Calculus of Variations

Multiple solutions for the nonlinear Choquard equation with even or odd nonlinearities

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Abstract

We prove existence of infinitely many solutions $u \in H_r^1(\mathbb{R}^N)$ for the nonlinear Choquard equation

$$-\Delta u + \mu u = (I_\alpha * F(u)) f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in (0, N)$, $I_\alpha(x) := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha} \frac{1}{|x|^{N-\alpha}}$, $x \in \mathbb{R}^N \setminus \{0\}$ is the Riesz potential, and F is an almost optimal subcritical nonlinearity, assumed odd or even. We analyze the two cases: μ is a fixed positive constant or μ is unknown and the L^2-norm of the solution is prescribed, i.e. $\int_{\mathbb{R}^N} |u|^2 = m > 0$. Since the presence of the nonlocality prevents to apply the classical approach, introduced by Berestycki and Lions (Arch Ration Mech Anal 82(4):347–375, 1983), we implement a new construction of multidimensional odd paths, where some estimates for the Riesz potential play an essential role, and we find a nonlocal counterpart of their multiplicity results. In particular we extend the existence results due to Moroz and Van Schaftingen (Trans Am Math Soc 367(9):6557–6579, 2015).

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1 Introduction

Given a nonlinearity $F \in C^1(\mathbb{R}, \mathbb{R})$ and set $f := F'$, we are interested to seek for multiple solutions $u \in H_r^1(\mathbb{R}^N)$ of the nonlocal equation

$$-\Delta u + \mu u = (I_\alpha * F(u)) f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $\alpha \in (0, N)$ and $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha} \frac{1}{|x|^{N-\alpha}};$$

here $H_r^1(\mathbb{R}^N)$ denotes the space of radially symmetric Sobolev functions.

We aim to analyze the two cases: μ is a fixed positive constant or μ is unknown and the mass of the solution, described by its L^2-norm, is prescribed.

In literature the semilinear Eq. (1.1) with nonlocal source has several physical motivations and it is usually called *nonlinear Choquard equation*.

In 1954 the Eq. (1.1) with $N = 3$, $\alpha = 2$ and $F(s) = \frac{1}{2}|s|^2$, that is

$$-\Delta u + \mu u = \left(\frac{1}{4\pi|x|} * |u|^2\right) u \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

was elaborated by Pekar in [40] to describe the quantum theory of a polaron at rest. Successively, in 1976 it was arisen in the work [28] suggested by Choquard on the modeling of an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one-component plasma (see also [16,17,49]).

In 1996 the Eq. (1.2) was derived by Penrose in his discussion on the self-gravitational collapse of a quantum mechanical wave-function [36,42–44] (see also [51,52]) and in that context it is referred as *Schrödinger-Newton system*.

Finally we notice that if u is a solution of (1.2), then the wave function

$$\psi(x, t) = e^{i\mu t}u(x), \quad (x, t) \in \mathbb{R}^3 \times [0, +\infty)$$

is a solitary wave of the time-dependent *Hartree equation* [20]

$$i\psi_t = -\Delta\psi - \left(\frac{1}{4\pi|x|} * |\psi|^2\right)\psi \quad \text{in } \mathbb{R}^3 \times (0, +\infty); \tag{1.3}$$

thus (1.2) represents the stationary nonlinear Hartree equation.

In literature the study of standing waves of (1.3) has been pursed in two main directions, which opened two different challenging research fields.

A first topic regards the search for solutions of (1.2) with a prescribed frequency μ and free mass, the so-called *unconstrained* problem. The second line of investigation of the problem (1.3) consists of prescribing the mass $m > 0$ of u, thus conserved by ψ in time

$$\int_{\mathbb{R}^3} |\psi(x, t)|^2 \, dx = m \quad \forall t \in [0, +\infty), \tag{1.4}$$

and letting the frequency μ to be free. Such problem is usually said *constrained*.

For the unconstrained problem, the first investigations for existence and symmetry of the solutions to (1.2) go back to the works of Lieb [29], Lions [32] and Menzala [34] and also to [7,36,49] by means of ordinary differential equations techniques. We mention also the recent papers by Lenzmann [26] and by Winter and Wei [54] about the nondegeneracy of the unique radial solution of (1.2).

Recently Moroz and Van Schaftingen considered the special model

$$-\Delta u + \mu u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

and they proved in [37, Theorem 1] that (1.5) has solutions if and only if

$$\frac{N + \alpha}{N} < p < \frac{N + \alpha}{N - 2}, \tag{1.6}$$

(see also [3,33,39]). Moreover in [37] they showed that all positive groundstates of (1.5) are radially symmetric and monotone decreasing about some point and derived the decay asymptotics at infinity of such groundstates (see [8] for $p \geq 2$). Furthermore, in [18,19,47] the authors study, for some values of p and α, least energy nodal solutions, odd with respect to a hyperplane; see also [8,14,53,55,56] for other results on sign-changing solutions with various symmetries and saddle type solutions.

Recently in [38] Moroz and Van Schaftingen considered the problem (1.1) when F is a Berestycki-Lions type function under the following general assumptions:

(F1) $F \in C^1(\mathbb{R}, \mathbb{R})$;
(F2) there exists $C > 0$ such that for every $s \in \mathbb{R}$,

$$|sf(s)| \leq C\left(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}\right);$$

(F3)

$$\lim_{s \to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = 0, \quad \lim_{s \to +\infty} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0;$$

(F4) $F(s) \not\equiv 0$, that is, there exists $s_0 \in \mathbb{R}$, $s_0 \neq 0$ such that $F(s_0) \neq 0$.

In [38, Theorem 1], they proved the existence of a ground state solution $u \in H^1(\mathbb{R}^N)$ of (1.1) and in [38, Theorem 4] they showed that, if F satisfies (F1), (F2) and, in addition, f is odd and has constant sign on $(0, \infty)$, then every ground state of (1.1) has constant sign and it is radially symmetric with respect to some point in $\mathbb{R}^N$.

To our knowledge it is still an open problem the existence of infinitely many radially symmetric solutions for the nonlinear Choquard Eq. (1.1) under the optimal assumptions (F1)–(F4) and symmetric conditions on the nonlocal source term $(I_\alpha * F(u))f(u)$. We note that this term is odd if F is even or odd. This issue requires the implementation of new ideas

since the approach due to Berestycki and Lions [5], dealing with scalar field equations with odd local nonlinearities, can not be directly adapted in presence of a nonlocal source, both if F is even or odd.

Concerning the constrained problem, we remark that it has a significant relevance in physics, not only for the quantum probability normalization, but also because the mass may also have specific meanings, such as the power supply in nonlinear optics, or the total number of atoms in Bose-Einstein condensation. Moreover, the investigation of constrained problems can give a better insight of the dynamical properties, as the orbital stability of solutions of (1.3). In a local framework the seminal contribution to the study of the constrained problem is due to Stuart [48] and Cazenave and Lions [6].

The existence of multiple radial standing wave solutions to (1.3) with prescribed L^2-norm has been faced by Lions in [31] and for the nonlinear Choquard Eq. (1.5) it has been obtained by Ye [57] (see also [11] for the planar logarithmic Choquard equation). We remark that all these multiplicity results deal with odd power nonlinearities f (see also [27] for odd powers-sum type functions). More recently, the first and the third author [12] obtained existence of a solution to

$$\begin{cases} -\Delta u + \mu u = (I_\alpha * F(u)) f(u) & \text{in } \mathbb{R}^N, \\[2mm] \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = m, \\[2mm] u \in H_r^1(\mathbb{R}^N), \end{cases} \tag{1.7}$$

assuming that F satisfies (F1), (F4) and it is L^2-subcritical, namely

(CF2) there exists $C > 0$ such that for every $s \in \mathbb{R}$,

$$|sf(s)| \le C\big(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha+2}{N}}\big);$$

(CF3)

$$\lim_{s \to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = 0, \qquad \lim_{s \to +\infty} \frac{F(s)}{|s|^{\frac{N+\alpha+2}{N}}} = 0.$$

The existence result in [12] relies on a new approach, based on a Lagrangian formulation of the problem.

As regards the problem of multiplicity, recently Bartsch et al. [2] obtained the existence of infinitely many solutions of (1.7), assuming that f is an odd function which satisfies monotonicity and Ambrosetti–Rabinowitz conditions [1]. We highlight that the restriction on odd functions is not just a matter of symmetry of the functional, but it is related also to some sign restriction on the function f. The authors in [2] rely on mountain pass and concentration-compactness arguments, together with the use of a *stretched* functional, i.e. a functional in an augmented space which takes into consideration scaling properties and the Pohozaev identity.

It remains open the challenging problem of the existence of infinitely many solutions for the constrained nonlinear Choquard Eq. (1.7) under optimal assumptions on the nonlinearity f, when monotonicity and Ambrosetti-Rabinowitz type conditions do not hold or f is not odd.

In the present paper we will give an affirmative answer to both the *unconstrained* and *constrained* problems when F satisfies the general Berestycki–Lions type assumptions (F1)–(F4) and (F1)–(CF2)–(CF3)–(F4) respectively, together with the symmetric condition

(F5) F is odd or even.

We begin to notice that despite [12], where existence is investigated, to gain multiplicity the symmetry of the function F plays a crucial role. In particular, we assume F to be *odd* or *even*, which guarantees the evenness of the energy functional associated to (1.1). We emphasize that the possibility to assume both the symmetries on F is a particular feature of the nonlocal source. Indeed, in the source-local case [5,9,22], the nonlinear term is usually assumed odd in order to get the symmetry of the functional. We mention the recent paper [15] where the existence of a single nonradial solution to (1.1) is obtained under the condition (F5).

We start our analysis by the constrained case, which appears, as usual, more delicate. By virtue of [41], radially symmetric solutions to (1.7) can be characterized as critical points of the C^1-functional $\mathcal{L} : H_r^1(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{L}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx,$$

constrained on the sphere

$$\mathcal{S}_m := \left\{ u \in H_r^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^2 \, dx = m \right\}.$$

A possible approach to problem (1.7) is to minimize $\mathcal{L}$ on the sphere $\mathcal{S}_m$, whenever the functional is here bounded. Nevertheless, in the spirit of [12], for the general class of nonlinearities related to [4,25,38], considered in this paper, we introduce a Lagrangian formulation of the nonlocal problem (1.7) and we extend a new approach introduced by Hirata and the third author [22] for the local case. One advantage of this method is that it can be suitably adapted to derive multiplicity results of normalized solutions in several different frameworks (see [9] for a fractional scalar field equation).

Namely, writing $\mathbb{R}_+ := (0, +\infty)$, a solution $(\mu, u) \in \mathbb{R}_+ \times H_r^1(\mathbb{R}^N)$ of (1.7) corresponds to a critical point of the functional $\mathcal{I}^m : \mathbb{R}_+ \times H_r^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{I}^m(\mu, u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx + \frac{\mu}{2} \left(\int_{\mathbb{R}^N} |u|^2 \, dx - m \right).$$

We seek for critical points $(\mu, u) \in \mathbb{R}_+ \times H_r^1(\mathbb{R}^N)$ of $\mathcal{I}^m$, namely weak solutions of $\partial_u \mathcal{I}^m(\mu, u) = 0$ and $\partial_\mu \mathcal{I}^m(\mu, u) = 0$.

Inspired by the Pohozaev's identity, we introduce the Pohozaev's functional $\mathcal{P} : \mathbb{R}_+ \times H_r^1(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{P}(\mu, u) := \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + N \frac{\mu}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx$$

and the *Pohozaev level set*

$$\Omega := \left\{ (\mu, u) \in \mathbb{R}_+ \times H_r^1(\mathbb{R}^N) \mid \mathcal{P}(\mu, u) > 0 \right\} \cup \left\{ (\mu, 0) \mid \mu \in \mathbb{R}_+ \right\}.$$

We note that $\{(\mu, 0) \mid \mu \in \mathbb{R}_+\} \subset int\, \Omega$ and thus

$$\partial\Omega = \left\{ (\mu, u) \in \mathbb{R}_+ \times H_r^1(\mathbb{R}^N) \mid \mathcal{P}(\mu, u) = 0, \ u \not\equiv 0 \right\},$$

where the interior and the boundary are taken with respect to the topology of $\mathbb{R}_+ \times H_r^1(\mathbb{R}^N)$. Therefore $(\mu, u) \in \partial\Omega$ if and only if $u \not\equiv 0$ satisfies the Pohozaev's identity $\mathcal{P}(\mu, u) = 0$. We recognize a mountain pass structure [1] for the functional $\mathcal{I}^m$ in $\mathbb{R}_+ \times H_r^1(\mathbb{R}^N)$, where the mountain is given by $\partial\Omega$. We call $\partial\Omega$ *a Pohozaev mountain* for $\mathcal{I}^m$. We emphasize that under assumptions (F1)-(F2), if $u \in H_r^1(\mathbb{R}^N)$ solves $\partial_u \mathcal{I}^m(\mu, u) = 0$ with $\mu \in \mathbb{R}_+$ fixed, then $\mathcal{P}(\mu, u) = 0$.

Using a variant of the Palais-Smale condition [22,23], which takes into account the Pohozaev's identity, we will prove a deformation theorem which enables us to apply minimax arguments in the product space $\mathbb{R}_+ \times H_r^1(\mathbb{R}^N)$. We will prove the existence of multiple L^2-normalized solutions detecting minimax structures in such product space.

We state our main results.

Theorem 1 *Suppose $N \geq 3$, $\alpha \in (0, N)$ and (F1)-(CF2)-(CF3)-(F4)-(F5).*

(i) *For any $k \in \mathbb{N}$ there exists $m_k \geq 0$ such that for every $m > m_k$, the problem (1.7) has at least k pairs of nontrivial, distinct, radially symmetric solutions.*

(ii) *Assume in addition an L^2-subcritical growth also at zero, i.e.*

(CF4)

$$\lim_{s \to 0} \frac{|F(s)|}{|s|^{\frac{N+\alpha+2}{N}}} = +\infty;$$

additionally, if F is odd, assume that $|F(s)|$ is non-decreasing in $[0, \delta_0]$ for some $\delta_0 > 0$.

Then $m_k = 0$ for each $k \in \mathbb{N}$, that is for any $m > 0$ the problem (1.7) has countably many pairs of solutions $(\mu_n, u_n)_{n \in \mathbb{N}}$ satisfying $\mathcal{L}(u_n) < 0$, $n \in \mathbb{N}$. Moreover we have

$$\mathcal{L}(u_n) \to 0 \quad as \ n \to +\infty.$$

We point out that a key point of our argument is the construction of multidimensional odd paths, which appears delicate in the case of nonlocal nonlinearities, especially when f is even. Differently from [9,22], the classical argument given by [5] cannot be applied directly in the context of nonlinear Choquard equations because of the presence of a nonlocal source, and we need to implement a new approach to gain the existence of an admissible odd path. To this aim we proceed by finding suitable annuli: using characteristic functions corresponding to the annuli, we construct our multidimensional odd paths. Here interactions between these characteristic functions produced by the Riesz potential play a crucial role, in particular the index α is related to the strength of interaction and the case $\alpha \in (0, 1]$ reveals to be more delicate. We use sharp estimates for the Riesz potential obtained by Thim [50] in an essential way.

We notice that Theorem 1 improves the multiplicity result found in [2] for L^2-subcritical odd nonlinearities f under stronger assumptions.

Remark 1 We point out that, for F odd, the monotonicity near the origin in (CF4) can be slightly weakened, with no change in the proof, with the following condition:

For some $\delta_0 > 0$, F has a constant sign in $(0, \delta_0]$ and

$$\sup_{s \in (0, \delta_0], \ h \in [0,1]} \frac{F(sh)}{F(s)} =: M < +\infty. \tag{1.8}$$

In particular, when $|F(s)|$ is non-decreasing, we have $M = 1$. As a nontrivial example one can consider $\beta \in (\frac{N+\alpha}{N}, \frac{N+\alpha+2}{N})$ and F oscillating near zero between $|s|^\beta$ and $2|s|^\beta$, so that $M \leq 2$; for instance the odd extension of

$$F(s) := s^\beta \left(2 + \sin(\tfrac{1}{s})\right) \quad \text{as } s \to 0^+.$$

If instead F oscillates (not strictly) between $|s|^{\beta_1}$ and $|s|^{\beta_2}$, with $\frac{N+\alpha}{N} < \beta_1 < \beta_2 < \frac{N+\alpha+2}{N}$, then $M = +\infty$; thus for instance the odd extension of

$$F(s) := s^{\beta_1}\left(1 + \sin(\tfrac{1}{s})\right) + s^{\beta_2}\left(1 - \sin(\tfrac{1}{s})\right) \quad \text{as } s \to 0^+$$

is not included in (1.8).

Remark 2 We highlight that we assume *a priori* the positivity of the Lagrange multiplier μ. As a matter of fact, this condition seems to be quite natural: one can indeed prove that if u is a minimum of $\mathcal{L}$ constrained on $\mathcal{S}_m$, and its energy is negative (that is, $\mathcal{L}(u) < 0$), then *a posteriori* the corresponding Lagrange multiplier μ is strictly positive (see [12]). This is actually the case of our paper: see Remark 7. In addition, from a physical point of view, in the study of standing waves the multiplier μ describes the frequency of the particle, and thus it is positive; moreover, this prescribed sign is characteristic also of chemical potentials in the description of ideal gases, see [30,45].

As a further byproduct of the construction of multidimensional odd paths we gain the existence of infinitely many solutions for the unconstrained problem. More precisely, defined the C^1-functional $\mathcal{J}_\mu : H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{J}_\mu(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{\mu}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx,$$

we will establish the following result.

Theorem 2 *Suppose $N \geq 3$, $\alpha \in (0, N)$ and $\mu > 0$ fixed. Assume that* (F1)–(F5) *hold. Then there exist countably many radial solutions $(u_n)_{n \in \mathbb{N}}$ of the nonlinear Choquard Eq. (1.1). Moreover we have*

$$\mathcal{J}_\mu(u_n) \to +\infty \quad as \; n \to +\infty.$$

Our multiplicity result is the counterpart of what done in [5] for the local case with odd nonlinearities and extend the existence result in [38], due to Moroz and Van Schaftingen.

Remark 3 Noticed that (F2)-(F3) are weaker than (CF2)-(CF3), we point out that in both cases we work in a subcritical setting: the exponent $2^m_\alpha := \frac{N+\alpha+2}{N}$ in condition (CF2)-(CF3) appears as a *mass* (or L^2) critical exponent for the Choquard equations, and it is strictly smaller than the *Hardy-Littlewood-Sobolev* upper-critical exponent $2^*_\alpha := \frac{N+\alpha}{N-2}$ appearing in (F2)-(F3). In both cases, as a peculiar feature of nonlocal sources (see [38]), we need also a lower-critical exponent $2^\#_\alpha := \frac{N+\alpha}{N}$. Different qualitative phenomena are related to sub and super L^2-critical cases: for instance, the sub or super L^2-criticality of the exponent influences the boundedness of the functional $\mathcal{J}$ on $\mathcal{S}_m$, and also the lifespan and the stability of the solutions in the time-dependent Choquard equation (see [6]).

The paper is organized as follows. Section 2 is dedicated to recalls and notations. In Sect. 3 we derive a Palais–Smale–Pohozaev condition and introduce an *augmented functional* which will be used in Sect. 3.3 to gain a deformation lemma. In Sect. 4 we first give some insights on the minimax geometry of the unconstrained case, building a multidimensional odd path and studying the behavior of the symmetric mountain pass values according to variable values of μ. Afterwards, we detect a mountain pass structure, built on the Pohozaev's mountain, for the constrained case. We study in addition suitable minimax values defined through the tool of the *genus*, and in Sect. 4.5 we prove the main Theorem 1. Section 4.6 is devoted for Proofs of Lemma 1 and 2, which give essential interaction estimates for non-local term. Finally in Sect. 5 we deal with the unconstrained case by proving Theorem 2.

2 Functional setting

In what follows we use the notation:

$$\|u\|_{H^1} := \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 \right) dx \right)^{1/2} \quad \text{for } u \in H^1(\mathbb{R}^N),$$

$$\|u\|_r := \left(\int_{\mathbb{R}^N} |u|^r \, dx \right)^{1/r} \quad \text{for } r \in [1, \infty) \text{ and } u \in L^r(\mathbb{R}^N),$$

$$\|u\|_\infty := \text{ess sup}_{\mathbb{R}^N} |u| \quad \text{for } u \in L^\infty(\mathbb{R}^N),$$

$$B(x_0, r) := \{ x \in \mathbb{R}^N \mid |x - x_0| < r \} \quad \text{for } x_0 \in \mathbb{R}^N \text{ and } r > 0,$$

$$D_n := \{ \xi \in \mathbb{R}^n \mid |\xi| \le 1 \} \quad \text{for } n \in \mathbb{N}^*,$$

and set

$$H_r^1(\mathbb{R}^N) := \{ u \in H^1(\mathbb{R}^N) \mid u \text{ radially symmetric} \};$$

moreover we briefly denote by q the lower-critical exponent $2_\alpha^\#$ and by p the L^2-critical exponent 2_α^m, i.e.

$$q := 2_\alpha^\# = \frac{N + \alpha}{N}, \quad p := 2_\alpha^m = \frac{N + \alpha + 2}{N}.$$

We recall the following generalized Hardy-Littlewood-Sobolev inequality [29].

Proposition 1 *Let $r, s \in (1, +\infty)$ such that $\frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}$, then the map*

$$L^r(\mathbb{R}^N) \to L^s(\mathbb{R}^N); \; f \mapsto I_\alpha * f$$

is continuous. In particular, if $r, t \in (1, +\infty)$ verify $\frac{1}{r} + \frac{1}{t} = \frac{N+\alpha}{N}$, then there exists a constant $C = C(N, \alpha, r, t) > 0$ such that

$$\left| \int_{\mathbb{R}^N} (I_\alpha * g) h \, dx \right| \le C \|g\|_r \|h\|_t$$

for all $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$.

For technical reasons we write, from now on,

$$\mu = e^\lambda \in (0, +\infty), \quad \lambda \in \mathbb{R}.$$

We consider the functional $\mathcal{I}^m : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{I}^m(\lambda, u) := \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2} \mathcal{D}(u) + \frac{e^\lambda}{2} \left(\|u\|_2^2 - m \right), \quad (\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N), \quad (2.1)$$

where we set

$$\mathcal{D}(u) := \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx.$$

Using Proposition 1 and (F1)–(F2), we notice that $\mathcal{D}$ is continuous on $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, where $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent, and thus continuous on $H_r^1(\mathbb{R}^N)$; notice that if we assume (CF2), then $\mathcal{D}$ is continuous also on $L^2(\mathbb{R}^N) \cap L^{2 + \frac{4}{N+\alpha}}(\mathbb{R}^N)$. Moreover, it is easy to see that $\mathcal{I}^m \in C^1(\mathbb{R} \times H_r^1(\mathbb{R}^N), \mathbb{R})$.

To deal with the unconstrained problem, we further define the C^1-functional $\mathcal{J} : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{J}(\lambda, u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\mathcal{D}(u) + \frac{e^\lambda}{2}\|u\|_2^2, \quad (\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N). \tag{2.2}$$

It is immediate that, for any $(\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N)$,

$$\mathcal{I}^m(\lambda, u) = \mathcal{J}(\mu, u) - \frac{e^\lambda}{2}m.$$

For a fixed $\lambda \in \mathbb{R}$, u is critical point of $\mathcal{J}(\lambda, \cdot)$ if and only if u solves

$$\begin{cases} -\Delta u + e^\lambda u = (I_\alpha * F(u))f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1_r(\mathbb{R}^N); \end{cases} \tag{2.3}$$

in this paper by *solution* we will always mean *weak solution*. If (F1)–(F2) hold, by [38, Theorems 2 and 3] we have that each solution u of (2.3) belongs to $W^{2,2}_{loc}(\mathbb{R}^N)$ and it satisfies the Pohozaev's identity

$$\frac{N-2}{2}\|\nabla u\|_2^2 + \frac{N}{2}e^\lambda\|u\|_2^2 = \frac{N+\alpha}{2}\mathcal{D}(u). \tag{2.4}$$

Inspired by this relation, we also introduce the Pohozaev's functional $\mathcal{P} : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{P}(\lambda, u) := \frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(u) + \frac{N}{2}e^\lambda\|u\|_2^2, \quad (\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N). \tag{2.5}$$

We consider the action of $\mathbb{Z}_2$ on $\mathbb{R}^n$, $n \in \mathbb{N}^*$, and on $\mathbb{R} \times H^1_r(\mathbb{R}^N)$, given by

$$\mathbb{Z}_2 \times \mathbb{R}^n \to \mathbb{R}^n; \ (\pm 1, \xi) \mapsto \pm\xi,$$
$$\mathbb{Z}_2 \times \left(\mathbb{R} \times H^1_r(\mathbb{R}^N)\right) \to \mathbb{R} \times H^1_r(\mathbb{R}^N); \ (\pm 1, \lambda, u) \mapsto (\lambda, \pm u).$$

We notice that, under the assumption (F5), $\mathcal{I}^m$, $\mathcal{J}$ and $\mathcal{P}$ are invariant under this action, i.e. they are even in u:

$$\mathcal{I}^m(\lambda, -u) = \mathcal{I}^m(\lambda, u), \quad \mathcal{J}(\lambda, -u) = \mathcal{J}(\lambda, u), \quad \mathcal{P}(\lambda, -u) = \mathcal{P}(\lambda, u).$$

In addition, we observe by the Principle of Symmetric Criticality of Palais [41] that every critical point of $\mathcal{I}^m$ (resp. $\mathcal{J}$) restricted to $\mathbb{R} \times H^1_r(\mathbb{R}^N)$ is actually a critical point of $\mathcal{I}^m$ (resp. $\mathcal{J}$) on the whole $\mathbb{R} \times H^1(\mathbb{R}^N)$. This observation justifies our restriction onto the radial setting. Finally, we denote by $P_2 : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to H^1_r(\mathbb{R}^N)$ the projection on the second component.

Remark 4 We observe that, by substituting F with $-F$, there is no loss of generality in assuming

$$F(s_0) > 0 \quad \text{for some } s_0 > 0$$

in (F4) and

$$\lim_{s \to 0^+} \frac{F(s)}{|s|^{\frac{N+\alpha+2}{N}}} = +\infty$$

in (CF4), together with F non-decreasing when it is odd. Thus, for the remaining of the paper we assume this positivity on the right-hand side of zero.

3 Palais-Smale-Pohozaev condition and deformation theory

3.1 Palais-Smale-Pohozaev condition

For every $b \in \mathbb{R}$ we set

$$K_b^m := \left\{ (\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) \mid \mathcal{I}^m(\lambda, u) = b, \ \partial_\lambda \mathcal{I}^m(\lambda, u) = 0, \ \partial_u \mathcal{I}^m(\lambda, u) = 0 \right\}.$$

As already observed, under (F1)-(F2) we have that $\mathcal{P}(\lambda, u) = 0$ for each $(\lambda, u) \in K_b^m$. We notice also that, assuming (F5), K_b^m is invariant under the following $\mathbb{Z}_2$-action, that is

$$(\lambda, u) \in K_b^m \implies (\lambda, -u) \in K_b^m.$$

Under our assumptions on F, it seems difficult to verify the standard Palais-Smale condition for the functional $\mathcal{I}^m$. Therefore we cannot recognize that K_b^m is compact.

Inspired by [13,22,23], we introduce the Palais-Smale-Pohozaev condition, which is a weaker compactness condition than the standard Palais-Smale one. Such condition takes into account the scaling properties of $\mathcal{I}^m$ through the Pohozaev's functional $\mathcal{P}$. Using this new condition we will show that K_b^m is compact when $b < 0$.

Definition 1 For $b \in \mathbb{R}$, we say that $(\lambda_n, u_n) \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ is a *Palais-Smale-Pohozaev sequence* for $\mathcal{I}^m$ at level b (shortly a $(PSP)_b$ sequence) if

$$\mathcal{I}^m(\lambda_n, u_n) \to b, \tag{3.1}$$

$$\partial_\lambda \mathcal{I}^m(\lambda_n, u_n) \to 0, \tag{3.2}$$

$$\|\partial_u \mathcal{I}^m(\lambda_n, u_n)\|_{(H_r^1(\mathbb{R}^N))^*} \to 0, \tag{3.3}$$

$$\mathcal{P}(\lambda_n, u_n) \to 0. \tag{3.4}$$

We say that $\mathcal{I}^m$ satisfies the *Palais-Smale-Pohozaev condition* at level b (shortly the $(PSP)_b$ condition) if every $(PSP)_b$ sequence has a strongly convergent subsequence in $\mathbb{R} \times H_r^1(\mathbb{R}^N)$.

We show now the following result.

Proposition 2 *Assume* (F1)-(CF2)-(CF3) *and let* $b < 0$. *Then* $\mathcal{I}^m$ *satisfies the* $(PSP)_b$ *condition.*

Proof Let $b < 0$ and let $(\lambda_n, u_n) \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ be a $(PSP)_b$ sequence, i.e. satisfying (3.1)–(3.4). First we note that by (3.2) we obtain

$$e^{\lambda_n} \left(\|u_n\|_2^2 - m \right) \to 0. \tag{3.5}$$

Step 1: λ_n is bounded from below and $\|u_n\|_2^2 \to m$ as $n \to +\infty$.

We have by (3.4), (3.1) and (3.5)

$$o(1) = \mathcal{P}(\lambda_n, u_n)$$

$$= -\frac{\alpha + 2}{2} \|\nabla u_n\|_2^2 + (N + \alpha) \left(\mathcal{I}^m(\lambda_n, u_n) - \frac{e^{\lambda_n}}{2} \left(\|u_n\|_2^2 - m \right) \right) + \frac{N}{2} e^{\lambda_n} \|u_n\|_2^2$$

$$= -\frac{\alpha + 2}{2} \|\nabla u_n\|_2^2 + (N + \alpha)(b + o(1)) + \frac{N}{2} e^{\lambda_n} m + o(1).$$

Here we used (3.5). From the above identity, we derive boundedness of λ_n from below, since $b < 0$. This result joined to (3.5) finally gives $\|u_n\|_2^2 \to m$.

Step 2: λ_n and $\|\nabla u_n\|_2^2$ are bounded.

Since, by (3.3), $\varepsilon_n := \|\partial_u \mathcal{I}^m(\lambda_n, u_n)\|_{(H_r^1(\mathbb{R}^N))^*} \to 0$, we have

$$\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) u_n dx + e^{\lambda_n} \|u_n\|_2^2 \leq \varepsilon_n \|u_n\|_{H^1}. \tag{3.6}$$

We observe that by (CF3) for $\delta > 0$ fixed, there exists $C_\delta > 0$ such that

$$|F(s)| \leq \delta |s|^p + C_\delta |s|^q$$

where we recall $p = \frac{N+\alpha+2}{N}$ and $q = \frac{N+\alpha}{N}$. Thus

$$\|F(u_n)\|_{\frac{2N}{N+\alpha}} \leq \delta \||u_n|^p\|_{\frac{2N}{N+\alpha}} + C_\delta \||u_n|^q\|_{\frac{2N}{N+\alpha}} = \delta \|u_n\|_{\frac{2Np}{N+\alpha}}^p + C_\delta \|u_n\|_2^q.$$

Therefore by (CF2), Proposition 1 and Young's inequality we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} (I_\alpha * |F(u_n)|) |f(u_n) u_n| \, dx \\
&\leq C \|F(u_n)\|_{\frac{2N}{N+\alpha}} \|f(u_n) u_n\|_{\frac{2N}{N+\alpha}} \\
&\leq C \left(\delta \|u_n\|_{\frac{2Np}{N+\alpha}}^p + C_\delta \|u_n\|_2^q \right) \cdot C' \left(\|u_n\|_{\frac{2Np}{N+\alpha}}^p + \|u_n\|_2^q \right) \\
&\leq CC'\delta \|u_n\|_{\frac{2Np}{N+\alpha}}^{2p} + CC'(\delta + C_\delta) \left(\frac{\delta}{2} \|u_n\|_{\frac{2Np}{N+\alpha}}^{2p} + \frac{1}{2\delta} \|u_n\|_2^{2q} \right) + CC'C_\delta \|u_n\|_2^{2q} \\
&\leq C''\delta \|u_n\|_{\frac{2Np}{N+\alpha}}^{2p} + C_\delta'' \|u_n\|_2^{2q}
\end{aligned}
$$

and thus, by the Gagliardo-Nirenberg inequality,

$$
\begin{aligned}
\|\nabla u_n\|_2^2 + e^{\lambda_n} \|u_n\|_2^2 &\leq \int_{\mathbb{R}^N} (I_\alpha * |F(u_n)|) |f(u_n) u_n| dx + \varepsilon_n \|u_n\|_{H^1} \\
&\leq C'''\delta \|\nabla u_n\|_2^2 \|u_n\|_2^{2(p-1)} + C_\delta'' \|u_n\|_2^{\frac{2(N+\alpha)}{N}} + \varepsilon_n \|u_n\|_{H^1}.
\end{aligned}
$$

Since by Step 1 $\|u_n\|_2^2 = m + o(1)$, we have

$$
\begin{aligned}
&(1 - C'''\delta(m + o(1))^{p-1}) \|\nabla u_n\|_2^2 + e^{\lambda_n}(m + o(1)) \\
&\leq C_\delta''(m + o(1))^{\frac{N+\alpha}{N}} + \varepsilon_n(\|\nabla u_n\|_2^2 + m + o(1))^{1/2}.
\end{aligned}
$$

For δ small enough, we have the boundedness of e^{λ_n} and $\|\nabla u_n\|_2$. Hence λ_n can not go to $+\infty$ and thus by Step 1 we infer that λ_n is bounded.

Step 3: λ_n and u_n strongly converge.

By Steps 1-2, the sequence (λ_n, u_n) is bounded in $\mathbb{R} \times H_r^1(\mathbb{R}^N)$ and thus after extracting a subsequence, denoted in the same way, we may assume that $\lambda_n \to \lambda_0$ and $u_n \rightharpoonup u_0$ weakly in $H_r^1(\mathbb{R}^N)$ for some $(\lambda_0, u_0) \in \mathbb{R} \times H_r^1(\mathbb{R}^N)$. Taking into account the assumptions (F1)–(F3) and the compact embedding of $H_r^1(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ for $r \in (2, 2^*)$, we have

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) u_0 \, dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) u_0 \, dx$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) u_n \, dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) u_0 \, dx.$$

Thus we derive that $\langle \partial_u \mathcal{I}^m(\lambda_n, u_n), u_n \rangle \to 0$ and $\langle \partial_u \mathcal{I}^m(\lambda_n, u_n), u_0 \rangle \to 0$, and hence

$$\|\nabla u_n\|_2^2 + e^{\lambda_0} \|u_n\|_2^2 \to \|\nabla u_0\|_2^2 + e^{\lambda_0} \|u_0\|_2^2$$

which implies $u_n \to u_0$ strongly in $H_r^1(\mathbb{R}^N)$.   □

As a straightforward consequence we obtain the following result.

Corollary 1 *Assume* (F1)-(CF2)-(CF3) *and let* $b < 0$. *Then* $K_b^m \cap (\mathbb{R} \times \{0\}) = \emptyset$ *and* K_b^m *is compact.*

Remark 5 We emphasize that the $(PSP)_b$ condition does not hold at level $b = 0$. Indeed we can consider a $(PSP)_0$ unbounded sequence $(\lambda_n, 0)$ with $\lambda_n \to -\infty$.

3.2 An augmented functional

Following [21,22,24] we define

$$M := \mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$$

and introduce the augmented functional $\mathcal{H}^m : M \to \mathbb{R}$

$$\mathcal{H}^m(\theta, \lambda, u) := \mathcal{I}^m(\lambda, u(e^{-\theta} \cdot)), \quad (\theta, \lambda, u) \in M. \tag{3.7}$$

By the scaling properties of $\mathcal{I}^m$ we can recognize that

$$\mathcal{H}^m(\theta, \lambda, u) = \frac{e^{(N-2)\theta}}{2} \|\nabla u\|_2^2 - \frac{e^{(N+\alpha)\theta}}{2} \mathcal{D}(u) + \frac{e^\lambda}{2} \left(e^{N\theta} \|u\|_2^2 - m \right)$$

for all $(\theta, \lambda, u) \in M$, and thus

$$\partial_\theta \mathcal{H}^m(\theta, \lambda, u) = \mathcal{P}(\lambda, u(\cdot/e^\theta)).$$

We point out that, considered the action of $\mathbb{Z}_2$ on M

$$\mathbb{Z}_2 \times M \to M; \ (\pm 1, \theta, \lambda, u) \mapsto (\theta, \lambda, \pm u)$$

and assumed (F5), it results that $\mathcal{H}^m$ is $\mathbb{Z}_2$-invariant, i.e. it is even in u:

$$\mathcal{H}^m(\theta, \lambda, -u) = \mathcal{H}^m(\theta, \lambda, u).$$

Introducing a metric on M by

$$\|(\alpha, v, h)\|_{(\theta, \lambda, u)}^2 := \left| \left(\alpha, v, \|h(e^{-\theta} \cdot)\|_{H^1} \right) \right|^2$$

for any $(\alpha, v, h) \in T_{(\theta, \lambda, u)} M \equiv \mathbb{R} \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$, we regard M as a Hilbert manifold. We also denote the dual norm on $T_{(\theta, \lambda, u)}^* M$ by $\| \cdot \|_{(\theta, \lambda, u), *}$, and observe that both $\| \cdot \|_{(\theta, \lambda, u)}$ and $\| \cdot \|_{(\theta, \lambda, u), *}$ actually depend only on θ.

Denote now

$$D := (\partial_\theta, \partial_\lambda, \partial_u)$$

the gradient with respect to all the variables; a direct computation shows that

$$D\mathcal{H}^m(\theta, \lambda, u)(\alpha, v, h)$$
$$= \mathcal{P}(\lambda, u(e^{-\theta} \cdot))\alpha + \partial_\lambda \mathcal{I}^m(\lambda, u(e^{-\theta} \cdot))v + \langle \partial_u \mathcal{I}^m(\lambda, u(e^{-\theta} \cdot)), h(e^{-\theta} \cdot) \rangle$$

for any $(\theta, \lambda, u) \in M$ and $(\alpha, v, h) \in T_{(\theta, \lambda, u)} M$, and thus we obtain

$$\|D\mathcal{H}^m(\theta, \lambda, u)\|_{(\theta, \lambda, u), *}^2$$
$$= |\mathcal{P}(\lambda, u(e^{-\theta} \cdot))|^2 + |\partial_\lambda \mathcal{I}^m(\lambda, u(e^{-\theta} \cdot))|^2 + \|\partial_u \mathcal{I}^m(\lambda, u(e^{-\theta} \cdot))\|_{(H_r^1(\mathbb{R}^N))^*}^2.$$

We furthermore define

$$\tilde{K}_b^m := \{(\theta, \lambda, u) \in M \mid \mathcal{H}^m(\theta, \lambda, u) = b, \ D\mathcal{H}^m(\theta, \lambda, u) = 0\}$$

the set of critical points of $\mathcal{H}^m$ at level b, and we deduce

$$\tilde{K}_b^m = \{(\theta, \lambda, u(e^\theta \cdot)) \mid (\lambda, u) \in K_b^m, \ \theta \in \mathbb{R}\}. \tag{3.8}$$

Finally we introduce the standard distance between two points as the infimum of the lengths of the curves connecting the two points, namely

$$\text{dist}_M\big((\theta_0, \lambda_0, h_0), (\theta_1, \lambda_1, h_1)\big)$$
$$:= \inf \left\{ \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} \, dt \mid \gamma \in C^1([0, 1], M), \ \gamma(0) = (\theta_0, \lambda_0, h_0), \gamma(1) = (\theta_1, \lambda_1, h_1) \right\}.$$

As a consequence of Proposition 2 we obtain the following.

Proposition 3 *Assume* (F1)–(CF2)–(CF3) *and let* $b < 0$. *Then* $\mathcal{H}^m$ *satisfies the following Palais–Smale-type condition* $\widetilde{(PSP)}_b$. *That is, for each sequence* $(\theta_n, \lambda_n, u_n) \subset M$ *such that*

$$\mathcal{H}^m(\theta_n, \lambda_n, u_n) \to b,$$
$$\|D\mathcal{H}^m(\theta_n, \lambda_n, u_n)\|_{(\theta_n, \lambda_n, u_n), *} \to 0$$

as $n \to +\infty$, *we have, up to a subsequence,*

$$\text{dist}_M((\theta_n, \lambda_n, u_n), \tilde{K}_b^m) \to 0.$$

Proof See [9, Proposition 4.6]. □

We notice that the $\widetilde{(PSP)}_b$ condition is different from the standard Palais-Smale condition and it ensures the compactness after a suitable scaling. We also highlight that, if $\tilde{K}_b^m \neq \emptyset$, then $\tilde{K}_b^m$ is not compact (see (3.8)).

3.3 Deformation theory

We write, for $b \in \mathbb{R}$

$$[\mathcal{I}^m \leq b] := \{(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) \mid \mathcal{I}^m(\lambda, u) \leq b\},$$
$$[\mathcal{H}^m \leq b]_M := \{(\theta, \lambda, u) \in M \mid \mathcal{H}^m(\theta, \lambda, u) \leq b\}.$$

We state the following result.

Proposition 4 *Assume* (F1)-(CF2)-(CF3). *Let* $b < 0$, *and let* $\mathcal{O}$ *be a neighborhood of* K_b^m *with respect to the standard distance of* $\mathbb{R} \times H_r^1(\mathbb{R}^N)$. *Let* $\bar{\varepsilon} > 0$, *then there exist* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\eta : [0, 1] \times (\mathbb{R} \times H_r^1(\mathbb{R}^N)) \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$ *continuous such that*

1. $\eta(0, \cdot, \cdot) = id_{\mathbb{R} \times H_r^1(\mathbb{R}^N)}$;
2. η *fixes* $[\mathcal{I}^m \leq b - \bar{\varepsilon}]$, *that is,* $\eta(t, \cdot, \cdot) = id_{[\mathcal{I}^m \leq b - \bar{\varepsilon}]}$ *for all* $t \in [0, 1]$;
3. $\mathcal{I}^m$ *is non-increasing along* η, *and in particular* $\mathcal{I}^m(\eta(t, \cdot, \cdot)) \leq \mathcal{I}^m(\cdot, \cdot)$ *for all* $t \in [0, 1]$;
4. *if* $K_b^m = \emptyset$, *then* $\eta(1, [\mathcal{I}^m \leq b + \varepsilon]) \subset [\mathcal{I}^m \leq b - \varepsilon]$;

5. if $K_b^m \neq \emptyset$, then

$$\eta(1, [\mathcal{I}^m \leq b + \varepsilon] \setminus \mathcal{O}) \subset [\mathcal{I}^m \leq b - \varepsilon]$$

and

$$\eta(1, [\mathcal{I}^m \leq b + \varepsilon]) \subset [\mathcal{I}^m \leq b - \varepsilon] \cup \mathcal{O};$$

6. if (F5) *holds, then* $\eta(t, \cdot, \cdot)$ *is* $\mathbb{Z}_2$-*equivariant, i.e. for* $\eta = (\eta_1, \eta_2)$ *we have* η_1 *even and* η_2 *odd in u.*

To prove this, we work first on the functional $\mathcal{H}$, for which we obtained a $(\widetilde{PSP})$ condition.

Proposition 5 *Assume* (F1)-(CF2)-(CF3). *Let* $b < 0$, *and let* $\tilde{\mathcal{O}}$ *be a neighborhood of* $\tilde{K}_b^m$ *with respect to* dist_M. *Let* $\bar{\varepsilon} > 0$, *then there exist* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\tilde{\eta} : [0, 1] \times M \to M$ *continuous such that*

1. $\tilde{\eta}(0, \cdot, \cdot, \cdot) = id_M$;
2. $\tilde{\eta}$ *fixes* $[\mathcal{H}^m \leq b - \bar{\varepsilon}]_M$, *that is* $\tilde{\eta}(t, \cdot, \cdot, \cdot) = id_{[\mathcal{H}^m \leq b - \bar{\varepsilon}]_M}$ *for all* $t \in [0, 1]$;
3. $\mathcal{H}^m$ *is non-increasing along* $\tilde{\eta}$, *and in particular* $\mathcal{H}^m(\tilde{\eta}(t, \cdot, \cdot, \cdot)) \leq \mathcal{H}^m(\cdot, \cdot, \cdot)$ *for all* $t \in [0, 1]$;
4. *if* $\tilde{K}_b^m = \emptyset$, *then* $\tilde{\eta}(1, [\mathcal{H}^m \leq b + \varepsilon]_M) \subset [\mathcal{H}^m \leq b - \varepsilon]_M$;
5. *if* $\tilde{K}_b^m \neq \emptyset$, *then*

$$\tilde{\eta}(1, [\mathcal{H}^m \leq b + \varepsilon]_M \setminus \tilde{\mathcal{O}}) \subset [\mathcal{H}^m \leq b - \varepsilon]_M$$

and

$$\tilde{\eta}(1, [\mathcal{H}^m \leq b + \varepsilon]_M) \subset [\mathcal{H}^m \leq b - \varepsilon]_M \cup \tilde{\mathcal{O}};$$

6. *if* (F5) *holds, then* $\tilde{\eta}(t, \cdot, \cdot)$ *is* $\mathbb{Z}_2$-*equivariant, i.e. for* $\tilde{\eta} = (\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2)$ *we have* $\tilde{\eta}_0, \tilde{\eta}_1$ *even and* $\tilde{\eta}_2$ *odd in u.*

Proof of Proposition 5 Under $(\widetilde{PSP})_b$, we observe that for any $b < 0$ there exists $\varepsilon, \delta, \nu > 0$ such that

$$\|D\mathcal{H}^m(\theta, \lambda, u)\|_{(\theta, \lambda, u), *} \geq \nu$$

for $(\theta, \lambda, u) \in M$ satisfying $\mathcal{H}^m(\theta, \lambda, u) \in [b - \varepsilon, b + \varepsilon]$ and $\mathrm{dist}_M((\theta, \lambda, u), \tilde{K}_b^m) \geq \delta$. We can prove Proposition 5 in a standard way. See e.g. [9, Theorem 7.2]. □

Proof of Proposition 4 We introduce the following notation:

$$\pi : M \to \mathbb{R} \times H_r^1(\mathbb{R}^N); \ (\theta, \lambda, u) \mapsto (\lambda, u(e^{-\theta} \cdot)),$$
$$\iota : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to M; \ (\lambda, u) \mapsto (0, \lambda, u),$$

which are a kind of rescaling projection and immersion. Observe that

$$\pi \circ \iota = id_{\mathbb{R} \times H_r^1(\mathbb{R}^N)}, \quad (\text{while } \iota \circ \pi \neq id_M),$$
$$\mathcal{H}^m \circ \iota = \mathcal{I}^m, \quad \mathcal{I}^m \circ \pi = \mathcal{H}^m,$$
$$\pi(\tilde{K}_b^m) = K_b^m.$$

By means of these operators we are able to prove that neighborhoods of $\tilde{K}_b^m$ are brought to neighborhoods of K_b^m. Moreover, for a deformation $\tilde{\eta}$ obtained in Proposition 5, define

$$\eta(t, \lambda, u) := \pi(\tilde{\eta}(t, \iota(\lambda, u))), \quad (t, \lambda, u) \in [0, 1] \times (\mathbb{R} \times H_r^1(\mathbb{R}^N)). \tag{3.9}$$

It is now a straightforward computation showing that η satisfies the requests of Proposition 4; we refer to [9,13,22,23] for details. □

We notice that the found deformation η does not generally satisfy other classical properties of deformations, such as semigroup property; see [22] for some comments.

4 Minimax methods

4.1 Construction of multidimensional odd paths

In this section we study geometry of

$$H_r^1(\mathbb{R}^N) \to \mathbb{R}; \ u \mapsto \mathcal{J}(\lambda, u)$$

for a fixed $\lambda \in \mathbb{R}$. We introduce a sequence of minimax values $a_n(\lambda)$, $n = 1, 2, \cdots$. These values play important roles to find multiple solutions for the constrained problem, i.e. for a Proof of Theorem 1, and those for the unconstrained problem, i.e. for a Proof of Theorem 2.

For $n \in \mathbb{N}^*$ and $\lambda \in \mathbb{R}$ we introduce the set of paths

$$\Gamma_n(\lambda) := \left\{ \gamma \in C(D_n, H_r^1(\mathbb{R}^N)) \mid \gamma \text{ odd}, \ \mathcal{J}(\lambda, \gamma_{|\partial D_n}) < 0 \right\}$$

and the minimax values

$$a_n(\lambda) := \inf_{\gamma \in \Gamma_n(\lambda)} \sup_{\xi \in D_n} \mathcal{J}(\lambda, \gamma(\xi)).$$

For $n \geq 2$ the nonemptiness of $\Gamma_n(\lambda)$ has to be checked; for $n = 1$ we refer to [38, claim 1 of Proposition 2.1]. Classically, in the local framework this fact was proved in [5] by constructing inductively piecewise affine paths. This construction does not fit the nonlocality interaction given by the Choquard term, thus we need another approach.

Proposition 6 *Assume* (F1)–(F4). *Let* $n \in \mathbb{N}^*$ *and* $\lambda \in \mathbb{R}$. *Then* $\Gamma_n(\lambda) \neq \emptyset$, *thus* $a_n(\lambda)$ *is well defined. Moreover,* $a_n(\lambda) > 0$ *and it is increasing with respect to* λ *and* n.

To deal with this proof we need a deep understanding of the Riesz potential on radial functions, and we rely on a result by [50, Theorem 1]. See also [35, Lemma 6.3] and references therein.

Theorem 3 ([50]). *Let* $u \in H_r^1(\mathbb{R}^N)$ *be radial and* $\alpha \in (0, N)$. *Then* $I_\alpha * u$ *is radial and*

$$(I_\alpha * u)(r) = r^\alpha \int_0^\infty F_\alpha\left(\frac{r}{\rho}\right) \left(\frac{\rho}{r}\right)^\alpha u(\rho) \frac{d\rho}{\rho}, \tag{4.1}$$

where F_α *is positive and it satisfies for some constants* $C_{N,0}$, $C_{N,\infty}$, $C_{N,\alpha} > 0$,

$$F_\alpha(s) \to C_{N,0} > 0 \text{ as } s \to 0, \quad \frac{F_\alpha(s)}{s^{\alpha-N}} \to C_{N,\infty} \text{ as } s \to +\infty$$

and

$$\frac{F_\alpha(s)}{G_\alpha(s)} \to 1 \quad \text{as } s \to 1, \tag{4.2}$$

with

$$G_\alpha(s) := \begin{cases} C_{N,\alpha} & \text{if } \alpha \in (1, N), \\ C_{N,\alpha}|\log|s-1|| & \text{if } \alpha = 1, \\ C_{N,\alpha}|s-1|^{\alpha-1} & \text{if } \alpha \in (0, 1). \end{cases} \tag{4.3}$$

For a Proof of Proposition 6, we prepare some notation and some estimates. We set

$$A(R, h) := \{x \in \mathbb{R}^N \mid |x| \in [R - h, R + h]\},$$

$$\chi(R, h; x) := \begin{cases} 1 & \text{for } x \in A(R, h), \\ 0 & \text{otherwise}, \end{cases}$$

for any $R \gg h > 0$. We have the following key estimates.

Lemma 1 *It results as $h \to 0$*

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)\chi(1, h; x)\chi(1, h; y)\,dxdy \sim \begin{cases} h^2 & \text{if } \alpha \in (1, N), \\ h^2|\log h| & \text{if } \alpha = 1, \\ h^{1+\alpha} & \text{if } \alpha \in (0, 1). \end{cases}$$

Here we write $f \sim g$ if there exist constants $C_1, C_2 > 0$ independent of h such that

$$C_1 g(h) \le f(h) \le C_2 g(h) \quad \text{for small } h.$$

We postpone a Proof of Lemma 1 and give it in Sect. 4.6.

We show how to use it to build a continuous odd map in $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$; by a regularization argument, we will obtain a map in $\Gamma_n(\lambda)$.

By scaling, we have

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)\chi(R, h; x)\chi(R, h; y)\,dxdy$$

$$= R^{N+\alpha} \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)\chi\left(1, \frac{h}{R}; x\right)\chi\left(1, \frac{h}{R}; y\right)\,dxdy$$

$$\sim \begin{cases} R^{N+\alpha}(\frac{h}{R})^2 & \text{if } \alpha \in (1, N), \\ R^{N+1}(\frac{h}{R})^2|\log \frac{h}{R}| & \text{if } \alpha = 1, \\ R^{N+\alpha}(\frac{h}{R})^{1+\alpha} & \text{if } \alpha \in (0, 1). \end{cases}$$

For $R \ge 2$, we set

$$h_R := \begin{cases} R^{-\frac{N-2+\alpha}{2}} & \text{if } \alpha \in (1, N), \\ R^{-\frac{N-1}{2}}(\log R)^{-1/2} & \text{if } \alpha = 1, \\ R^{-\frac{N-1}{1+\alpha}} & \text{if } \alpha \in (0, 1). \end{cases}$$

Then we have

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)\chi(R, h_R; x)\chi(R, h_R; y)\,dxdy \in [C_{01}, C_{02}] \quad \text{for } R \ge 2, \tag{4.4}$$

where $C_{01}, C_{02} > 0$ are independent of $R \ge 2$. We check (4.4) only for $\alpha = 1$. We have

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)\chi\left(1, \frac{h_R}{R}; x\right)\chi\left(1, \frac{h_R}{R}; y\right)\,dxdy$$

$$\sim R^{N+1} \left(\frac{h_R}{R} \right)^2 \left| \log \left(\frac{h_R}{R} \right) \right|$$

$$= R^{N+1} \left(\frac{R^{-\frac{N-1}{2}} |\log R|^{-1/2}}{R} \right)^2 \left| \log \left(\frac{R^{-\frac{N-1}{2}} |\log R|^{-1/2}}{R} \right) \right|$$

$$= (\log R)^{-1} \left| \log \left(R^{-\frac{N+1}{2}} (\log R)^{-1/2} \right) \right|$$

$$= (\log R)^{-1} \left(\frac{N+1}{2} \log R + \frac{1}{2} \log(\log R) \right)$$

$$\to \frac{N+1}{2} \quad \text{as } R \to \infty.$$

Next we compute the interaction effect between $\chi(R^i, h_{R^i}; \cdot)$ and $\chi(R^j, h_{R^j}; \cdot)$ with $i, j \in \mathbb{N}, i \neq j$ and $R \gg 1$.

Lemma 2 *For $i < j$ we have*

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) \chi(R^i, h_{R^i}; x) \chi(R^j, h_{R^j}; y) \, dx dy \to 0 \quad \text{as } R \to \infty.$$

We postpone the Proof of Lemma 2 and we will give it in Sect. 4.6.

Proof of Proposition 6 For $s_0 > 0$ with $F(s_0) > 0$, which is given in (F4) or (CF4), we will construct a path $\gamma \in \Gamma_n(\lambda)$ such that

$$\max_{\xi \in D_n, \, x \in \mathbb{R}^N} |\gamma(\xi)(x)| \leq s_0.$$

Step 1: Construction **of an odd path in L^r**.

For $n \geq 2$ (for $n = 1$ the construction is simpler), we consider the polyhedron

$$\Sigma := \left\{ t = (t_1, \cdots, t_n) \mid \max_{i=1,2,\cdots,n} |t_i| = 1 \right\}$$

and we recall that Σ is homeomorphic to ∂D_n. For a large $R \gg 1$, which we will choose later, we define

$$\gamma_R(t)(x) := \sum_{i=1}^{n} \text{sgn}(t_i) \chi \left(R^i, |t_i| h_{R^i}; x \right) : \Sigma \to L^r(\mathbb{R}^N)$$

where $r \in [1, +\infty]$. Here we regard $\chi(R^i, 0; x) \equiv 0$. For $s_0 > 0$ with $F(s_0) > 0$, which is given in (F4) or (CF4), we have

$$\mathcal{D}(s_0 \gamma(t)) = \sum_{i,j} F(\text{sgn}(t_i) s_0) F(\text{sgn}(t_j) s_0) \times$$

$$\times \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) \chi(R^i, |t_i| h_{R^i}; x) \chi(R^i, |t_j| h_{R^i}; y) \, dx dy.$$

We note that

(i) For any $t = (t_1, \cdots, t_n) \in \Sigma$, there exists at least one t_k such that $|t_k| = 1$.

(ii) By Lemma 1,

$$F(\pm s_0)^2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) \chi(R^k, h_{R^k}; x) \chi(R^k, h_{R^k}; y) \, dx dy \geq C_0.$$

(iii) By (i) and (ii),

$$\sum_{i=1}^{n} F(\pm s_0)^2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)\chi(R^i, h_{R^i}; x)\chi(R^i, h_{R^i}; y)\,dxdy \geq C_0.$$

(iv) If $i \neq j$, by Lemma 2,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)\chi(R^i, h_{R^i}; x)\chi(R^j, h_{R^j}; x)\,dxdy \to 0 \quad \text{as } R \to \infty.$$

By (i)–(iv), we have for sufficiently large $R \gg 1$,

$$\mathcal{D}(s_0\gamma(t)) > 0 \quad \text{for all } t \in \Sigma. \tag{4.5}$$

In what follows we fix $R \gg 1$ so that (4.5) holds.

Step 2: **Construction of an odd path in H_r^1.**

For $0 \leq h \ll R$ and $\varepsilon > 0$, we set

$$\chi_\varepsilon(R, h; x) := \begin{cases} 1 & \text{if } x \in A(R, h), \\ 1 - \frac{1}{\varepsilon}\text{dist}(x, A(R, h)), & \text{if } \text{dist}(x, A(R, h)) \in (0, \varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

Here we regard

$$A(R, 0) = \{x \in \mathbb{R}^N \,|\, |x| = R\}.$$

We note that

$$\chi_\varepsilon(R, h; \cdot) \in H_r^1(\mathbb{R}^N) \text{ for } \varepsilon > 0,$$
$$\chi_\varepsilon(R, h; \cdot) \to \chi(R, h; \cdot) \text{ in } L^r(\mathbb{R}^N) \text{ as } \varepsilon \to 0 \text{ for all } r \in [1, \infty),$$
$$\text{supp}\chi_\varepsilon(R^i, h_{R^i}; \cdot) \cap \text{supp}\chi_\varepsilon(R^j, h_{R^j}; \cdot) = \emptyset \text{ for } i \neq j \text{ for } \varepsilon \text{ small.}$$

We set

$$\gamma_{\varepsilon, R}(t) := \sum_{i=1}^{n} \text{sgn}(t_i)\chi_\varepsilon(R^i, |t_i|h_{R^i}; \cdot) : \Sigma \to H_r^1(\mathbb{R}^N). \tag{4.6}$$

We note that for $\varepsilon > 0$, $\gamma_{\varepsilon, R}(t) : \Sigma \to H_r^1(\mathbb{R}^N)$ is continuous and by (4.5) and the continuity of $\mathcal{D}$ on $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, we have for $\varepsilon > 0$ small

$$\mathcal{D}(s_0\gamma_{\varepsilon, R}(t)) > 0 \quad \text{for all } t \in \Sigma.$$

Since

$$\mathcal{J}(\lambda, u(\cdot/\theta)) = \frac{1}{2}\theta^{N-2}\|\nabla u\|_2^2 + \frac{e^\lambda}{2}\theta^N \|u\|_2^2 - \frac{1}{2}\theta^{N+\alpha}\mathcal{D}(u),$$

we have for large $\theta \gg 1$

$$\mathcal{J}(\lambda, s_0\gamma_{\varepsilon, R}(t)(\cdot/\theta)) < 0 \quad \text{for all } t \in \Sigma \simeq \partial D_n.$$

Regarding $D_n = \{st \mid s \in [0, 1],\ t \in \Sigma\}$ and extending $s_0\gamma_{\varepsilon, R}(t)(\cdot/\theta)$ to D_n by

$$\widetilde{\gamma}(st) := ss_0\gamma_{\varepsilon, R}(t)(\cdot/\theta),$$

finally we obtain a path $\widetilde{\gamma} \in \Gamma_n(\lambda)$.

Step 3: **Conclusion.** What remains to prove is the monotonicity and positivity of $a_n(\lambda)$. Since $D_n \subset D_{n+1}$, we may regard for $\gamma \in \Gamma_{n+1}(\lambda)$,

$$\gamma_{|D_n} \in \Gamma_n(\lambda).$$

Thus we have $a_n(\lambda) \le a_{n+1}(\lambda)$. Since $\mathcal{J}(\lambda, u)$ is monotone in λ, we also have the monotonicity with respect to λ.

The positivity of $a_1(\lambda)$ is essentially obtained in [38] (see also [12]). Thus

$$a_n(\lambda) \ge a_1(\lambda) > 0.$$

$\square$

Remark 6 We notice that the construction of an odd map in L^r gets much easier when F is an even function. Indeed there is no negative contribution given by the mixed interactions. We give only an outline of the proof, highlighting that in this case we do not need to use the fine Theorem 3 given by [50].

Define for every $i = 1, \ldots n$ and $s \in [0, 1]$ the annuli

$$A_i(s) := \left\{ x \in \mathbb{R}^N \mid |x| \in [2ni - s, 2ni + s] \right\}.$$

For every $t = (t_1, \ldots, t_n) \in \Sigma$ we have that $A_1(t_1), \ldots, A_n(t_n)$ are disjoint. Moreover, if $t_i = 0$, then $\mathrm{meas}(A_i(t_i)) = 0$. Thus we define a continuous, odd map by

$$\gamma(t)(x) := \sum_{i=1}^{n} \mathrm{sgn}(t_i) \chi_{A_i(t_i)}(x) : \; \Sigma \to L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N).$$

Since F is even, we obtain

$$\begin{aligned}
&\mathcal{D}(s_0\gamma(t)) \\
&= \sum_{i,j} \iint_{A_i(t_i) \times A_j(t_j)} I_\alpha(x - y) F(s_0\mathrm{sgn}(t_i)\chi_{A_i(t_i)}(x)) F(s_0\mathrm{sgn}(t_j)\chi_{A_j(t_j)}(y)) \, dx dy \\
&= F(s_0)^2 \sum_{i,j} \iint_{A_i(t_i) \times A_j(t_j)} I_\alpha(x - y) \, dx dy \ge C > 0,
\end{aligned}$$

where C does not depend on the specific t. The regularization to a H_r^1-path can be done as in the general case (or by mollification), as well as the extension to D_n.

We highlight that this construction can be adapted also to the local case, and thus it gives a simplified construction of a multidimensional path in the setting of Berestycki and Lions [5].

4.2 Asymptotic of symmetric mountain pass values

We end this section with some key estimates on the asymptotic behavior of $a_n(\lambda)$ as $\lambda \to \pm\infty$.

Proposition 7 *Assume* (F1)–(F4) *and let* $n \in \mathbb{N}^*$.

(i) *If* (CF3) *holds, then* $\lim_{\lambda \to +\infty} \frac{a_n(\lambda)}{e^\lambda} = +\infty$.
(ii) *If* (CF4) *holds, then* $\lim_{\lambda \to -\infty} \frac{a_n(\lambda)}{e^\lambda} = 0$.

Proof of (i) of Proposition 7 We write $q = \frac{N+\alpha}{N}$, $p = \frac{N+\alpha+2}{N}$ and $\mu = e^\lambda$ (and consequently adapt the notations) for the sake of simplicity.

Since $a_n(\mu) \geq a_1(\mu)$ for each $n \in \mathbb{N}^*$, it is sufficient to show the claim for $n = 1$. By (CF3), for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|F(s)| \leq \delta |s|^p + C_\delta |s|^q \quad \text{for all } s \in \mathbb{R}.$$

For $v \in H^1_r(\mathbb{R}^N)$, setting $u_s := s^{N/2} v(s\cdot)$, we have

$$
\begin{aligned}
\mathcal{D}(u_s) &= s^{-N-\alpha} \mathcal{D}(s^{N/2} v) \\
&\leq s^{-N-\alpha} \int_{\mathbb{R}^N} \left(I_\alpha * (\delta s^{\frac{N}{2}p} |v|^p + C_\delta s^{\frac{N}{2}q} |v|^q) \right) (\delta s^{\frac{N}{2}p} |v|^p + C_\delta s^{\frac{N}{2}q} |v|^q)\, dx \\
&= s^2 \int_{\mathbb{R}^N} \left(I_\alpha * (\delta |v|^p + C_\delta s^{-1} |v|^q) \right) (\delta |v|^p + C_\delta s^{-1} |v|^q)\, dx \\
&=: s^2 D_{\delta, C_\delta s^{-1}}(v),
\end{aligned}
\tag{4.7}
$$

where we write for $\delta > 0$ and $A \geq 0$,

$$D_{\delta, A}(v) := \int_{\mathbb{R}^N} \left(I_\alpha * (\delta |v|^p + A |v|^q) \right) (\delta |v|^p + A |v|^q)\, dx,$$

$$\mathcal{J}_{\delta, A}(v) := \frac{1}{2} \|\nabla v\|_2^2 + \frac{1}{2} \|v\|_2^2 - \frac{1}{2} D_{\delta, A}(v).$$

We also denote by $b(\delta, A)$ the MP value of $\mathcal{J}_{\delta, A}$. Taking into account the continuity and monotonicity property of $b(\delta, A)$ with respect to each variable δ and A and observing that $\mathcal{J}_{\delta, A}$ satisfies the (PS) condition, we have

$$b(\delta, A) \to b(\delta, 0) \quad \text{as } A \to 0^+,$$
$$b(\delta, 0) \to +\infty \quad \text{as } \delta \to 0^+.$$

Thus, from (4.7) we have that

$$\mathcal{J}(\mu, u_s) \geq s^2 \left(\frac{1}{2} \|\nabla v\|_2^2 + \frac{1}{2} \mu s^{-2} \|v\|_2^2 - \frac{1}{2} D_{\delta, C_\delta s^{-1}}(v) \right).$$

Setting $s := \sqrt{\mu}$, we obtain

$$\mathcal{J}(\mu, u_{\sqrt{\mu}}) \geq \mu \mathcal{J}_{\delta, C_\delta \mu^{-1/2}}(v)$$

and thus $\frac{a_1(\mu)}{\mu} \geq b(\delta, C_\delta \mu^{-1/2})$, which implies

$$\liminf_{\mu \to \infty} \frac{a_1(\mu)}{\mu} \geq \lim_{A \to 0} b(\delta, A) = b(\delta, 0).$$

Since $\delta > 0$ is arbitrary, we gain

$$\lim_{\mu \to +\infty} \frac{a_1(\mu)}{\mu} = +\infty.$$

$\square$

We deal now with the proof of (ii) of Proposition 7. We highlight that, when F is even, the proof can be simplified (see [12]).

We start noticing that, by (CF4) and Remark 4, for some $\delta_0 > 0$

$$F(s) > 0 \quad \text{for } s \in (0, \delta_0],$$

which implies

(i) when F is even, $F(s) > 0$ for all $s \in [-\delta_0, \delta_0] \setminus \{0\}$;
(ii) when F is odd, $F(s) < 0$ for all $s \in [-\delta_0, 0)$.

By (CF4), we also note that there exists $L_s > 0$ with $L_s \to \infty$ as $s \to 0^+$ such that

$$F(\sigma) \leq L_s \sigma^p \quad \text{for all } \sigma \in [0, s]. \tag{4.8}$$

First we observe that the path $\gamma_{R,\varepsilon} : \Sigma \to H_r^1(\mathbb{R}^N)$, defined in (4.6), has the following property.

Lemma 3 *There exists a constant $A > 0$ independent of $s \in (0, \delta_0]$ such that*

$$\mathcal{D}(s\gamma_{R,\varepsilon}(t)) \geq F(s)^2(A + o(1)) \quad \text{as } \varepsilon \to 0.$$

Here $o(1)$ is a quantity which goes to 0 as $\varepsilon \to 0$ uniformly in $t \in \Sigma$ and $s \in (0, \delta_0]$.

Proof We prove Lemma 3 in 2 steps.
Step 1: For $t \in \Sigma$, set

$$a_{ij}(t) := \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) \chi(R^i, |t_i| h_{R^i}; x) \chi(R^j, |t_j| h_{R^j}; y) \, dx dy.$$

Then for sufficiently large $R > 0$, we have

$$A := \inf_{t \in \Sigma} \left(\sum_{i=1}^n a_{ii}(t) - \sum_{i \neq j} a_{ij}(t) \right) > 0. \tag{4.9}$$

This fact follows from (4.4) and Lemma 2. We fix $R \gg 1$ so that (4.9) holds.
Step 2: $\mathcal{D}(s\gamma_{R,\varepsilon}(t)) \geq \frac{1}{2} F(s)^2 A$ as $\varepsilon \to 0$.
We note that for $\varepsilon > 0$ small

$$\text{supp}\chi_\varepsilon(R^i, |t_i| h_{R^i}; \cdot) \cap \text{supp}\chi_\varepsilon(R^j, |t_j| h_{R^j}; \cdot) = \emptyset \quad \text{for } i \neq j.$$

Thus we have

$$\mathcal{D}(s\gamma_{R,\varepsilon}(t))$$
$$= \sum_{i,j} \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) F(s \, \text{sgn}(t_i) \chi_\varepsilon(R^i, |t_i| h_{R^i}; x)) F(s \, \text{sgn}(t_j) \chi_\varepsilon(R^j, |t_j| h_{R^j}; y)) \, dx dy$$
$$\equiv \sum_{i,j} B_{ij}(s, t). \tag{4.10}$$

We consider cases $i = j$ and $i \neq j$ separately.
First we focus on the case $i = j$. For both of even and odd F

$$B_{ii}(s, t)$$
$$= \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) F(s \, \text{sgn}(t_i) \chi_\varepsilon(R^i, |t_i| h_{R^i}; x)) F(s \, \text{sgn}(t_i) \chi_\varepsilon(R^j, |t_i| h_{R^i}; y)) \, dx dy$$
$$= \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) F(s \chi_\varepsilon(R^i, |t_i| h_{R^i}; x)) F(s \chi_\varepsilon(R^j, |t_i| h_{R^i}; y)) \, dx dy$$
$$\geq \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) F(s \chi(R^i, |t_i| h_{R^i}; x)) F(s \chi(R^j, |t_i| h_{R^i}; y)) \, dx dy$$

$$= F(s)^2 a_{ii}(t) \tag{4.11}$$

where we used the positivity of F and the monotonicity of the integral. Next we consider the case $i \neq j$ for even F. Since $F(s) \geq 0$ for $s \in [-\delta_0, \delta_0]$,

$$B_{ij}(t) \geq 0 \quad \text{for all } t \in \Sigma. \tag{4.12}$$

Finally we consider the case $i \neq j$ for odd F. Since $|F(s)| = F(|s|)$ for $s \in [-\delta_0, \delta_0]$

$$
\begin{aligned}
& B_{ij}(s, t) \\
&= \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) F(s\, \mathrm{sgn}(t_i) \chi_\varepsilon(R^i, |t_i| h_{R^i}; x)) F(s\, \mathrm{sgn}(t_j) \chi_\varepsilon(R^j, |t_j| h_{R^j}; y))\, dx dy \\
&\geq -\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) F(s \chi_\varepsilon(R^i, |t_i| h_{R^i}; x)) F(s \chi_\varepsilon(R^j, |t_j| h_{R^j}; y))\, dx dy. \tag{4.13}
\end{aligned}
$$

Setting $C_i(t, \varepsilon) := \{x \mid \mathrm{dist}(x, A(R^i, |t_i| h_{R^i})) \in (0, \varepsilon)\}$, we have

$$
\begin{aligned}
&\chi_\varepsilon(R^i, |t_i| h_{R^i}; x) \in (0, 1) \quad \text{for } x \in C_i(t_i, \varepsilon), \\
&\chi_\varepsilon(R^i, |t_i| h_{R^i}; x) = \chi(R^i, |t_i| h_{R^i}; x) \quad \text{for } x \notin C_i(t_i, \varepsilon), \\
&\mathrm{meas}(C_i(t_i, \varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0, \text{ uniformly in } t \in \Sigma.
\end{aligned}
$$

Thus for $r \in [1, \infty)$ and $s \in (0, \delta]$

$$
\begin{aligned}
& \left\| \frac{1}{F(s)} F(s \chi_\varepsilon(R^i, |t_i| h_{R^i}; \cdot)) - \chi(R^i, |t_i| h_{R^i}; \cdot) \right\|_r^r \\
&\leq \int_{C_i(t_i, \varepsilon)} \left| \frac{1}{F(s)} F(s \chi_\varepsilon(R^i, |t_i| h_{R^i}; x)) \right|^r dx \\
&= \left(\max_{h \in [0,1]} \frac{|F(hs)|}{|F(s)|} \right)^r \mathrm{meas}(C_i(t_i, \varepsilon)) \\
&\to 0 \quad \text{as } \varepsilon \to 0 \text{ uniformly in } t \in \Sigma. \tag{4.14}
\end{aligned}
$$

Here we use the fact that $\max_{h \in [0,1]} \frac{F(hs)}{F(s)} \leq 1$, which follows from the local monotonicity assumption in (CF4). We note that (4.14) implies, exploiting again the local monotonicity

$$
\left| \frac{1}{F(s)^2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y) F(s \chi_\varepsilon(R^i, |t_i| h_{R^i}; x)) F(s \chi_\varepsilon(R^j, |t_j| h_{R^j}; y))\, dx dy - a_{ij}(t) \right|
$$
$$
\to 0 \quad \text{as } \varepsilon \to 0. \tag{4.15}
$$

By (4.13) and (4.15),

$$B_{ij}(s, t) \geq -F(s)^2 (a_{ij}(t) + o(1)) \quad \text{as } \varepsilon \to 0. \tag{4.16}$$

Thus, it follows from (4.10)–(4.12) and (4.16) that

$$
\begin{aligned}
\mathcal{D}(s \gamma_{R, \varepsilon}(t)) &\geq F(s)^2 \left(\sum_{i=1}^n a_{ii}(t) - \sum_{i \neq j} a_{ij} + o(1) \right) \\
&\geq \frac{1}{2} F(s)^2 A \quad \text{for } \varepsilon > 0 \text{ small.}
\end{aligned}
$$

This concludes the proof. □

Proof of (ii) of Proposition 7 For $s_0 \in (0, \delta_0]$ and $\mu > 0$, we consider a path

$$D_n \to H_r^1(\mathbb{R}^N); \ st \mapsto ss_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}}).$$

We have

$$\mu^{-1}\mathcal{J}(\mu, ss_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}}))$$
$$= \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\nabla\gamma_{R,\varepsilon}(t)\|_2^2 + \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\gamma_{R,\varepsilon}(t)\|_2^2 - \frac{1}{2}\mu^{-\frac{N}{2}p}\mathcal{D}(ss_0\gamma_{R,\varepsilon}(t))$$
$$\leq \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\gamma_{R,\varepsilon}(t)\|_{H^1}^2 - \frac{1}{4}\mu^{-\frac{N}{2}p}F(ss_0)^2A.$$

Thus for μ small

$$\mathcal{J}(\mu, s_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}})) < 0 \quad \text{for } t \in \Sigma,$$

which implies $s_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}}) \in \Gamma_n(\lambda)$. Moreover by (4.8)

$$\mu^{-1}a_n(\mu)$$
$$\leq \max_{s \in [0,1], t \in \Sigma} \mu^{-1}\mathcal{J}(\mu, ss_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}}))$$
$$\leq \max_{s \in [0,1], t \in \Sigma} \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\gamma_{R,\varepsilon}(t)\|_{H^1}^2 - \frac{1}{4}\mu^{-\frac{N}{2}p}F(ss_0)^2A$$
$$\leq \max_{s \in [0,1], t \in \Sigma} \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\gamma_{R,\varepsilon}(t)\|_{H^1}^2 - \frac{1}{4}L_{s_0}(\mu^{-\frac{N}{2}}(ss_0)^2)^pA$$
$$\leq C_{s_0},$$

where

$$C_{s_0} := \sup_{\tau \geq 0, t \in \Sigma}\left(\frac{1}{2}\tau\|\gamma_{R,\varepsilon}(t)\|_{H^1}^2 - \frac{1}{4}L_{s_0}A\tau^p\right) \in \mathbb{R}.$$

Thus we have

$$\limsup_{\mu \to 0^+}\mu^{-1}a_n(\mu) \leq C_{s_0}.$$

Since $C_{s_0} \to 0$ as $s_0 \to 0$, we have (ii) of Proposition 7. $\qquad\square$

4.3 The Pohozaev mountain

We consider the *Pohozaev level set*

$$\Omega := \left\{(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) > 0\right\} \cup \left\{(\lambda, 0) \mid \lambda \in \mathbb{R}\right\}.$$

We notice that, under the assumption (F5), Ω is symmetric with respect to the axis $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$, that is,

$$(\lambda, u) \in \Omega \implies (\lambda, -u) \in \Omega.$$

We start showing the following property.

Lemma 4 *We have*

$$\{(\lambda, 0) \mid \lambda \in \mathbb{R}\} \subset int(\Omega). \tag{4.17}$$

Proof Since $\mathcal{D}(u) = o(\|u\|_{H^1}^2)$ as $u \to 0$, the conclusion follows from the definition of $\mathcal{P}(\lambda, u)$. □

By (4.17) we detect the *Pohozaev's mountain*

$$\partial\Omega = \big\{(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) = 0, \ u \not\equiv 0\big\}.$$

We observe that $\partial\Omega \neq \emptyset$, for instance by [38, Theorems 1 and 3].

Proposition 8 *Assume* (F1)–(F4). *We have the following properties.*

(i) $\mathcal{J}(\lambda, u) \geq 0$ *for all* $(\lambda, u) \in \Omega$.
(ii) $\mathcal{J}(\lambda, u) \geq a_1(\lambda) > 0$ *for all* $(\lambda, u) \in \partial\Omega$.
(iii) *Assume* (CF3). *For any* $m > 0$, *we set*

$$E^m := \inf_{(\lambda, u) \in \partial\Omega} \mathcal{I}^m(\lambda, u), \quad \text{and} \quad B^m := \inf_{\lambda \in \mathbb{R}} \left(a_1(\lambda) - \frac{e^\lambda}{2} m \right).$$

Then $E^m \geq B^m > -\infty$. *In particular* $B^m \in \mathbb{R}$ *and*

$$\mathcal{I}^m(\lambda, u) \geq B^m \quad \text{for every } (\lambda, u) \in \partial\Omega.$$

Proof We notice that for all $(\lambda, u) \in \Omega$

$$\mathcal{J}(\lambda, u) \geq \mathcal{J}(\lambda, u) - \frac{\mathcal{P}(\lambda, u)}{N + \alpha} = \frac{\alpha + 2}{2(N + \alpha)}\|\nabla u\|_2^2 + \frac{\alpha}{2(N + \alpha)} e^\lambda \|u\|_2^2 \geq 0$$

and thus (i) follows. Point (ii) follows from the fact that for each λ the mountain pass level $a_1(\lambda)$ coincides with the ground state energy level (see [38, Sect. 4.2], and [10, Proposition 2] for details). Focus on (iii): the fact that $E^m \geq B^m$ is a direct consequence of (ii), while the fact that $B^m > -\infty$ comes from Proposition 7 (i). □

From now on we assume (CF3) to give sense to the quantity B^m. In view of Proposition 8 (iii), we set for $m > 0$ and $n \in \mathbb{N}^*$

$$\Gamma_n^m := \big\{\Theta \in C(D_n, \mathbb{R} \times H_r^1(\mathbb{R}^N)) \mid \Theta \text{ is } \mathbb{Z}_2\text{-equivariant}, \ \mathcal{I}^m(\Theta(0)) \leq B^m - 1,$$
$$\Theta|_{\partial D_n} \notin \Omega, \ \mathcal{I}^m(\Theta|_{\partial D_n}) \leq B^m - 1\big\}$$

and

$$b_n^m := \inf_{\Theta \in \Gamma_n^m} \sup_{\xi \in D_n} \mathcal{I}(\Theta(\xi));$$

we point out that asking $\Theta = (\Theta_1, \Theta_2) \in \Gamma_n^m$ to be $\mathbb{Z}_2$-equivariant means that Θ_1 is even and Θ_2 is odd, and in particular $\Theta_2(0) = 0$ which implies $\Theta(0) \in \Omega$.

Proposition 9 *Assume* (F1)-(F2)-(CF3)-(F4)-(F5). *We have the following properties.*

(i) *For any* $m > 0$ *and* $n \in \mathbb{N}^*$, *we have* $\Gamma_n^m \neq \emptyset$ *and*

$$b_n^m \leq a_n(\lambda) - e^\lambda \frac{m}{2}, \tag{4.18}$$

for each $\lambda \in \mathbb{R}$. *Moreover,* b_n^m *increases with respect to* n.

(ii) *For any $k \in \mathbb{N}^*$ there exists $m_k \geq 0$, namely given by*

$$m_k := 2 \inf_{\lambda \in \mathbb{R}} \frac{a_k(\lambda)}{e^\lambda}, \tag{4.19}$$

such that for $m > m_k$

$$b_n^m < 0 \quad \text{for } n = 1, 2, \ldots, k.$$

Moreover, m_k is increasing with respect to k.

(iii) *If* (CF4) *holds, then $m_k = 0$ for each $k \in \mathbb{N}^*$. That is, for each $m > 0$ we have*

$$b_n^m < 0 \quad \text{for all } n \in \mathbb{N}^*.$$

Proof For given $\lambda \in \mathbb{R}$ and $\zeta \in \Gamma_n(\lambda)$, we will find a $\psi \in \Gamma_n^m$ such that

$$\max_{\xi \in D_n} \mathcal{J}(\psi(\xi)) \leq \max_{\xi \in D_n} \mathcal{J}(\lambda, \zeta(\xi)), \tag{4.20}$$

so that we have

$$b_n^m \leq \max_{\xi \in D_n} \mathcal{I}^m(\psi(\xi)) \leq \max_{\xi \in D_n} \mathcal{J}(\lambda, \zeta(\xi)) - \frac{e^\lambda}{2} m$$

and, passing to the infimum over $\Gamma_n(\lambda)$, we gain (4.18).

To find $\psi \in \Gamma_n^m$ with (4.20), observe that, by definition of $\Gamma_n(\lambda)$ and compactness of $\zeta(\partial D_n)$, there exists $C > 0$ such that $\mathcal{D}(\zeta(\xi)) \geq C > 0$ for $\xi \in \partial D_n$. Thus, we have $\mathcal{I}^m(\lambda, \zeta(\xi)(\cdot/L)) \to -\infty$ and $\mathcal{P}(\lambda, \zeta(\xi)(\cdot/L)) \to -\infty$ as $L \to +\infty$, uniformly for $\xi \in \partial D_n$. Thus, for $L \gg 1$ we obtain, for every $\xi \in \partial D_n$,

$$\mathcal{I}^m(\lambda, \zeta(\xi)(\cdot/L)) \leq B^m - 1 \quad \text{and} \quad \mathcal{P}(\lambda, \zeta(\xi)(\cdot/L)) < 0. \tag{4.21}$$

We also note that $\mathcal{I}^m(\lambda + L, 0) = -\frac{e^{\lambda+L}}{2} m \to -\infty$ as $L \to +\infty$. Thus, for $L \gg 1$, we find that the path $\psi : D_n \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$

$$\psi(\xi) := \begin{cases} (\lambda + L(1 - 2|\xi|), 0) & \text{if } |\xi| \in [0, 1/2], \\ \left(\lambda, \zeta\left(\frac{\xi}{|\xi|}(2|\xi| - 1)\right)(\cdot/L)\right) & \text{if } |\xi| \in (1/2, 1] \end{cases}$$

satisfies $\psi(0) = (\lambda + L, 0) \in \mathbb{R} \times \{0\}$, $\mathcal{I}^m(\psi(0)) \leq B^m - 1$ and $\mathcal{I}^m(\psi(\xi)) \leq B^m - 1$ for $\xi \in \partial D_n$. Thus, by (4.21), we obtain $\psi \in \Gamma_n^m$ and (4.20) holds.

The monotonicity of b_n^m with respect to n is a consequence of the definition. Point (ii) follows from (4.18) and (iii) follows from Proposition 7 (ii). $\square$

As a corollary to Proposition 9, we have the following result.

Corollary 2 *For any $m > 0$, we have $B^m = E^m = b_1^m$, i.e. the first minimax value b_1^m equals the Pohozaev minimum E^m on the product space.*

Proof Since any path in Γ_n^m passes through $\partial \Omega$, we have $b_n^m \geq E^m \geq B^m$ for each n. On the other hand, passing to the infimum (4.18) we obtain $b_1^m \leq B^m$ and thus the claim. $\square$

By Propositions 2 and 4, $\mathcal{I}^m$ satisfies the $(PSP)_b$ condition for $b < 0$ and the deformation lemma holds. Let $m_k \geq 0$ be a number given in Proposition 9. For $m > m_k$ we can see that $b_n^m < 0$ for $n = 1, 2, \cdots, k$ are critical values of $\mathcal{I}^m$. If b_n^m are different, we can see the multiplicity of solutions. To deal with the case $b_n^m = b_{n'}^m$ for some $n \neq n'$, we need another family of minimax methods, which we consider in the following section.

4.4 Existence of multiple critical points

Let us define now new minimax families which enable us to find multiple solutions. We use an idea from [46], in which the genus theory is developed effectively in a general setting, where $\mathbb{Z}_2$-action $\mathbb{Z}_2 \times X \to X; (\pm 1, u) \mapsto \pm u$ is considered in a general Banach space X. The genus theory is applied for our $\mathbb{Z}_2$-action in [22].

We recall the *genus* of a closed symmetric set $A \subset X$ with $0 \notin A$. We write $\text{genus}(A) = n$ if n is the least integer $n \in \mathbb{N}$ such that there exists a continuous odd map $\beta : A \to \mathbb{R}^n \setminus \{0\}$; if such n does not exists, we set $\text{genus}(A) := +\infty$.

For each $n \in \mathbb{N}^*$, define

$$\Lambda_n^m := \{A = \Theta(\overline{D_{n+l} \setminus Y}) \mid l \in \mathbb{N}^*, \ \Theta \in \Gamma_{n+l}^m,$$
$$Y \subset D_{n+l} \setminus \{0\} \text{ is closed,}$$
$$\text{symmetric and genus}(Y) \leq l\}$$

and

$$c_n^m := \inf_{A \in \Lambda_n^m} \sup_A \mathcal{I}^m.$$

We notice that $\{\Theta(D_n)\}_{\Theta \in \Gamma_n^m} \subset \Lambda_n^m$. In the following lemma, we observe that Λ_n^m and c_n^m inherits the properties of Γ_n^m and b_n^m and they enjoy an extra property (v).

Proposition 10 *Assume* (F1)-(F2)-(CF3)-(F4). *Let* $n \in \mathbb{N}^*$ *and* $m > 0$. *Then*

(i) $\Lambda_n^m \neq \emptyset$.
(ii) $\Lambda_{n+1}^m \subset \Lambda_n^m$, *and thus* $c_n^m \leq c_{n+1}^m$.
(iii) $c_n^m \leq b_n^m$.
(iv) $B^m = E^m \leq c_1^m$.
(v) *Let* $A \in \Lambda_n^m$ *and* $Z \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ *be* $\mathbb{Z}_2$-*invariant, closed, and such that* $0 \notin \overline{P_2(Z)}$ *and* $\text{genus}(\overline{P_2(Z)}) \leq i < n$. *Then* $\overline{A \setminus Z} \in \Lambda_{n-i}^m$.

Proof The proof is essentially given in [22] and [9, Proposition 7.7]. In particular, (iv) follows from the fact that

$$A \cap \partial\Omega \neq \emptyset \quad \text{for all } A \in \Lambda_1^m.$$

$\square$

4.5 Proof of Theorem 1

Fix $n \in \mathbb{N}^*$ and let Λ_n^m and c_n^m satisfying the properties of Proposition 10. We build now multiple solutions.

Proposition 11 *Assume* (F1)–(CF2)–(CF3)–(F4)–(F5). *Fix* $k \in \mathbb{N}^*$ *and assume* $m > m_k$ *(see* (4.19)*). Then*

$$c_1^m \leq c_2^m \leq \cdots \leq c_k^m < 0$$

are critical values of $\mathcal{I}^m$. *Moreover*

(i) *If, for some* $q \in \mathbb{N}^*$,

$$c_n^m < c_{n+1}^m < \cdots < c_{n+q}^m < 0$$

then we have $q + 1$ different nonzero critical values, and thus $q + 1$ different pairs of nontrivial solutions of (1.7);

(ii) *If instead, for some $q \in \mathbb{N}^*$,*

$$c_n^m = c_{n+1}^m = \cdots = c_{n+q}^m =: b < 0 \tag{4.22}$$

then

$$\mathrm{genus}(P_2(K_b^m)) \geq q + 1 \tag{4.23}$$

and thus $\# P_2(K_b^m) = +\infty$, which means that we have infinite different solutions of (1.7).

Summing up, we have at least k different pairs of nontrivial solutions of (1.7).

Proof Since the $(PSP)_b$ condition holds for $b < 0$ by Proposition 2, we can develop deformation theory given in Proposition 5. We can also observe that the minimax classes Λ_n^m are stable under the deformation. Thus Proposition 11 follows from Proposition 10. See [9, Theorem 7.8] for details. □

Proof of Theorem 1 Theorem 1 follows from Proposition 11 easily. See also [9, Theorem 1.4]. □

Remark 7 It is shown in [12] that the first solution corresponding to b_1^m is also a *ground state*, that is, it attains a minimum on the L^2-sphere $\mathcal{S}_m$.

4.6 Proofs of Lemmas 1 and 2

Here we give Proof of Lemmas 1 and 2.

Proof of Lemma 1 We apply Theorem 3 to $u(|x|) = \chi(1, h; |x|)$. In particular, by (4.1) we have

$$
\begin{aligned}
S_h &:= \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)u(x)u(y)\, dx dy \\
&= c \int_0^\infty (I_\alpha * u)(r)u(r)r^{N-1}\, dr \\
&= c \int_0^\infty \int_0^\infty F_\alpha\left(\frac{r}{\rho}\right)\rho^{\alpha-1}r^{N-1}u(\rho)u(r)\, d\rho dr \\
&= c \iint_{[1-h,1+h]^2} F_\alpha\left(\frac{r}{\rho}\right)\rho^{\alpha-1}r^{N-1}\, d\rho dr.
\end{aligned}
$$

First we note that

$$\sup_{\rho,r\in[1-h,1+h]}\left|\frac{r}{\rho} - 1\right| \to 0 \quad \text{as } h \to 0.$$

We consider the following three cases separately:

$$(i)\ \alpha \in (1, N), \quad (ii)\ \alpha = 1, \quad (iii)\ \alpha \in (0, 1).$$

(i) When $\alpha \in (1, N)$ we may assume $F(\frac{r}{\rho}) \sim C_{N,\alpha} > 0$. Thus

$$S_h \sim \iint_{[1-h,1+h]^2} \rho^{\alpha-1}r^{N-1}\, d\rho dr$$

$$\sim h^2.$$

(ii) When $\alpha = 1$

$$F_\alpha\left(\frac{r}{\rho}\right) \sim G_1\left(\frac{r}{\rho}\right) = C_{N,1}\left|\log\left|\frac{r}{\rho} - 1\right|\right|$$
$$\sim \left|\log|r - \rho| - \log\rho\right|$$
$$= -\log|r - \rho| + \log\rho.$$

Thus

$$S_h \sim \iint_{[1-h,1+h]^2} (-\log|r - \rho| + \log\rho)r^{N-1}\,d\rho dr.$$

We set

$$A_h := \left\{(\rho, r)|\ |\rho - r| \le \tfrac{1}{2}h,\ |r - 1| \le \tfrac{1}{2}h\right\},$$
$$B_h := \left\{(\rho, r)|\ |\rho - r| \le 2h,\ |r - 1| \le h\right\}.$$

Then

$$A_h \subset [1 - h, 1 + h]^2 \subset B_h.$$

Thus for some $C, C' > 0$

$$C\iint_{A_h} (-\log|r - \rho| + \log\rho)r^{N-1}\,d\rho dr \le S_h \le$$
$$C'\iint_{B_h} (-\log|r - \rho| + \log\rho)r^{N-1}\,d\rho dr. \tag{4.24}$$

We compute

$$\iint_{B_h} (-\log|r - \rho| + \log\rho)r^{N-1}\,d\rho dr$$
$$\le \iint_{B_h} (-\log|r - \rho| + \log(1 + h))(1 + h)^{N-1}\,d\rho dr$$
$$= \iint_{[-2h,2h]\times[1-h,1+h]} (-\log|\tau| + \log(1 + h))(1 + h)^{N-1}\,d\tau dr$$
$$= 4h(1 + h)^{N-1}\int_0^{2h} (-\log\tau)\,d\tau + 8h^2(1 + h)^{N-1}\log(1 + h)$$
$$= 4h(1 + h)^{N-1}\left(-2h\log(2h) + 2h\right) + 8h^2(1 + h)^{N-1}\log(1 + h)$$
$$\le C''h^2|\log h| \quad \text{as } h \to 0.$$

Similarly we have

$$\iint_{A_h} (\cdots)r^{N-1}\,d\rho dr \ge C'''h^2|\log h|,$$

from which we obtain

$$S_h \sim h^2|\log h| \quad \text{as } h \to 0.$$

(iii) When $\alpha \in (0, 1)$

$$F_\alpha\left(\frac{r}{\rho}\right) \sim G_\alpha\left(\frac{r}{\rho}\right) = C_{N,\alpha}\left|\frac{r}{\rho} - 1\right|^{\alpha-1}.$$

Thus

$$S_h \sim \iint_{[1-h,1+h]^2}\left|\frac{r}{\rho} - 1\right|^{\alpha-1}\rho^{\alpha-1}r^{N-1}\,d\rho dr$$
$$= \iint_{[1-h,1+h]^2}|r - \rho|^{\alpha-1}r^{N-1}\,d\rho dr.$$

Since

$$C\iint_{A_h}|r - \rho|^{\alpha-1}(1-h)^{N-1}\,d\rho dr \leq S_h \leq C'\iint_{B_h}|r - \rho|^{\alpha-1}(1+h)^{N-1}\,d\rho dr,$$

we have as in (4.24)

$$S_h \sim h^{1+\alpha} \quad \text{as } h \to 0.$$

This completes the proof. $\qquad\square$

Proof of Lemma 2 Since $\text{supp}\chi(R, h_R; \cdot) = \left\{x \in \mathbb{R}^N \mid |x| \in [R - h_R, R + h_R]\right\}$ we have

$$\text{dist}\big(\text{supp}\chi(R^i, h_{R^i}; \cdot), \text{supp}\chi(R^j, h_{R^j}; \cdot)\big) = (R^j - h_{R^j}) - (R^i + h_{R^i})$$
$$= R^j - O(R^i).$$

Thus

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x - y)\chi(R^i, h_{R^i}; x)\chi(R^j, h_{R^j}; y)\,dxdy$$
$$\leq C(R^j + O(R^i))^{-(N-\alpha)}\|\chi(R^i, h_{R^i}; \cdot)\|_1\|\chi(R^j, h_{R^j}; \cdot)\|_1.$$

Here

$$\|\chi(R, h_R; \cdot)\|_1 = \text{meas}(A(R, h_R)) \sim CR^{N-1}h_R.$$

Thus

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N}(\cdots)\,dxdy \leq C(R^j - O(R^i))^{-(N-\alpha)}R^{(N-1)i}h_{R^i}R^{(N-1)j}h_{R^j}$$
$$\leq C'R^{(\alpha-1)j+(N-1)i}h_{R^i}h_{R^j}.$$

When $\alpha \in (1, N)$, we have by the definition of h_R

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N}(\cdots)\,dxdy \leq CR^{(\alpha-1)j+(N-1)i}R^{-\frac{1}{2}(N-2+\alpha)(i+j)}$$
$$= C'R^{-\frac{1}{2}(N-\alpha)(j-i)}$$
$$\to 0 \quad \text{as } R \to \infty.$$

When $\alpha = 1$, we obtain

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N}(\cdots)\,dxdy \leq C'R^{(N-1)i}R^{-\frac{1}{2}(N-1)(i+j)}(\log R^i)^{-\frac{1}{2}}(\log R^j)^{-\frac{1}{2}}$$
$$= C'R^{-\frac{1}{2}(N-1)(j-i)}(ij)^{-\frac{1}{2}}(\log R)^{-1}$$

$$\to 0 \quad \text{as } R \to \infty.$$

When $\alpha \in (0, 1)$,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} (\cdots) \, dx dy \leq C' R^{(\alpha-1)j+(N-1)i} R^{-\frac{N-1}{1+\alpha}(i+j)}$$

$$= C' R^{-\frac{1}{1+\alpha}((N-\alpha^2)j-\alpha(N-1)i)}$$

$$\to 0 \quad \text{as } R \to \infty.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5 Unconstrained problem

In this section we sketch how to obtain infinitely many radial solutions for the *unconstrained problem* (1.1) and give a proof of Theorem 2. Here we assume (F1)–(F5). We fix $\lambda \in \mathbb{R}$ and write $\mu = e^\lambda$; omitting λ, we denote $\mathcal{J}(\cdot) := \mathcal{J}(\lambda, \cdot) : H_r^1(\mathbb{R}^N) \to \mathbb{R}$, i.e.

$$\mathcal{J}(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\mathcal{D}(u) + \frac{\mu}{2}\|u\|_2^2, \quad u \in H_r^1(\mathbb{R}^N). \tag{5.1}$$

Similarly we write $\mathcal{P}(\cdot) := \mathcal{P}(\lambda, \cdot)$. For every $b \in \mathbb{R}$ we set

$$K_b := \{u \in H_r^1(\mathbb{R}^N) \mid \mathcal{J}(u) = b, \ \mathcal{J}'(u) = 0\}.$$

We have the following result.

Proposition 12 *Assume* (F1)–(F3) *and let* $b \in \mathbb{R}$. *Then* $\mathcal{J}$ *satisfies the Palais-Smale-Pohozaev condition at level b (shortly* $(PSP)_b$*), that is every sequence* $(u_n) \subset H_r^1(\mathbb{R}^N)$ *satisfying*

$$\mathcal{J}(u_n) \to b, \tag{5.2}$$

$$\|\mathcal{J}'(u_n)\|_{(H_r^1(\mathbb{R}^N))^*} \to 0, \tag{5.3}$$

$$\mathcal{P}(u_n) \to 0, \tag{5.4}$$

admits a strongly convergent subsequence in $H_r^1(\mathbb{R}^N)$. *In particular,* $K_b(\lambda)$ *is compact in* $H_r^1(\mathbb{R}^N)$.

Proof First observe that, by (5.2) and (5.4) we obtain

$$\frac{\alpha+2}{2}\|\nabla u_n\|_2^2 + \frac{\alpha}{2}\mu\|u_n\|_2^2 = (N+\alpha)b + o(1). \tag{5.5}$$

We observe that $b \geq 0$ and the boundedness of u_n in $H_r^1(\mathbb{R}^N)$. Thus by (F2)–(F3), $\mathcal{D}'(u_n)$ has a strongly convergent subsequence in $(H_r^1(\mathbb{R}^N))^*$ and by (5.3), u_n has a strongly convergent subsequence in $H_r^1(\mathbb{R}^N)$. $\qquad\qquad\qquad\qquad\qquad\square$

Set $[\mathcal{J} \leq b] := \{u \in H_r^1(\mathbb{R}^N) \mid \mathcal{J}_\lambda(u) \leq b\}$. Following the arguments of Sects. 3.2 and 3.3, we prove the following deformation result by means of an augmented functional. See also [12,13].

Proposition 13 *Assume* (F1)–(F3). *Let* $b \in \mathbb{R}$ *and let* $\mathcal{O}$ *be a neighborhood of* $K_b(\lambda)$. *Let* $\bar{\varepsilon} > 0$, *then there exist* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\eta : [0, 1] \times H_r^1(\mathbb{R}^N) \to H_r^1(\mathbb{R}^N)$ *continuous such that*

1. $\eta(0, \cdot) = id_{H_r^1(\mathbb{R}^N)}$;

2. η fixes $[\mathcal{J} \leq b - \bar{\varepsilon}]$, that is, $\eta(t, u) = u$ for all $t \in [0, 1]$ and $\mathcal{J}(u) \leq b$;
3. $\mathcal{J}$ is non-increasing along η, and in particular $\mathcal{J}(\eta(t, \cdot)) \leq \mathcal{J}(\cdot)$ for all $t \in [0, 1]$;
4. if $K_b = \emptyset$, then $\eta(1, [\mathcal{J} \leq b + \varepsilon]) \subset [\mathcal{J} \leq b - \varepsilon]$;
5. if $K_b \neq \emptyset$, then

$$\eta(1, [\mathcal{J} \leq b + \varepsilon] \setminus \mathcal{O}) \subset [\mathcal{J} \leq b - \varepsilon]$$

   and

$$\eta(1, [\mathcal{J} \leq b + \varepsilon]) \subset [\mathcal{J} \leq b - \varepsilon] \cup \mathcal{O};$$

6. if (F5) holds, then $\eta(t, \cdot)$ is $\mathbb{Z}_2$-equivariant, i.e. it is odd.

As in Sect. 4.4, for any $n \in \mathbb{N}^*$ we define $\Gamma_n := \Gamma_n(\lambda)$. We note that $\Gamma_n \neq \emptyset$ is shown in Proposition 6. Now our Theorem 2 can be obtained through the arguments given in [46]. Here we just give the definition of another minimax classes Λ_n^m, which ensures the multiplicity of solutions. We set for $n \in \mathbb{N}^*$

$$\Lambda_n := \{A = \Theta(\overline{D_{n+l} \setminus Y}) \mid l \in \mathbb{N}^*, \ \Theta \in \Gamma_{n+l}(\lambda),$$
$$Y \subset D_{n+l} \setminus \{0\} \text{ is closed,}$$
$$\text{symmetric and genus}(Y) \leq l\}$$

and

$$c_n := \inf_{A \in \Lambda_n(\lambda)} \sup_A \mathcal{J}.$$

Then we have $\{\gamma(D_n) \mid \gamma \in \Gamma_n\} \subset \Lambda_n$ and we can also see that

$$0 < c_1 \leq c_2 \leq \cdots \leq c_n \leq \cdots.$$

Moreover we have the following result.

Proposition 14 *Assume* (F1)–(F5). *Let* $n \in \mathbb{N}^*$ *and* $m > 0$. *Then*

(i) $\Lambda_n \neq \emptyset$ *and* $c_n \leq c_{n+1}$.
(ii) *Let* $A \in \Lambda_n$ *and* $Z \subset H_r^1(\mathbb{R}^N)$ *be* $\mathbb{Z}_2$-*invariant, closed, and such that* $0 \notin \overline{Z}$ *and* genus$(\overline{Z}) \leq i < n$. *Then* $\overline{A \setminus Z} \in \Lambda_{n-i}$.
(iii) c_n *is a critical value of* $\mathcal{J}$. *Moreover*

$$c_n \to +\infty \quad \text{as } n \to +\infty.$$

In particular, $\mathcal{J}$ has an unbounded sequence of critical values.

Proof Using Proposition 13, the proof can be given along the lines in [46]. See also [13]. □

Proof of Theorem 2 Theorem 2 follows from Proposition 14. □

Declarations

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