p \geq 1$ , we can find  $s_1, s_2, s_3 > 1$  such that

$$\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} + \frac{1}{s} = \frac{1}{p}.$$

Then Hölder’s inequality implies that

$$\|\lambda e^{|\omega_{\mathbf{a}, \delta, \xi}|} e^{|\varphi_3|} |\varphi_1 - \varphi_2| \varphi_3\|_{L^p(I)} \leq \|\lambda e^{|\omega_{\mathbf{a}, \delta, \xi}|}\|_{L^s(I)} \|e^{\varphi_3}\|_{L^{s_1}(I)} \|\varphi_1 - \varphi_2\|_{L^{s_2}(I)} \|\varphi_3\|_{L^{s_3}(I)}. \tag{3.8}$$

Now, using Lemma 3.1, we see that  $\lambda e^{|\omega_{\mathbf{a}, \delta, \xi}|} = O(\lambda)$  in  $I \setminus \bigcup_{i=1}^k (\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})$ , and that

$$\lambda e^{|\omega_{\mathbf{a}, \delta, \xi}|} = \lambda e^{PU_{\delta_i, \xi_i} + O(1)} = O(e^{U_{\delta_i, \xi_i}})$$

in  $(\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})$  for  $i = 1, \dots, k$ . Therefore

$$\|\lambda e^{|\omega_{\mathbf{a}, \delta, \xi}|}\|_{L^s(I)}^s = \sum_{i=1}^k \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} O(e^{sU_{\delta_i, \xi_i}}) dx + O(\lambda^s) = O(\lambda^{1-s}) + O(\lambda^s) = O(\lambda^{1-s}), \tag{3.9}$$

where we used (3.6) and  $1 - s < s$ . Note that the quantity  $O(\lambda^{1-s})$  depends on  $\eta$  and  $s$ .

Using the Moser–Trudinger inequality (see [24]), we get that

$$\int_{\mathbb{R}} e^{s_1|\varphi_3|} dx \leq e^{\frac{s_1^2}{4\pi} \|\varphi_3\|^2} \int_I e^{\pi \frac{\varphi_3^2}{\|\varphi_3\|^2}} dx \leq C e^{\frac{s_1^2}{4\pi} \|\varphi_3\|^2} \leq C(s_1). \tag{3.10}$$

Finally, thanks to Sobolev’s inequality, we have the estimates

$$\|\varphi_1 - \varphi_2\|_{L^{s_2}(I)} \leq C(s_2) \|\varphi_1 - \varphi_2\| \quad \text{and} \quad \|\varphi_3\|_{L^{s_3}(I)} \leq C(s_3) \|\varphi_3\| \leq C(s_3) (\|\varphi_1\| + \|\varphi_2\|). \tag{3.11}$$

Thus, replacing (3.8)–(3.11) into (3.7), we obtain

$$\|N(\varphi_1) - N(\varphi_2)\|_{L^p(I)} \leq C\lambda^{\frac{1-s}{s}} \|\varphi_1 - \varphi_2\| (\|\varphi_1\| + \|\varphi_2\|),$$

with  $C$  depending only on  $\eta, s, s_1, s_2$  and  $s_3$ . Since the choice of  $s_1, s_2$  and  $s_3$  depends only on  $s$  and  $p$ , we get the conclusion.  $\square$

**Remark 3.6.** Repeating the argument of the above proof, we can show that, for any  $s, s_1 > p \geq 1$  such that  $\frac{1}{s_1} + \frac{1}{s} < \frac{1}{p}$ , there exists a constant  $C = C(p, s, s_1, \eta)$  such that

$$\|f_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \varphi) - f_\lambda(\omega_{\mathbf{a}, \delta, \xi})\|_{L^p(I)} + \|f'_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \varphi) - f'_\lambda(\omega_{\mathbf{a}, \delta, \xi})\|_{L^p(I)} \leq C\lambda^{\frac{1-s}{s}} e^{\frac{s_1^2}{4\pi} \|\varphi\|^2} \|\varphi\|$$

for any  $\varphi \in X_0^{1/2}(I), \xi \in \mathcal{P}_{k, \eta}$  and  $\lambda \in (0, 1)$ . Note that whenever  $s > p$ , it is possible to choose  $s_1 > p$  large enough so that  $\frac{1}{s_1} + \frac{1}{s} < \frac{1}{p}$ .

### 4 Properties of the Linearized Operator

This section is devoted to the study of the linear operator  $L : X_0^{1/2}(I) \rightarrow X_0^{1/2}(I)$ , defined as

$$L\varphi = \varphi - (-\Delta)^{-\frac{1}{2}} f_\lambda'(\omega_{\mathbf{a}, \delta, \xi}) \varphi.$$

In particular, we are interested in exhibiting an approximate kernel of  $L$ . As a first step we describe the behavior of the term  $f_\lambda'(\omega_{\mathbf{a}, \delta, \xi})$ .

**Lemma 4.1.** *For any  $i = 1, \dots, k$ , we have the expansion*

$$f_\lambda'(\omega_{\mathbf{a}, \delta, \xi}) = \begin{cases} e^{U_{\delta_i, \xi_i}} (1 + O(|\cdot - \xi_i|) + O(\lambda^2)) & \text{uniformly in } (\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2}), \\ O(\lambda) & \text{uniformly in } I \setminus \bigcup_{i=1}^k (\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2}). \end{cases}$$

In particular,  $\|f_\lambda'(\omega_{\mathbf{a}, \delta, \xi})\|_{L^1(I)} = O(1)$ .

*Proof.* Indeed, arguing as in (3.1), we get

$$\begin{aligned} f_\lambda'(\omega_{\mathbf{a}, \delta, \xi}) &= \frac{\lambda}{(2\delta_i)^{a_i}} e^{a_i U_{\delta_i, \xi_i} + 2\pi a_i H(\xi_i, \cdot) + 2\pi \sum_{j \neq i} a_j G_{\xi_j} + O(\lambda^2)} + \lambda (2\delta_i)^{a_i} e^{-a_i U_{\delta_i, \xi_i} - 2\pi a_i H(\xi_i, \cdot) - 2\pi \sum_{j \neq i} a_j G_{\xi_j} + O(\lambda^2)} \\ &= \frac{\lambda}{2\delta_i} e^{U_{\delta_i, \xi_i} + F_i(\xi) + O(|\cdot - \xi_i|) + O(\lambda^2)} + O(\lambda \delta_i e^{-U_{\delta_i, \xi_i}}) \\ &= e^{U_{\delta_i, \xi_i} + O(|\cdot - \xi_i|) + O(\lambda^2)} + O(\lambda) \\ &= e^{U_{\delta_i, \xi_i}} (1 + O(|\cdot - \xi_i|) + O(\lambda^2)) + O(\lambda) \\ &= e^{U_{\delta_i, \xi_i}} (1 + O(|\cdot - \xi_i|) + O(\lambda^2)) \end{aligned}$$

in  $(\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})$ , where in the last equality we used (3.2) to estimate  $O(\lambda)$  as

$$O(\lambda) = e^{U_{\delta_i, \xi_i}(x)} O(\lambda e^{-U_{\delta_i, \xi_i}(x)}) = e^{U_{\delta_i, \xi_i}(x)} (O(\lambda^2) + O(|x - \xi_i|^2)).$$

Moreover,  $f_\lambda'(\omega_{\mathbf{a}, \delta, \xi}) = O(\lambda)$  in  $I \setminus \bigcup_{i=1}^k (\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})$ , since on this set we have

$$\omega_{\mathbf{a}, \delta, \xi} = 2\pi \sum_{j=1}^k a_j G_{\xi_j} + O(\lambda^2) = O(1). \quad \square$$

Next, we focus on the kernel of  $L$ . Observe that if  $\varphi \in X_0^{1/2}(I)$  and  $L\varphi = 0$ , then the scaled functions

$$\Phi_i(x) := \varphi(\xi_i + \delta_i x)$$

are weak solutions to

$$(-\Delta)^{\frac{1}{2}} \Phi_i + \delta_i f_\lambda'(\omega_{\mathbf{a}, \delta, \xi}(\xi_i + \delta_i \cdot)) = 0, \quad i = 1, \dots, k,$$

in the expanding intervals  $(\frac{-\xi_i - 1}{\delta_i}, \frac{1 - \xi_i}{\delta_i})$ . According to Lemma 4.1, for any fixed  $y \in \mathbb{R}$ , we have

$$\delta_i f_\lambda'(\omega_{\mathbf{a}, \delta, \xi}(\xi_i + \delta_i y)) \sim \delta_i e^{U_{\delta_i, \xi_i}(\xi_i + \delta_i y)} = \frac{2}{1 + y^2}.$$

Then  $\Phi_i$  should behave locally as a solution of the problem

$$(-\Delta)^{\frac{1}{2}} \Phi = \frac{2\Phi}{1 + |\cdot|^2} \quad \text{in } \mathbb{R}. \tag{4.1}$$

This equation was studied by many authors. In particular, Santra [32, Theorem 1.4] (see also [11]) proved that the only bounded solutions to (1.10) are linear combinations of the functions

$$Z_0(y) := \frac{1 - y^2}{1 + y^2} \quad \text{and} \quad Z_1(y) := \frac{2y}{1 + y^2}. \tag{4.2}$$

Here we will need a small modification of this classification result. Let us consider the spaces

$$\mathcal{L} := \{u \in L^1_{\text{loc}}(\mathbb{R}) : |u|^2(1 + x^2)^{-1} \in L^1(\mathbb{R})\}, \quad \mathcal{H} := \{u \in \mathcal{L} : (-\Delta)^{\frac{1}{4}} u \in L^2(\mathbb{R})\}. \tag{4.3}$$

These spaces are endowed with the norms

$$\|u\|_{\mathcal{L}}^2 = \int_{\mathbb{R}} \frac{u(x)^2}{1 + |x|^2} dx \quad \text{and} \quad \|u\|_{\mathcal{H}}^2 = \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})}^2 + \|u\|_{\mathcal{L}}^2.$$

It is known that one can construct an isometry between  $L^2(\mathbb{R})$  and  $\mathcal{L}$  and between  $H^{\frac{1}{2}}(S^1)$  and  $\mathcal{H}$  via the standard stereographic projection. In particular,  $\mathcal{H}$  is compactly embedded into  $\mathcal{L}$ .

**Lemma 4.2.** *Let  $\Phi \in \mathcal{H}$  be a solution to (4.1). Then there exist  $\kappa_0, \kappa_1 \in \mathbb{R}$  such that  $\Phi = \kappa_0 Z_0 + \kappa_1 Z_1$ , where  $Z_0$  and  $Z_1$  are the functions in (4.2).*

*Proof.* First of all, we observe that any solution  $\varphi$  to (4.1) is smooth. This follows by standard regularity results (see [22, Theorem 13], the appendix in [19], and [31, Corollaries 2.4 and 2.5])

Using the density of  $C_c^\infty(\mathbb{R})$  in  $\mathcal{H}$  (which can be proved using the arguments of [16, Lemmas 11 and 12], since  $\|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})}$  is equivalent to the Gagliardo seminorm), we can find a sequence  $\psi_n \in C_c^\infty(\mathbb{R})$  such that  $\psi_n \rightarrow 1$  in  $\mathcal{H}$  (note that constant functions belong to  $\mathcal{H}$ ). Then, for any  $n$ , have

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \varphi (-\Delta)^{\frac{1}{4}} \psi_n dx = \int_{\mathbb{R}} \varphi (-\Delta)^{\frac{1}{2}} \psi_n dx = \int_{\mathbb{R}} f_\varphi \psi_n dx,$$

where  $f_\varphi(x) := 2 \frac{\varphi(x)}{1+x^2}$ . Passing to the limit as  $n \rightarrow \infty$ , we get

$$\int_{\mathbb{R}} f_\varphi dx = 0. \tag{4.4}$$

Let us now consider the functions

$$\Gamma(x, y) = \frac{1}{\pi} \log\left(\frac{1 + |y|}{|x - y|}\right) \quad \text{and} \quad \Phi(x) := \int_{\mathbb{R}} \Gamma(x, y) f_\varphi(y) dy.$$

Since  $\varphi \in \mathcal{L} \subseteq L^{\frac{1}{2}}(\mathbb{R})$ , according to [18, Lemma 2.4], we have  $\varphi = \Phi + c$  for some  $c \in \mathbb{R}$ . Now, observing that  $\Gamma(\frac{1}{x}, y) = \frac{\log|x|}{\pi} + \Gamma(x, \frac{1}{y})$  for any  $x, y \in \mathbb{R} \setminus \{0\}$  with  $x \neq \frac{1}{y}$ , we have that

$$\varphi\left(\frac{1}{x}\right) = \Phi\left(\frac{1}{x}\right) + c = \underbrace{\frac{\log|x|}{\pi} \int_{\mathbb{R}} f_\varphi(y) dy}_{= 0 \text{ by (4.4)}} + \int_{\mathbb{R}} \Gamma\left(x, \frac{1}{y}\right) f_\varphi(y) dy + c = 2 \int_{\mathbb{R}} \Gamma(x, z) \frac{\varphi(\frac{1}{z})}{1+z^2} dz + c \tag{4.5}$$

for a.e.  $x \in \mathbb{R} \setminus \{0\}$ . Denoting  $\tilde{\varphi}(x) := \varphi(\frac{1}{x})$  and  $f_{\tilde{\varphi}}(x) := 2 \frac{\tilde{\varphi}(x)}{1+x^2}$ , via a simple change of variable we can show that  $\tilde{\varphi} \in \mathcal{L}$  and  $f_{\tilde{\varphi}} \in L^1(\mathbb{R})$ . Since  $f_{\tilde{\varphi}} \in L^1(\mathbb{R})$ , [18, Lemma 2.3] implies that

$$\tilde{\Phi}(x) := \int_{\mathbb{R}} \Gamma(x, z) f_{\tilde{\varphi}}(z) dz$$

is a distributional solution to  $(-\Delta)^{\frac{1}{2}} \tilde{\Phi} = f_{\tilde{\varphi}}$  in  $\mathbb{R}$ . Moreover, using that  $\tilde{\varphi} \in \mathcal{L}$  (and in particular  $f_{\tilde{\varphi}} \in L^2_{\text{loc}}(\mathbb{R})$ ), we can repeat the first part of the proof and show that  $\tilde{\Phi} \in C^\infty(\mathbb{R})$ . By (4.5), we infer that  $\tilde{\varphi}$  can be extended to a smooth function on  $\mathbb{R}$ . In particular, this gives that  $\varphi \in L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$ . Then we can conclude using directly the classification result in [32, Theorem 1.4].  $\square$

In the following, for  $\xi \in \mathcal{P}_{k,\eta}$  and  $\lambda > 0$ , we shall denote  $Z_{i,j}(x) := Z_i(\frac{x-\xi_j}{\delta_j})$ ,  $i = 0, 1, j = 1, \dots, k$ , where  $\delta_j$  is defined as in (3.5). Namely, we consider

$$Z_{0,j}(x) := \frac{\delta_j^2 - (x - \xi_j)^2}{\delta_j^2 + (x - \xi_j)^2} \quad \text{and} \quad Z_{1,j}(x) := \frac{2\delta_j(x - \xi_j)}{\delta_j^2 + (x - \xi_j)^2}, \tag{4.6}$$

which are solutions of the problem

$$(-\Delta)^{\frac{1}{2}} \varphi = e^{U_{\delta_j, \xi_j}} \varphi \quad \text{in } \mathbb{R}.$$

We let  $PZ_{i,j} := (-\Delta)^{-\frac{1}{2}} (e^{U_{\delta_j, \xi_j}} Z_{i,j})$  be the projection of  $Z_{i,j}$  on  $X_0^{1/2}(I)$ . Then we have the following expansions.

**Lemma 4.3.** *As  $\lambda \rightarrow 0$ , we have*

$$\begin{aligned} PZ_{0,j} &= Z_{0,j} + 1 + O(\lambda^2), \\ PZ_{1,j} &= Z_{1,j} + 2\delta_j \frac{\partial H}{\partial \xi}(\xi_j, \cdot) + O(\lambda^3), \end{aligned}$$

*uniformly in  $\mathbb{R}$ , for  $j = 1, \dots, k$ . In particular,  $PZ_{0,j} = O(\lambda^2)$  and  $PZ_{1,j} = O(\lambda)$  in  $\mathbb{R} \setminus (\xi_j - \frac{\eta}{2}, \xi_j + \frac{\eta}{2})$ .*

*Proof.* First, note that for any  $x \in \mathbb{R}$  the function  $\xi \rightarrow H(\xi, x)$  belongs to  $C^1(I)$ , with derivative

$$\frac{\partial H}{\partial \xi}(\xi, x) = \begin{cases} -\frac{1}{\pi} \frac{x + \frac{\xi\sqrt{1-x^2}}{\sqrt{1-\xi^2}}}{1-x\xi + \sqrt{(1-\xi^2)(1-x^2)}} & \text{for } x \in I, \\ \frac{1}{\pi} \frac{1}{\xi-x} & \text{for } x \in \mathbb{R} \setminus I. \end{cases}$$

We claim that  $\frac{\partial H}{\partial \xi}(\xi, \cdot)$  is  $\frac{1}{2}$ -harmonic in  $I$ , for any  $\xi \in I$ . We prove that this is true in the sense of distributions. To show this, we observe that

$$\int_{\mathbb{R}} \frac{\partial H}{\partial \xi}(\xi, x) (-\Delta)^{\frac{1}{2}} \varphi(x) dx = 0 \quad \text{for all } \varphi \in C_c^\infty(I).$$

Indeed, if we take  $\psi \in C_c^\infty(-1, 1)$ , we have

$$\begin{aligned} \int_{\mathbb{R}} \psi(\xi) \int_{\mathbb{R}} \frac{\partial H}{\partial \xi}(\xi, x) (-\Delta)^{\frac{1}{2}} \varphi(x) dx d\xi &= \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} \varphi(x) \int_{\mathbb{R}} \psi(\xi) \frac{\partial H}{\partial \xi}(\xi, x) d\xi dx \\ &= - \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} \varphi(x) \int_{\mathbb{R}} \psi'(\xi) H(\xi, x) d\xi dx \\ &= - \int_{\mathbb{R}} \psi'(\xi) \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} \varphi(x) H(\xi, x) dx d\xi = 0, \end{aligned}$$

where the last equality follows from  $(-\Delta)^{\frac{1}{2}} H(\xi, x) = 0$ . Since  $\varphi$  and  $\psi$  are arbitrary, we have proved the claim.

Now, the statement can be proved as in Lemma 3.1. Let us fix  $1 \leq j \leq k$ . Since  $\frac{\partial H}{\partial \xi}(\xi_j, \cdot)$  is  $\frac{1}{2}$ -harmonic in  $I$ , the definitions of  $PZ_{0,j}$  and  $PZ_{1,j}$  imply that also the functions

$$v_{0,j} := PZ_{0,j} - Z_{0,j} - 1 \quad \text{and} \quad v_{1,j} := PZ_{1,j} - Z_{1,j} - 2\pi\delta_j \frac{\partial H}{\partial \xi}(\xi_j, \cdot)$$

are  $\frac{1}{2}$ -harmonic in  $I$ . Additionally, for  $x \in \mathbb{R} \setminus I$ , we have that

$$\begin{aligned} v_{0,j}(x) &= -\frac{\delta_j^2 - (x - \xi_j)^2}{\delta_j^2 + (x - \xi_j)^2} - 1 = O(\delta_j^2) = O(\lambda^2), \\ v_{1,j}(x) &= -\frac{2\delta_j(x - \xi_j)}{\delta_j^2 + (x - \xi_j)^2} + \frac{2\delta_j}{(x - \xi_j)} = O(\delta_j^3) = O(\lambda^3). \end{aligned}$$

Thus, we conclude via the maximum principle as in the proof of Lemma 3.1. □

**Remark 4.4.** For  $i, j \in \{0, 1\}$  and  $h, l \in \{1, \dots, k\}$ , we have the orthogonality condition

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} PZ_{i,h} \cdot (-\Delta)^{\frac{1}{4}} PZ_{j,l} dx = \int_{\mathbb{R}} e^{U_{\delta_h, \xi_h}} Z_{i,h} PZ_{j,l} dx = \pi \delta_{i,j} \delta_{h,l} + O(\lambda),$$

where  $\delta_{i,j}$  denotes the Kronecker delta symbol. Indeed, for  $h \neq l$  we have  $PZ_{j,l} = O(\lambda)$  in  $\mathbb{R} \setminus (\xi_l - \frac{\eta}{2}, \xi_l + \frac{\eta}{2})$  and  $e^{U_{\delta_h, \xi_h}} = O(\lambda)$  in  $(\xi_l - \frac{\eta}{2}, \xi_l + \frac{\eta}{2})$ , while for  $h = l$ , we have

$$\int_{\mathbb{R}} e^{U_{\delta_i, \xi_i}} Z_{i,h} PZ_{j,l} dx = \int_{\mathbb{R}} e^{U_{\delta_i, \xi_i}} Z_{i,h} Z_{j,l} dx + O(\lambda) = \int_{\mathbb{R}} \frac{2Z_i(y)Z_j(y)}{1+y^2} dy + O(\lambda) = \pi \delta_{i,j} + O(\lambda).$$

A standard procedure consists in inverting the operator  $L$  on the orthogonal of the space generated by the functions  $PZ_{i,j}$ ,  $i = 0, 1$ ,  $j = 1, \dots, k$ , which can be considered as an approximate kernel for  $L$ . However, Lemma 4.3 shows that  $PZ_{0,j}$  is not close to  $Z_{0,j}$ , as their difference approaches 1 as  $\lambda \rightarrow 0$ . For this reason we can construct a smaller approximate kernel for  $L$  using only the functions  $PZ_{1,j}$ ,  $j = 1, \dots, k$ .

In the following, for  $\delta = (\delta_1, \dots, \delta_k)$  defined as in (3.5) and for any  $\xi \in \mathcal{P}_{k,\eta}$ , we shall denote

$$K_{\delta,\xi} = \langle \{PZ_{1,j}, j \in \{1, \dots, k\}\} \rangle.$$

Let also  $\pi$  and  $\pi^\perp$  be the projections of  $X_0^{1/2}(I)$  respectively into  $K_{\delta,\xi}$  and  $K_{\delta,\xi}^\perp$ . We now establish the invertibility of  $L$  on  $K_{\delta,\xi}^\perp$ .

**Lemma 4.5.** *There exist  $\bar{\lambda}, C > 0$  such that*

$$\|\psi\| \leq C|\log \lambda| \|\pi^\perp L\psi\|$$

for any  $\lambda \in (0, \bar{\lambda})$ ,  $\xi \in \mathcal{P}_{k,\eta}$  and  $\psi \in K_{\delta,\xi}^\perp$ , with  $\delta$  given by (3.5).

*Proof.* We argue by contradiction. Suppose that there exist sequences  $\lambda_n \rightarrow 0$ ,  $\xi_n = (\xi_{1,n}, \dots, \xi_{k,n}) \in \mathcal{P}_{k,\eta}$ , and  $\psi_n \in K_{\delta_n,\xi_n}$  (where  $\delta_n = (\delta_{1,n}, \dots, \delta_{k,n})$  with  $\delta_{i,n} = \delta_{i,n}(\lambda_n, \xi_n)$  given by (3.5)) such that

$$\|\psi_n\| = 1 \quad \text{and} \quad |\log \lambda_n| \|h_n\| \rightarrow 0, \quad \text{where } h_n := \pi^\perp L\psi_n.$$

Throughout this proof we will write  $f_n := f_{\lambda_n}$ ,  $\omega_n := \omega_{\mathbf{a},\delta_n,\xi_n}$  and  $U_{i,n} = U_{\delta_{i,n},\xi_{i,n}}$ . For any  $i = 1, \dots, k$ , we also let  $Z_{0,i,n}$  and  $Z_{1,i,n}$  denote the functions in (4.6) with  $\xi_i = \xi_{i,n}$  and  $\delta_i = \delta_{i,n}$ .

By the definition of  $\pi^\perp$  there exists  $\zeta_n \in K_{\delta_n,\xi_n}$  such that  $L\psi_n = h_n + \zeta_n$ . This means that

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \psi_n (-\Delta)^{\frac{1}{4}} v \, dx = \int_{\mathbb{R}} f'_n(\omega_n) \psi_n v \, dx + \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} h_n (-\Delta)^{\frac{1}{4}} v \, dx + \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \zeta_n (-\Delta)^{\frac{1}{4}} v \, dx \tag{4.7}$$

for any  $v \in X_0^{1/2}(I)$ . Note that taking  $v = \psi_n \in K_{\delta_n,\xi_n}^\perp$ , one finds

$$\|\psi_n\|^2 = \int_{\mathbb{R}} f'_n(\omega_n) \psi_n^2 \, dx + \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \psi_n (-\Delta)^{\frac{1}{4}} h_n \, dx = \int_{\mathbb{R}} f'_n(\omega_n) \psi_n^2 \, dx + O(\|h_n\|),$$

from which we get

$$\int_{\mathbb{R}} f'_n(\omega_n) \psi_n^2 \, dx \rightarrow 1, \tag{4.8}$$

as  $n \rightarrow \infty$ . Since  $f'_n(\omega_n)$  is bounded in  $L^1(I)$  by Lemma 4.1, Hölder’s inequality also gives

$$\int_{\mathbb{R}} f'_n(\omega_n) |\psi_n| \, dx \leq \left( \int_{\mathbb{R}} f'_n(\omega_n) \psi_n^2 \, dx \right)^{\frac{1}{2}} \left( \int_I f'_n(\omega_n) \, dx \right)^{\frac{1}{2}} = O(1). \tag{4.9}$$

Keeping in mind the relations above, we split the rest of the proof into several steps.

**Step 1.** Since  $\zeta_n \in K_{\delta_n,\xi_n}$ , we can write  $\zeta_n = \sum_{i=1}^k c_{i,n} PZ_{1,i,n}$ . We have  $c_{i,n} = O(\|\zeta_n\|)$  for  $i = 1, \dots, k$ . In particular,  $\|\zeta_n\|_{L^\infty(\mathbb{R})} = O(\|\zeta_n\|)$ .

Indeed, setting  $\bar{c}_n = \max\{|c_{i,n}| : 0 \leq i \leq k\}$  and using Remark 4.4, we find that

$$\|\zeta_n\|^2 = \sum_{i=1}^k \sum_{j=1}^k c_{i,n} c_{j,n} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} PZ_{1,i,n} (-\Delta)^{\frac{1}{4}} PZ_{1,j,n} \, dx = \pi \sum_{i=1}^k c_{i,n}^2 + O(\lambda_n \bar{c}_n^2) \geq \pi \bar{c}_n^2 + O(\lambda_n \bar{c}_n^2).$$

Since  $\lambda_n \rightarrow 0$ , this implies that  $\bar{c}_n = O(\|\zeta_n\|)$ .

**Step 2.** For  $i = 1, \dots, k$ , and  $s = 1, 2$ , we have that

$$\int_{\xi_{i,n} - \frac{\eta}{2}}^{\xi_{i,n} + \frac{\eta}{2}} |f'_n(\omega_n) - e^{U_{i,n}}| |\psi_n|^s \, dx = O(\sqrt{\lambda_n}). \tag{4.10}$$

Moreover, we have

$$\int_{\mathbb{R}} e^{U_{i,n}} |\psi_n|^s \, dx = O(1) \quad \text{and} \quad \int_{\mathbb{R}} |f'_n(\omega_n) - e^{U_{i,n}}| |\psi_n|^s |PZ_{j,i,n}| \, dx = O(\sqrt{\lambda_n}), \quad j = 0, 1. \tag{4.11}$$

Indeed, in view of Lemma 4.1, we have

$$\int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} |f'_n(\omega_n) - e^{U_{i,n}}| |\psi_n|^s dx = \int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} e^{U_{i,n}} |\psi_n|^s O(|x - \xi_{i,n}|) dx + \int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} e^{U_{i,n}} |\psi_n|^s O(\lambda_n^2) dx. \tag{4.12}$$

By Hölder’s inequality, estimate (3.6) and Sobolev’s inequality, we get

$$\int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} e^{U_{i,n}} |x - \xi_{i,n}| |\psi_n|^s dx \leq \left( \int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} e^{2U_{i,n}} |x - \xi_{i,n}|^2 dx \right)^{\frac{1}{2}} \|\psi_n^s\|_{L^2(I)} = O(\sqrt{\lambda_n}). \tag{4.13}$$

Furthermore, using again Lemma 4.1 and (4.8)–(4.9), we find that

$$\begin{aligned} \int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} e^{U_{i,n}} |\psi_n|^s dx &= \int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} f'_n(\omega_n) |\psi_n|^s dx + O(\sqrt{\lambda_n}) + O\left(\lambda_n^2 \int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} e^{U_{i,n}} |\psi_n|^s dx\right) \\ &= O(1) + O\left(\lambda_n^2 \int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} e^{U_{i,n}} |\psi_n|^s dx\right), \end{aligned}$$

which implies that

$$\int_{\xi_{i,n}-\frac{\eta}{2}}^{\xi_{i,n}+\frac{\eta}{2}} e^{U_{i,n}} |\psi_n|^s dx = O(1). \tag{4.14}$$

Then we get (4.10) by substituting (4.13) and (4.14) in (4.12). The first estimate in (4.11) follows by (4.14) and the bound  $e^{U_{i,n}} = O(\lambda_n)$  in  $\mathbb{R} \setminus (\xi_{i,n} - \frac{\eta}{2}, \xi_{i,n} + \frac{\eta}{2})$ . Similarly, the second estimate in (4.11) is a consequence of (4.10) and the bounds  $PZ_{0,i,n} = O(\lambda_n^2)$ ,  $PZ_{1,i,n} = O(\lambda_n)$  in  $\mathbb{R} \setminus (\xi_{i,n} - \frac{\eta}{2}, \xi_{i,n} + \frac{\eta}{2})$ .

**Step 3.** We have  $\|\zeta_n\| = o(|\log \lambda_n|^{-1})$  as  $n \rightarrow \infty$ .

Taking  $v = \zeta_n$  in (4.7) and recalling that  $\zeta_n \in K_{\delta_n, \xi_n}$ ,  $\psi_n, h_n \in K_{\frac{1}{\delta_n}, \xi_n}$  we find that

$$0 = \int_{\mathbb{R}} f'_n(\omega_n) \psi_n \zeta_n dx + \|\zeta_n\|^2 = \sum_{i=1}^k c_{i,n} \int_{\mathbb{R}} f'_n(\omega_n) \psi_n PZ_{1,i,n} dx + \|\zeta_n\|^2. \tag{4.15}$$

Now, for  $i = 1, \dots, k$ , Step 2 and Lemma 4.3 give

$$\int_{\mathbb{R}} f'_n(\omega_n) \psi_n PZ_{1,i,n} dx = \int_{\mathbb{R}} e^{U_{i,n}} PZ_{1,i,n} \psi_n dx + O(\sqrt{\lambda_n}) = \underbrace{\int_{\mathbb{R}} \psi_n (-\Delta)^{\frac{1}{2}} PZ_{1,i,n} dx}_{= 0 \text{ by } \psi_n \in K_{\frac{1}{\delta_n}, \xi_n}} + O(\sqrt{\lambda_n}).$$

Then, using also Step 1, we can rewrite (4.15) as

$$\|\zeta_n\| = O(\sqrt{\lambda_n}) = o(|\log \lambda_n|^{-1}).$$

**Step 4.** For  $i = 1, \dots, k$ , we have that

$$\int_{\mathbb{R}} e^{U_{i,n}} \psi_n dx = o(|\log \lambda_n|^{-1}) \quad \text{and} \quad \int_{\mathbb{R}} e^{U_{i,n}} U_{i,n} \psi_n dx \rightarrow 0.$$

First of all, taking  $v = PZ_{0,i,n}$  in (4.7), and using Steps 2–3 and Lemma 4.3, we find that

$$\begin{aligned} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} PZ_{0,i,n} \cdot (-\Delta)^{\frac{1}{4}} \psi_n dx &= \int_{\mathbb{R}} f'_n(\omega_n) \psi_n PZ_{0,i,n} dx + O(\|PZ_{0,i,n}\| \|h_n\|) + O(\|PZ_{0,i,n}\| \|\zeta_n\|) \\ &= \int_{\mathbb{R}} e^{U_{i,n}} \psi_n PZ_{0,i,n} dx + o(|\log \lambda_n|^{-1}) \\ &= \int_{\mathbb{R}} e^{U_{i,n}} \psi_n Z_{0,i,n} dx + \int_{\mathbb{R}} e^{U_{i,n}} \psi_n dx + o(|\log \lambda_n|^{-1}). \end{aligned}$$

Besides, by the definition of  $PZ_{0,i,n}$ , we have

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} PZ_{0,i,n} \cdot (-\Delta)^{\frac{1}{4}} \psi_n \, dx = \int_{\mathbb{R}} e^{U_{i,n}} \psi_n Z_{0,i,n} \, dx.$$

If we combine the two estimates above, we find that

$$\int_{\mathbb{R}} e^{U_{i,n}} \psi_n \, dx = o(|\log \lambda_n|^{-1}).$$

Since  $\|U_{i,n}\|_{L^\infty(I)} = O(|\log \lambda_n|)$  (in fact  $0 \leq |x - \xi_i| \leq 2$  implies  $\frac{2\delta_{i,n}}{\delta_{i,n}^2 + 4} \leq e^{U_{i,n}} \leq \frac{2}{\delta_{i,n}}$  in  $I$ ) and  $\psi_n = 0$  in  $\mathbb{R} \setminus I$ , we get the conclusion.

**Step 5.** For  $i = 1, \dots, k$ , the function  $\Psi_{i,n} := \psi_n(\xi_{i,n} + \delta_{i,n} \cdot)$  satisfies  $\Psi_n \rightarrow 0$  in  $\mathcal{L}$ , where  $\mathcal{L}$  is defined in (4.3).

First of all, we observe that

$$\|\Psi_{i,n}\| = \|\psi_n\| = 1 \quad \text{and} \quad 2 \int_{\mathbb{R}} \frac{|\Psi_{i,n}|^2}{1 + |x|^2} \, dx = \int_{\mathbb{R}} e^{U_{i,n}} \psi_n^2 \, dx \leq C,$$

by Step 2. Then  $\Psi_{i,n}$  is uniformly bounded in the space  $\mathcal{H}$  (see (4.3)), which is compactly embedded in  $\mathcal{L}$ . Thus, there exists  $\Psi_\infty \in \mathcal{H}$  such that, up to subsequences, we have  $\Psi_{i,n} \rightharpoonup \Psi_\infty$  weakly in  $\mathcal{H}$  and  $\Psi_{i,n} \rightarrow \Psi_\infty$  in  $\mathcal{L}$  as  $n \rightarrow +\infty$ . The weak convergence in  $\mathcal{H}$  implies that

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \Psi_{i,n} \cdot (-\Delta)^{\frac{1}{4}} w \, dx \rightarrow \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \Psi_\infty \cdot (-\Delta)^{\frac{1}{4}} w \, dx$$

for any  $w \in C_c^\infty(\mathbb{R})$ . Besides, using (4.7) with  $v = v_n := w(\frac{\cdot - \xi_{i,n}}{\delta_{i,n}})$ , we get

$$\begin{aligned} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \Psi_{i,n} \cdot (-\Delta)^{\frac{1}{4}} w \, dx &= \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \psi_n \cdot (-\Delta)^{\frac{1}{4}} v_n \, dx \\ &= \int_{\mathbb{R}} f'_n(\omega_n) \psi_n v_n \, dx + \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \tilde{h}_n (-\Delta)^{\frac{1}{4}} v_n \, dx, \end{aligned}$$

where  $\tilde{h}_n = h_n + \zeta_n$ . Since  $\tilde{h}_n \rightarrow 0$  (by Step 3), we get that

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \tilde{h}_n (-\Delta)^{\frac{1}{4}} v_n \, dx \leq \|\tilde{h}_n\| \|v_n\| = \|\tilde{h}_n\| \|w\| \rightarrow 0.$$

Moreover, noting that  $v_n$  is supported in  $(\xi_{i,n} - R\delta_{i,n}, \xi_{i,n} + \delta_{i,n}R)$  for some  $R > 0$ , we have

$$\begin{aligned} \int_{\mathbb{R}} f'_n(\omega_n) \psi_n v_n \, dx &= \int_{\xi_{i,n} - R\delta_{i,n}}^{\xi_{i,n} + R\delta_{i,n}} e^{U_{i,n}} (1 + O(|x - \xi_{i,n}|) + O(\lambda_n)) \psi_n v_n \\ &= \int_{-R}^R \frac{2}{1 + y^2} (1 + O(\lambda_n(|y| + 1))) \Psi_{i,n} w \, dy \rightarrow \int_{\mathbb{R}} \frac{2}{1 + y^2} \Psi_\infty w \, dy, \end{aligned}$$

where the convergence in the last line follows by the convergence of  $\Psi_{i,n}$  in  $\mathcal{L}$ . Then it follows that  $\Psi_\infty$  is a solution in  $\mathcal{H}$  to the problem

$$(-\Delta)^{\frac{1}{2}} \Psi_\infty = \frac{2}{1 + x^2} \Psi_\infty \quad \text{in } \mathbb{R}.$$

Then, by Lemma 4.2, there exist  $\kappa_0, \kappa_1 \in \mathbb{R}$  such that  $\Psi_\infty = \kappa_0 Z_0 + \kappa_1 Z_1$ . But using again the convergence in  $\mathcal{L}$  and recalling that  $\psi_n \in K_{\delta_{i,n}, \xi_{i,n}}^\perp$ , we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} \psi_n (-\Delta)^{\frac{1}{4}} PZ_{1,i,n} \, dx = \int_{\mathbb{R}} \psi_n e^{U_{i,n}} Z_{1,i,n} \, dx \\ &= \int_{\mathbb{R}} \frac{2\Psi_{i,n} Z_1}{1 + y^2} \, dy \rightarrow \kappa_0 \int_{\mathbb{R}} \frac{2Z_0 Z_1}{1 + y^2} \, dy + \kappa_1 \int_{\mathbb{R}} \frac{2Z_1^2}{1 + y^2} \, dy = \pi\kappa_1, \end{aligned}$$

Hence,  $\kappa_1 = 0$ . Similarly, thanks to Step 4, we know that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \psi_n e^{U_{\delta_{i,n}, \xi_{i,n}}} (U_{\delta_{i,n}, \xi_{i,n}} + \log \delta_{i,n}) dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{2\Psi_{i,n}}{1+y^2} \log\left(\frac{2}{1+y^2}\right) dy \\ &= \kappa_0 \int_{\mathbb{R}} \frac{2Z_0}{1+y^2} \log\left(\frac{2}{1+y^2}\right) dy \\ &= \pi\kappa_0, \end{aligned}$$

which implies  $\kappa_0 = 0$  and  $\Psi_\infty \equiv 0$ .

**Step 6.** Conclusion of the proof.

We know that

$$1 + o(1) = \int_{\mathbb{R}} f'_n(\omega_n) \psi_n^2 dx = \sum_{i=1}^k \int_{\xi_{i,n} - \frac{\eta}{2}}^{\xi_{i,n} + \frac{\eta}{2}} e^{U_{i,n}} \psi_n^2 dx + O(\sqrt{\lambda_n})$$

by (4.8), Lemma 4.1 and Step 2. But, using Step 5, one gets

$$\int_{\xi_{i,n} - \frac{\eta}{2}}^{\xi_{i,n} + \frac{\eta}{2}} e^{U_{i,n}} \psi_n^2 dx \leq \int_{\mathbb{R}} e^{U_{i,n}} \psi_n^2 dx = \int_{\mathbb{R}} \frac{2\Psi_{i,n}^2}{1+y^2} dy \rightarrow 0$$

for any  $i = 1, \dots, k$ . This gives a contradiction. □

The a priori estimates of Lemma 4.5 imply the following invertibility property.

**Corollary 4.6.** For  $\lambda \in (0, \bar{\lambda})$  and  $\xi \in \mathcal{P}_{k,\eta}$ , the operator  $A := \pi^\perp L : K_{\delta,\xi}^\perp \rightarrow K_{\delta,\xi}^\perp$  is invertible and

$$\|A^{-1}\|_{\mathcal{L}(K_{\delta,\xi}^\perp)} = O(|\log \lambda|),$$

where  $\|F\|_{\mathcal{L}(K_{\delta,\xi}^\perp)} := \sup_{\{h \in K_{\delta,\xi}^\perp : \|h\| \leq 1\}} \|Fh\|$ .

*Proof.* By Lemma 4.5, for any  $\varphi \in K_{\delta,\xi}^\perp$ , we have  $\|\varphi\| \leq C|\log \lambda| \|A\varphi\|$ . In particular,  $A$  is injective. Since  $K_{\delta,\xi}^\perp$  is a Hilbert space, and since  $A$  is a Fredholm operator of index 0 (indeed it decomposes as the identity of  $K_{\delta,\xi}^\perp$  plus a compact operator), we can assert that  $A$  is invertible. Moreover, we have

$$\|A^{-1}\|_{\mathcal{L}(K_{\delta,\xi}^\perp)} = \sup_{\{h \in K_{\delta,\xi}^\perp : \|h\| \leq 1\}} \|A^{-1}h\| \leq C|\log \lambda|. \quad \square$$

## 5 Fix Point Argument

As we have outlined in Section 2, equation (1.1) can be reduced to the couple of non-linear problems (2.6) and (2.7). With the notation of the previous section, let us consider the operator  $A = \pi^\perp L : K_{\delta,\xi}^\perp \rightarrow K_{\delta,\xi}^\perp$ . Thanks to Corollary 4.6, we can rewrite equation (2.6) as

$$\varphi = A^{-1}(\pi^\perp(-\Delta)^{-\frac{1}{2}} E + \pi^\perp(-\Delta)^{-\frac{1}{2}} N(\varphi)).$$

We now prove that this equation admits a solution for any small  $\lambda$  and any  $\xi \in \mathcal{P}_{k,\eta}$ .

**Lemma 5.1.** Let  $p \in (1, 2)$  be fixed. Then there exist  $\kappa = \kappa(p, \eta) > 0$  and  $\lambda_0 = \lambda_0(p, \eta) > 0$  such that, for any  $\xi \in \mathcal{P}_{k,\eta}$  and  $\lambda \in (0, \lambda_0)$ , the operator

$$T(\varphi) := A^{-1}(\pi^\perp(-\Delta)^{-\frac{1}{2}} E + \pi^\perp(-\Delta)^{-\frac{1}{2}} N(\varphi))$$

has a fixed point on

$$B := \{\varphi \in K_{\delta,\xi}^\perp : \|\varphi\| \leq \kappa |\log \lambda| \lambda^{\frac{1}{p}}\}. \quad (5.1)$$

*Proof.* By Corollary 4.6, Lemma 3.2, estimate (2.2), and the fact that the projection  $\pi^\perp$  reduces the norm, we can find constants  $C_A$  and  $C_E$  such that

$$\|A^{-1}\|_{\mathcal{L}(K_{\delta,\xi}^\perp)} \leq C_A |\log \lambda| \quad \text{and} \quad \|\pi^\perp(-\Delta)^{\frac{1}{2}} E\| \leq C_E \lambda^{\frac{1}{p}} \tag{5.2}$$

for any  $\xi \in \mathcal{P}_{k,\eta}$  and any small  $\lambda$ . Note that  $C_A$  and  $C_E$  do not depend neither on  $\xi$  nor on  $\lambda$ . Similarly, for any  $s > p$ , Lemma 3.5 and estimate (2.2) imply the existence of a constant  $C_N$ , depending only on  $s, p, \eta$  such that

$$\|\pi^\perp(-\Delta)^{\frac{1}{2}}(N(\varphi_1) - N(\varphi_2))\| \leq C_N \lambda^{\frac{1}{s}-1} |\log \lambda| \|\varphi_1 - \varphi_2\| (\|\varphi_1\| + \|\varphi_2\|), \tag{5.3}$$

for any  $\xi \in \mathcal{P}_{k,\eta}$ ,  $\lambda$  small enough and any  $\varphi_1, \varphi_2 \in X_0^{1/2}(I)$  with  $\|\varphi_i\| \leq 1$  for  $i = 1, 2$ .

Let us set  $\kappa := 2C_A C_E$ . We shall prove that  $T$  is a contraction on  $B$ . First, taking  $\lambda$  small enough so that  $\kappa \lambda^{\frac{1}{p}} |\log \lambda| \leq 1$ , we get  $\|\varphi\| \leq 1$  for any  $\varphi \in B$ . Hence, (5.2) and (5.3) give

$$\begin{aligned} \|T(\varphi)\| &\leq \|A^{-1}\|_{\mathcal{L}(K_{\delta,\xi}^\perp)} \|\pi^\perp(-\Delta)^{-\frac{1}{2}} E\| + \|A^{-1}\|_{\mathcal{L}(K_{\delta,\xi}^\perp)} \|\pi^\perp(-\Delta)^{-\frac{1}{2}} N(\varphi)\| \\ &\leq C_A C_E \lambda^{\frac{1}{p}} |\log \lambda| + C_A C_N \lambda^{\frac{1}{s}-1} \|\varphi\|^2 \\ &\leq C_A C_E \lambda^{\frac{1}{p}} |\log \lambda| (1 + 4C_A^2 C_E C_N \lambda^{\frac{1}{p} + \frac{1}{s} - 1} |\log \lambda|), \end{aligned}$$

where last inequality follows from the definition of  $B$  in (5.1) and our choice of  $\kappa$ . Since  $p \in (1, 2)$ , it is enough to take  $s \in (p, \frac{p}{p-1})$  and  $\lambda$  such that  $4C_A^2 C_E C_N |\log \lambda|^2 \lambda^{\frac{1}{p} + \frac{1}{s} - 1} \leq 1$  to get  $T(\varphi) \in B$  for all  $\varphi \in B$ .

Arguing as above, we now prove that  $\|T\varphi_1 - T\varphi_2\| \leq \frac{1}{2} \|\varphi_1 - \varphi_2\|$  for all  $\varphi_1, \varphi_2 \in B$ . Indeed, thanks to (5.3), it is sufficient to choose  $\lambda$  small enough such that

$$4C_A^2 C_E C_N |\log \lambda|^2 \lambda^{\frac{1}{p} + \frac{1}{s} - 1} \leq \frac{1}{2}$$

to get

$$\begin{aligned} \|T(\varphi_1) - T(\varphi_2)\| &= \|A^{-1} \pi^\perp(-\Delta)^{-\frac{1}{2}}(N(\varphi_1) - N(\varphi_2))\| \\ &\leq C_A |\log \lambda| C_N \lambda^{\frac{1}{s}-1} \|\varphi_1 - \varphi_2\| (\|\varphi_1\| + \|\varphi_2\|) \\ &\leq \underbrace{4C_A^2 C_E C_N |\log \lambda|^2 \lambda^{\frac{1}{p} + \frac{1}{s} - 1}}_{\leq \frac{1}{2}} \|\varphi_1 - \varphi_2\|. \end{aligned}$$

Thus we have proved that  $T$  is a contraction on the ball  $B$ , so it has a unique fix point in  $B$ . □

For  $\xi \in \mathcal{P}_{k,\eta}$  and  $\lambda$  small enough, let  $\varphi_{\lambda,\xi}$  be the fix point for the operator  $T$  constructed in Lemma 5.1. By definition,  $\varphi_{\lambda,\xi}$  satisfies (2.6). Then, since  $K_{\delta,\xi}$  is spanned by  $PZ_{1,1}, \dots, PZ_{1,k}$ , as a consequence of Lemma 5.1, we get the following proposition:

**Proposition 5.2.** Fix  $p \in (1, 2)$  and let  $\lambda_0$  and  $\kappa$  be as in Lemma 5.1. Then, for any  $\lambda \in (0, \lambda_0)$  and any  $\xi \in \mathcal{P}_{k,\eta}$ , there exists a unique function  $\varphi_{\lambda,\xi} \in K_{\delta,\xi}^\perp$  such that  $\|\varphi_{\lambda,\xi}\| \leq \kappa \lambda^{\frac{1}{p}} |\log \lambda|$  and such that we can find  $k$  coefficients  $c_i = c_i(\lambda, \xi)$ ,  $i = 1, \dots, k$ , such that

$$(-\Delta)^{\frac{1}{2}} \varphi_{\lambda,\xi} - f\lambda'(\omega_{\mathbf{a},\delta,\xi}) \varphi_{\lambda,\xi} = E + N(\varphi_{\lambda,\xi}) + \sum_{i=1}^k c_i e^{U_{\delta_i,\xi_i}} Z_{1,i}. \tag{5.4}$$

Moreover, by the definition of  $E, N$  and  $L$ , we also have

$$(-\Delta)^{\frac{1}{2}}(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}) = f\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}) + \sum_{i=1}^k c_i e^{U_{\delta_i,\xi_i}} Z_{1,i}. \tag{5.5}$$

**Remark 5.3.** By testing equation (5.4) against  $PZ_{1,i}$ ,  $i = 1, \dots, k$ , we get that

$$c_i(\lambda, \xi) = - \sum_{j=1}^k b^{ij} \int_{\mathbb{R}} (f\lambda'(\omega_{\mathbf{a},\delta,\xi}) \varphi_{\lambda,\xi} + E + N(\varphi_{\lambda,\xi})) PZ_{1,j} dx,$$

where the  $b^{ij} = b^{ij}(\lambda, \xi)$  are the coefficients of the inverse of the matrix  $(b_{ij})_{1 \leq i, j \leq k}$  with

$$b_{ij} = \int_{\mathbb{R}} e^{U_{\delta_i,\xi_i}} Z_{1,i} PZ_{1,j} dx.$$

The matrix  $(b_{ij})_{1 \leq i, j \leq k}$  is symmetric and invertible by Remark 4.4.

We conclude this section by proving the regularity of  $\varphi_{\lambda,\xi}$  with respect to  $\xi$ . From now on, with some abuse of notation we will use the notation  $\lambda_0$  to refer different constants possibly smaller than the one given by Lemma 5.1 and Proposition 5.2.

**Lemma 5.4.** *For any  $\lambda \in (0, \lambda_0)$ , the map  $\xi \rightarrow \varphi_{\lambda,\xi}$  is a  $C^1$  map from  $\mathcal{P}_{k,\eta}$  into  $X_0^{1/2}(I)$ .*

*Proof.* For the study of the regularity of  $\varphi_{\lambda,\xi}$  it is important to recall that  $\pi, \pi^\perp, L$  and  $T$  depend on  $\lambda$  and  $\xi$ . For this reason, throughout this proof these operators will be denoted respectively by  $\pi_{\lambda,\xi}, \pi_{\lambda,\xi}^\perp, L_{\lambda,\xi}$  and  $T_{\lambda,\xi}$ . For a fixed  $\lambda \in (0, \lambda_0)$ , let us consider the  $C^1$  map  $G_\lambda : \mathcal{P}_{k,\eta} \times X_0^{1/2}(I) \rightarrow X_0^{1/2}(I)$  defined by

$$G_\lambda(\xi, \varphi) = \varphi + \pi_{\lambda,\xi}^\perp[\omega_{\mathbf{a},\delta,\xi} - (-\Delta)^{-\frac{1}{2}}f_\lambda(\omega_{\mathbf{a},\delta,\xi} + \pi_{\lambda,\xi}^\perp\varphi)].$$

Note that

$$\frac{\partial G_\lambda}{\partial \varphi}(\xi, \varphi)[v] = v - \pi_{\lambda,\xi}^\perp(-\Delta)^{-\frac{1}{2}}f'_\lambda(\omega_{\mathbf{a},\delta,\xi} + \pi_{\lambda,\xi}^\perp\varphi)\pi_{\lambda,\xi}^\perp v$$

for any  $v \in X_0^{1/2}(I)$ . In particular,  $\frac{\partial G_\lambda}{\partial \varphi}(\xi, \varphi)$  is a Fredholm operator of index 0 and thus, it is invertible if and only if it is injective. By definition, we have that  $G_\lambda(\xi, \varphi) = 0$  if and only if  $\varphi \in K_{\delta,\xi}^\perp$  is a fix point for  $T_{\lambda,\xi}$ . In particular,  $G_\lambda(\xi, \varphi_{\lambda,\xi}) = 0$ . Moreover,

$$\begin{aligned} \frac{\partial G_\lambda}{\partial \varphi}(\xi, \varphi_{\lambda,\xi})[v] &:= v - \pi_{\lambda,\xi}^\perp(-\Delta)^{-\frac{1}{2}}f'_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi})\pi_{\lambda,\xi}^\perp v \\ &= \pi_{\lambda,\xi} v + \pi_{\lambda,\xi}^\perp[\pi_{\lambda,\xi}^\perp v - (-\Delta)^{-\frac{1}{2}}(f'_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi})\pi_{\lambda,\xi}^\perp v)] \\ &= \pi_{\lambda,\xi} v + \pi_{\lambda,\xi}^\perp L_{\lambda,\xi}(\pi_{\lambda,\xi}^\perp v) - \pi_{\lambda,\xi}^\perp[(-\Delta)^{-\frac{1}{2}}((f'_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}) - f'_\lambda(\omega_{\mathbf{a},\delta,\xi}))\pi_{\lambda,\xi}^\perp v)]. \end{aligned}$$

For any  $p \in (1, 2)$  and  $s > p$  such that  $\frac{1}{p} + \frac{1}{s} > 1$ , Remark 3.6 gives

$$\|f'_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}) - f'_\lambda(\omega_{\mathbf{a},\delta,\xi})\|_{L^p(I)} = O(\lambda^{\frac{1-s}{s}}\|\varphi_{\lambda,\xi}\|) = O(\lambda^{\frac{1}{p} + \frac{1}{s} - 1}|\log \lambda|).$$

Hence, using Sobolev's inequality, we can find  $\alpha > 0$  such that

$$\|\pi_{\lambda,\xi}^\perp[(-\Delta)^{-\frac{1}{2}}((f'_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}) - f'_\lambda(\omega_{\mathbf{a},\delta,\xi}))\pi_{\lambda,\xi}^\perp v)]\| = O(\lambda^\alpha\|v\|).$$

Then we have

$$\begin{aligned} \left\| \frac{\partial G_\lambda}{\partial \varphi}(\xi, \varphi_{\lambda,\xi})[v] \right\| &\geq \|\pi_{\lambda,\xi} v + \pi_{\lambda,\xi}^\perp L_{\lambda,\xi}(\pi_{\lambda,\xi}^\perp v)\| + O(\lambda^\alpha\|v\|) \\ &\geq \frac{1}{\sqrt{2}}\|\pi_{\lambda,\xi} v\| + \frac{1}{\sqrt{2}}\|\pi_{\lambda,\xi}^\perp L_{\lambda,\xi}(\pi_{\lambda,\xi}^\perp v)\| + O(\lambda^\alpha\|v\|) \\ &\geq \frac{1}{\sqrt{2}}\|\pi_{\lambda,\xi} v\| + \frac{c}{\sqrt{2}}|\log \lambda|^{-1}\|\pi_{\lambda,\xi}^\perp v\| + O(\lambda^\alpha\|v\|) \\ &\geq c(\|\pi_{\lambda,\xi} v\| + \|\pi_{\lambda,\xi}^\perp v\|) + O(\lambda^\alpha\|v\|) \\ &\geq (c + O(\lambda^\alpha))\|v\|. \end{aligned}$$

This implies that  $\frac{\partial G_\lambda}{\partial \varphi}(\xi, \varphi_{\lambda,\xi})[v]$  is invertible. Then the implicit function theorem gives that  $\varphi_{\lambda,\xi}$  is of class  $C^1$ . □

## 6 Choice of the Concentration Points

Let  $\varphi_{\lambda,\xi}$  be as in Proposition 5.2. It is clear that if we find  $\xi = (\xi_1, \dots, \xi_k)$  (depending on  $\lambda$ ) such that

$$c_i(\lambda, \xi) = 0 \quad \text{for all } i = 1, \dots, k, \tag{6.1}$$

then the function  $u_\lambda := \omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}$  is solution for our initial problem (1.1). In this section, we will prove that (6.1) is satisfied when  $\xi$  is a critical point of the reduced energy functional

$$\mathfrak{F}_\lambda(\xi) := J_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}), \tag{6.2}$$

where

$$J_\lambda(u) := \frac{1}{2} \|u\|^2 - \int_I g_\lambda(u) \, dx \quad \text{with } g_\lambda(t) := \int_0^t f_\lambda(s) \, ds.$$

In order to prove this, we will need the following preliminary estimate.

**Lemma 6.1.** *Let  $F_1, \dots, F_k$  be as in (3.3). As  $\lambda \rightarrow 0$ , we have*

$$\|\omega_{\mathbf{a}, \delta, \xi}\|^2 = -4\pi k \log \lambda - 2\pi \sum_{i=1}^k F_i(\xi) + O(\lambda |\log \lambda|), \tag{6.3}$$

uniformly with respect to  $\xi \in \mathcal{P}_{k, \eta}$ .

*Proof.* By the definition of  $\omega_{\mathbf{a}, \delta, \xi}$ , in order to prove (6.3), it is sufficient to show that

$$\|PU_{\delta_i, \xi_i}\|^2 = -4\pi \log \lambda - 2\pi F_i(\xi) - 4\pi^2 a_i \sum_{j \neq i} a_j G_{\xi_j}(\xi_i) + O(\lambda |\log \lambda|), \quad \text{for } i = 1, \dots, k, \tag{6.4}$$

and

$$\langle PU_{\delta_i, \xi_i}, PU_{\delta_j, \xi_j} \rangle = 4\pi^2 G_{\xi_j}(\xi_i) + O(\lambda |\log \lambda|) \quad \text{for } i, j = 1, \dots, k \text{ with } i \neq j. \tag{6.5}$$

Let us prove (6.4) first. For  $i = 1, \dots, k$ , since  $e^{U_{\delta_i, \xi_i}} = O(\lambda)$  in  $\mathbb{R} \setminus (\xi_i - \frac{\eta}{2}, \xi_i + \frac{\eta}{2})$ ,  $PU_{\delta_i, \xi_i} = 0$  in  $\mathbb{R} \setminus I$  and, by Lemma 3.1,  $\|PU_{\delta_i, \xi_i}\|_{L^\infty(I)} = O(|\log \lambda|)$ , we have

$$\|PU_{\delta_i, \xi_i}\|^2 = \int_{\mathbb{R}} PU_{\delta_i, \xi_i} e^{U_{\delta_i, \xi_i}} \, dx = \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} PU_{\delta_i, \xi_i} e^{U_{\delta_i, \xi_i}} \, dx + O(\lambda |\log \lambda|).$$

Moreover, thanks to the estimates

$$\int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} e^{U_{\delta_i, \xi_i}} \, dx = 2\pi + O(\lambda) \quad \text{and} \quad \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} U_{\delta_i, \xi_i} e^{U_{\delta_i, \xi_i}} \, dx = -2\pi \log(2\delta_i) + O(\lambda |\log \lambda|),$$

the expansion of  $PU_{\delta_i, \xi_i}$  from Lemma 3.1 yields

$$\begin{aligned} \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} PU_{\delta_i, \xi_i} e^{U_{\delta_i, \xi_i}} \, dx &= \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} (-\log(2\delta_i) + U_{\delta_i, \xi_i} + 2\pi H(\xi_i, x)) e^{U_{\delta_i, \xi_i}} \, dx + O(\lambda^2) \\ &= -4\pi \log(2\delta_i) + 4\pi^2 H(\xi_i, \xi_i) + O(\lambda |\log \lambda|) \\ &= -4\pi \log(2\delta_i) + 2\pi F_i(\xi) - 4\pi^2 \sum_{j \neq i} G_{\xi_j}(\xi_i) + O(\lambda |\log \lambda|). \end{aligned}$$

Recalling that  $\delta_i$  is chosen as in (3.5), we have  $\log(2\delta_i) = \log \lambda + F_i(\xi)$ , and we obtain (6.4).

With similar arguments, for  $i \neq j$  we get

$$\begin{aligned} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} PU_{\delta_i, \xi_i} (-\Delta)^{\frac{1}{4}} PU_{\delta_j, \xi_j} \, dx &= \int_{\mathbb{R}} e^{U_{\delta_i, \xi_i}} PU_{\delta_j, \xi_j} \, dx \\ &= \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} e^{U_{\delta_i, \xi_i}} PU_{\delta_j, \xi_j} \, dx + O(\lambda |\log \lambda|) \\ &= 2\pi \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} e^{U_{\delta_i, \xi_i}} G_{\xi_j} \, dx + O(\lambda |\log \lambda|) \\ &= 4\pi^2 G_{\xi_j}(\xi_i) + O(\lambda |\log \lambda|), \end{aligned}$$

so that (6.5) holds. □

**Proposition 6.2.** For  $\lambda \in (0, \lambda_0)$  and  $\xi \in \mathcal{P}_{k,\eta}$ , the following conditions are equivalent:

- (i)  $c_i(\lambda, \xi) = 0$ , for  $i = 1, \dots, k$ .
- (ii)  $\nabla \mathfrak{F}_\lambda(\xi) = 0$ .

*Proof.* By the definition of  $J_\lambda$ , we have that

$$\nabla J_\lambda(u) = u - (-\Delta)^{-\frac{1}{2}} f_\lambda(\omega_{\mathbf{a},\delta,\xi})$$

for any  $u \in X_0^{1/2}(I)$ . Then, recalling that  $\varphi_\lambda$  satisfies (5.4)–(5.5), we get

$$\nabla J_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}) = \omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi} - (-\Delta)^{-\frac{1}{2}} f_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}) = \sum_{j=1}^k c_j PZ_{1,j}.$$

For  $i = 1, \dots, k$ , by the chain rule, we find

$$\frac{\partial \mathfrak{F}_\lambda}{\partial \xi_i}(\xi) = \langle \nabla J_\lambda(\omega_{\mathbf{a},\delta,\xi} + \varphi_{\lambda,\xi}), \frac{d}{d\xi_i} \omega_{\mathbf{a},\delta,\xi} + \frac{d}{d\xi_i} \varphi_{\lambda,\xi} \rangle = \sum_{j=1}^k \beta_{ij} c_j,$$

where

$$\beta_{ij} = \left\langle PZ_{1,j}, \frac{d}{d\xi_i} \omega_{\mathbf{a},\delta,\xi} \right\rangle + \left\langle PZ_{1,j}, \frac{d}{d\xi_i} \varphi_{\lambda,\xi} \right\rangle.$$

Then it suffices to show that the matrix  $(\beta_{ij})$  is invertible. Indeed, this gives

$$\nabla \mathfrak{F}_\lambda(\xi) = 0 \iff c_i(\lambda, \xi) = 0, \quad i = 1, \dots, k.$$

Let us then estimate the coefficients  $\beta_{ij}$ . First, for  $i, h = 1, \dots, k$ , we observe that

$$\begin{aligned} \frac{d}{d\xi_i} e^{U_{\delta_h, \xi_h}} &= \frac{\delta_{i,h}}{\delta_i} e^{U_{\delta_i, \xi_i}} Z_{1,i} - \frac{1}{\delta_h} e^{U_{\delta_h, \xi_h}} Z_{0,h} \frac{\partial \delta_h}{\partial \xi_i} \\ &= \frac{\delta_{i,h}}{\delta_i} e^{U_{\delta_i, \xi_i}} Z_{1,i} - e^{U_{\delta_h, \xi_h}} Z_{0,h} \frac{\partial F_h}{\partial \xi_i}(\xi), \end{aligned}$$

where  $\delta_{i,h}$  denotes the Kronecker delta and we have used that  $\delta = (\delta_1, \dots, \delta_k)$  is given by (3.5). Consequently,

$$\frac{d}{d\xi_i} \omega_{\mathbf{a},\delta,\xi} = \sum_{h=1}^k a_h \frac{d}{d\xi_i} P U_{\delta_h, \xi_h} = \frac{a_i}{\delta_i} PZ_{1,i} - \sum_{h=1}^k a_h PZ_{0,h} \frac{\partial F_h}{\partial \xi_i}(\xi).$$

Then we infer

$$\left\langle PZ_{1,j}, \frac{d\omega_{\mathbf{a},\delta,\xi}}{d\xi_i} \right\rangle = \frac{a_i}{\delta_i} \langle PZ_{1,j}, PZ_{1,i} \rangle - \sum_{h=1}^k a_h \frac{\partial F_h}{\partial \xi_i}(\xi) \langle PZ_{1,j}, PZ_{0,h} \rangle = \frac{\pi a_i}{\delta_i} \delta_{i,j} + O(1). \tag{6.6}$$

Now, for  $i, j = 1, \dots, k$ , observe that

$$\varphi_{\lambda,\xi} \in K_{\delta,\xi}^\perp \implies \langle PZ_{1,j}, \varphi_{\lambda,\xi} \rangle = 0 \implies \left\langle \frac{d}{d\xi_i} PZ_{1,j}, \varphi_{\lambda,\xi} \right\rangle + \left\langle PZ_{1,j}, \frac{d}{d\xi_i} \varphi_{\lambda,\xi} \right\rangle = 0.$$

Note further that we have the identity

$$\begin{aligned} \frac{d}{d\xi_i} e^{U_{\delta_j, \xi_j}} Z_{1,j} &= \delta_{i,j} \left( \frac{1}{\delta_i} e^{U_{\delta_i, \xi_i}} Z_{1,i}^2 - e^{2U_{\delta_i, \xi_i}} Z_{0,i} \right) - 2e^{U_{\delta_j, \xi_j}} Z_{1,j} Z_{0,j} \frac{\partial F_j(\xi)}{\partial \xi_i} \\ &= O\left( \frac{1}{\delta_i} e^{3U_{\delta_i, \xi_i}} |x - \xi_i|^2 \right) + O(e^{2U_{\delta_i, \xi_i}}) + O(e^{2U_{\delta_j, \xi_j}}), \end{aligned}$$

where we have used that  $|Z_{0,i}|, |Z_{0,j}| \leq 1$ .

Then, since  $\frac{d}{d\xi_i} PZ_{1,j} = (-\Delta)^{-\frac{1}{2}} \frac{d}{d\xi_i} e^{U_{\delta_j, \xi_j}} Z_{1,j}$ , by (2.2) and (3.6) we get that

$$\left\| \frac{d}{d\xi_i} PZ_{1,j} \right\| = O\left( \left\| \frac{d}{d\xi_i} e^{U_{\delta_j, \xi_j}} Z_{1,j} \right\|_{L^2(I)} \right) = O(\lambda^{-\frac{3}{2}}).$$

In particular, recalling that for  $p \in (1, 2)$  we have  $\|\varphi_{\lambda, \xi}\| = O(\lambda^{\frac{1}{p}} |\log \lambda|)$ , we get

$$\left\langle PZ_{1,j}, \frac{d}{d\xi_i} \varphi_{\lambda, \xi} \right\rangle = - \left\langle \frac{d}{d\xi_i} PZ_{1,j}, \varphi_{\lambda, \xi} \right\rangle = O(\lambda^{-\frac{3}{2}} \|\varphi_{\lambda, \xi}\|) = O(\lambda^{\frac{1}{p} - \frac{3}{2}} |\log \lambda|) = o(\lambda^{-1}), \tag{6.7}$$

uniformly for  $\xi \in \mathcal{P}_{k, \eta}$ . For  $\lambda$  small enough, using (3.5), (6.6) and (6.7), we conclude that the matrix  $(\beta_{ij})$  is dominant diagonal and thus invertible. This concludes the proof.  $\square$

The following lemma describes the asymptotic behavior of  $\mathfrak{F}_\lambda$  as  $\lambda \rightarrow 0$ .

**Lemma 6.3.** *We have*

$$\mathfrak{F}_\lambda(\xi) = -2\pi k \log \lambda - 2\pi k - \pi \sum_{i=1}^k F_i(\xi) + o(1),$$

where  $o(1) \rightarrow 0$  as  $\lambda \rightarrow 0$ , uniformly for  $\xi \in \mathcal{P}_{k, \eta}$ .

*Proof.* According to Lemma 6.1, we have  $\|\omega_{\mathbf{a}, \delta, \xi}\|^2 = O(|\log \lambda|)$ , so that

$$\begin{aligned} \|\omega_{\mathbf{a}, \delta, \xi} + \varphi_{\lambda, \xi}\|^2 &= \|\omega_{\mathbf{a}, \delta, \xi}\|^2 + O(\|\omega_{\mathbf{a}, \delta, \xi}\| \|\varphi_{\lambda, \xi}\|) + O(\|\varphi_{\lambda, \xi}\|^2) \\ &= \|\omega_{\mathbf{a}, \delta, \xi}\|^2 + O(\lambda^{\frac{1}{p}} |\log \lambda|^{\frac{3}{2}}). \end{aligned}$$

Noting that  $g_\lambda = f'_\lambda - 2\lambda$ , by Remark 3.6, for any  $p \in (1, 2)$  and  $s > p$  such that  $\frac{1}{s} + \frac{1}{p} > 1$ , one has

$$\|g_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \varphi_{\lambda, \xi}) - g_\lambda(\omega_{\mathbf{a}, \delta, \xi})\|_{L^1(I)} \leq C \lambda^{\frac{1-s}{s}} \|\varphi_{\lambda, \xi}\| = O(\lambda^{\frac{1}{p} + \frac{1}{s} - 1} |\log \lambda|) = o(1)$$

as  $\lambda \rightarrow 0$ . Thus

$$J_\lambda(\omega_{\mathbf{a}, \delta, \xi} + \varphi_{\lambda, \xi}) = J_\lambda(\omega_{\mathbf{a}, \delta, \xi}) + o(1) = \frac{1}{2} \|\omega_{\mathbf{a}, \delta, \xi}\|^2 - \int_I g_\lambda(\omega_{\mathbf{a}, \delta, \xi}) dx + o(1). \tag{6.8}$$

Using again that  $g_\lambda = f'_\lambda - 2\lambda$  together with Lemma 4.1 and (3.6), we find

$$\int_I g_\lambda(u) dx = \sum_{i=1}^k \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} e^{U_{\delta_i, \xi_i}} dx + O(\lambda |\log \lambda|) = 2\pi k + O(\lambda |\log \lambda|).$$

Then the conclusion follows by Lemma 6.1 and (6.8).  $\square$

The previous lemma shows that, up to constant terms that do not depend on  $\xi$ , the functional  $\mathfrak{F}_\lambda$  converges uniformly to a multiple of the function

$$\mathfrak{F}(\xi) := \frac{1}{2\pi} \sum_{i=1}^k F_i(\xi) = \sum_{i=1}^k H(\xi_i, \xi_j) + \sum_{i,j=1, i \neq j}^k a_i a_j G(\xi_i, \xi_j). \tag{6.9}$$

In the next subsection, we shall study the properties of  $\mathfrak{F}$  and exploit them to show that  $\mathfrak{F}_\lambda$  has a critical point (a local minimum) in  $\mathcal{P}_{k, \eta}$ , provided  $\eta$  is fixed small enough and the coefficients  $a_i$  have alternating sign.

### 6.1 Existence of a Critical Point

Let us now assume  $a_i = -a_{i+1}$  for all  $i \in \{1, \dots, k-1\}$ . We refer to the appendix for some considerations concerning different possible choices of the coefficients  $a_i$ . With this assumption, the function  $\mathfrak{F}$  defined in (6.9) becomes

$$\begin{aligned} \mathfrak{F}(\xi) &= \sum_{i=1}^k H(\xi_i, \xi_i) + \sum_{i,j=1, i \neq j}^k (-1)^{i+j} G(\xi_i, \xi_j) \\ &= \sum_{i=1}^k \frac{1}{\pi} \log(2(1 - \xi_i^2)) + \sum_{i,j=1, i \neq j}^k (-1)^{i+j} \frac{1}{\pi} \log\left(\frac{1 - \xi_i \xi_j + \sqrt{(1 - \xi_i^2)(1 - \xi_j^2)}}{|\xi_i - \xi_j|}\right). \end{aligned}$$

The goal of this subsection is to show that the set of maximum points for  $\mathfrak{F}$  on the set

$$\mathcal{P}_k := \{\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) : -1 < \xi_i < \xi_{i+1} < 1, i = 1, \dots, k - 1\} \tag{6.10}$$

is a non-empty compact subset of  $\mathcal{P}_k$ , independently of the value of  $k \in \mathbb{N}$ . Combining this with Lemma 6.3, we will prove that the functional  $\mathfrak{F}_\lambda$  defined in (6.2) has a critical point in  $\mathcal{P}_k$  (in fact in  $\mathcal{P}_{k,\eta}$ , if  $\eta$  is small enough). The proof of this result is inspired by [4, proof of Theorem 3.3]. We will provide some details here, since having the explicit expression for the Green function of our operator simplifies considerably many steps of the proof. For example, we easily get the following properties.

**Lemma 6.4.** *The following properties hold:*

- (i)  $H(\xi, \xi) \rightarrow -\infty$  as  $\xi \rightarrow \partial I = \{-1, 1\}$ .
- (ii) Let  $\varepsilon > 0$  there exists a constant  $c(\varepsilon)$  such that  $|H(\xi, \xi)| \leq c(\varepsilon)$  if  $\text{dist}(\xi, \partial I) \geq \varepsilon$ .
- (iii) Let  $\varepsilon > 0$  be small enough, there exist a constant  $c(\varepsilon)$  such that  $|G(x, y)| \leq c(\varepsilon)$  if  $|x - y| \geq \varepsilon$ .
- (iv) For any  $x, \xi \in I, x \neq \xi$ , we have

$$\frac{d}{dx} G(\xi, x) = -\frac{1}{\pi} \frac{\sqrt{1 - \xi^2}}{(x - \xi)\sqrt{1 - x^2}}.$$

- (v) Given any three points  $x < y < z \in I$ , we have  $G(x, z) - G(x, y) \leq 0$ .

The following lemma provides upper bounds on  $\mathfrak{F}$ .

**Lemma 6.5.** *For any  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \mathcal{P}_k$ , we have*

$$\sum_{i,j=1, i \neq j}^k (-1)^{i+j} G(\xi_i, \xi_j) \leq 0,$$

*Proof.* For any  $1 \leq i \leq k - 1$ , we set

$$G_i(\boldsymbol{\xi}) := \sum_{j=i+1}^k (-1)^{i+j} G(\xi_i, \xi_j),$$

so that

$$\sum_{i,j=1, i \neq j}^k (-1)^{i+j} G(\xi_i, \xi_j) = 2 \sum_{i=1}^{k-1} G_i(\boldsymbol{\xi}).$$

Then it is sufficient to observe that  $G_i(\boldsymbol{\xi}) \leq 0$  for any  $1 \leq i \leq k - 1$ . Indeed, if  $k - i$  is even, we have that

$$G_i(\boldsymbol{\xi}) = \sum_{j=1}^{k-i} (-1)^j G(\xi_i, \xi_{i+j}) = \sum_{j=2, j \text{ even}}^{k-i} G(\xi_i, \xi_{i+j}) - G(\xi_i, \xi_{i+j-1}) \leq 0,$$

where the last inequality follows by (v) of Lemma 6.4. If instead  $k - i$  is odd, then we have

$$G_i(\boldsymbol{\xi}) = \sum_{j=1}^{k-i} (-1)^j G(\xi_i, \xi_{i+j}) = -G(\xi_i, \xi_k) + \sum_{j=2, j \text{ even}}^{k-i-1} G(\xi_i, \xi_{i+j}) - G(\xi_i, \xi_{i+j-1}) \leq 0,$$

where we used again property (v) of Lemma 6.4 together with the inequality  $G(\xi_i, \xi_k) \geq 0$ . □

**Proposition 6.6.** *For any  $k \in \mathbb{N}$ , we have  $\mathfrak{F}(\boldsymbol{\xi}) \rightarrow -\infty$  when  $\text{dist}(\boldsymbol{\xi}, \partial \mathcal{P}_k) \rightarrow 0$ . In particular,  $\mathfrak{F}$  has a maximum point in  $\mathcal{P}_k$ . Moreover, the set  $M_{\mathfrak{F}}$  of global maxima for  $\mathfrak{F}$  in  $\mathcal{P}_k$  is compact.*

*Proof.* It suffices to show that, for any sequence  $\boldsymbol{\xi}^n = (\xi_1^n, \dots, \xi_k^n) \in \mathcal{P}_k$  with  $\text{dist}(\boldsymbol{\xi}^n, \partial \mathcal{P}_k) \rightarrow 0$  as  $n \rightarrow +\infty$ , up to extracting a subsequence, one has  $\mathfrak{F}(\boldsymbol{\xi}^n) \rightarrow -\infty$  as  $n \rightarrow +\infty$ . If there exists  $i \in \{1, \dots, k\}$  such that  $\xi_i^n \rightarrow \partial I$ , then (i) of Lemma 6.4 implies that  $H(\xi_i^n, \xi_i^n) \rightarrow -\infty$  and, thanks to Lemma 6.5,

$$\mathfrak{F}(\boldsymbol{\xi}^n) \leq \sum_{j=1}^k H(\xi_j^n, \xi_j^n) \leq H(\xi_i^n, \xi_i^n) + \frac{k-1}{\pi} \log 2 \rightarrow -\infty$$

as  $n \rightarrow +\infty$ . Thus, up to a subsequence, we may assume that there exists an  $\varepsilon > 0$  such that  $|\xi_i^n| \leq 1 - \varepsilon$ ,  $1 \leq i \leq k$ . Then  $d(\boldsymbol{\xi}^n, \partial \mathcal{P}_k) \rightarrow 0$  implies  $\xi_i^n - \xi_{i+1}^n \rightarrow 0$  for some  $1 \leq i \leq k - 1$ . Let  $i_0$  be the maximal index  $i \in \{1, \dots, k - 1\}$  such that this property holds. Then, up to extracting another subsequence, we may assume

$\varepsilon \leq \xi_{i+1} - \xi_i \leq 2$  for any  $i_0 < i \leq k - 1$ . Note that (ii) and (iii) of Lemma 6.4 give

$$\begin{aligned} \mathfrak{F}(\xi^n) &= -\frac{1}{\pi} \sum_{i \neq j} (-1)^{i+j} \log |\xi_i^n - \xi_j^n| + O(1) \\ &= -\frac{2}{\pi} \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} (-1)^j \log |\xi_i^n - \xi_{i+j}^n| + O(1) \\ &= -\frac{2}{\pi} \sum_{i=1}^{i_0} \underbrace{\sum_{j=1}^{k-i} (-1)^j \log |\xi_i^n - \xi_{i+j}^n|}_{=: \zeta_i^n} + O(1). \end{aligned}$$

If  $k - i$  is even, then we have

$$e^{\zeta_i^n} = \prod_{j=1}^{\frac{k-i}{2}} \frac{|\xi_i^n - \xi_{i+2j}^n|}{|\xi_i^n - \xi_{i+2j-1}^n|} \geq 1,$$

while if  $k - i$  is odd, we have that

$$e^{\zeta_i^n} = \frac{1}{|\xi_k^n - \xi_i^n|} \prod_{j=1}^{\frac{k-i-1}{2}} \frac{|\xi_i^n - \xi_{i+2j}^n|}{|\xi_i^n - \xi_{i+2j-1}^n|} \geq \frac{1}{2}.$$

Then all the sequences  $(\zeta_i^n)_{n \in \mathbb{N}}$ ,  $1 \leq i \leq k - 1$  are bounded from below and we obtain

$$\begin{aligned} \mathfrak{F}(\xi^n) &\leq -\frac{2}{\pi} \zeta_{i_0}^n + O(1) \\ &= -\frac{2}{\pi} \sum_{j=1}^{k-i_0} (-1)^j \log |\xi_{i_0}^n - \xi_{i_0+j}^n| + O(1) \\ &= \frac{2}{\pi} \log |\xi_{i_0}^n - \xi_{i_0+1}^n| + O(1) \rightarrow -\infty \end{aligned}$$

as  $n \rightarrow \infty$ . □

**Corollary 6.7.** *Let  $\mathfrak{F}_\lambda$  be as in (6.2). Then there exists  $\eta_0 \in (0, \frac{2}{k+1})$  such that  $\mathfrak{F}_\lambda$  has a critical point  $\xi(\lambda) \in \mathcal{P}_{k,\eta_0}$  for any small  $\lambda$ .*

*Proof.* By Proposition 6.6, we can fix  $\eta_0$  such that all the maxima of  $\mathfrak{F}$  in  $\mathcal{P}_k$  belong to  $\mathcal{P}_{k,\eta_0}$ . In particular, since  $\mathcal{P}_{k,\eta_0}$  is open, we have

$$\max_{\partial \mathcal{P}_{k,\eta_0}} \mathfrak{F} < \max_{\overline{\mathcal{P}_{k,\eta_0}}} \mathfrak{F}. \tag{6.11}$$

According to Lemma 6.3, we have that

$$S_\lambda := \frac{1}{\pi^2} (2\pi k \log \lambda + 2\pi k - \mathfrak{F}_\lambda) \rightarrow \mathfrak{F},$$

uniformly in  $\overline{\mathcal{P}_{k,\eta_0}}$  (in fact, in  $\mathcal{P}_{k,\eta}$  for any fixed  $\eta < \eta_0$ ) as  $\lambda \rightarrow 0$ . Then, by (6.11), we must have

$$\max_{\partial \mathcal{P}_{k,\eta_0}} S_\lambda < \max_{\overline{\mathcal{P}_{k,\eta_0}}} S_\lambda,$$

which implies that  $S_\lambda$  has a maximum point  $\xi(\lambda)$  in  $\mathcal{P}_{k,\eta_0}$ . In particular,  $\xi(\lambda)$  is a critical point for  $S_\lambda$  and  $\mathfrak{F}_\lambda$ . □

**Remark 6.8.** By construction, we also have that  $\text{dist}(\xi(\lambda), M_{\mathfrak{F}}) \rightarrow 0$ . In particular, for any sequence  $\lambda_n \rightarrow 0$ , we have  $\xi(\lambda_n) \rightarrow \xi$  up to extracting a subsequence, where  $\xi$  is a maximum point for  $\mathfrak{F}$ .

## 7 Proof of the Main Theorems

We now collect the results of the previous sections to complete the proof of our main results.

*Proof of Theorem 1.2.* Let  $\omega_{\mathbf{a},\delta,\xi}$  be as in (2.4), with  $\delta = (\delta_1, \dots, \delta_k)$  as in (3.5). For a given  $p \in (1, 2)$ , let  $\lambda_0$  and  $\varphi_{\lambda,\xi}$  be as in Proposition 5.2 and let  $\mathfrak{F}_\lambda$  be as in (6.2). By Corollary 6.7, there exists  $\eta_0$  small

enough such that  $\mathfrak{F}_\lambda$  has a critical point in  $\mathcal{P}_{k,\eta_0}$ . By Propositions 6.2 and 5.2, setting  $\varphi_\lambda := \varphi_{\lambda,\xi(\lambda)}$  and  $\mathbf{\delta}(\lambda) := (\delta_1(\lambda, \xi(\lambda)), \dots, \delta_k(\lambda, \xi(\lambda)))$ , we get that  $u_\lambda := \omega_{\mathbf{a},\mathbf{\delta}(\lambda),\xi(\lambda)} + \varphi_\lambda$  is a solution of equation (1.1), as claimed in Theorem 1.2. Proposition 5.2 also gives  $\|\varphi_\lambda\| = O(\lambda^{\frac{1}{p}} |\log \lambda|) \rightarrow 0$  as  $\lambda \rightarrow 0$ . It remains to prove that  $\|\varphi_\lambda\|_{L^\infty(I)} \rightarrow 0$ . Let us recall that  $\varphi_\lambda \in X_0^{1/2}(I)$  is a weak solution to

$$(-\Delta)^{\frac{1}{2}} \varphi_\lambda = f_\lambda'(\omega_{\mathbf{a},\mathbf{\delta}(\lambda),\xi(\lambda)})\varphi_\lambda + E + N(\varphi_\lambda)$$

in  $I$ . Thanks to Lemma 4.1, we have

$$\int_I (f_\lambda'(\omega_{\mathbf{a},\mathbf{\delta}(\lambda),\xi(\lambda)})|\varphi_\lambda|)^p dx = \sum_{i=1}^k \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} O(e^{pU_{\delta_i,\xi_i}})|\varphi_\lambda|^p dx + o(1)$$

as  $\lambda \rightarrow 0$ . But by Hölder’s inequality (with any  $q > 1$ ) and (3.6), we get that

$$\begin{aligned} \int_{\xi_i - \frac{\eta}{2}}^{\xi_i + \frac{\eta}{2}} (e^{U_{\delta_i,\xi_i}}|\varphi_\lambda|)^p dx &\leq \left( \int_{\xi_i - \frac{\eta_0}{2}}^{\xi_i + \frac{\eta_0}{2}} e^{pqU_{\delta_i,\xi_i}} dx \right)^{\frac{1}{q}} \left( \int_I |\varphi_\lambda|^{\frac{pq}{q-1}} dx \right)^{\frac{q-1}{q}} \\ &= (O(\lambda^{1-qp}))^{\frac{1}{q}} O(\|\varphi_\lambda\|^p) \\ &= O(\lambda^{\frac{1}{q}+1-p} |\log \lambda|^p). \end{aligned}$$

Since  $p < 2$ , we can take  $q > 1$  such that  $\frac{1}{q} + 1 - p > 0$ , so that  $\|f_\lambda'(\omega_{\mathbf{a},\mathbf{\delta}(\lambda),\xi(\lambda)})\varphi_\lambda\|_{L^p(I)} \rightarrow 0$  as  $\lambda \rightarrow 0$ . In addition, Lemma 3.2 and Lemma 3.5 give  $\|E\|_{L^p(I)} \rightarrow 0$  and  $\|N(\varphi_\lambda)\|_{L^p(I)} \rightarrow 0$  as  $\lambda \rightarrow 0$ . Thus  $(-\Delta)^{\frac{1}{2}} \varphi_\lambda \rightarrow 0$  in  $L^p(I)$  and elliptic estimates (see [22, Theorem 13]) imply  $\|\varphi_\lambda\|_{L^\infty(I)} \rightarrow 0$ , as desired.  $\square$

We now turn to the proof of Theorem 1.1. From now on, we let  $u_\lambda$  be the solution constructed in Theorem 1.2. Note that  $u_\lambda$  has the form

$$u_\lambda := \sum_{i=1}^k (-1)^{i-1} P U_{\delta_i(\lambda),\xi_i(\lambda)} + \varphi_\lambda,$$

with  $\xi(\lambda) = (\xi_1(\lambda), \dots, \xi_k(\lambda)) \in \mathcal{P}_{k,\eta_0}$  for some small  $\eta_0$ ,  $\mathbf{\delta}(\lambda) = (\delta_1(\lambda), \dots, \delta_k(\lambda))$  such that  $\delta_i = O(\lambda)$  as  $\lambda \rightarrow 0$ , and  $\varphi_\lambda \in X_0^{1/2}(I)$  satisfying  $\|\varphi_\lambda\| + \|\varphi_\lambda\|_{L^\infty(I)} \rightarrow 0$ . Up to extracting a subsequence, we may also assume that

$$\xi_i(\lambda) \rightarrow \xi_i \quad \text{with } -1 < \xi_1 < \dots < \xi_k < 1.$$

**Lemma 7.1.** *The following properties hold:*

- $u_\lambda$  blows-up with alternating sign at  $\xi_1, \dots, \xi_k$  as  $\lambda \rightarrow 0$ , that is (1.6) holds for any small  $\varepsilon > 0$ .
- $u_\lambda \in C^\infty(I)$  and  $u_\lambda \rightarrow u_0 := \sum_{i=1}^k (-1)^{i-1} G_{\xi_i}$  in  $C^\infty_{\text{loc}}(I \setminus \{\xi_1, \dots, \xi_k\})$  as  $\lambda \rightarrow 0$ .
- For any  $\varepsilon > 0$ , and  $\alpha \in (0, \frac{1}{2})$ , we have

$$\left\| \frac{u_\lambda - u_0}{\sqrt{d}} \right\|_{C^{0,\alpha}((-1,\xi_1-\varepsilon))} + \left\| \frac{u_\lambda - u_0}{\sqrt{d}} \right\|_{C^{0,\alpha}((\xi_k+\varepsilon,1))} \rightarrow 0$$

as  $\lambda \rightarrow 0$ , where  $d(x) := 1 - |x|$  is the distance of  $x$  from  $\partial I$ .

*Proof.* In order to get the first property, it is sufficient to observe that Lemma 3.1 implies

$$P U_{\delta_i(\lambda),\xi_i(\lambda)}(\xi_i(\lambda)) \rightarrow +\infty \quad \text{and} \quad P U_{\delta_j(\lambda),\xi_j(\lambda)}(\xi_i(\lambda)) \rightarrow G_{\xi_j(\xi_i)}(\xi_i) \quad \text{for } j \neq i.$$

Since  $\|\varphi_\lambda\|_{L^\infty(I)} \rightarrow 0$  as  $\lambda \rightarrow 0$ , this gives the conclusion.

Similarly, the second property follows by the boundedness of  $u_\lambda$  in  $L^\infty_{\text{loc}}(I \setminus \{\xi_1, \dots, \xi_k\})$  and elliptic estimates for  $(-\Delta)^{\frac{1}{2}}$  (see e.g. [31]).

It remains to prove the third property. We focus first on the case  $x \in [\xi_k + \varepsilon, 1)$  and we let  $\psi$  be a smooth cut-off function such that  $\psi \equiv 0$  on  $(-\infty, \xi_k + \frac{\varepsilon}{2})$  and  $\psi \equiv 1$  on  $[\xi_k + \varepsilon, \infty)$ . By construction, we have  $u_\lambda \psi \equiv 0$  in  $\mathbb{R} \setminus (\xi_k + \frac{\varepsilon}{2}, 1)$ . Moreover, for  $x \in (\xi_k + \frac{\varepsilon}{2}, 1)$  we have

$$(-\Delta)^{\frac{1}{2}}(u_\lambda \psi) = \psi(-\Delta)^{\frac{1}{2}} u_\lambda + u_\lambda (-\Delta)^{\frac{1}{2}} \psi + \int_{\mathbb{R}} \frac{(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x - y|^2} dy. \tag{7.1}$$

Note that the last integral is well defined since  $u_\lambda \in C^{0, \frac{1}{2}}(\mathbb{R})$  and  $\psi \in C^\infty(\mathbb{R})$ . Moreover,  $\psi \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$  implies  $(-\Delta)^{\frac{1}{2}} \psi \in C^\infty(\mathbb{R})$  (see for example [34, Proposition 2.1.4]). Then, since  $u_\lambda$  is uniformly bounded in  $L^\infty_{\text{loc}}(\mathbb{R} \setminus \{\xi_1, \dots, \xi_k\})$ , and  $u_\lambda$  solves (1.1), we have that

$$\|\psi(-\Delta)^{\frac{1}{2}} u_\lambda\|_{L^\infty((\xi_k + \frac{\varepsilon}{2}, 1))} + \|u_\lambda(-\Delta)^{\frac{1}{2}} \psi\|_{L^\infty((\xi_k + \frac{\varepsilon}{2}, 1))} \leq C$$

for some  $C > 0$ , depending only on  $\varepsilon$ . Now, if  $x \in [\xi_k + 2\varepsilon, 1)$ , then

$$\begin{aligned} \left| \int_{\mathbb{R}} \frac{(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x - y|^2} dy \right| &= \left| \int_{-\infty}^{\xi_k + \varepsilon} \frac{(u_\lambda(x) - u_\lambda(y))(1 - \psi(y))}{|x - y|^2} dy \right| \\ &\leq \frac{1}{\varepsilon^2} (\|u_\lambda\|_{L^\infty(\xi_k + 2\varepsilon, 1)} + \|u_\lambda\|_{L^1(\mathbb{R})}) \leq C. \end{aligned}$$

If instead  $x \in (\xi_k + \frac{\varepsilon}{2}, \xi_k + 2\varepsilon)$ , then

$$\begin{aligned} &\int_{\mathbb{R}} \frac{(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x - y|^2} dy \\ &= \int_{x - \frac{\varepsilon}{3}}^{x + \frac{\varepsilon}{3}} \frac{(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x - y|^2} dy + O(\|\psi\|_{L^\infty(\mathbb{R})} (\|u_\lambda\|_{L^1(\mathbb{R})} + \|u_\lambda\|_{L^\infty((\xi_k + \frac{\varepsilon}{2}, 1))})), \end{aligned}$$

with

$$\left| \int_{x - \frac{\varepsilon}{3}}^{x + \frac{\varepsilon}{3}} \frac{(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x - y|^2} dy \right| \leq \frac{2\varepsilon}{3} \|u'_\lambda\|_{L^\infty(\xi_k + \frac{\varepsilon}{6}, \xi_k + \frac{2}{3}\varepsilon)} \|\psi'\|_{L^\infty(\mathbb{R})}.$$

We have so proved that the right-hand side of (7.1) is bounded in  $L^\infty((\xi_k + \frac{\varepsilon}{2}, 1))$ . Then the regularity results of Ros-Oton and Serra ([31, Theorem 1.2]) implies that

$$\left\| \frac{u_\lambda}{\sqrt{d}} \right\|_{C^{0,\beta}((\xi_k + \varepsilon, 1))} \leq \left\| \frac{u_\lambda \psi}{\sqrt{d}} \right\|_{C^{0,\beta}((\xi_k + \frac{\varepsilon}{2}, 1))} \leq C$$

for any  $\beta < \frac{1}{2}$ . In particular, we have that  $\| \frac{u_\lambda - u_0}{\sqrt{d}} \|_{C^{0,\alpha}} \rightarrow 0$  for any  $\alpha < \beta < \frac{1}{2}$ . With similar arguments, we prove an analogue convergence result in  $(-1, \xi_1 - \varepsilon)$ .  $\square$

The main step in the proof on Theorem 1.1 consists in showing that the limit profile  $u_0$  has exactly  $k - 1$  zeroes in  $I \setminus (\xi_1, \dots, \xi_k)$ . In fact, we shall prove that  $u_0$  is strictly monotone in each of the intervals  $(\xi_i, \xi_{i+1})$ . In the following it is useful to denote  $\xi_0 = -1$  and  $\xi_{k+1} = 1$ .

**Proposition 7.2.** *For any given  $k \in \mathbb{N}$  and  $\xi = (\xi_1, \dots, \xi_k)$  with  $-1 = \xi_0 < \xi_1 < \dots < \xi_k < \xi_{k+1} = 1$ , consider the function*

$$u_0 = \sum_{i=1}^k (-1)^{i-1} G_{\xi_i}.$$

Then there exists a constant  $c = c(k, \xi)$  such that, for any  $j = 0, \dots, k$ , we have

$$(-1)^j u'_0(x) \sqrt{1 - x^2} \geq c > 0 \quad \text{for } x \in (\xi_j, \xi_{j+1}).$$

*Proof.* Throughout the proof we denote  $\alpha(x) := \sqrt{1 - x^2}$ ,  $x \in I$ .

**Step 1.** *There exists  $c_1 = c_1(\xi_1, \dots, \xi_k)$  such that  $G'_{\xi_i} \alpha \geq c_1$  in  $(-1, \xi_i)$  and  $G'_{\xi_i} \alpha \leq -c_1$  in  $(\xi_i, 1)$ .*

Fix  $i \in \{1, \dots, k\}$ . According to (iv) of Lemma 6.4, we have that

$$G'_{\xi_i}(x) = -\frac{1}{\pi} \frac{\sqrt{1 - \xi_i^2}}{(x - \xi_i)\alpha(x)}.$$

In particular,

$$|G'_{\xi_i}(x)\alpha(x)| = \frac{\sqrt{1 - \xi_i^2}}{\pi|x - \xi_i|} \geq \frac{\sqrt{1 - \xi_i^2}}{2\pi} \geq \frac{\sqrt{1 - \max_{1 \leq j \leq k} |\xi_j|^2}}{2\pi},$$

where we used  $|x - \xi_i| \leq 2$ . Since  $G'_{\xi_i} > 0$  in  $(\xi_0, \xi_i)$  and  $G'_{\xi_i} < 0$  in  $(\xi_i, \xi_{k+1})$ , the inequality above gives the conclusion.

**Step 2.** Assume  $k \geq 2$  and for any  $1 \leq i \leq k - 1$  set  $g_i := G_{\xi_i} - G_{\xi_{i+1}}$ . There exists a constant  $c_2 > 0$  such that  $g'_i \alpha \geq c_2$  in  $(\xi_0, \xi_i) \cup (\xi_{i+1}, \xi_{k+1})$  and  $g'_i \alpha \leq -c_2$  in  $(\xi_i, \xi_{i+1})$  for any  $1 \leq i \leq k$ .

By Step 1, we know that  $G'_{\xi_i} \alpha \leq -c_1$  and  $G'_{\xi_{i+1}} \alpha \geq c_1$  in  $(\xi_i, \xi_{i+1})$ . This immediately gives  $g'_i \alpha \leq -2c_1$  in  $(\xi_i, \xi_{i+1})$ . Let us now assume  $x < \xi_i$  or  $x > \xi_{i+1}$ . As in Step 1, we have the explicit expression

$$g'_i(x) \geq G'_{\xi_i}(x) - G'_{\xi_{i+1}}(x) = \frac{1}{\pi} \frac{\sqrt{1 - \xi_{i+1}^2}}{(x - \xi_{i+1})\alpha(x)} - \frac{1}{\pi} \frac{\sqrt{1 - \xi_i^2}}{(x - \xi_i)\alpha(x)} = \frac{f_x(\xi_{i+1}) - f_x(\xi_i)}{\pi\alpha(x)},$$

where  $f_x(t) := \frac{\sqrt{1-t^2}}{x-t}$ . If  $x < \xi_i$  or  $x > \xi_{i+1}$ , using that  $f_x \in C^1((-1, 1) \setminus \{x\})$ , we get that

$$f_x(\xi_{i+1}) - f_x(\xi_i) = f'_x(\bar{\xi})(\xi_{i+1} - \xi_i),$$

where  $\bar{\xi}$  is a point between  $\xi_i$  and  $\xi_{i+1}$ . In particular,

$$f'_x(\bar{\xi}) = \frac{1 - x\bar{\xi}}{(x - \bar{\xi})^2 \sqrt{1 - \bar{\xi}^2}} \geq \frac{1 - |\bar{\xi}|}{(x - \bar{\xi})^2 \sqrt{1 - \bar{\xi}^2}} \geq \frac{1 - \max\{|\xi_i|, |\xi_{i+1}|\}}{(x - \bar{\xi})^2} \geq \frac{1 - M(\xi)}{4},$$

where  $M(\xi) := \max_{1 \leq j \leq k} |\xi_j| \in (0, 1)$ . We can so conclude that

$$g'_i(x)\alpha(x) \geq \frac{(1 - M(\xi))(\xi_{i+1} - \xi_i)}{4\pi} \geq \frac{(1 - M(\xi))\sigma(\xi)}{4\pi},$$

where  $\sigma(\xi) := \min_{1 \leq j \leq k} \xi_{j+1} - \xi_j > 0$ . The right-hand side is a constant depending only on  $k$  and  $\xi$ .

**Step 3.** Conclusion of the proof.

If  $k = 1$  or  $k = 2$ , the conclusion follows directly from Steps 1 and 2.

Assume  $k \geq 3$ ,  $k$  odd. For  $1 \leq i \leq k - 1$  let  $g_i$  be as in Step 2. We can write

$$u_0 = \sum_{i=1, i \text{ odd}}^{k-2} g_i + G_{\xi_k}, \tag{7.2}$$

$$u_0 = G_{\xi_1} - \sum_{i=2, i \text{ even}}^{k-1} g_i. \tag{7.3}$$

Note that if  $1 \leq i \leq k - 2$  is odd, and if  $0 \leq j \leq k - 1$  is even, the interval  $(\xi_j, \xi_{j+1})$  is contained in  $(-1, \xi_i) \cup (\xi_{i+1}, 1)$  and in  $(-1, \xi_k)$ . In particular, Steps 1 and 2 guarantee the product of  $\alpha$  with any of the functions appearing in (7.2) is increasing in  $(\xi_j, \xi_{j+1})$ . In fact, we get

$$u'_0 \alpha \geq \frac{k-1}{2} c_2 + c_1 \quad \text{for any } j \text{ even.}$$

Similarly, when  $2F \leq i \leq k - 1$  is even and  $1 \leq j \leq k$  is odd, then  $(\xi_j, \xi_{j+1})$  is contained in  $(-1, \xi_i) \cup (\xi_{i+1}, 1)$  and in  $(\xi_1, 1)$ . Therefore, (7.3) together with Steps 1 and 2 yields

$$-u'_0 \alpha \geq c_1 + \frac{k-1}{2} c_2 \quad \text{in } (\xi_j, \xi_{j+1}), \quad j \text{ odd.}$$

Finally, assume  $k$  even and  $k \geq 4$ . Then we can decompose

$$u_0 = \sum_{i=1, i \text{ odd}}^{k-1} g_i, \tag{7.4}$$

$$u_0 = G_{\xi_1} - \sum_{i=2, i \text{ even}}^{k-2} g_i - G_{\xi_k}. \tag{7.5}$$

As before, for  $0 \leq j \leq k$  even, we have  $(\xi_j, \xi_{j+1}) \subseteq (-1, \xi_i) \cup (\xi_{i+1}, 1)$  and  $g'_i \alpha \geq c_2$  in  $(\xi_j, \xi_{j+1})$  (by Step 2), for any odd  $1 \leq i \leq k - 1$ . Then (7.4) yields

$$u'_0 \alpha \geq \frac{k}{2} c_2 \quad \text{in } (\xi_j, \xi_{j+1}) \quad \text{for any } j \text{ even.}$$

If instead  $j$  is odd, one has  $(\xi_j, \xi_{j+1}) \subseteq (-1, \xi_i) \cup (\xi_{i+1}, 1)$  and  $g'_i \alpha \geq c$  in  $(\xi_j, \xi_{j+1})$  for any  $2 \leq i \leq k - 2$  even. Moreover, since  $(\xi_j, \xi_{j+1}) \subseteq (\xi_1, 1) \cap (-1, \xi_k)$ , we also get  $G'_{\xi_1} \alpha \leq -c_1$  and  $G'_{\xi_k} \alpha \geq c_1$ . Then thanks to (7.5) we find that

$$-u'_0 \alpha \geq 2c_1 + \frac{k-2}{2}c_2 \quad \text{in } (\xi_j, \xi_{j+1}) \quad \text{for any } j \text{ odd.} \quad \square$$

We can now complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Let  $u_\lambda$  be the solution constructed in Theorem 1.2. In view of Lemma 7.1, we only need to prove that, for  $\lambda$  small enough,  $u_\lambda$  has exactly  $k$  nodal regions in  $I$  or, equivalently, exactly  $k - 1$  zeroes in  $I$ . Let us fix  $\varepsilon > 0$  small enough so that

$$(-1)^{i-1}u_0(\xi_i + \varepsilon) > 0, \quad (-1)^{i-1}u_0(\xi_i - \varepsilon) > 0 \quad \text{and} \quad (\xi_i - 2\varepsilon, \xi_i + 2\varepsilon) \subseteq I \setminus \bigcup_{j \neq i} (\xi_j - \varepsilon, \xi_j + \varepsilon) \quad (7.6)$$

for  $i = 1, \dots, k$ . Let us split  $I := I_\varepsilon^1 \cup I_\varepsilon^2 \cup I_\varepsilon^3$ , where

$$I_\varepsilon^1 := (-1, \xi_1 - \varepsilon] \cup [\xi_k + \varepsilon, 1), \quad I_\varepsilon^2 := \bigcup_{i=1}^k (\xi_i - \varepsilon, \xi_i + \varepsilon), \quad I_\varepsilon^3 = I \setminus (I_\varepsilon^1 \cup I_\varepsilon^2).$$

First, we observe that  $u_\lambda$  has no zeroes in  $I_\varepsilon^1$ . Using Proposition 7.2, in  $[\xi_k + \varepsilon, 1)$ , we can write

$$(-1)^{k-1}u_0(x) = \int_x^1 (-1)^k u'_0(t) dt \geq \int_x^1 \frac{c}{\sqrt{1-t^2}} dt \geq \frac{c}{\sqrt{2}} \int_x^1 \frac{1}{\sqrt{1-t}} dt = c\sqrt{2(1-x)} = \sqrt{2d(x)},$$

where  $d(x) = 1 - |x| = \text{dist}(x, \partial I)$ . Similarly, for  $x \in (-1, \xi_1 - \varepsilon]$ , we can write

$$u_0(x) = \int_{-1}^x u'_0(t) dt \geq \int_x^1 \frac{c}{\sqrt{1-t^2}} dt \geq \frac{c}{\sqrt{2}} \int_x^1 \frac{1}{\sqrt{1+t}} dt = c\sqrt{2(1+x)} = \sqrt{2d(x)}.$$

Thanks to Lemma 7.1, we get  $|u_\lambda(x)| \geq \sqrt{d(x)}$  in  $I_\varepsilon^1$ , provided  $\lambda$  is sufficiently small. This shows that  $u_\lambda$  has no zeroes in  $I_\varepsilon^1$ .

Next, we observe that  $u_\lambda$  has no zeroes in  $I_\varepsilon^2$ . Let us fix  $1 \leq i \leq k$ . Lemma 3.1 gives that

$$\begin{aligned} u_\lambda &= (-1)^{i-1}PU_{\delta_i(\lambda), \xi_i(\lambda)} + 2\pi \sum_{j \neq i} (-1)^{j-1}G(\xi_j(\lambda), \xi_i) + O(|\cdot - \xi_i|) \\ &= (-1)^{i-1} \log\left(\frac{1}{\delta_i(\lambda)^2 + |x - \xi_i(\lambda)|^2}\right) + 2\pi(-1)^{i-1}H(\xi_i, \xi_i) + 2\pi \sum_{j \neq i} (-1)^{j-1}G(\xi_j, \xi_i) + O(\varepsilon) \end{aligned}$$

in  $(\xi_i - \varepsilon, \xi_i + \varepsilon)$  if  $\lambda$  is small enough. Moreover, we may assume that  $|\xi_i(\lambda) - \xi_i| \leq \varepsilon$  and  $\delta_i(\lambda) \leq \varepsilon$ . In particular, we have that  $|x - \xi_i(\lambda)|^2 + \delta_i(\lambda)^2 \leq 5\varepsilon^2$  for any  $x \in (\xi_i - \varepsilon, \xi_i + \varepsilon)$ . Then we get

$$|u_\lambda| \geq \log \frac{1}{5\varepsilon^2} - O(1).$$

Thus, we have  $|u_\lambda| \geq 1$  in  $(\xi_i - \varepsilon, \xi_i + \varepsilon)$  if  $\varepsilon$  is fixed small enough.

Finally, let us consider the interval  $I_\varepsilon^3$ . Note that  $I_\varepsilon^3$  has exactly  $k - 1$  connected components, namely we have

$$I_\varepsilon^3 = \bigsqcup_{i=1}^{k-1} J_{i,\varepsilon}, \quad \text{where } J_{i,\varepsilon} = [\xi_i + \varepsilon, \xi_{i+1} - \varepsilon].$$

By (7.6) and Proposition 7.2, we know that for any  $1 \leq i \leq k - 1$ , if  $\varepsilon$  is small enough, we have

$$(-1)^i u_0(\xi_i + \varepsilon) < 0, \quad (-1)^i u_0(\xi_{i+1} - \varepsilon) > 0 \quad \text{and} \quad (-1)^i u'_0 \geq c \text{ in } J_{i,\varepsilon}.$$

Since  $u_\lambda \rightarrow u_0$  in  $C^1(\bar{I}_\varepsilon^3)$  by Lemma 7.1, this implies that

$$(-1)^i u_\lambda(\xi_i + \varepsilon) < 0, \quad (-1)^i u_\lambda(\xi_{i+1} - \varepsilon) > 0 \quad \text{and} \quad (-1)^i u'_\lambda \geq c \text{ in } J_{i,\varepsilon}.$$

Then  $u_\lambda$  has exactly one zero in  $J_{i,\varepsilon}$  for any  $1 \leq i \leq k - 1$ . We can so conclude that  $u_\lambda$  has exactly  $k - 1$  zeroes in  $I_\varepsilon^3$  (and thus in  $I$ ), as claimed.  $\square$

## A Some Special Cases

In the proof of Theorem 1.2, we had to assume that the coefficients  $a_1, \dots, a_k \in \{-1, 1\}$  appearing in front of the bubbles  $PU_{\delta_i, \xi_i}$  in the expression of the approximate solution  $\omega_{\alpha, \delta, \xi}$  are sign-alternating i.e.  $a_i = -a_{i+1}$  for  $1 \leq i \leq k-1$ . This condition has been used in order to ensure the existence of a maximum point for the functional

$$\mathfrak{F}(\xi) = \sum_{i=1}^k H(\xi_i, \xi_i) + \sum_{i \neq j} a_i a_j G(\xi_i, \xi_j),$$

in the set  $\mathcal{P}_k$  defined in (6.10), as well as the validity of Proposition 6.6. It is simple to see that this strategy cannot be used for different choices of the  $a_i$ 's. In fact, if there exists  $i \in \{1, \dots, k\}$  such that  $a_i = a_{i+1}$ , then  $\mathfrak{F}$  is not bounded from above.

However, it is interesting to investigate whether one can find different kinds of critical points. Indeed, since it is possible to show that the convergence in Lemma 6.3 holds in the  $C^1$ -sense, we can construct solutions to (1.1) whenever we can find a  $C^1$ -stable critical point for  $\mathfrak{F}$ . A complete answer to this question can be given for  $k = 1$  or  $k = 2$ , since one can explicitly find all the critical points of  $\mathfrak{F}$ . In fact, we have the following:

- In the case  $k = 1$ , we have

$$\mathfrak{F}(\xi_1) = H(\xi_1, \xi_1) = \frac{1}{\pi} \log 2(1 - \xi_1^2).$$

Then  $\mathfrak{F}$  does not depend on the choice of  $a_1$  and has only one critical point at  $\xi_1 = 0$  (a non-degenerate maximum point).

- In the case  $k = 2$ , we should find critical points of

$$\begin{aligned} \mathfrak{F}(\xi_1, \xi_2) &= H(\xi_1, \xi_1) + H(\xi_2, \xi_2) + 2a_1 a_2 G(\xi_1, \xi_2) \\ &= \frac{1}{\pi} \log(1 - \xi_1^2)(1 - \xi_2^2) + \frac{2a_1 a_2}{\pi} \log \frac{1 - \xi_1 \xi_2 + \sqrt{(1 - \xi_1^2)(1 - \xi_2^2)}}{|\xi_1 - \xi_2|}. \end{aligned}$$

This leads to two possible configurations:

- If we choose  $a_1 = -a_2$ , we can easily see that  $\mathfrak{F}$  has only one critical point in  $\mathcal{P}_2$ , located at  $(\xi_1, \xi_2) = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ . This point is a non-degenerate global maximum.
- If we choose  $a_1 = a_2$ , we can easily see that  $\mathfrak{F}$  has no critical points in  $\mathcal{P}_2$ .

We conjecture that for  $k \geq 3$ , the function  $\mathfrak{F}$  has a unique critical point (the global maximum) if the coefficients  $a_i$  have alternating sign, and has no critical point otherwise.

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## References

- [1] G. Alessandrini and E. Sincich, Detecting nonlinear corrosion by electrostatic measurements, *Appl. Anal.* **85** (2006), no. 1–3, 107–128.
- [2] G. Alessandrini and E. Sincich, Solving elliptic Cauchy problems and the identification of nonlinear corrosion, *J. Comput. Appl. Math.* **198** (2007), no. 2, 307–320.
- [3] D. Bartolucci and A. Pistoia, Existence and qualitative properties of concentrating solutions for the sinh-Poisson equation, *IMA J. Appl. Math.* **72** (2007), no. 6, 706–729.
- [4] T. Bartsch, A. Pistoia and T. Weth,  $N$ -vortex equilibria for ideal fluids in bounded planar domains and new nodal solutions of the sinh-Poisson and the Lane–Emden–Fowler equations, *Comm. Math. Phys.* **297** (2010), no. 3, 653–686.
- [5] K. Bryan and M. Vogelius, Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling, *Quart. Appl. Math.* **60** (2002), no. 4, 675–694.
- [6] C. Bucur, Some observations on the Green function for the ball in the fractional Laplace framework, *Comm. Pure Appl. Math.* **15** (2016), no. 2, 657–699.
- [7] A. J. Chorin, *Vorticity and Turbulence*, Appl. Math. Sci. 103, Springer, New York, 1994.

- [8] A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, 2nd ed., Texts Appl. Math. 4, Springer, New York, 1990.
- [9] F. Da Lio and L. Martinazzi, The nonlocal Liouville-type equation in  $\mathbb{R}$  and conformal immersions of the disk with boundary singularities, *Calc. Var. Partial Differential Equations* **56** (2017), no. 5, Paper No. 152.
- [10] J. Dávila, M. del Pino, M. Musso and J. Wei, Singular limits of a two-dimensional boundary value problem arising in corrosion modelling, *Arch. Ration. Mech. Anal.* **182** (2006), no. 2, 181–221.
- [11] J. Dávila, M. del Pino and Y. Sire, Nondegeneracy of the bubble in the critical case for nonlocal equations, *Proc. Amer. Math. Soc.* **141** (2013), no. 11, 3865–3870.
- [12] M. del Pino, M. Kowalczyk and M. Musso, Singular limits in Liouville-type equations, *Calc. Var. Partial Differential Equations* **24** (2005), no. 1, 47–81.
- [13] J. Deconinck, *Current Distributions and Electrode Shape Changes in Electrochemical Systems*, Lecture Notes in Eng. 75, Springer, Berlin, 1992.
- [14] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), no. 5, 521–573.
- [15] P. Esposito, M. Grossi and A. Pistoia, On the existence of blowing-up solutions for a mean field equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), no. 2, 227–257.
- [16] A. Fiscella, R. Servadei and E. Valdinoci, Density properties for fractional Sobolev spaces, *Ann. Acad. Sci. Fenn. Math.* **40** (2015), no. 1, 235–253.
- [17] M. Grossi and A. Pistoia, Multiple blow-up phenomena for the sinh-Poisson equation, *Arch. Ration. Mech. Anal.* **209** (2013), no. 1, 287–320.
- [18] A. Hyder, Structure of conformal metrics on  $\mathbb{R}^n$  with constant  $Q$ -curvature, *Differential Integral Equations* **32** (2019), no. 7–8, 423–454.
- [19] T. Jin, A. Maalaoui, L. Martinazzi and J. Xiong, Existence and asymptotics for solutions of a non-local  $Q$ -curvature equation in dimension three, *Calc. Var. Partial Differential Equations* **52** (2015), no. 3–4, 469–488.
- [20] G. R. Joyce and D. Montgomery, Negative temperature states for a two-dimensional guiding center plasma, *J. Plasma Phys.* **10** (1973), 107–121.
- [21] O. Kavian and M. Vogelius, On the existence and “blow-up” of solutions to a two-dimensional nonlinear boundary-value problem arising in corrosion modelling, *Proc. Roy. Soc. Edinburgh Sect. A* **133** (2003), no. 1, 119–149.
- [22] T. Leonori, I. Peral, A. Primo and F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, *Discrete Contin. Dyn. Syst.* **35** (2015), no. 12, 6031–6068.
- [23] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Appl. Math. Sci. 96, Springer, New York, 1994.
- [24] L. Martinazzi, Fractional Adams–Moser–Trudinger type inequalities, *Nonlinear Anal.* **127** (2015), 263–278.
- [25] K. Medville and M. S. Vogelius, Blow-up behavior of planar harmonic functions satisfying a certain exponential Neumann boundary condition, *SIAM J. Math. Anal.* **36** (2005), no. 6, 1772–1806.
- [26] D. Montgomery and G. Joyce, Statistical mechanics of “negative temperature” states, *Phys. Fluids* **17** (1974), 1139–1145.
- [27] L. Onsager, Statistical hydrodynamics, *Nuovo Cimento (9)* **6** (1949), no. 2, 279–287.
- [28] C. D. Pagani and D. Pierotti, A three dimensional Steklov eigenvalue problem with exponential nonlinearity on the boundary, *Nonlinear Anal.* **79** (2013), 28–40.
- [29] C. D. Pagani, D. Pierotti, A. Pistoia and G. Vaira, Concentration along geodesics for a nonlinear Steklov problem arising in corrosion modeling, *SIAM J. Math. Anal.* **48** (2016), no. 2, 1085–1108.
- [30] A. Pistoia and T. Ricciardi, Sign-changing tower of bubbles for a sinh-Poisson equation with asymmetric exponents, *Discrete Contin. Dyn. Syst.* **37** (2017), no. 11, 5651–5692.
- [31] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary, *J. Math. Pures Appl. (9)* **101** (2014), no. 3, 275–302.
- [32] S. Santra, Existence and shape of the least energy solution of a fractional Laplacian, *Calc. Var. Partial Differential Equations* **58** (2019), no. 2, Paper No. 48.
- [33] R. Servadei and E. Valdinoci, Weak and viscosity solutions of the fractional Laplace equation, *Publ. Mat.* **58** (2014), no. 1, 133–154.
- [34] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007), no. 1, 67–112.
- [35] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton Lect. Anal. 1, Princeton University Press, Princeton, 2003.
- [36] M. Vogelius and J.-M. Xu, A nonlinear elliptic boundary value problem related to corrosion modeling, *Quart. Appl. Math.* **56** (1998), no. 3, 479–505.