This is the preprint of the paper
Title: Fiscal policy delays and the classical growth cycle
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Published in Applied Mathematics and Computation, 354, 2019, 9-31
https://doi.org/10.1016/j.amc.2019.02.030

Fiscal policy delays and the classical growth cycle

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Abstract

This paper deals with the impact of fiscal policy delays on the national income adjustment process. Here we reconsidered the pioneering work by Wolfstetter, who introduced the public sector in the well-known Goodwin's classical growth cycle model, where the conflict between capital and labor on the distribution of income is formalized. Unlike Wolfstetter, we take into account two finite time delays characterizing the public economic activity. The former delay concerns the structure of the tax system and the government tax revenues; the latter pertains the political process governing the public purchase decisions and the actual expenditures. The result is a system of delayed differential equations (DDEs). Choosing delay terms as bifurcation parameters, we proved the existence of Hopf bifurcations. Therefore, we studied the stability and the direction of the bifurcating periodic solutions by using the first Lyapunov coefficient. Some numerical simulations carried out to support theoretical results show that, in the basic model, which coincides with the one by Wolfstetter. The effectiveness of policies (pro-cyclical and

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counter-cyclical) are strictly dependent on the length of the lags and on their particular combinations. As the basic model lacks an investment function, because investments passively equals the saving, we add that function taking into account the profit expectations. Furthermore, we assumed that the size of the public expenditure decreases quickly with the rise of the employment rate. These new hypotheses are such that to yield an extended model, where, unlike the basic model, we proved that, without lags, a pro-cyclical policy does not assure the stabilization of the economy if the government adopts weak reduction of the public expenditure. In this case, regular cycles around the equilibrium arise. When the lags are positive, the government might stabilize the system only by a low discretional expenditure, if the policy is counter-cyclical, and by low reduction of its expenditure, if the policy is pro-cyclical. This on condition that some particular pairs of the two delays subsist.

Keywords: DDEs, Hopf bifurcation, Growth cycle, Limit cycle.

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1. Introduction

Recently, many papers concerning the impact of fiscal policy delays on the national income adjustment processes enriched the economic modelling. As we know, finite lags either in the tax revenues or in the government expenditure characterize fiscal policy measures.

As far as the delay in the taxation revenues is concerned, we can observe that any economic system has collection lags in its tax system, because the accrual and the payment of taxes are not simultaneous. This implies that, at every time t, tax revenues come from a sum of finite sequence of differently lagged incomes, whose tax rate may be different. Furthermore, looking at the public expenditure, we know that it responds to the change of national income with a delay, because of the political process underlying the public purchase decisions. Recently, several models in this field [1, 2, 4, 9, 10, 11, 13, 14, 20] show that the income dynamic path may evolve either as regular cycle or as an erratic behavior.

With the exception of the work by Yoshida and Asada[1] [20], never the consequence of policy lags has been studied in a theoretical framework like the one by Goodwin, where the conflict between capital and labor on the distribution of income is modelled. Here we filled up this gap reconsidering the pioneering work by Wolfstetter [16], who introduced the public sector in the Goodwin [5] growth cycle model. His main purpose was "to provide counterexample against both Keynesian and classical view of fiscal stabilization". To do this, with reference to the basic version of the model, which coincides with the one by Goodwin, Wolfstetter proved that only the pro-cyclical fiscal policy rule (neoclassical view) assured a globally asymptotic stability of the equilibrium, while a counter-cyclical policy rule (Keynesian view) gave rise without escape to instability. Furthermore, by setting two slight changes in the original model (i.e. by adding the labor market reaction to the inflation rate, and the degree of capacity utilization in the employment rate dynamics) Wolfstetter suggested a three dimensional system that might be locally stable independently of the kind of policy adopted by the government. The system stability was only dependent on the strength of the public expenditure.

[1] In their paper, the authors studied a three dimensional system in the Goodwin's tradition, but they assume that only the lag in the public expenditure exists.

Our paper takes up Wolfstetter's basic model, which coincides with the one by Goodwin, and adds two finite delays in the public sector. The one dealing with the structure of the tax system is justified because all workers pay their taxes when they perceive the wage. Whereas, capitalists are taxed at fixed date institutionally fixed. The one concerning the public expenditure simply derives from the political process governing all the expenditures of the public sector. These two hypotheses allow us to prove that the effectiveness of fiscal policy is strictly dependent either on the length of the lags or on their particular combinations. This implies that, whatever the kind of policy may be, without controlling completely the lags, the outcomes of stabilization policy take the risk of being ineffectual to direct the GDP dynamics. Furthermore, we suggest an extension of the model, which is still two dimensional to preserve the elegant simplicity of the one by Goodwin. This extension adds to the basic model an investment function, defined by an accelerator coefficient that considers the profit expectations, and a nonlinear function of the discretionary public expenditure. Without lags, the result is that only a pro-cyclical policy adopted with sufficiently strength might stabilize the economy, otherwise the system is still unstable and cyclical movements arise. Instead, the introduction of the time delays requires a weak discretional public expenditure, together with particular pairs of the lags, to achieve the system stability. This last conclusion is true whatever the kind of policy may be.

We organized the paper as follows. Sections 2 and 3 contain a formal description of the basic and the extended model respectively. Sections 4 and 6 investigate the qualitative behavior of the DDEs systems, where the existence of Hopf bifurcations is proved. Section 5 studies the stability and the direction of the bifurcating periodic solutions by using the first Lyapunov coefficient. By means of different sets of fiscal policy parameters and different combinations of the lags, section 7 provides the model with some numerical simulations and some comments on the economic implications of the results. Section 8 is devoted to the conclusions.

2. The basic model

Following Wolfstetter, we add to the Goodwnin model the public sector and leave any other Goodwin's assumption unchanged. Therefore, we set:

$a_t = a_0 e^{\alpha t} \quad (\alpha > 0)$	labor productivity;
$N_t = N_0 e^{\beta t} \quad (\beta > 0)$	labor force;
$Q_t = a_t L_t = \sigma K_t$	real GDP, where $\sigma = Q_t/K_t < 1$ (constant) is the capital productivity;
$\nu_t = L_t/N_t = Q_t/(a_t N_t)$	employment rate;
$\dfrac{\dot{w}}{w_t} = -\gamma + \dfrac{\rho}{1-\nu_t}$	real wage dynamics (Phillips curve), where $\gamma > \rho > 0$;
$u_t = w_t L_t / Q_t$	wage share in national income.

Logarithmic differentiation of ν_t and u_t yields[2] :

$$\frac{\dot{\nu}}{\nu_t} = \frac{\dot{Q}}{Q_t} - (\alpha + \beta) \tag{1}$$

$$\frac{\dot{u}}{u_t} = \frac{\rho}{1-\nu_t} - (\alpha + \gamma). \tag{2}$$

[2]As usual, a dot on the variable will indicate the operator d/dt.

We assume an extreme classical saving function and a public debt financed by state bonds[3]. With reference to the tax system, we postulate that tax revenues come from the current labor incomes and, with a lag τ_1 from the profit incomes. Furthermore, as the government purchase decisions are subjected to political processes, we introduce a lag τ_2 in the public actual expenditure. These assumptions imply that the budget constraints of the government sector, capitalists and workers are as follows:

$$\dot{B} = G_t - T_t + iB_t$$

$$S_t = (1 - u_t)Q_t - (1 - u_{t-\tau_1})\delta_k Q_{t-\tau_1} + (1 - \delta_k)iB_t$$

$$C_t = (1 - \delta_w)u_t Q_t$$

where G_t, T_t and iB_t are the public expenditure, the tax revenues and the interests on the debt[4] respectively; S_t the aggregate saving that equals the after tax income of capitalists ($0 < \delta_k < 1$ is the tax rate) and $Q_{t-\tau_1}$ is referred to the time $t - \tau_1$; C_t the aggregate consumption that equals the after tax labor incomes ($0 < \delta_w < \delta_k$)[5].

The market equilibrium condition requires

$$\dot{K} + G_t = S_t + (T_t - iB_t) \tag{3}$$

[3]This assumption implies that the aggregate saving finances both the public debt and investments.

[4]i = rate of interest constant and exogenous.

[5]As it happens in the real world, the tax rates on the profit and labor incomes are different. However, the distinction has no implication on the system dynamics. Furthermore, we have to point out that, like Wolfstetter, we assume that the government formulates and administers policies without regard for their effects upon the market price of debt instruments.

where $\dot{K}$ is the investment and $G_t = T_t - \delta_k iB + \mu(\nu^* - \nu_{t-\tau_2})Q_{t-\tau_2}$ the equality between the public expenditure and the income tax receipts plus a discretionary component linked with the deviation of the actual $(1-\nu_t)$ from the natural $(1-v^*)$ rate of unemployment. Because of the political process underlying the discretionary purchases, the national income and the actual rate of unemployment are really referred to the time $t-\tau_2$.

Taking into account that $T_t = \delta_w u_t Q_t + \delta_k(1-u_{t-\tau_1})Q_{t-\tau_1} + \delta_k iB_t$, after some rearrangement, equation (3) can be rewritten as

$$\dot{K} = (1-u_t)Q_t - \delta_k(1-u_{t-\tau_1})Q_{t-\tau_1} - \mu(\nu^* - \nu_{t-\tau_2})Q_{t-\tau_2}$$

so that

$$\frac{\dot{K}}{K_t} = \frac{\dot{Q}}{Q_t} = \sigma\frac{\dot{K}}{Q_t} = \sigma\left[(1-u_t) - \delta_k(1-u_{t-\tau_1})\frac{Q_{t-\tau_1}}{Q_t} - \mu(\nu^* - \nu_{t-\tau_2})\frac{Q_{t-\tau_2}}{Q_t}\right].$$

Since

$$\frac{Q_{t-\tau_j}}{Q_t} = \frac{a_0 e^{\alpha(t-\tau_j)}L_{t-\tau_j}}{a_0 e^{\alpha t}L_t} = \frac{e^{\alpha(t-\tau_j)}N_0 e^{\beta\tau_j}\nu_{t-\tau_j}}{e^{\alpha t}N_0 e^{\beta t}\nu_t} = e^{-(\alpha+\beta)\tau_j}\ \frac{\nu_{t-\tau_j}}{\nu_t},\ \text{where } j = 1, 2$$

it follows that

$$\frac{\dot{Q}}{Q_t} = \sigma\left[(1-u_t) - \delta_k(1-u_{t-\tau_1})e^{-(\alpha+\beta)\tau_1}\ \frac{\nu_{t-\tau_1}}{\nu_t} - \mu(\nu^* - \nu_{t-\tau_2})e^{-(\alpha+\beta)\tau_2}\ \frac{\nu_{t-\tau_2}}{\nu_t}\right].$$

Therefore, the equation (1) becomes

$$\frac{\dot{\nu}}{\nu_t} = \sigma\left[(1-u_t) - \delta_k(1-u_{t-\tau_1})e^{-g\tau_1}\ \frac{\nu_{t-\tau_1}}{\nu_t} - \mu(\nu^* - \nu_{t-\tau_2})e^{-g\tau_2}\ \frac{\nu_{t-\tau_2}}{\nu_t}\right] - g \tag{4}$$

where we set $g = \alpha + \beta$ as the natural rate of growth.

3. The extended model

Unlike Wolfstetter, our extension of the model is still two dimensional to keep unchanged the elegant simplicity of the one by Goodwin. Our first new assumption replaces the “naive”hypothesis making the investment passively equal to the saving. So that we introduce a simple investment function taking into account the profit expectations. As it is logical to suppose here, these expectations are linked with the expected labor cost dynamics. Therefore, we set:

$$\dot{K} = \phi(\nu_t)\dot{Q} \tag{5}$$

such that $\phi' < 0$, $\phi(\nu^*) = \sigma^{-1}$ because that expected cost is mainly dependent on the current rate of employment, which explains the insiders bargaining power.

Our second new assumption concerns the discretionary public expenditure. We assume that the size of the public sector in the economy decreases quickly with the rise of the employment rate. Therefore, the previous equation of G_t becomes

$$G_t = T_t - \delta_k iB_t + \mu \frac{\nu^* - \nu_{t-\tau_2}}{1 - \nu_{t-\tau_2}} Q_{t-\tau_2}.$$

For the sake of simplicity, by assuming a linear approximation of equation (5), i.e.

$$\dot{K} = \left[\sigma^{-1} - \zeta\left(\frac{\nu_t}{\nu^*} - 1\right)\right]\dot{Q},$$

and the ex-post equality between savings and investments, the equation (3) can be rewritten as:

$$\left[\sigma^{-1} - \zeta\left(\frac{\nu_t}{\nu^*} - 1\right)\right]\dot{Q} + G_t = S_t + (T_t - iB_t).$$

The necessary substitutions yield

$$\left[\sigma^{-1} - \zeta\left(\frac{\nu_t}{\nu^*} - 1\right)\right]\dot{Q} + \mu\frac{\nu^* - \nu_{t-\tau_2}}{1-\nu_{t-\tau_2}}Q_{t-\tau_2} = (1-u_t)Q_t - \delta_k(1-u_{t-\tau_1})Q_{t-\tau_1}.$$

As $\frac{Q_{t-\tau_j}}{Q_t} = e^{g\tau_j}\frac{\nu_{t-\tau_j}}{\nu_t}$, dividing both sides by Q_t and rearranging, we have:

$$\frac{\dot{Q}}{Q_t} = \left[(1-u_t) - \delta_k(1-u_{t-\tau_1})\frac{\nu_{t-\tau_1}}{e^{g\tau_1}\nu_t} - \mu\frac{\nu^* - \nu_{t-\tau_2}}{1-\nu_{t-\tau_2}}\frac{\nu_{t-\tau_2}}{e^{g\tau_2}\nu_t}\right]\frac{\sigma\nu^*}{\nu^* - \zeta\sigma(\nu_t - \nu^*)}.$$

Therefore, the new system becomes:

$$\begin{cases} \dot{\nu} = \left\{\left[(1-u_t) - \delta_k(1-u_{t-\tau_1})\frac{\nu_{t-\tau_1}}{e^{g\tau_1}\nu_t} - \mu\frac{\nu^* - \nu_{t-\tau_2}}{1-\nu_{t-\tau_2}}\frac{\nu_{t-\tau_2}}{e^{g\tau_2}\nu_t}\right]\frac{\sigma\nu^*}{\nu^* - \zeta\sigma(\nu_t - \nu^*)} - g\right\}\nu_t \\ \dot{u} = \left[\frac{\rho}{1-\nu_t} - (\alpha+\gamma)\right]u_t \end{cases} \tag{6}$$

4. Qualitative analysis 1

The equations (4) and (2) make up a system of delay differential equations (DDEs)

$$\begin{cases} \dot{\nu} = \left\{\sigma\left[(1-u_t) - \delta_k(1-u_{t-\tau_1})e^{-g\tau_1}\frac{\nu_{t-\tau_1}}{\nu_t} - \mu(\nu^* - \nu_{t-\tau_2})e^{-g\tau_2}\frac{\nu_{t-\tau_2}}{\nu_t}\right] - g\right\}\nu_t \\ \dot{u} = \left[\frac{\rho}{1-\nu_t} - (\alpha+\gamma)\right]u_t \end{cases} \tag{7}$$

which has three equilibrium points if $\mu \neq 0$:

- $\nu_1 = 0, \quad u_1 = 0,$
- $\nu_2 = \nu^*, \quad u_2 = u^*,$
- $\nu_3 = \nu^* - \frac{\sigma(1-\delta_k e^{-g\tau_1}) - g}{\sigma\mu e^{-g\tau_2}}, \quad u_3 = 0,$

where $\nu^* = 1 - \frac{\rho}{\alpha+\gamma}$, $u^* = 1 - \frac{g}{\sigma(1-\delta_k e^{-g\tau_1})}$. If $\mu = 0$ the equilibrium points are only $(0,0)$ and (ν^*, u^*).

We set $g < \sigma(1-\delta_k)$, so that $0 < u^* < 1$. Furthermore, as $\rho < \gamma < \alpha + \gamma$ it results $0 < \nu^* < 1$. Whereas, we have that

$$\begin{cases} v_3 < 0 & \text{if} \quad 0 < \mu < e^{g\tau_2} \dfrac{\sigma(1-\delta_k e^{-g\tau_1}) - g}{\sigma \nu^*}, \\ v_3 > 1 & \text{if} \quad -e^{g\tau_2} \dfrac{\sigma(1-\delta_k e^{-g\tau_1}) - g}{\sigma(1-\nu^*)} < \mu < 0, \\ 0 \le v_3 \le 1 & \text{otherwise.} \end{cases} \tag{8}$$

It is clear that the first and the second inequality have no economic meaning.

4.0.1. The case $\tau_1 = \tau_2 = 0$

In this case the system (7) becomes

$$\begin{cases} \dot{\nu} = \{\sigma\left[(1-\delta_k)(1-u_t) - \mu(\nu^* - \nu_t)\right] - g\}\, \nu_t \\ \dot{u} = \left[\frac{\rho}{1-\nu_t} - (\alpha+\gamma)\right] u_t \end{cases} \tag{9}$$

As usual, we investigate the local dynamics of the system (9) analytically by means of the linear approximation method. Expanding the system (9) in a Taylor series around an equilibrium point (ν_r, u_r), $r = 1, 2, 3$, and neglecting the terms of higher order than the first order, we have the following linear approximation:

$$\begin{pmatrix} \dot{\nu} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \nu_t - \nu_r \\ u_t - u_r \end{pmatrix}$$

where

$$
\begin{aligned}
a_{11} &= \{\sigma\,[(1-\delta_k)(1-u_r)-\mu(\nu^*-\nu_r)]-g\}+\sigma\mu\nu_r,\\
a_{12} &= -\sigma(1-\delta_k)\nu_r,\\
a_{21} &= \frac{\rho u_r}{(1-\nu_r)^2},\\
a_{22} &= \frac{\rho}{1-\nu_r}-(\alpha+\gamma).
\end{aligned}
$$

The Jacobian of the system (9) at the equilibrium point $(0,0)$ is a diagonal matrix

$$
\begin{pmatrix} \sigma[(1-\delta_k)-\mu\nu^*]-g & 0 \\ 0 & \rho-(\alpha+\gamma) \end{pmatrix}
$$

hence the eigenvalues are the entries on its main diagonal so that this equilibrium is unstable iff

$$
\mu < \overline{\overline{\mu}} := \frac{\sigma(1-\delta_k)-g}{\sigma\nu^*}.
$$

The Jacobian of the system (9) at the equilibrium point (ν^*, u^*) is

$$
\begin{pmatrix} \sigma\mu\nu^* & -\sigma(1-\delta_k)\nu^* \\ \frac{\rho u^*}{(1-\nu^*)^2} & 0 \end{pmatrix}
$$

Its characteristic equation is

$$
\lambda^2-\sigma\mu\nu^*\lambda+\sigma(1-\delta_k)\frac{\rho\nu^*u^*}{(1-\nu^*)^2}=0
$$

which admits two solutions with negative real part iff $\mu < 0$. As shown by Wolfstetter, the equilibrium point (ν^*, u^*) is stable for $\mu < 0$ and unstable for $\mu > 0$.

We now examine the Jacobian of the system (9) at the equilibrium point $(\nu_3, 0)$ completely disregarded by Wolfstetter. In this case we have a triangular matrix

$$
\begin{pmatrix} \sigma\mu\nu_3 & -\sigma(1-\delta_k)\nu_3 \\ 0 & \frac{\rho}{1-\nu_3}-(\alpha+\gamma) \end{pmatrix}
$$

whose eigenvalues are the entries on its main diagonal. It follows that the equilibrium point $(\nu_3, 0)$ is locally stable if and only if

$$\begin{cases} \sigma\mu\nu_3 < 0 \\ \dfrac{\rho}{1-\nu_3} - (\alpha+\gamma) < 0 \end{cases}$$

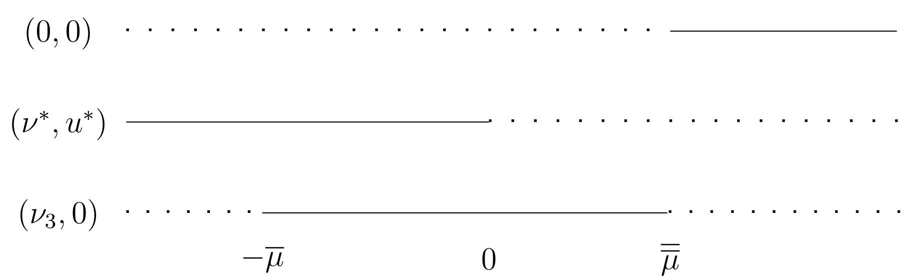
This implies that, at $(\nu_3, 0)$ with $\mu \neq 0$, the following inequalities have to be satisfied to make that point stable:

$$-\overline{\mu} < \mu < \overline{\overline{\mu}}$$

where

$$\overline{\mu} = \frac{\sigma(1-\delta_k) - g}{\sigma(1-\nu^*)}.$$

From conditions (8) we have that if $-\overline{\mu} < \mu < 0$ then $v_3 > 1$ and if $0 < \mu < \overline{\overline{\mu}}$ then $v_3 < 0$. It follows that, if equilibrium point $(\nu_3, 0)$ is stable, it has no economic meaning. We summarize the stability of the three equilibrium points as follows:



where the continuous lines show the interval where the equilibria are stable and the dotted lines the interval where the equilibria are unstable.

4.1. The case $\tau_1 > 0$ and $\tau_2 = 0$

The linearized system around an equilibrium point (ν_r, u_r), $r = 1, 2, 3$, is

$$\begin{pmatrix} \dot{\nu} \\ \dot{u} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \nu_t - \nu_r \\ u_t - u_r \end{pmatrix} + \mathcal{B} \begin{pmatrix} \nu_{t-\tau_1} - \nu_r \\ u_{t-\tau_1} - u_r \end{pmatrix} \tag{10}$$

where

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix} \tag{11}$$

$$a_{11} = \sigma(1 - u_r + 2\mu\nu_r - \mu\nu^*) - g, \quad a_{12} = -\sigma\nu_r,$$

$$a_{21} = \frac{\rho u_r}{(1-\nu_r)^2}, \quad a_{22} = \frac{\rho}{1-\nu_r} - (\alpha + \gamma),$$

$$b_{11} = -\sigma\delta_k(1 - u_r)e^{-g\tau_1}, \quad b_{12} = \sigma\delta_k\nu_r e^{-g\tau_1}.$$

To investigate the linear stability, we examine the eigenvalues of the system (10) which are the solutions of the characteristic equation

$$\det(\lambda I - \mathcal{A} - \mathcal{B}e^{-\lambda\tau_1}) = 0. \tag{12}$$

It is well known that an equilibrium point is locally stable if all the eigenvalues have negative real part and unstable if at least one eigenvalue has positive real part. Looking at the corollary by Ruan and Wei [12], if τ_1 varies, the sum of the orders of the zeros with positive real part of equation (12) can change only when a zero appears on or crosses the imaginary axis. An equilibrium point may change its stability from stable to unstable or vice versa, and the system exhibits a Hopf bifurcation if a purely imaginary root crosses the imaginary axis with nonzero speed.

We look for imaginary roots $\lambda = i\omega$ of characteristic equation (12) which has the form

$$P(\lambda, \tau_1) + Q(\lambda, \tau_1)e^{-\lambda\tau_1} = 0$$

where P and Q are respectively second and first order degree polynomials in λ with real coefficients dependent on τ_1. Because of the coefficients of P and Q are real, if $\lambda = i\omega$ is an eigenvalue, even $\lambda = -i\omega$ is an eigenvalue. Without loss of generality, we can assume that $\omega > 0$. If P and Q have a common root $\lambda = \overline{\lambda}$, then we can rewrite the characteristic equation as $(\lambda - \overline{\lambda})(P_1(\lambda, \tau_1) + Q_1(\lambda, \tau_1)e^{-\lambda\tau_1}) = 0$, where P_1 and Q_1 are co-prime. If the real part of $\overline{\lambda}$ is positive, the equilibrium point is always unstable. Otherwise, if the real part of $\overline{\lambda}$ is negative, the characteristic equation can be reduced to $P_1(\lambda, \tau_1) + Q_1(\lambda, \tau_1)e^{-\lambda\tau_1} = 0$.

4.1.1. Linear stability of (0,0)

Here we have that P and Q have a common root $\overline{\lambda} = \rho - (\alpha + \gamma) < 0$, hence we can study the reduced characteristic equation, which, after some calculations, is

$$\lambda - \sigma(1 - \mu\nu^*) + g + \sigma\delta_k e^{-g\tau_1} e^{-\lambda\tau_1} = 0. \tag{13}$$

In the Appendix we prove that, if

$$\frac{\sigma\delta_k}{g} < \frac{e\pi}{2} \approx 4.27$$

then equation (13) has no purely imaginary roots. Hence the stability of equilibrium point $(0, 0)$ with no delay does not change for $\tau_1 > 0$.

If $\dfrac{\sigma\delta_k}{g} \geq \dfrac{e\pi}{2}$ and $\dfrac{\sigma - g}{\sigma\nu^*} < \mu < \dfrac{\sigma(1 + \delta_k) - g}{\sigma\nu^*}$, some stability switches may occur. Let us notice that if μ is negative the previous condition does not hold and no stability switches exist.

4.1.2. Linear stability of $(\nu_3, 0)$

Also in this case P and Q have a common root $\overline{\lambda} = \frac{\rho}{1-\nu_3} - (\alpha + \gamma)$. Note that the eigenvalue $\overline{\lambda}$ is positive, iff $\nu^* < \nu_3 < 1$. From (8), this condition holds if $\mu < -\frac{\sigma(1-\delta_k e^{-g\tau_1})-g}{\sigma(1-\nu^*)}$. In this case the equilibrium point is unstable.

The reduced characteristic equation becomes

$$\lambda - \sigma(1 + 2\mu\nu_3 - \mu\nu^*) + g + \sigma\delta_k e^{-g\tau_1} e^{-\lambda\tau_1} = 0. \tag{14}$$

In the Appendix we prove that if $0 < \nu_3 < \nu^*$, that is $\mu > \frac{\sigma(1-\delta_k e^{-g\tau_1})-g}{\sigma\nu^*}$, the equation (14) has no purely imaginary roots. We conclude that the positive delay $\tau_1 > 0$ does not change the stability of equilibrium point $(\nu_3, 0)$ with no delay.

4.1.3. Linear stability of (ν^*, u^*)

By substitution of the equilibrium point (ν^*, u^*) in the Jacobian matrices (11), the characteristic equation (12) becomes

$$\lambda^2 - a_{11}\lambda - a_{12}a_{21} - (b_{11}\lambda + a_{21}b_{12})e^{-\tau_1\lambda} = 0. \tag{15}$$

Let us note that a_{12} and b_{11} are negative and a_{21}, b_{12} are positive. Furthermore, the coefficients a_{21}, b_{11} and b_{12} depend on the delay τ_1 and $a_{11} + b_{11} = \sigma\mu\nu^*$. We define $P(\lambda, \tau_1) = \lambda^2 - a_{11}\lambda - a_{12}a_{21}$ and $Q(\lambda, \tau_1) = -b_{11}\lambda - a_{21}b_{12}$ respectively the second and the first order polynomial in λ of the characteristic equation (15).

We note that $\lambda = 0$ is not an eigenvalue because

$$P(0, \tau_1) + Q(0, \tau_1) = \frac{\rho\sigma\nu^* u^*}{(1-\nu^*)^2}(1 - \delta_k e^{-g\tau_1}) > 0.$$

If we assume $\mu \neq 0$, because of $P(i\omega, \tau_1) + Q(i\omega, \tau_1) = -\omega^2 - a_{21}(a_{12} + b_{12}) - i\omega\sigma\mu\nu^*$, then $P(i\omega, \tau_1) + Q(i\omega, \tau_1) \neq 0$, for any $\tau_1 > 0$. This ensures that P and Q have no common imaginary roots.

A necessary condition for a purely imaginary root $\lambda = i\omega$, $\omega > 0$, of characteristic equation (15), is $|P(i\omega, \tau_1)|^2 = |Q(i\omega, \tau_1)|^2$. It follows that we have to find the positive solutions of

$$F(\omega, \tau_1) := |P(i\omega, \tau_1)|^2 - |Q(i\omega, \tau_1)|^2 = \omega^4 + p_1\omega^2 + p_0 = 0 \qquad (16)$$

where

$$p_1 = 2a_{12}a_{21} + a_{11}^2 - b_{11}^2 = (\sigma\nu^*)^2\mu^2 + 2\sigma\nu^*(1 - u^*)\delta_k e^{-g\tau_1}\mu - 2\frac{\rho\nu^* u^*}{(1 - \nu)^2}$$

$$p_0 = a_{21}^2(a_{12}^2 - b_{12}^2) = \frac{(\sigma\rho\nu^* u^*)^2(1 - \delta_k e^{-g\tau_1})}{(1 - \nu^*)^4} > 0.$$

Equation (16) admits positive solutions if $\Delta(\mu, \tau_1) := p_1^2 - 4p_0 \geq 0$ and $p_1 < 0$.

As p_1 is a second order polynomial in μ with a negative root $\overline{\mu}_-$ and a positive root $\overline{\mu}_+$ depending on τ_1, it follows that, if $\mu \leq \overline{\mu}_-$ or $\mu \geq \overline{\mu}_+$, then $p_1 \geq 0$ and no stability switch exists. Let us note that, if $\mu > 0$, then $\Delta(\mu, \tau_1) > 0$ is always satisfied.

Suppose that equation (16) admits positive solutions and let $\omega(\tau_1)$ be one of that solutions. Substituting $\lambda = i\omega(\tau_1)$ in equation (15) and considering the real and the imaginary parts, a purely imaginary root of characteristic equation must satisfy

$$\begin{cases} \cos\omega(\tau_1)\tau_1 = -\frac{(a_{21}b_{12} + a_{11}b_{11})\omega^2(\tau_1) + a_{12}a_{21}^2 b_{12}}{b_{11}^2\omega^2(\tau_1) + a_{21}^2 b_{12}^2} \\ \sin\omega(\tau_1)\tau_1 = \omega(\tau_1)\frac{-b_{11}\omega^2(\tau_1) + a_{21}(a_{11}b_{12} - b_{11}a_{12})}{b_{11}^2\omega^2(\tau_1) + a_{21}^2 b_{12}^2} \end{cases}$$

Suppose that τ_1^* is a solution of the previous system. Let us consider

$$\delta(\tau_1^*) := \left.\frac{d\,\mathrm{Re}\,\lambda}{d\tau_1}\right|_{\lambda=i\omega(\tau_1^*)}$$

If $\delta(\tau_1^*) > 0$, a pair of conjugate pure imaginary roots crosses the imaginary axis from left to the right; if $\delta(\tau_1^*) < 0$, then a pair of conjugate pure imaginary roots crosses the imaginary axis from right to the left.

If the equilibrium point changes its stability when τ_1 crosses τ_1^* and $\delta(\tau_1^*) \neq 0$, the system (7) exhibits an Hopf bifurcation when $\tau_1 = \tau_1^*$.

4.2. The case $\tau_1 = 0$ and $\tau_2 > 0$

The linearization of the system (7) around an equilibrium point (ν_r, u_r), $r = 1, 2, 3$, is

$$\begin{pmatrix} \dot{\nu} \\ \dot{u} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \nu_t - \nu_r \\ u_t - u_r \end{pmatrix} + \mathcal{B} \begin{pmatrix} \nu_{t-\tau_2} - \nu_r \\ u_{t-\tau_2} - u_r \end{pmatrix} \tag{17}$$

where

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix} \tag{18}$$

$$a_{11} = \sigma(1 - u_r)(1 - \delta_k) - g, \quad a_{12} = -\sigma\nu_r(1 - \delta_k),$$

$$a_{21} = \frac{\rho u_r}{(1 - \nu_r)^2}, \quad a_{22} = \frac{\rho}{1 - \nu_r} - (\alpha + \gamma),$$

$$b_{11} = -\sigma\mu(\nu^* - 2\nu_r)e^{-g\tau_2}.$$

To investigate the linear stability we need the eigenvalues of the system (17) which are the solutions of the characteristic equation

$$\det(\lambda I - \mathcal{A} - \mathcal{B}e^{-\lambda\tau_2}) = 0. \tag{19}$$

Similar considerations made earlier for characteristic equation (12) when $\tau_1 > 0$ and $\tau_2 = 0$ also apply in this case. If $\mu = 0$, the characteristic equation (19) becomes $\det(\lambda I - \mathcal{A}) = 0$ which is a second order polynomial with no delay.

4.2.1. Linear stability of (0,0)

After some calculations, the characteristic equation at equilibrium point $(0,0)$ becomes

$$[\lambda - \rho + (\alpha + \gamma)](\lambda - \sigma(1 - \delta_k) + g + \sigma\mu\nu^* e^{-g\tau_2} e^{-\lambda\tau_2}) = 0.$$

Here we have a real negative eigenvalue $\lambda_1 = \rho - (\alpha + \gamma)$ and infinite complex eigenvalues given by

$$\lambda - \sigma(1 - \delta_k) + g + \sigma\mu\nu^* e^{-g\tau_2} e^{-\lambda\tau_2} = 0. \tag{20}$$

Equation (20) correspond to equation (30) in the Appendix where $c_0 = g - \sigma(1 - \delta_k) < 0$ and $c_1 = \sigma\mu\nu$. If we suppose that $\frac{2g}{-c_0} \leq 3$, by Theorem 6, no stability switches exist for $\sigma\nu^*|\mu| \leq (\sigma(1-\delta_k) - g)e^{\frac{g}{\sigma(1-\delta_k)-g}}$. If $\frac{2g}{-c_0} > 3$, then no stability switches exist for $\sigma^2(\nu^*)^2\mu^2 \leq 2e^3[\sigma(1 - \delta_k) - g][2g - \sigma(1 - \delta_k)]$.

In all other cases some stability switches may occur.

4.2.2. Linear stability of $(\nu_3, 0)$

The characteristic equation at equilibrium point $(\nu_3, 0)$ becomes

$$[\lambda - \frac{\rho}{1 - \nu_r} + (\alpha + \gamma)](\lambda - \sigma(1 - \delta_k) + g - (\sigma\mu\nu^* e^{-g\tau_2} - 2\sigma(1 - \delta_k) + 2g)e^{-\lambda\tau_2}) = 0$$

Here we have a real eigenvalue $\lambda_1 = \frac{\rho}{1-\nu_3} - (\alpha + \gamma)$ and infinite complex eigenvalues given by

$$\lambda - \sigma(1 - \delta_k) + g - (\sigma\mu\nu^* e^{-g\tau_2} - 2\sigma(1 - \delta_k) + 2g)e^{-\lambda\tau_2} = 0. \tag{21}$$

If $\lambda_1 > 0$, that is $\nu^* < \nu_3 < 1$, the equilibrium point $(\nu_3, 0)$ is unstable. This condition holds if $\mu < -\frac{\sigma(1-\delta_k)-g}{\sigma(1-\nu^*)e^{-g\tau_2}}$.

If $0 < \nu_3 < \nu^*$, that is $\mu > \frac{\sigma(1-\delta_k)-g}{\sigma\nu^* e^{-g\tau_2}}$, we prove that the equation (21) has no purely imaginary roots for $\mu \leq 3\frac{\sigma(1-\delta_k)-g}{\sigma\nu^* e^{-g\tau_2}}$. We conclude that the positive delay $\tau_2 > 0$ does not change the stability of equilibrium point $(\nu_3, 0)$ with no delay.

4.2.3. Linear stability of (ν^*, u^*)

By substitution of the equilibrium point (ν^*, u^*) in the Jacobian matrices (18), the characteristic equation becomes

$$\lambda^2 + c_0 - 2c_1\lambda e^{-g\tau_2}e^{-\tau_2\lambda} = 0 \tag{22}$$

where $c_0 = -a_{12}a_{21} > 0$ does not depend on τ_2 and μ and $c_1 = \sigma\mu\nu^*/2$.

We define $P(\lambda, \tau_2) = \lambda^2 + c_0$ and $Q(\lambda, \tau_2) = -2c_1 e^{-g\tau_2}\lambda$ respectively the second and the first order polynomial in λ of the characteristic equation (22).

We note that $\lambda = 0$ is not an eigenvalue, indeed $P(0, \tau_2) + Q(0, \tau_2) = c_0 > 0$. Because of $P(i\omega, \tau_2) + Q(i\omega, \tau_2) = -\omega^2 - c_0 - 2i\omega c_1 e^{-g\tau_2} \neq 0$, P and Q have no common imaginary roots.

A necessary condition for a purely imaginary root $\lambda = i\omega$, $\omega > 0$, of characteristic equation (22), is $|P(i\omega, \tau_2)|^2 = |Q(i\omega, \tau_2)|^2$ which leads to $(-\omega^2 + c_0)^2 = 4c_1^2 e^{-2g\tau_2}\omega^2$. The positive solutions of the last equation are

$$\omega_{\pm} = \mp c_1 e^{-g\tau_2} + \sqrt{c_1^2 e^{-2g\tau_2} + c_0}.$$

Substituting $\lambda = i\omega_{\pm}$ in equation (22) and considering the real and the imaginary parts, a purely imaginary root of characteristic equation must satisfy

$$\begin{cases} \cos \omega_{\pm} \tau_2 = 0 \\ \sin \omega_{\pm} \tau_2 = \pm 1 \end{cases}$$

Hence, in order to find purely imaginary root of (22), we can solve the following equations

$$\begin{aligned} f_{+,n} &:= \omega_+ \tau_2 - \tfrac{\pi}{2} - 2n\pi = 0 \quad n = 0, 1, \dots \\ f_{-,n} &:= \omega_- \tau_2 - \tfrac{3\pi}{2} - 2n\pi = 0 \quad n = 0, 1, \dots \end{aligned} \tag{23}$$

We prove that:

Lemma 1. *Let us assume $\mu \neq 0$. There exists $\bar{m} \in \mathbb{N}$ such that every equation $f_{\pm,n}(\tau_2) = 0$ admits a unique solution $\tau_{\pm,n}$, for all $n = \bar{m}, \bar{m}+1, \dots$. Moreover:*

a) if $c_0 > \frac{1+\sqrt{5}}{2} e^{-(1+\sqrt{5})} \sigma^2(\nu^*) \frac{\mu^2}{4}$, then $\bar{m} = 0$ and every equation (23) admits a unique simple solution;

b) If $c_0 \leq \frac{1+\sqrt{5}}{2} e^{-(1+\sqrt{5})} \sigma^2(\nu^*) \frac{\mu^2}{4}$, then a finite number[6] of equations (23) admit three simple solutions or a double solution and a simple solution.

Proof. Note that $f_{\pm,n}(0) < 0$ and $f_{\pm,n}(+\infty) = +\infty$. After some calculations, we set

$$\varphi_{\pm}(\tau_2) := \frac{\sqrt{c_1^2 e^{-2g\tau_2} + c_0}}{\omega_{\pm}} f'_{\pm,n}(\tau_2) = \sqrt{c_1^2 e^{-2g\tau_2} + c_0} \pm c_1 g \tau_2 e^{-g\tau_2}.$$

We can prove that if

$$\frac{1+\sqrt{5}}{2} e^{-(1+\sqrt{5})} < \frac{c_0}{c_1^2}, \tag{24}$$

[6] Note that there might be no equations.

then

$$\sqrt{c_1^2 e^{-2g\tau_2} + c_0} > |c_1| g\tau_2 e^{-g\tau_2}$$

that is the sign of derivative of $f_{\pm,n}$ is strictly positive for all $\tau_2 \geq 0$. Hence, statement a) holds.

In order to prove statement b), let us suppose $c_1 > 0$. We note that if condition (24) does not hold, there exist $\bar{\tau}_2$ and $\bar{\bar{\tau}}_2$ ($\bar{\tau}_2 \leq \bar{\bar{\tau}}_2$) such that $f_{-,n}$ is increasing on intervals $[0, \bar{\tau}_2]$ and $[\bar{\bar{\tau}}_2, +\infty[$ and decreasing on interval $[\bar{\tau}_2, \bar{\bar{\tau}}_2]$. Hence, statement b) holds. If $c_1 < 0$, we obtain the same result but for the functions $f_{+,n}$. □

Lemma 2. *Let $\tau_{\pm,n}$ be a simple solution of $f_{\pm,n}(\tau_2) = 0$. Then*

$$\delta(\tau_{\pm,n}) := \operatorname{sign} \left.\frac{d \operatorname{Re} \lambda}{d\tau_2}\right|_{\lambda = i\omega_{\pm}(\tau_{\pm,n})} = \mp \operatorname{sign}(c_1) \operatorname{sign}(\varphi_{\pm})$$

Proof. By implicit differentiation of the function $F(\lambda, \tau_2) = \lambda^2 + c_0 - 2c_1\lambda e^{-(g+\lambda)\tau_2}$, we have

$$\left(\frac{\partial \lambda}{\partial \tau_2}\right)^{-1} = -\frac{e^{(g+\lambda)\tau_2}\lambda - c_1(1 - \lambda\tau_2)}{c_1\lambda(g + \lambda)}$$

Substituting $\lambda = i\omega_{\pm}(\tau_{\pm,n})$, we can prove that

$$\operatorname{sign} \left.\frac{d \operatorname{Re} \lambda}{d\tau_2}\right|_{\lambda = i\omega_{\pm}(\tau_{\pm,n})} = \operatorname{sign}(c_1) \operatorname{sign}\left(\mp\sqrt{c_1^2 e^{-2g\tau_2} + c_0} - c_1 g\tau_2 e^{-g\tau_2}\right) =$$

$$= \mp \operatorname{sign}(c_1) \operatorname{sign}(\varphi_{\pm})$$

□

Theorem 1. *We assume that $\mu \neq 0$ and every equation (23) admits a unique simple solution $\tau_{\pm,n}$, for all $n = 0, 1, \ldots$. There exists $\bar{n} \in \mathbb{N}$, such that $0 < \tau_{+,\bar{n}} < \tau_{-,\bar{n}} < \tau_{+,\bar{n}+1} < \tau_{-,\bar{n}+1} < \ldots$*

a) If $\mu > 0$, the equilibrium point (ν^*, u^*) is stable when $\tau_2 \in]\tau_{+,\bar{n}}, \tau_{-,\bar{n}}[\cup]\tau_{+,\bar{n}+1}, \tau_{-,\bar{n}+1}[\cup \ldots$ and unstable when $\tau_2 \in]0, \tau_{+,\bar{n}}[\cup]\tau_{-,\bar{n}}, \tau_{+,\bar{n}+1}] \cup \ldots$.

b) If $\mu < 0$, the equilibrium point (ν^*, u^*) is stable when $\tau_2 \in]0, \tau_{+,\bar{n}}[\cup]\tau_{-,\bar{n}}, \tau_{+,\bar{n}+1}] \cup \ldots$ and unstable when $\tau_2 \in]\tau_{+,\bar{n}}, \tau_{-,\bar{n}}[\cup]\tau_{+,\bar{n}+1}, \tau_{-,\bar{n}+1}[\cup \ldots$.

Moreover a Hopf bifurcation occurs when τ_2 passes through $\tau_{\pm,n}$.

If a finite number of equation (23) admit three simple solutions, then the theorem still holds but a finite number of additional stability switches and Hopf bifurcations may occur.

Proof. From

$$f_{+,n}(\tau_2) - f_{-,n}(\tau_2) = (\omega_+ - \omega_-)\tau_2 + \pi = -2c_1 e^{-g\tau_2} + \pi$$

$$f_{-,n}(\tau_2) - f_{+,n+1}(\tau_2) = (\omega_- - \omega_+)\tau_2 + \pi = 2c_1 e^{-g\tau_2} + \pi$$

after some calculation, we have that, if $2c_1 < ge\pi$, then $f_{+,n}(\tau_2) - f_{-,n}(\tau_2) > 0$, for all τ_2. On the other hand, if $2c_1 > -ge\pi$ then $f_{-,n}(\tau_2) - f_{+,n+1}(\tau_2) > 0$, for all τ_2. It follows that, if $c_1 \neq 0$ and $-eg\pi < 2c_1 < eg\pi$, we have $\tau_{+,n} < \tau_{-,n} < \tau_{+,n+1} < \tau_{-,n+1}$.

If $2c_1 \geq ge\pi$, then there exists $\bar{\tau}$ such that $f_{+,n}(\tau_2) - f_{-,n}(\tau_2) > 0$, for all $\tau_2 > \bar{\tau}$. If $2c_1 \leq -ge\pi$, then there exists $\bar{\bar{\tau}}$ such that $f_{-,n}(\tau_2) - f_{+,n+1}(\tau_2) > 0$, for all $\tau_2 > \bar{\bar{\tau}}$. Hence, there exists $\bar{n} \in \mathbb{N}$, such that $0 < \tau_{+,\bar{n}} < \tau_{-,\bar{n}} < \tau_{+,\bar{n}+1} < \tau_{-,\bar{n}+1} < \ldots$

If $\delta(\tau_{\pm,n}) > 0$ a pair of conjugate pure imaginary roots crosses the imaginary axis from left to the right; if $\delta(\tau_{\pm,n}) < 0$, then a pair of conjugate pure imaginary roots crosses the imaginary axis from right to the left. The statements of the Theorem follow by Lemmas 1, 2 and by the Hopf bifurcation Theorem. □

4.3. The case $\tau_1 > 0$ and $\tau_2 > 0$

Expanding the system (7) in a Taylor series around an equilibrium point (ν_r, u_r), $r = 1, 2, 3$, and neglecting the terms of higher order than the first order, we have the following linear approximation:

$$\begin{pmatrix} \dot{\nu} \\ \dot{u} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \nu_t - \nu_r \\ u_t - u_r \end{pmatrix} + \mathcal{B} \begin{pmatrix} \nu_{t-\tau_1} - \nu_r \\ u_{t-\tau_1} - u_r \end{pmatrix} + \mathcal{C} \begin{pmatrix} \nu_{t-\tau_2} - \nu_r \\ u_{t-\tau_2} - u_r \end{pmatrix} \qquad (25)$$

where

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

$$a_{11} = \sigma(1 - u_r) - g, \quad a_{12} = -\sigma\nu_r,$$

$$a_{21} = \frac{\rho u_r}{(1 - \nu_r)^2}, \quad a_{22} = \frac{\rho}{1 - \nu_r} - (\alpha + \gamma),$$

$$b_{11} = -\sigma\delta_k(1 - u_r)e^{-g\tau_1}, \quad b_{12} = \sigma\delta_k\nu_r e^{-g\tau_1},$$

$$c_{11} = \sigma(2\nu_r - \nu^*)\mu e^{-g\tau_2}.$$

To investigate the linear stability, we examine the eigenvalues of the system (25) which are the solutions of the characteristic equation

$$\det(\lambda I - \mathcal{A} - \mathcal{B}e^{-\lambda\tau_1} - \mathcal{C}e^{-\lambda\tau_2}) = 0. \qquad (26)$$

It follows that

$$\lambda^2 - (a_{11} + a_{22})\lambda + \det(\mathcal{A}) +$$

$$+(b_{11}\lambda + b_{11}a_{22} - a_{21}b_{12})e^{-\lambda\tau_1} + c_{11}(\lambda + a_{22})e^{-\lambda\tau_2} = 0.$$

For the equilibrium (ν^*, u^*) we have that $a_{22} = 0$. In this case the characteristic equation (26) becomes

$$\lambda^2 - a_{11}\lambda - a_{12}a_{21} + (b_{11}\lambda - a_{21}b_{12})e^{-\lambda\tau_1} + c_{11}\lambda e^{-\lambda\tau_2} = 0.$$

Following Li and Wei [8], we suppose that our system is stable for a given $\bar{\tau}_2$. We consider τ_1 as a parameter. A necessary condition for a purely imaginary root $\lambda = i\omega$, $\omega > 0$, of characteristic equation (26), is that ω is a solution of

$$F(\omega) := | - \omega^2 - a_{12}a_{21} - (a_{11} + c_{11}e^{i\omega\bar{\tau}_2})\omega i|^2 - |b_{11}\omega i - a_{21}b_{12}|^2 = 0$$

After some calculations, we have

$$F(\omega) = \omega^4 + d_3\omega^3 + d_2\omega^2 + d_1\omega + d_0 \tag{27}$$

where $d_3 = -2c_{11}\sin\omega\bar{\tau}_2$, $d_2 = 2a_{12}a_{21} - 2a_{11}c_{11}\cos\omega\bar{\tau}_2 + a_{11}^2 + c_{11}^2 - b_{11}^2$, $d_1 = -2a_{12}a_{21}c_{11}\sin\omega\bar{\tau}_2$, $d_0 = a_{12}^2a_{21}^2 - a_{21}^2b_{12}^2$. Let us suppose that equation (27) has finite positive roots ω_1, $\omega_2, \ldots,$ ω_n and for all $k = 1, \ldots, n$ the following system[7]

$$\begin{cases} \cos\omega\tau_1 = -\frac{(a_{11}b_{11}+a_{21}b_{12})\omega^2+a_{12}a_{21}^2b_{12}}{b_{11}^2\omega^2+a_{21}^2b_{12}^2} \\ \sin\omega\tau_1 = \omega\frac{-b_{11}\omega^2+a_{21}(a_{11}b_{12}-a_{12}b_{11})}{b_{11}^2\omega^2+a_{21}^2b_{12}^2} \end{cases}$$

admits a sequence of solutions $\tau_{1,k,j}$, $j = 1, 2, \ldots$. If $\tau_1^* = \min\{\tau_{1,k,j} | k = 1, \ldots, n; j = 1, 2, \ldots\}$ and

$$\left.\frac{d\,\mathrm{Re}\,\lambda}{d\tau_1}\right|_{\lambda = i\omega(\tau_1^*)} \neq 0$$

by Hopf bifurcation Theorem, the equilibrium point (ν^*, u^*) is asymptotically stable for $\tau_2 = \bar{\tau}_2$ and $\tau_1 < \tau_1^*$ and the system undergoes a Hopf bifurcation when $\tau_1 = \tau_1^*$.

[7] For the sake of simplicity, we neglected the ω functional dependence on τ_2.

For equilibria $(0,0)$ and $(\nu_3, 0)$ $u_r = 0$, as $a_{21} = 0$, the characteristic equation (26) becomes

$$(\lambda - a_{22})(\lambda - a_{11} + b_{11}e^{-\lambda\tau_1} + c_{11}e^{-\lambda\tau_2}) = 0.$$

The qualitative analysis of these points is similar to the one we previously made.

5. Direction of the Hopf bifurcation

Let us assume that the system (7) undergoes a Hopf bifurcation at the equilibrium point (ν^*, u^*) for $\tau_1 = \tau_1^*$ and $\tau_2 = \tau_2^*$ and the corresponding simple purely imaginary roots of the characteristic equation at (ν^*, u^*) are $\lambda = \pm i\omega_0$ $(\omega_0 > 0)$. We also assume that no other eigenvalues exist on the imaginary axis. We do not consider degenerate Hopf bifurcation. To investigate the direction of the Hopf bifurcation, we consider the critical normal form coefficients of a local bifurcation that can be calculated with different methods (see among others [6, 17, 18, 19]). We follow the approach in [15] in order to calculate the first Lyapunov coefficient for DDE defined by Kuznetsov [7] for ordinary differential equation.

We translate the critical point to the origin by an affine transformation of coordinates and for convenience we rewrite the system (7) as

$$\dot{x} = f(x_{t-\tau_0}, x_{t-\tau_1}, x_{t-\tau_2})$$

where $\tau_0 = 0$, $x_{t-\tau_j} = (x_{t-\tau_j,1}, x_{t-\tau_j,2})^T = (\nu_{t-\tau_j}, u_{t-\tau_j})^T$, $j = 0, 1, 2$,

$$f(x_{t-\tau_0}, x_{t-\tau_1}, x_{t-\tau_2}) = g(x_{t-\tau_0}+(\nu^*, u^*)^T, x_{t-\tau_1}+(\nu^*, u^*)^T, x_{t-\tau_2}+(\nu^*, u^*)^T)$$

and

$$g(x_{t-\tau_0}, x_{t-\tau_1}, x_{t-\tau_2}) =$$

$$\begin{pmatrix} \left\{\sigma\left[(1-x_{t,2})-\delta_k(1-x_{t-\tau_1,2})e^{-g\tau_1}\ \frac{x_{t-\tau_1,1}}{x_{t,1}}-\mu(\nu^*-x_{t-\tau_2,1})e^{-g\tau_2}\ \frac{x_{t-\tau_2,1}}{x_{t,1}}\right]-g\right\}x_{t,1} \\ \left[\frac{\rho}{1-x_{t,1}}-(\alpha+\gamma)\right]x_{t,2} \end{pmatrix}$$

According to the Central Manifold Theorem, the restriction of the system (7) to the two-dimensional critical center manifold is equivalent to the Poincaré normal form

$$\dot{z} = i\omega_o z + c_1 z^2\overline{z} + O(|z|^4) \qquad z \in \mathbb{C}$$

where c_1 is the critical normal form coefficient and the first Lyapunov coefficient is given by

$$l_1 = \frac{1}{\omega_0}\text{Re } c_1.$$

In the following, we give a brief overview of the computation of an explicit formula for the first Lyapunov coefficient in terms of f and its second and third order derivatives at the critical equilibrium point (see [15] for more details).

The Taylor expansion of $f(x_{t-\tau_0}, x_{t-\tau_1}, x_{t-\tau_2})$ at the origin is

$$f(X) = (Df)(X)+\frac{1}{2}(D^2 f)(X,X)+\frac{1}{6}(D^3 f)(X,X,X)+O(||X||^4) \quad X \in \mathbb{R}^{2\times 3}$$

where the components of the linear/multilinear forms $(Df)(X)$, $(D^2 f)(X,Y)$ and $(D^3 f)(X,Y,Z)$ are, for $i = 1, 2$,

$$(Df)_i(X) = \sum_{k=1}^{2}\sum_{j=0}^{3}\frac{\partial f_i}{\partial x_{t-\tau_j,k}}X_{k,j} \quad X \in \mathbb{R}^{2\times 3}$$

$$(D^2 f)_i(X,Y) = \sum_{k,r=1}^{2}\sum_{j,l=0}^{3}\frac{\partial^2 f_i}{\partial x_{t-\tau_j,k}\partial x_{t-\tau_l,r}}X_{k,j}Y_{r,l} \quad X,Y \in \mathbb{R}^{2\times 3}$$

$$(D^3 f)_i(X,Y,Z) = \sum_{k,r,s=1}^{2} \sum_{j,l,m=0}^{3} \frac{\partial^3 f_i}{\partial x_{t-\tau_j,k} \partial x_{t-\tau_l,r} \partial x_{t-\tau_m,s}} X_{k,j} Y_{r,l} Z_{s,m} \quad X,Y,Z \in \mathbb{R}^{2\times 3}$$

By the linear approximation (25), we define the characteristic matrix $\Delta(\lambda) = \lambda I - \mathcal{A} - \mathcal{B}e^{-\lambda\tau_1} - \mathcal{C}e^{-\lambda\tau_2}$ where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are evaluated at the critical equilibrium point. Let $p, q \in \mathbb{R}^2$ be two vector in the nullspaces respectively of $\Delta(i\omega_0)$ and $\Delta^T(i\omega_0)$ such that, assumed Δ' as the derivative with respect to λ, the normalization condition $p^T \Delta'(i\omega_0) q = 1$ holds. The coefficient c_1 is given by

$$c_1 = \frac{1}{2} p^T \left[(D^2 f)(\overline{\Phi}, H_{20}) + 2(D^2 f)(\Phi, H_{11}) + (D^3 f)(\Phi, \Phi, \overline{\Phi}) \right]$$

with

$$\Phi = (q, e^{-i\omega_0\tau_1} q, e^{-i\omega_0\tau_2} q)$$

$$h_{20}^j = e^{-2i\omega_0\tau_j} \Delta(2i\omega_0)^{-1} (D^2 f)(\Phi, \Phi) \quad j = 0,1,2$$

$$h_{11} = \Delta(0)^{-1} (D^2 f)(\Phi, \overline{\Phi})$$

$$H_{20} = (h_{20}^0, h_{20}^1, h_{20}^2), \quad H_{11} = (h_{11}, h_{11}, h_{11})$$

For the basic model, the multilinear forms in the Taylor expansion are

$$(D^2 f)_1(X,Y) = \delta_k \sigma e^{-g\tau_1} (X_{1,2}Y_{2,2} + X_{2,2}Y_{1,2}) - \sigma(X_{2,1}Y_{1,1} + X_{1,1}Y_{2,1}) + 2\mu\sigma e^{-g\tau_2} X_{1,3}Y_{1,3}$$

$$(D^2 f)_2(X,Y) = \frac{\rho}{(1-\nu^*)^2} (X_{1,1}Y_{2,1} + X_{2,1}Y_{1,1}) + \frac{2\rho u^*}{(1-\nu^*)^3} X_{1,1}Y_{1,1}$$

$$(D^3 f)_1(X,Y,Z) = 0$$

$$(D^3 f)_2(X,Y,Z) = \frac{6\rho u^*}{(1-\nu^*)^4} X_{1,1}Y_{1,1}Z_{1,1} + \frac{2\rho}{(1-\nu^*)^3} (X_{1,1}Y_{2,1}Z_{1,1} + X_{2,1}Y_{1,1}Z_{1,1} + X_{1,1}Y_{1,1}Z_{2,1})$$

To calculate the direction of the Hopf bifurcation, we can use the following result (see e.g. [7]):

Theorem 2. *A non-degenerate Hopf bifurcation is supercritical if $l_1 < 0$ and subcritical if $l_1 > 0$.*

6. Qualitative analysis 2

The basic model and the one we defined extended have two common equilibria: $(0,0)$ and (ν^*, u^*). Instead, if it exists, the third is different and may be a twofold equilibrium if $\mu < 0$.

6.1. The case $\tau_1 = \tau_2 = 0$

At the origin the characteristic equation of the system (6) is

$$\lambda^2 - \mathrm{tr}J_0\lambda + \det J_0 = 0$$

where J_0 is the Jacobian matrix evaluated at $(0,0)$, $\mathrm{tr}J_0 = [(1-\delta_k) - \mu\nu^*]\frac{\sigma}{1+\sigma\zeta} - g + [\rho - (\alpha+\gamma)]$ and $\det J_0 = \left\{[(1-\delta_k) - \mu\nu^*]\frac{\sigma}{1+\sigma\zeta} - g\right\}[\rho - (\alpha+\gamma)]$. Let us note that the sign of $\det J_0$ depends on the sign and the value of μ and that $\rho - (\alpha+\gamma) < 0$. Therefore, $\det J_0 \gtreqless 0$. From the economic point of view, this means that, if μ is such that $\mathrm{tr}J_0 < 0$ and $\det J_0 > 0$, fiscal policy might generate a great economic crash if the system is in the basin of attraction of the origin. This whatever may be the kind of policy. However, here we shall consider as a normal case value of μ such that $\det J_0 < 0$, which make the origin unstable.

At the point (ν^*, u^*) the Jacobian matrix (J^*) yields

$$\lambda^2 - \mathrm{tr}J^*\lambda + \det J^* = 0 \tag{28}$$

where $\mathrm{tr}J^* = \mu\frac{\sigma\nu^*}{1-\nu^*} + g\sigma\zeta$ and $\det J^* = (1-\delta_k)\sigma\nu^*\frac{(\alpha+\gamma)^2}{\rho}u^*$. Easily it can be verified that $\mu > 0$ implies an unstable equilibrium, because $\mathrm{tr}\ J^* > 0$ and

$\det J^* > 0$. Whereas, if $\mu < 0$,

$$\mathrm{tr} J^* \le 0 \Leftrightarrow \mu \le -\frac{g\zeta(1-\nu^*)}{\nu^*}.$$

Let $\bar{\mu} < 0$ be the value of μ such that the characteristic equation (28) has pure imaginary roots. According to the Hopf bifurcation Theorem, it can be shown that for $\mu \in]\bar{\mu}, 0[$, the equilibrium point (ν^*, u^*) is an unstable focus surrounded by a stable limit cycle, whose size increases with μ; while for $\mu < \bar{\mu}$, the equilibrium point is a stable focus inclined to become a stable node for values of μ sufficiently low[8].

The additional equilibria are characterized, if they exist, by $u = 0$. The Jacobian matrix evaluated at generic point $(\nu, 0)$ is triangular

$$J_3 = \begin{pmatrix} j_{11}^{(3)} & j_{12}^{(3)} \\ 0 & j_{22}^{(3)} \end{pmatrix}$$

where

$$j_{11}^{(3)} = \left\{ \frac{\mu}{1-\nu} \left[\frac{\nu^* - 1}{1-\nu} + (\nu^* - \nu) \frac{\sigma\zeta}{\nu^* - \sigma\zeta(\nu - \nu^*)} \right] + \frac{(1-\delta_k)\sigma\zeta}{\nu^* - \sigma\zeta(\nu - \nu^*)} \right\} \frac{\sigma\nu^*\nu}{\nu^* - \sigma\zeta(\nu - \nu^*)}$$

and

$$j_{22}^{(3)} = \frac{\rho}{1-\nu} - (\alpha + \gamma).$$

Therefore, the eigenvalues of J_3 are $j_{11}^{(3)}$ and $j_{22}^{(3)}$.

To study the existence of that additional equilibria, we think that it is useful to consider the shape assumed by the isocline $\dot{\nu} = 0$ shown in the figure 1, i.e.

$$u(\nu) = 1 - \frac{1}{1-\delta_k} \left(\mu \frac{\nu^* - \nu}{1-\nu} + \frac{g}{\sigma} - g\zeta \frac{\nu - \nu^*}{\nu^*} \right) \tag{29}$$

[8] It is easy to prove that transversality condition holds.

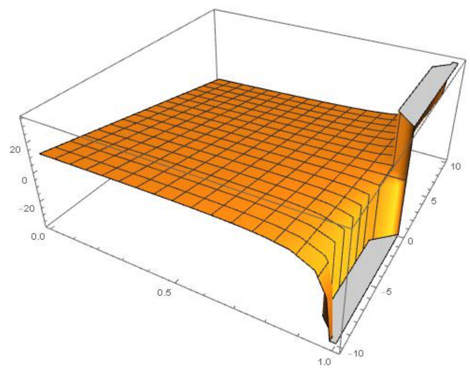


Figure 1: isocline $\dot{\nu} = 0$

Obviously, we restrict our analysis to the domain $[0, 1[$. Differentiation of (29) with respect to ν yields

$$\frac{\partial u}{\partial \nu} = \frac{1}{1-\delta_k}\left(\mu \frac{1-\nu^*}{(1-\nu)^2} + \frac{g\zeta}{\nu^*}\right)$$

If $\mu > 0$, always $\frac{\partial u}{\partial \nu} > 0$, so that, because of $u(1) = +\infty$, the additional equilibrium inside $[0, 1[$ exists and is unique iff $u(0) < 0$. This is equivalent to say that

$$\mu > \frac{\sigma(1-\delta_k) - g(1+\sigma\zeta)}{\sigma\nu^*}$$

If this condition holds, let $\nu^{(3)}$ be the unique solution of $u(\nu) = 0$. If the eigenvalue $j_{11}^{(3)}$ is positive[9], the equilibrium point $(\nu^{(3)}, 0)$ is unstable.

If $\mu < 0$, $\frac{\partial u}{\partial \nu} > 0$ iff $(1-\nu)^2 > h^2$, where $h^2 = -\mu\frac{(1-\nu^*)\nu^*}{g\zeta}$. If $h^2 \geq 1$, then always $\frac{\partial u}{\partial \nu} \leq 0$. Note that $u(\nu^*) > 0$ and $u(1) = -\infty$. It follows that exists a unique additional equilibrium $(\nu^{(3)}, 0)$.

If $h^2 < 1$, then the function $u(\nu)$ has a global maximum when $\nu_m = 1 - h$.

[9] The set of parameters we used assures that this inequality always holds.

Because of $u(\nu_m) \geq u(\nu^*) > 0$, then the equation $u(\nu) = 0$ has a unique root ($\nu_+^{(3)} > \nu^*$) if $u(0) \geq 0$, otherwise two roots ($\nu_-^{(3)} < \nu^* < \nu_+^{(3)}$) exist. Therefore, the equilibrium $(\nu_-^{(3)}, 0)$ has the eigenvalue $j_{22}^{(3)} < 0$ and, given the set of parameters here used, the eigenvalue $j_{11}^{(3)} > 0$. So that this equilibrium is unstable. With reference to the equilibrium point $(\nu_+^{(3)}, 0)$, we have that $j_{22}^{(3)} > 0$. This implies that this equilibrium is always unstable.

6.2. The case $\tau_1 > 0$ and $\tau_2 > 0$

The linearization of the system (6) around an equilibrium point (ν_r, u_r), $r = 1, 2, 3$, is

$$\begin{pmatrix} \dot{\nu} \\ \dot{u} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \nu_t - \nu_r \\ u_t - u_r \end{pmatrix} + \mathcal{B} \begin{pmatrix} \nu_{t-\tau_1} - \nu_r \\ u_{t-\tau_1} - u_r \end{pmatrix} + \mathcal{C} \begin{pmatrix} \nu_{t-\tau_2} - \nu_r \\ u_{t-\tau_2} - u_r \end{pmatrix}$$

where

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} c_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

$$a_{11} = \left[(1-u_r)(1-\delta_k e^{-g\tau_1} - \mu\frac{\nu^* - \nu_r}{1-\nu_r}e^{-g\tau_2}\right]\frac{\sigma(\nu^*)^2(\sigma\zeta+1)}{[\nu^* - \zeta\sigma(\nu_r - \nu^*)]^2} - g \quad a_{12} = -\frac{\sigma\nu^*\nu_r(\sigma\zeta+1)}{\nu^* - \zeta\sigma(\nu_r - \nu^*)}$$

$$a_{21} = \frac{\rho u_r}{(1-\nu_r)^2}, \quad a_{22} = \frac{\rho}{1-\nu_r} - (\alpha+\gamma),$$

$$b_{11} = -\frac{\sigma\nu^*}{\nu^* - \zeta\sigma(\nu_r - \nu^*)}\delta_k(1-u_r)e^{-g\tau_1}, \quad b_{12} = \frac{\sigma\nu^*}{\nu^* - \zeta\sigma(\nu_t - \nu^*)}\delta_k\nu_r e^{-g\tau_1},$$

$$c_{11} = \frac{2\nu_r - \nu_r^2 - \nu^*}{(1-\nu_r)^2}\mu e^{-g\tau_2}\frac{\sigma\nu^*}{\nu^* - \zeta\sigma(\nu_t - \nu^*)}.$$

To investigate the linear stability, we examine the eigenvalues of the system (25), which are the solutions of the characteristic equation

$$\det(\lambda I - \mathcal{A} - \mathcal{B}e^{-\lambda\tau_1} - \mathcal{C}e^{-\lambda\tau_2}) = 0.$$

Following the identical procedure explained in the subsection 4.3, we have that some stability switches and some Hopf bifurcations may exist.

To investigate the direction of the Hopf bifurcation, we can define and calculate the first Lyapunov coefficient in the way similar to the one of section (5).

7. Parameters and numerical simulations

In this section, we used the following set of parameters:

$$\begin{array}{lll} \alpha = 0.02 & \beta = 0.01 & g = \alpha + \beta = 0.03 \\ \gamma = 1.23 & \rho = 0.075 & \sigma = 0.2 \\ \delta_k = 0.4 & \zeta = 80 & \mu \in [-10, 10] \end{array}$$

Like Wolfstetter, we assumed the intensity of the discretional public expenditure μ as critical parameter.

The simulations have been performed by means of the bifurcation analysis software "DDE-BIFTOOL" which is a MATLAB package developed by K. Engelborghs et al. [3].

7.1. Numerical simulations of the basic model

Directly by the Routh-Hurwitz criterion, with $\tau_1 = \tau_2 = 0$, we showed that the equilibrium $(\nu^*, u^*) = (0.94, 0.75)$ of the system (9) is stable when $\mu < 0$ and unstable when $\mu > 0$. This is the Wolfstetter result. Nevertheless, as soon the time delays become positive, Wolstetters conclusion becomes inconsistent: time delays are such that instability comes out from stability and vice versa.

HERE FIGURE 2

As figures 2 show, particular pairs of τ_1 and τ_2 are able to stabilize the system independently of the kind of policy adopted by the government. Specifically, as long as $-1.5 \leq \mu \leq 1.5$, there are some intervals of τ_1 where stability (green area in the figures) prevails whatever τ_2 may be. Conversely, when $|\mu|$ increases, the intervals of τ_2 consistent with the system stability show the tendency to get narrower as τ_1 increases. Looking at figures 3, we can see that the Keynesian policy rule yields similar results to the one of the pro-cyclical policy. Further increases of $|\mu|$ confirm this result for both the kind of policies. But, when $\mu \leq -5$, with a classical policy the stability area tends to come down at a neighborhood of $\tau_2 \simeq 0.8$, whatever τ_1 may be (see figures 4).

HERE FIGURE 3

HERE FIGURE 4

Conversely, a Keynesian policy together with higher values of μ requires specific combinations of the two delays to be effective. As it is clear in the figures 5, values of $\mu > 6$ contract the stability area to an island where $0.1 < \tau_1 < 2.8$ and $0.8 < \tau_2 < 2$ approximately.

To conclude, we can say that both pro-cyclical and counter-cyclical stabilization policies are highly destabilizing if high time delays dealing with the public sector match with a high intensity of the public expenditure. The real problem for the policy makers is to reconcile the strength of the public policies with the lags of the tax system and the ones of the political process governing the public purchase decisions.

HERE FIGURE 5

7.2. Numerical simulations of the extended model

Unlike the basic model, the one we defined extended makes the equilibrium (ν^*, u^*) stable if and only if the tax revenues have about a lag $0.2 < \tau_1 < 1$ and the government's budget is balanced trough taxes ($\mu = 0$). This result is true whatever τ_2 may be (see green area in the figure 6).

HERE FIGURE 6

When there are no delays, the convergence to a stable equilibrium is possible only if the government adopts a pro-cyclical policy with sufficiently strength ($\mu < -0.1532$). This result agrees with the one Wolfstetter ([16], p. 388) obtains in his three dimensional version of the model. Nevertheless, there are two issues making our results different from Wolfstetters conclusions. Firstly, if a weak classical policy rule is adopted, the equilibrium (ν^*, u^*) becomes unstable and a stable limit cycle emerges. In this case, the system experiments wide oscillations in the income distribution with $-0.1532 < \mu < 0$ (see figure 7). Secondly, if a Keynesian policy rule is adopted ($\mu > 0$), without escape the government expenditure aggravates the business cycle, whatever may be its strength.

HERE FIGURE 7

When we take into account the fiscal policy delays, if the policy is of the Keynesian type, the government's discretional expenditure is able to stabilize the system only with particular pairs of the two delays (green area in the figures 8) and values of $0 < \mu < 0.5$. Specifically, looking at the lag τ_2 and given τ_1 that depends on the tax system, only by applying a low discretional expenditure the policy makers might stabilize the economic activity. However, it is a hard task to reconcile the strength of policy μ with the lag τ_2 underlying the political decisions. Surely, as figures 9 show, if μ approaches and exceeds 1, the tendency is to a complete instability of the system.

HERE FIGURE 8

HERE FIGURE 9

If the policy is pro-cyclical, then the local stability requires again low values of μ (i.e. $\mu < -0.12$) and particular pairs of τ_1 and τ_2 as it is shown in the figures 10[10].

HERE FIGURE 10

From the economic point of view, when the time delay of the tax revenues is brief enough, then a classical policy rule may be effective in fighting a recession. This because "the government's withdrawal from the capital market"(Wolfstetter [16], p. 388) and the reduced labor cost, due to the decreasing employment levels, stimulate the private investments and restore the job openings in the system. In the case of a Keynesian policy, to achieve stable equilibria, not only the government needs a careful control of fiscal policy delays, but also it must weigh carefully the strength of policy measures.

Finally, we have to notice that, always preserving the economic meaning, different parameter values yield qualitative results similar to the ones previously discussed.

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[10]When μ reaches values less than -0.5, lower values of μ tend to make the system always unstable.

7.3. The direction of the Hopf bifurcation

We calculated all the partial derivatives up to third order of the right-hand side of the system both for the basic model (7) and the extended model (6). By using the set of parameters given above, the values of these derivatives allow us to obtain the first Lyapunov coefficient of the Hopf bifurcation points [11]. The results are plotted in the figures 11 and 12, where the green line highlights supercritical Hopf bifurcations ($l_1 < 0$), whereas the red line highlights subcritical Hopf bifurcation ($l_1 > 0$). Also we plotted (black circle) degenerate Hopf bifurcation, which have $l_1 = 0$ (generalized Hopf bifurcation) or two pair of simple eigenvalues on imaginary axes (double Hopf bifurcation).

HERE FIGURE 11

HERE FIGURE 12

From the economic point of view, either in the basic model or in the extended one, the direction of Hopf bifurcations confirms the difficult to reconcile the intensity of public expenditure with a right combination of the time delays. This whatever the kind of policy may be. When $|\mu|$ increases (especially in the extended model) regular cycles suddenly may degenerate in a complete instability.

8. Concluding remarks

In this paper, we reconsidered the Wolfstetter (Goodwin) classical growth cycle model, taking into account the fiscal policy delays that characterize the government activity in the economic system.

[11] We note that an Hopf bifurcation does not correspond necessarily to a stability switch.

With reference to the basic model, unlike Wolfstetter's results, we showed that the outcomes of the pro-cyclical as well of the counter-cyclical policies are dependent either on the length of the lags or on their particular combinations. Therefore, the effectiveness of fiscal policy on the GDP dynamics becomes extremely hard. This because the policy makers cannot control with preciseness the lags, especially the ones dealing with the political process underlying the public expenditure. Furthermore, it is difficult to reconcile the strength of policy measures with the length of the lags. Let us point out that, if the intensity of the discretional public expenditure becomes high enough, the tendency of the system is toward a complete instability, whatever the kind of policy may be. The only one certainty is that a pro-cyclical policy may stabilize the economy only if there are no lags. Instead, the counter-cyclical policy with no lags remains always destabilizing. This in accordance with Wolfstetter's conclusions.

As far as the extended model is concerned, our results are different. If the government's budget is balanced trough taxes ($\mu = 0$), a low delay of the tax receipts is able to stabilize the economy whatever the lag of the public expenditure may be. In any other case, with particular pairs of the two delays the government may stabilize the economy only by applying a low discretional expenditure if the policy is of the Keynesian type and low reductions of its expenditure if the policy is of the classical type.

Finally, an important outcome of our extended model is that concerning the case with no lags. Unlike Wolfstetter's result, a pro-cyclical policy does not assure the stabilization of the economy. This because a weak reduction of the public expenditure makes unstable the equilibrium (ν^*, u^*) by giving rise to a stable cycle, whose size increases when μ approaches zero from the left. From the economic point of view, this means that if the policy makers are lacking in resoluteness, the social cost of a periodic high unemployment might become very high.

To conclude we can say that our analysis might have a link with the specific functions we choose to formalize the model; nevertheless, it has the merit to shed new light on the complex problems dealing with the effectiveness of fiscal policy.

Appendix

In this Appendix we investigate the existence of a pair of purely imaginary roots $\lambda = \pm i\omega$ (with $\omega > 0$) of the characteristic equation

$$\lambda + c_0 + c_1 e^{-g\tau_1} e^{-\lambda\tau_1} = 0 \tag{30}$$

where c_0 and $c_1 \neq 0$ are real constants. Substituting $\lambda = i\omega$ in (30) and taking into account that the real and the imaginary parts must be zero, equation (30) has purely imaginary roots if the following equations are satisfied:

$$\begin{cases} c_0 + c_1 e^{-g\tau_1} \cos\omega\tau_1 = 0 \\ \omega - c_1 e^{-g\tau_1} \sin\omega\tau_1 = 0 \end{cases} \tag{31}$$

Isolating trigonometric terms, squaring each equation and summing, we obtain that $\omega^2 = c_1^2 e^{-2g\tau_1} - c_0^2$. Hence, if $c_1^2 e^{-2g\tau_1} - c_0^2 \leq 0$, the system (31) has no solutions and no purely imaginary roots exist for the characteristic equation (30). We set the following:

Theorem 3. *If* $c_0 \neq 0$ *and* $\tau_1 \geq \frac{1}{g} \log \frac{|c_1|}{|c_0|}$ *equation (30) has no purely imaginary roots.*

Note that if $|c_1| \leq |c_0|$, there are no purely imaginary roots for all $\tau_1 \geq 0$.

For $c_1^2 e^{-2g\tau_1} - c_0^2 > 0$, we distinguish three cases: $c_0 = 0$, $c_0 > 0$ and $c_0 < 0$.

Theorem 4. *Suppose that $c_0 = 0$. Then equation (30) admits purely imaginary roots if*

$$\frac{|c_1|}{ge} \geq \frac{\pi}{2} \tag{32}$$

otherwise no purely imaginary roots exist.

Proof. If $c_0 = 0$, from system (31), we have that $\cos \omega\tau_1 = 0$ and $\omega = |c_1|e^{g\tau_1}$. Hence ω is a solution of the system (31) if $\tau_1 > 0$ is a solution of the equation

$$f_n(\tau_1) := |c_1|\tau_1 e^{-g\tau_1} - \frac{\pi}{2} - n\pi = 0 \quad n = 0, 1, \ldots$$

Note that $f_n(0) < 0$, $f_n(+\infty) < 0$ and, by the sign of derivative, f_n is an increasing function for $\tau_1 \leq \frac{1}{g}$ and a decreasing function for $\tau_1 \geq \frac{1}{g}$. Hence the equation $f_n(\tau_1) = 0$ admits no solution if $f_n(\frac{1}{g}) \leq f_0(\frac{1}{g}) = \frac{|c_1|}{ge} - \frac{\pi}{2} < 0$ otherwise for all positive integer n such that $\frac{|c_1|}{ge} - \frac{\pi}{2} \geq n\pi$ one or two solutions exist[12]. □

Theorem 5. *Suppose that $c_0 > 0$.*

If $c_1^2 - c_0^2 - gc_0 \leq 0$ the equation (30) has no purely imaginary roots, otherwise this equation may have some purely imaginary roots.

Proof. From system (31), we have to solve the following equations

$$f_n(\tau_1) := \omega\tau_1 + \arctan\frac{\omega}{c_0} - \pi - 2n\pi = 0 \quad n = 0, 1, \ldots$$

[12] We have one solution if the previous inequality becomes an equality. Otherwise, we have two solutions.

We set $\varphi(\tau_1) = \omega f_n'(\tau_1)$. Differentiation of with respect to τ_1 yields

$$\varphi(\tau_1) = c_1^2 e^{-2g\tau_1}(1 - g\tau_1) - gc_0 - c_0^2.$$

It is easy to show that for $\tau_1 \geq 0$ the function φ has a global maximum at $\tau_1 = 0$. It follows that if $\varphi(0) = c_1^2 - c_0^2 - gc_0 \leq 0$, then φ is negative for all $\tau_1 > 0$. Consequently, the function f_n decreases for $\tau_1 \geq 0$. Observing that $f_n(0) = \arctan \frac{\omega}{c_0} - \pi - 2n\pi < 0$, we obtain that f_n has no roots for $\tau_1 \geq 0$.

In order to prove the second statement of the theorem, we can show that if $\varphi(0) > 0$, then there exists a unique $\hat{\tau}_1 \in]0, \frac{1}{g} \log \frac{|c_1|}{|c_0|}[$ such that $\varphi(\hat{\tau}_1) = 0$. Hence the function f_n is increasing on the interval $[0, \hat{\tau}_1]$ and decreasing on the interval $[\hat{\tau}_1, \frac{1}{g} \log \frac{|c_1|}{|c_0|}]$. It follows that, if $f_n(\hat{\tau}_1) > 0$, two f_n simple roots ($\tau_{n,1}^*$ and $\tau_{n,2}^*$ s.t. $\varphi(\tau_{n,1}^*) > 0$ and $\varphi(\tau_{n,1}^*) < 0$) exist. If $f_n(\hat{\tau}_1) = 0$, then $\hat{\tau}_1$ is the unique double root of f_n, otherwise, if $f_n(\hat{\tau}_1) < 0$, there are no roots. □

Obviously, if $f_0(\hat{\tau}_1) < 0$, no roots exist for any $n = 0, 1, \ldots$. Hence, equation (30) has no purely imaginary roots. If $f_0(\hat{\tau}_1) > 0$, then there exists n_0 such that $f_n(\hat{\tau}_1) < 0$ for $n > n_0$.

Theorem 6. *Suppose that $c_0 < 0$ and $|c_1| > |c_0|$.*

a) If $c_1^2 > c_0^2 \, e^{\frac{2g}{-c_0}}$, then equation (30) has purely imaginary roots.

b) If $c_1^2 \leq c_0^2 \, e^{\frac{2g}{-c_0}}$ and $c_1^2 \leq c_0^2 \, e^3$, then equation (30) has no purely imaginary roots.

c) If $c_1^2 \leq c_0^2 \, e^{\frac{2g}{-c_0}}$, $c_1^2 > c_0^2 \, e^3$ and $c_1^2 < 2e^3(-gc_0 - c_0^2)$, then equation (30) has no purely imaginary roots.

d) If $c_1^2 \leq c_0^2 \, e^{\frac{2g}{-c_0}}$, $c_1^2 > c_0^2 \, e^3$ and $c_1^2 \geq 2e^3(-gc_0 - c_0^2)$ then equation (30) may have purely imaginary roots.

Proof. The proof is quite similar to the previous one. In this case we have to solve the equations

$$f_n(\tau_1) := \omega\tau_1 + \arctan\frac{\omega}{c_0} - 2n\pi = 0 \quad n = 0, 1, \ldots$$

This functions differ from the similar one in the previous theorem by an additive constant. So that, they have the same derivative. We note that $\varphi(0) > 0$ and $\varphi(\frac{1}{g}\log\frac{|c_1|}{|c_0|}) = -c_0(c_0\log\frac{|c_1|}{|c_0|} + g)$. If $\varphi(\frac{1}{g}\log\frac{|c_1|}{|c_0|}) < 0$, which is equivalent to the hypothesis a), we can prove that there exists a unique $\hat{\tau}_1 \in]0, \frac{1}{g}\log\frac{|c_1|}{|c_0|}[$ such that $\varphi(\hat{\tau}_1) = 0$. Moreover $f_0(\hat{\tau}_1) > 0$ because of $f_0(\frac{1}{g}\log\frac{|c_1|}{|c_0|}) = 0$. Hence equation (30) has at least one purely imaginary root.

If $\varphi(\frac{1}{g}\log\frac{|c_1|}{|c_0|}) \geq 0$ and $\log\frac{|c_1|}{|c_0|}) \leq \frac{3}{2}$, which is equivalent to the hypothesis b), we can prove that the function φ is positive on the interval $]0, \frac{1}{g}\log\frac{|c_1|}{|c_0|}[$, hence on this interval no roots exist for f_n.

If hypothesis c) hold, we can prove the same thing for the function φ.

If hypothesis d) hold, we can prove that there exist two roots, $\hat{\tau}_1$ and $\hat{\hat{\tau}}_1$, of the function φ. We can show that $\hat{\tau}_1$ is a local maximum for f_n and $\hat{\hat{\tau}}_1$ is a local minimum for f_n.

It follows that if $f_n(\hat{\tau}_1) > 0$, as we said above, two simple roots exist such that $\varphi(\tau^*_{n,1}) > 0$ and $\varphi(\tau^*_{n,2}) < 0$. If $f_n(\hat{\tau}_1) = 0$, then $\hat{\tau}$ is the unique double root of f_n, otherwise, if $f_n(\hat{\tau}_1) < 0$, there are no roots. □

We note that if $\frac{2g}{-c_0} \leq 3$ then $c_1^2 \leq c_0^2\, e^{\frac{2g}{-c_0}}$ implies $c_1^2 \leq c_0^2\, e^3$. Hence only conditions a) and b) of theorem 6 can hold. On the other hand, if $\frac{2g}{-c_0} > 3$, we can prove that $c_0^2 e^3 < 2e^3(-gc_0 - c_0^2) < c_0^2 e^{\frac{2g}{-c_0}}$. By Theorem 6 we have that if $c_1^2 < 2e^3(-gc_0 - c_0^2)$, then equation (30) has no purely imaginary roots.

Theorem 7. *If $\lambda = i\omega(\tau_1^*)$ is a simple purely imaginary root of equation (30) then*

$$\left.\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\right|_{\lambda=i\omega(\tau_1^*)} = \mathrm{sign}(\varphi(\tau_1^*))$$

where $\varphi(\tau_1^*) = c_1^2 e^{-2g\tau^*}(1 - g\tau^*) - gc_0 - c_0^2$.

Proof. The proof follows by implicit differentiation of the function $F(\lambda, \tau_1) = \lambda + c_0 + c_1 e^{-(g+\lambda)\tau_1}$. Note that if $\lambda = i\omega(\tau_1^*)$ is a simple root then $F'_\lambda(i\omega, \tau_1^*) \neq 0$. □

Lemma 3. *If*

$$\mu \leq \frac{\sigma - g}{\sigma\nu^*} \quad \text{or} \quad \frac{\sigma(1+\delta_k) - g}{\sigma\nu^*} \leq \mu,$$

then the characteristic equation (13) has no purely imaginary roots.

Proof. Let $c_0 = g - \sigma(1 - \mu\nu^*)$ and $c_1 = \sigma\delta_k$ be. We proved (see Theorem 3) that if $|c_1| \leq |c_0|$ no purely imaginary roots exist. This happens if

$$\mu \leq \frac{\sigma(1-\delta_k) - g}{\sigma\nu^*} \quad \text{or} \quad \frac{\sigma(1+\delta_k) - g}{\sigma\nu^*} \leq \mu.$$

We can show that if $c_0 < 0$ and $|c_1| > |c_0|$, we have also $c_0 \log\frac{|c_1|}{|c_0|} + g \geq 0$ and $|c_1| \leq |c_0| e^{\frac{3}{2}}$. By Theorem 6-b) no purely imaginary roots exist. Note that $c_0 < 0$ is equivalent to $\mu < \frac{\sigma - g}{\sigma\nu^*}$. □

Theorem 8. *If* $\dfrac{\sigma\delta_k}{g} < \dfrac{e\pi}{2}$, *then the characteristic equation (13) has no purely imaginary roots. Otherwise, they could exist some purely imaginary roots if*

$$\frac{\sigma - g}{\sigma\nu^*} < \mu < \frac{\sigma(1+\delta_k) - g}{\sigma\nu^*}.$$

Proof. □

If $c_0 = 0$, that is $\mu = \frac{\sigma - g}{\sigma\nu^*}$, there may exist a purely imaginary root if condition (32) is satisfied (see Theorem 4). If $c_0 > 0$ and $c_0^2 + gc_0 < c_1^2$, by Theorem 5, some purely imaginary roots may exist. In the proof of Theorem 5, we prove that the function f_n, in the interval $[0, \frac{1}{g}\log\frac{|c_1|}{|c_0|}]$, has a global maximum at $\tau^*_{c_0}$ depending on c_0. We define the function $h(c_0) = f_n(\tau^*_{c_0})$, for $c_0 > 0$, which compute the maximum value of f_n in the interval $[0, \frac{1}{g}\log\frac{|c_1|}{|c_0|}]$. For c_0 which goes to zero, by Theorem 4, we define $h(0) = \frac{|c_1|}{ge} - \frac{\pi}{2} - n\pi$. After some calculation, we can prove that $h'(c_0)$ is negative, that is h is decreasing. We can conclude that if $h(0) < 0$, then h remains negative and there are no purely imaginary roots. If $h(0) \geq 0$, some purely imaginary roots may exist.

Theorem 9. *If $0 < \nu_3 < \nu^*$, then the characteristic equation (21) has no purely imaginary roots.*

Proof. We can rewrite the characteristic equation (21) as

$$\lambda - \sigma\nu^*\mu + \sigma - g - 2\sigma\delta_k e^{-g\tau_1} + \sigma\delta_k e^{-g\tau_1} e^{-\lambda\tau_1} = 0.$$

By substitution of $\lambda = i\omega$, $(\omega > 0)$, taking into account that real and imaginary parts must be zero, we have the following system

$$\begin{cases} -\sigma\nu^*\mu + \sigma - g - 2\sigma\delta_k e^{-g\tau_1} + \sigma\delta_k e^{-g\tau_1}\cos\omega\tau_1 = 0 \\ \omega - \sigma\delta_k e^{-g\tau_1}\sin\omega\tau_1 = 0 \end{cases}$$

Isolating trigonometric terms, squaring each equation and summing we obtain

$$\omega^2 = -b^2 - 4\sigma\delta_k e^{-g\tau_1} b - 3\sigma^2\delta_k^2 e^{-2g\tau_1}$$

where $b = \sigma\nu^*\mu - \sigma + g$. It is easy to show that if $0 < \nu_3 < \nu^*$, then $b > -\sigma\delta_k e^{-g\tau_1}$ and by this condition we have that $-b^2 - 4\sigma\delta_k e^{-g\tau_1} b - 3\sigma^2\delta_k^2 e^{-2g\tau_1}$ is negative. Hence, no purely imaginary eigenvalues exist. □

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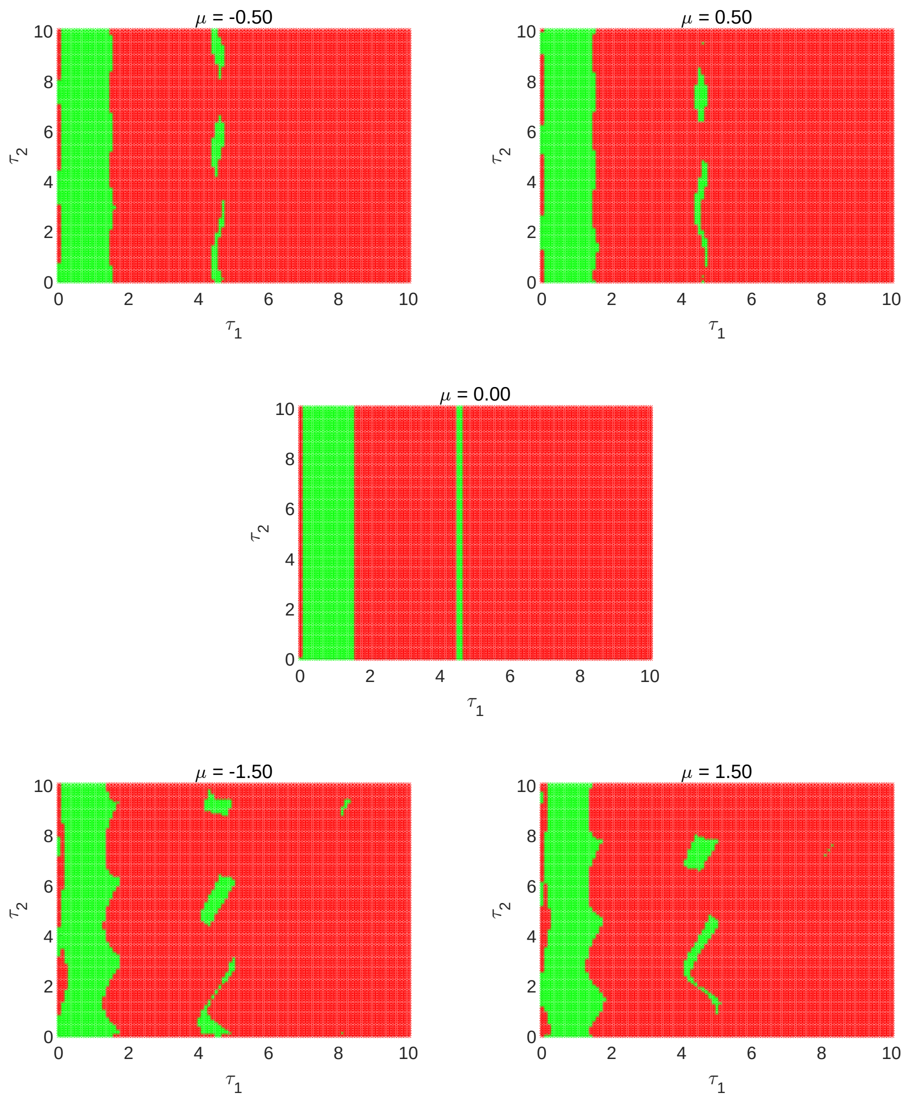


Figure 2: Stability and instability areas.

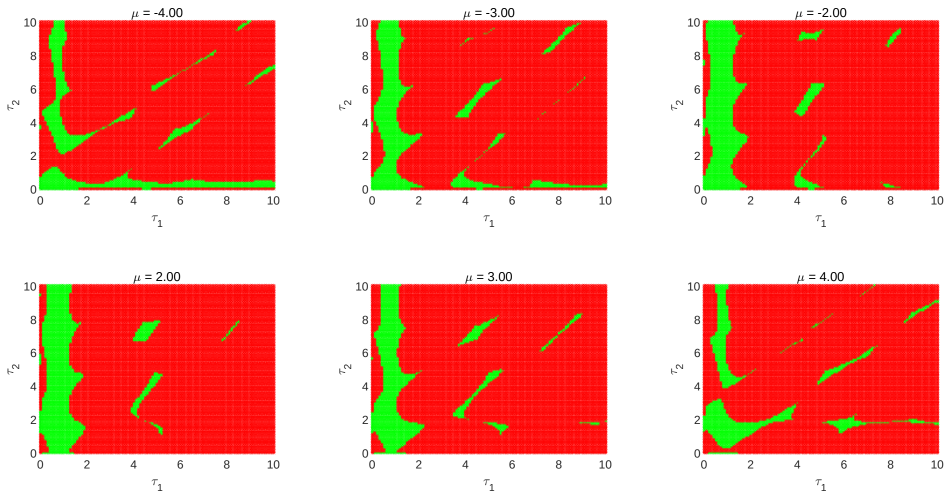


Figure 3: Stability and instability areas.

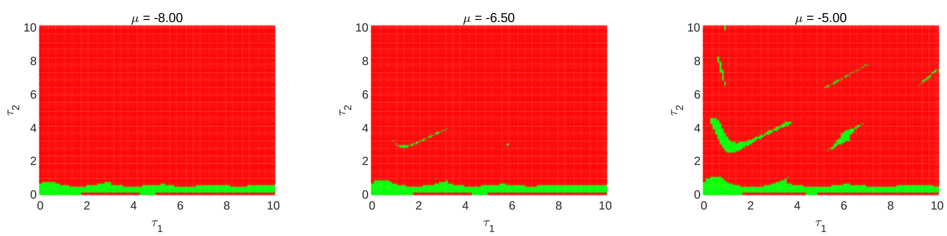
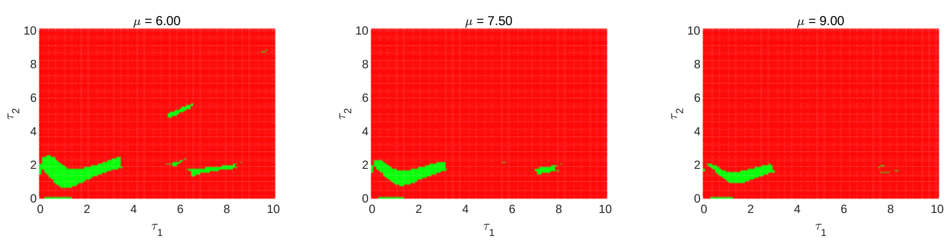


Figure 4: Stability and instability areas.



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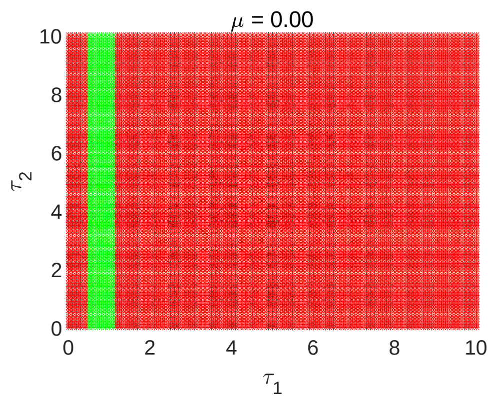

Figure 6: Stability and instability areas.

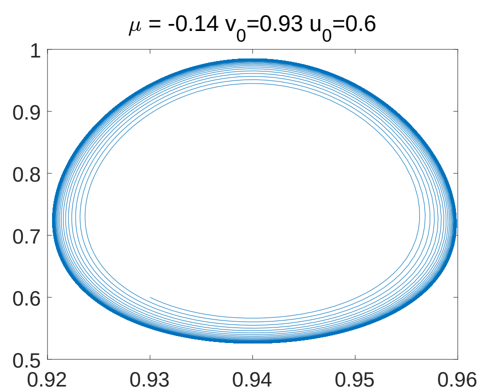

Figure 7: Limit cycle with no delays and $\mu < 0$.

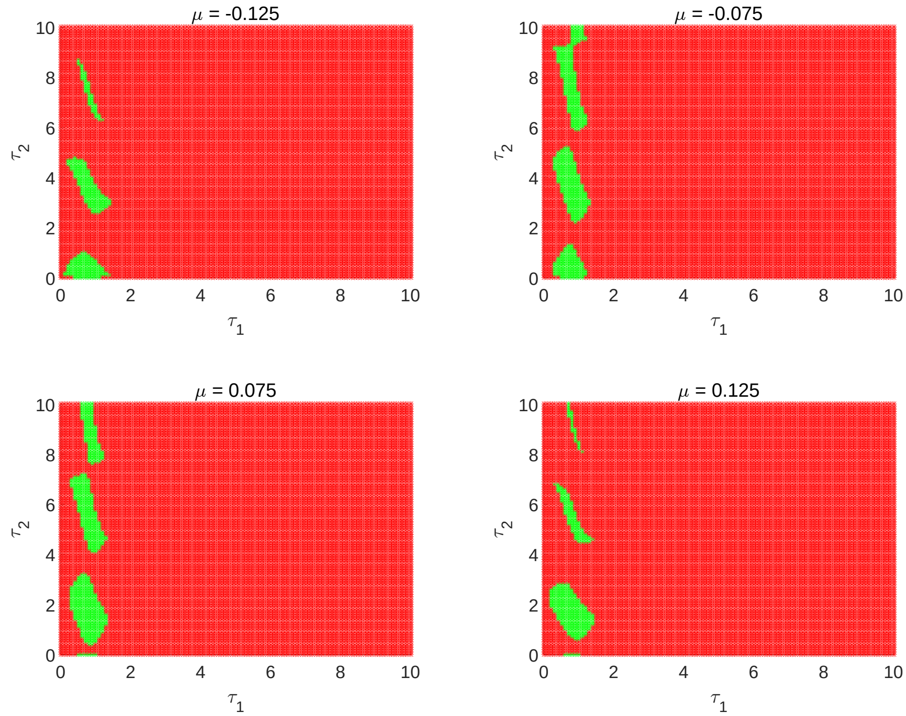

Figure 8: Stability and instability areas.

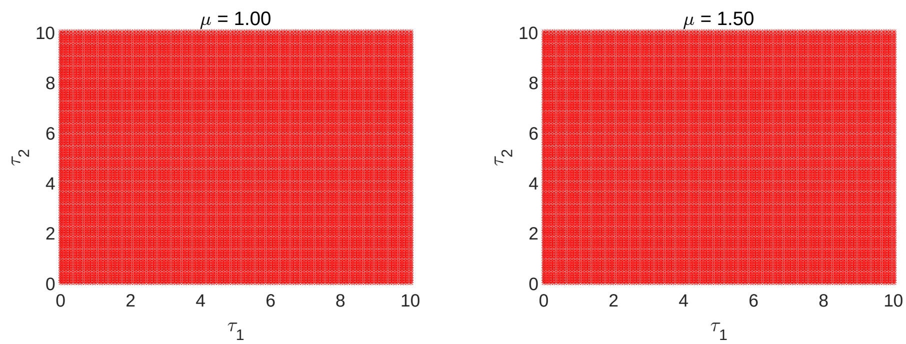

Figure 9: Stability and instability areas.

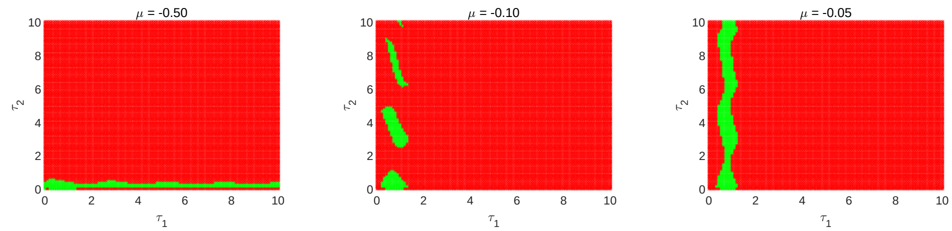

Figure 10: Stability and instability areas.

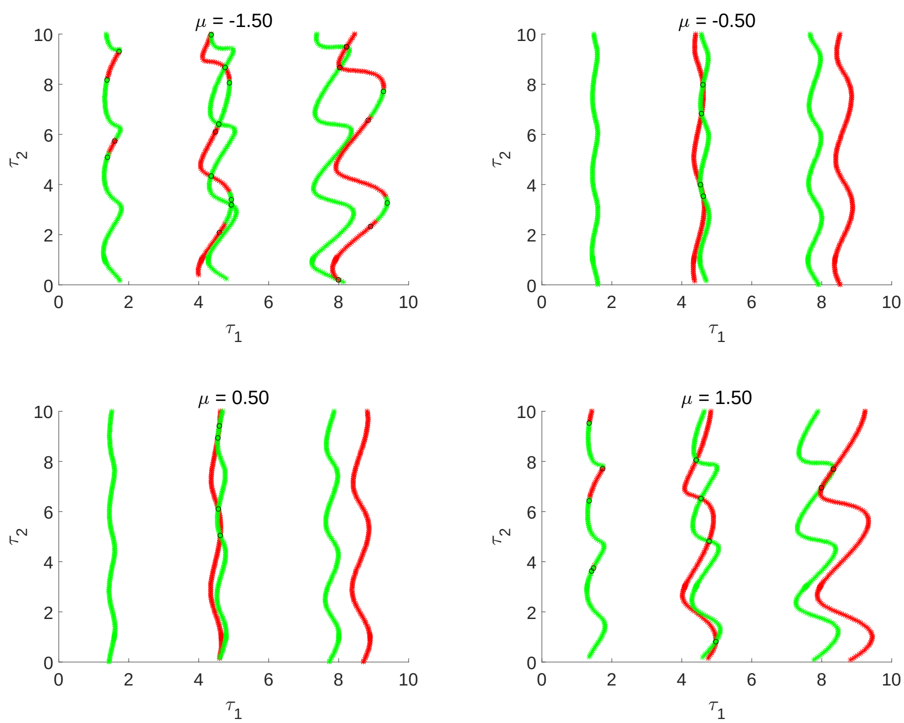

Figure 11: Direction of the Hopf bifurcation.

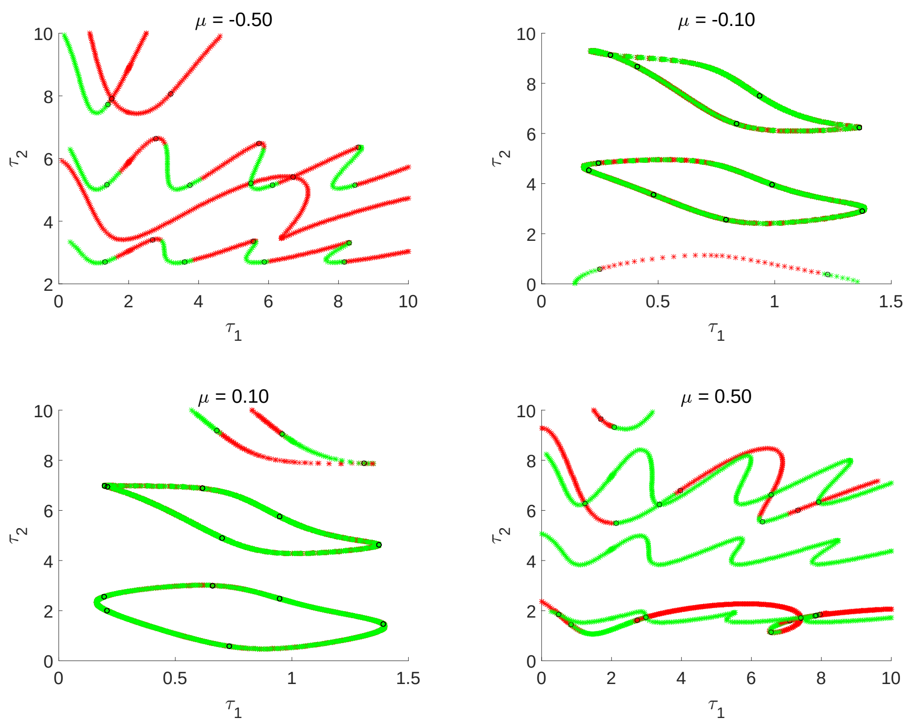

Figure 12: Direction of the Hopf bifurcation.

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