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Research Article

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Extremals for Fractional Moser–Trudinger Inequalities in Dimension 1 via Harmonic Extensions and Commutator Estimates

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Abstract: We prove the existence of extremals for fractional Moser–Trudinger inequalities in an interval and on the whole real line. In both cases we use blow-up analysis for the corresponding Euler–Lagrange equation, which requires new sharp estimates obtained via commutator techniques.

Keywords: Moser–Trudinger, Fractional Laplacian, Extremals, Blow-Up, Commutator Estimates

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1 Introduction

The celebrated Moser–Trudinger inequality [39] states that for $\Omega \subset \mathbb{R}^n$ with finite measure $|\Omega|$ we have

$$\sup_{u\in W_0^{1,n}(\Omega),\,\|\nabla u\|_{L^n(\Omega)}\le 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}}\,dx \le C|\Omega|, \qquad \alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}, \tag{1.1}$$

where ω_{n-1} is the volume of the unit sphere in $\mathbb{R}^n$. The constant α_n is sharp in the sense that the supremum in (1.1) becomes infinite if α_n is replaced by any $\alpha > \alpha_n$. In the case $\Omega = \mathbb{R}^2$, Ruf [46] proved a similar inequality, using the full $W^{1,2}$-norm instead of the L^2-norm of the gradient, which was then generalized to $\mathbb{R}^n$, $n \ge 2$, by Li and Ruf [31] by

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\,\|u\|^n_{L^n(\mathbb{R}^n)}+\|\nabla u\|^n_{L^n(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} (e^{\alpha_n |u|^{\frac{n}{n-1}}} - 1)\,dx < \infty. \tag{1.2}$$

Higher-order versions of (1.1) were proven by Adams [2] on the space $W_0^{k,n/k}(\Omega)$ for $n > k \in \mathbb{N}$. The proofs of (1.1) and (1.2) in [39] and [31] rely on symmetrization arguments which cannot be applied when the Pólya-Szegö inequality fails. A rearrangement-free approach was proposed by Lam and Lu to prove Adams-type inequalities for high-order Sobolev spaces on $\mathbb{R}^n$ (see [24]). This approach was also used to obtain inequalities on the Heisenberg group with applications to sub-elliptic PDEs (see, e.g., [23, 25]).

In [21], the authors proved the following 1-dimensional fractional extension of the previous results (for the definition of $H^{1/2,2}(\mathbb{R})$ and $(-\Delta)^{1/4}$, see (A.4) in Appendix A).

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Theorem A. *Set* $I := (-1,1) \subset \mathbb{R}$ *and* $\tilde{H}^{1/2,2}(I) := \{u \in H^{1/2,2}(\mathbb{R}) : u \equiv 0 \text{ on } \mathbb{R} \setminus I\}$. *Then we have*

$$\sup_{u \in \tilde{H}^{\frac{1}{2},2}(I),\, \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(I)} \leq 1} \int_I (e^{\alpha u^2} - 1)\, dx = C_\alpha < \infty \quad \text{for } \alpha \leq \pi, \tag{1.3}$$

and

$$\sup_{u \in H^{\frac{1}{2},2}(\mathbb{R}),\, \|u\|_{H^{\frac{1}{2},2}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, dx = D_\alpha < \infty \quad \text{for } \alpha \leq \pi, \tag{1.4}$$

where

$$\|u\|^2_{H^{\frac{1}{2},2}(\mathbb{R})} := \|(-\Delta)^{\frac{1}{4}} u\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}.$$

The constant π *is sharp in* (1.3) *and* (1.4).

More general results have recently appeared (see, e.g., [1, 16, 24, 38, 47, 50]) in which both the dimension and the (fractional) order of differentiability have been generalized. For instance, (1.3) and (1.4) can be seen as 1-dimensional cases of the more general results of [16, 24, 38] that hold in arbitrary dimension n.

The existence of extremals for this kind of inequalities is a challenging question. Existence of extremals for (1.1) was originally proven by Carleson and Chang [5] in the case of the unit ball, a fundamental result later extended by Struwe [49] and Flucher [15] to the case of general bounded domains in $\mathbb{R}^2$, and by Lin [32] to the case of bounded domains in $\mathbb{R}^n$. In the case of the Li–Ruf inequality (1.2), the existence of extremals appears in [31] when $n \geq 3$, and was proven by Ishiwata [20] when $n = 2$. For the higher-order Adams inequality the existence of extremals has been proven in various cases by, e.g., Li and Ndiaye [30] on a 4-dimensional closed manifold, by Lu and Yang [33] (see also [40]) for a 4-dimensional bounded domain and by DelaTorre and Mancini [9] for a bounded domain in $\mathbb{R}^{2m}$, $m \geq 1$ arbitrary. In recent years, there have been many other papers studying the existence of extremals for similar inequalities on $\mathbb{R}^n$ (see, e.g., [12, 13, 34, 41, 42] and the references therein). Most of the results we mentioned are based on a blow-up analysis approach, but a different method has been recently proposed in [27], where the authors exploit the exact relation between critical and subcritical Moser–Trudinger suprema (see [6, 26]) to prove the existence of extremals.

On the other hand, the existence of extremals for the fractional Moser–Trudinger inequality has remained open until now, with the exception of Takahashi [50] considering a subcritical version of (1.4) of Adachi–Tanaka type [1], and Li and Liu [29] treating the case of a fractional Moser–Trudinger on $H^{1/2,2}(\partial M)$ with M being a compact Riemann surface with boundary. The idea of Li and Liu is that by working on the boundary of a compact manifold one can localize the $H^{1/2,2}$-norm.

Applying the same method for an interval $I \subset \mathbb{R}$ creates problems near ∂I, which require additional care in the estimate, and the problem becomes even more challenging when working on the whole $\mathbb{R}$. The main purpose of this paper is to handle these two cases and prove that the suprema in (1.3) and (1.4) are attained.

Theorem 1.1. *For any* $0 < \alpha \leq \pi$, *the inequality* (1.3) *has an extremal, i.e. there exists* $u_\alpha \in \tilde{H}^{1/2,2}(I)$ *such that*

$$\|(-\Delta)^{\frac{1}{4}} u_\alpha\|_{L^2(\mathbb{R})} \leq 1 \quad \text{and} \quad \int_I (e^{\alpha u_\alpha^2} - 1)\, dx = C_\alpha.$$

Theorem 1.1 is rather simple to prove for $\alpha \in (0, \pi)$, while the case $\alpha = \pi$ relies on a delicate blow-up analysis for subcritical extremals.

A similar analysis can be carried out for the Ruf-type inequality (1.4). However, working on the whole real line, we need to face additional difficulties due to the lack of compactness of the embedding of $H = H^{1/2,2}(\mathbb{R})$ into $L^2(\mathbb{R})$: vanishing at infinity might occur for maximizing sequences, even in the sub-critical case $\alpha \in (0, \pi)$. This issue is not merely technical. Indeed, Takahashi [50] proved that (1.4) has no extremal when α is small enough. Here, in analogy with the results in dimension $n \geq 2$, we prove that the supremum in (1.4) is attained if α is sufficiently close to π.

Theorem 1.2. *There exists $\alpha^* \in (0,\pi)$ such that for $\alpha^* \le \alpha \le \pi$ inequality (1.4) has an extremal, namely there exists $\bar{u}_\alpha \in H^{1/2,2}(\mathbb{R})$ such that*

$$\|\bar{u}_\alpha\|_{H^{\frac{1}{2},2}(\mathbb{R})} \le 1 \quad \text{and} \quad \int_{\mathbb{R}} (e^{\alpha \bar{u}_\alpha^2} - 1)\,dx = D_\alpha.$$

As for Theorem 1.1, the proof of Theorem 1.2 for $\alpha = \pi$ is based on blow-up analysis. In fact, we need to study the blow-up of a non-local equation on the whole real line (no boundary conditions), as is done in the following theorem.

Theorem 1.3. *Let $(u_k) \subset H = H^{1/2,2}(\mathbb{R})$ be a sequence of non-negative solutions to*

$$(-\Delta)^{\frac{1}{2}} u_k + u_k = \lambda_k u_k e^{\alpha_k u_k^2} \quad \text{in } \mathbb{R}, \tag{1.5}$$

where $\alpha_k \to \pi$ and $\lambda_k \to \lambda_\infty \ge 0$. Assume u_k even and decreasing ($u_k(-x) = u_k(x) \le u_k(y)$ for $x \ge y \ge 0$) for every k and set $\mu_k := \sup_{\mathbb{R}} u_k = u_k(0)$. Assume also that

$$\Lambda := \limsup_{k\to\infty} \|u_k\|_H^2 < \infty. \tag{1.6}$$

Then, up to extracting a subsequence, we have that one of the following assertions holds:

(i) $\mu_k \le C$, $u_k \to u_\infty$ *in $C^\ell_{\mathrm{loc}}(\mathbb{R})$ for every $\ell \ge 0$, where $u_\infty \in C^\ell_{\mathrm{loc}}(\mathbb{R}) \cap H$ solves*

$$(-\Delta)^{\frac{1}{2}} u_\infty + u_\infty = \lambda_\infty u_\infty e^{\pi u_\infty^2} \quad \text{in } \mathbb{R}. \tag{1.7}$$

(ii) $\mu_k \to \infty$, $u_k \to u_\infty$ *weakly in H and strongly in $C^0_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$ where u_∞ is a solution to (1.7). Moreover, setting r_k such that*

$$\lambda_k r_k \mu_k^2 e^{\alpha_k \mu_k^2} = \frac{1}{\alpha_k} \tag{1.8}$$

and

$$\eta_k(x) := 2\alpha_k \mu_k (u_k(r_k x) - \mu_k), \qquad \eta_\infty(x) := -\log(1 + |x|^2), \tag{1.9}$$

one has $\eta_k \to \eta_\infty$ in $C^\ell_{\mathrm{loc}}(\mathbb{R})$ for every $\ell \ge 0$, $\sup_k \|\eta_k\|_{L_s(\mathbb{R})} < \infty$ for any $s > 0$ (cf. (A.2)), and $\Lambda \ge \|u_\infty\|_H^2 + 1$.

The proof of Theorem 1.3 is quite delicate because local elliptic estimates of a non-local equation depend on global bounds as we shall prove in Lemma 3.6. This will be based on sharp commutator estimates (Lemma 3.3), as developed in [35] for the case of a bounded domain in $\mathbb{R}^n$, extending the approach of [37] to the fractional case.

We expect similar existence results to hold for a perturbed version of inequalities (1.3)–(1.4), as in [36, 51] (see also the recent results in [19]), but we will not investigate this issue here.

2 Proof of Theorem 1.1

2.1 Strategy of the Proof

We will focus on the case $\alpha = \pi$ since the existence of extremals for (1.3) with $\alpha \in (0,\pi)$ follows easily by Vitali's convergence theorem; see, e.g., the argument in [36, Proposition 6].

Let u_k be an extremal of (1.3) for $\alpha = \alpha_k = \pi - \frac{1}{k}$. By replacing u_k with $|u_k|$, we can assume that $u_k \ge 0$. Moreover, $\|(-\Delta)^{1/4} u_k\|_{L^2(\mathbb{R})} = 1$, and u_k satisfies the Euler–Lagrange equation

$$(-\Delta)^{\frac{1}{2}} u_k = \lambda_k u_k e^{\alpha_k u_k^2}, \tag{2.1}$$

with bounds on the Lagrange multipliers λ_k (see (2.4)).

Using the monotone convergence theorem, we also get

$$\lim_{k\to\infty} \int_I (e^{\alpha_k u_k^2} - 1)\,dx = \lim_{k\to\infty} C_{\alpha_k} = C_\pi, \tag{2.2}$$

where C_{α_k} and C_π are as in (1.3).

If $\mu_k := \max_I u_k = O(1)$ as $k \to \infty$, then up to a subsequence $u_k \to u_\infty$ locally uniformly, where by (2.2), u_∞ maximizes (1.3) with $\alpha = \pi$. Therefore, we will work by contradiction, assuming

$$\lim_{k\to\infty} \mu_k = \infty. \tag{2.3}$$

By studying the blow-up behavior of u_k (see in particular Propositions 2.2 and 2.9), we will show that (2.3) implies $C_\pi \le 4\pi$ (Proposition 2.10), but with suitable test functions we will also prove that $C_\pi > 4\pi$ (Proposition 2.11), hence contradicting (2.3) and completing the proof of Theorem 1.1.

2.2 The Blow-Up Analysis

The following proposition is well known in the local case, and its proof in the present setting is similar to the local one. We give it for completeness.

Proposition 2.1. *We have $u_k \in C^\infty(I) \cap C^{0,1/2}(\bar{I})$, $u_k > 0$ in I, and u_k is symmetric with respect to 0 and decreasing with respect to $|x|$. Moreover,*

$$0 < \lambda_k < \lambda_1(I). \tag{2.4}$$

Up to a subsequence, we have $\lambda_k \to \lambda_\infty$ and $u_k \to u_\infty$ weakly in $\tilde{H}^{1/2,2}(I)$ and strongly in $L^2(I)$, where u_∞ solves

$$(-\Delta)^{\frac{1}{2}} u_\infty = \lambda_\infty u_\infty e^{\pi u_\infty^2}. \tag{2.5}$$

Proof. For the first claim, see [35, Remark 1.4]. The positivity follows from the maximum principle, and the symmetry and monotonicity follow from the moving point technique; see, e.g., [8, Theorem 11].

Now testing (2.1) with φ_1, the first eigenfunction of $(-\Delta)^{1/2}$ in $\tilde{H}^{1/2,2}(I)$, positive and with eigenvalue $\lambda_1(I) > 0$, we obtain

$$\lambda_1(I) \int_I u_k \varphi_1 \, dx = \lambda_k \int_I u_k e^{\alpha_k u_k^2} \varphi_1 \, dx > \lambda_k \int_I u_k \varphi_1 \, dx,$$

hence proving (2.4). By the theorem of Banach–Alaoglu and the compactness of the Sobolev embedding of $\tilde{H}^{1/2,2}(I) \hookrightarrow L^2(I)$, we obtain the claimed convergence of u_k to u_∞. Finally, to show that u_∞ solves (2.5), test with $\varphi \in C_c^\infty(I)$:

$$\begin{aligned}
\int_I u_\infty (-\Delta)^{\frac{1}{2}} \varphi \, dx &= \lim_{k\to\infty} \int_I u_k (-\Delta)^{\frac{1}{2}} \varphi \, dx \\
&= \lim_{k\to\infty} \int_I \lambda_k u_k e^{\alpha_k u_k^2} \varphi \, dx \\
&= \int_I \lambda_\infty u_\infty e^{\pi u_\infty^2} \varphi \, dx,
\end{aligned}$$

where the convergence of the last integral is justified by splitting I into

$$I_1 := \{x \in I : u_k(x) \le L\} \quad \text{and} \quad I_2 := \{x \in I : u_k(x) > L\},$$

applying the dominated convergence on I_1 and bounding

$$\begin{aligned}
\int_{I_2} \lambda_k u_k e^{\alpha_k u_k^2} \varphi \, dx &\le \frac{\sup_I |\varphi|}{L} \int_I \lambda_k u_k^2 e^{\alpha_k u_k^2} \, dx \\
&= \frac{\sup_I |\varphi|}{L} \int_I u_k (-\Delta)^{\frac{1}{2}} u_k \, dx \\
&= \frac{\sup_I |\varphi|}{L} \|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

and letting $L \to \infty$. □

Let $\tilde{u}_k$ be the harmonic extension of u_k to $\mathbb{R}^2_+$ given by the Poisson integral; see (A.5) in Appendix A. Notice that

$$\int_I \lambda_k u_k^2 e^{\alpha_k u_k^2}\,dx = \|(-\Delta)^{\frac{1}{4}} u_k\|^2_{L^2(\mathbb{R})} = \|\nabla \tilde{u}_k\|^2_{L^2(\mathbb{R}^2_+)} = 1. \tag{2.6}$$

Let

$$r_k = \frac{1}{\alpha_k \lambda_k \mu_k^2 e^{\alpha_k \mu_k^2}} \quad \text{and} \quad \eta_k(x) := 2\alpha_k \mu_k (u_k(r_k x) - \mu_k)$$

be as in (1.8) and (1.9), and set

$$\tilde{\eta}_k(x,y) := 2\alpha_k \mu_k (\tilde{u}_k(r_k x, r_k y) - \mu_k).$$

Note that $\tilde{\eta}_k$ is the Poisson integral of η_k.

Proposition 2.2. *We have $r_k \to 0$ and $\tilde{\eta}_k \to \tilde{\eta}_\infty$ in $C^\ell_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ for every $\ell \geq 0$, where*

$$\tilde{\eta}_\infty(x,y) = -\log((1+y)^2 + x^2)$$

is the Poisson integral (compare to (A.5)*) of $\eta_\infty := -\log(1+x^2)$, and*

$$(-\Delta)^{\frac{1}{2}} \eta_\infty = 2e^{\eta_\infty}, \qquad \int_{\mathbb{R}} e^{\eta_\infty}\,dx = \pi. \tag{2.7}$$

Proof. According to [35, Lemma 2.2, Theorem 1.5 and Proposition 2.7], we have $r_k \to 0$ and $\eta_k \to \eta_\infty$ in $C^\ell_{\mathrm{loc}}(\mathbb{R})$ for every $\ell \geq 0$ and (η_k) is uniformly bounded in $L_{1/2}(\mathbb{R})$ (see (A.2)).

To obtain the local convergence of $\tilde{\eta}_k$, fix $R > 0$ and split the integral in the Poisson integral (A.5) of $\tilde{\eta}_k$ into an integral over $(-R, R)$ and an integral over $\mathbb{R} \setminus (-R, R)$, for R large. The former is bounded by the convergence of η_k locally, the latter by the boundedness of η_k in $L_{1/2}(\mathbb{R})$, provided

$$(x,y) \in B_{\frac{R}{2}} \cap \mathbb{R}^2_+.$$

As a consequence, we get that $\tilde{\eta}_k$ is locally uniformly bounded in $\overline{\mathbb{R}^2_+}$. Since $\tilde{\eta}_k$ is harmonic, we conclude by elliptic estimates. □

Remark 2.3. As $L \to \infty$, we have

$$\int_{\mathbb{R}^2_+ \cap B_L} |\nabla \tilde{\eta}_\infty|^2\,dx\,dy = 4\pi \log\Big(\frac{L}{2}\Big) + O\Big(\frac{\log L}{L}\Big). \tag{2.8}$$

Moreover, the same estimate holds if B_L is replaced by $B_L(0,-1)$.

Proof. As $L \to \infty$, we have

$$\tilde{\eta}_\infty(x,y) = -2\log L + O(L^{-1}) \quad \text{and} \quad \nabla \eta_\infty(x,y) \cdot \frac{(x,y)}{|(x,y)|} = -\frac{2}{L} + O(L^{-2})$$

for $(x,y) \in \mathbb{R}^2_+ \cap \partial B_L$. Then, integrating by parts and using (2.7), we get that

$$\begin{aligned}\int_{\mathbb{R}^2_+ \cap B_L} |\nabla \tilde{\eta}_\infty|^2\,dx\,dy &= \int_{\mathbb{R}^2_+ \cap \partial B_L} \tilde{\eta}_\infty \frac{\partial \tilde{\eta}_\infty}{\partial \nu}\,d\sigma + 2\int_{-L}^{L} \eta_\infty e^{\eta_\infty}\,dx \\ &= 4\pi \log L + 2\int_{\mathbb{R}} \eta_\infty e^{\eta_\infty}\,dx + O\Big(\frac{\log L}{L}\Big).\end{aligned}$$

The definition of the Poisson integral (see (A.5)) gives

$$2\int_{\mathbb{R}} \eta_\infty e^{\eta_\infty}\,dx = 2\int_{\mathbb{R}} \frac{\eta_\infty}{1+x^2}\,dx = 2\pi \tilde{\eta}_\infty(0,1) = -4\pi \log 2.$$

This proves (2.8). Finally, observe that $B_L \setminus B_L(0,-1)$ and $B_L(0,-1) \setminus B_L$ are contained in $A_L := B_{L+1} \setminus B_{L-1}$. Since $|\nabla\tilde\eta_\infty|^2 = O(L^{-2})$ in $\mathbb{R}^2_+ \cap A_L$, we get

$$\Bigg|\int_{\mathbb{R}^2_+\cap B_L} |\nabla\tilde\eta_\infty|^2\,dx\,dy - \int_{\mathbb{R}^2_+\cap B_L(0,-1)} |\nabla\tilde\eta_\infty|^2\,dx\,dy\Bigg| \le \int_{\mathbb{R}^2_+\cap A_L} |\nabla\tilde\eta_\infty|^2\,dx\,dy = O(L^{-1}). \qquad \square$$

Corollary 2.4. *For $R > 0$ and $i = 0, 1, 2$, we have*

$$\lim_{k\to\infty} \int_{-Rr_k}^{Rr_k} \lambda_k \mu_k^i u_k^{2-i} e^{\alpha_k u_k^2}\,dx = \frac{1}{\pi}\int_{-R}^{R} e^{\eta_\infty}\,dx. \tag{2.9}$$

Moreover, $u_\infty \equiv 0$, i.e. up to a subsequence $u_k \to 0$ in $L^2(I)$, weakly in $\tilde H^{1/2,2}(I)$, and a.e. in I.

Proof. With the change of variables $\xi = \frac{x}{r_k}$, writing $u_k(r_k\cdot) = \mu_k + \frac{\eta_k}{2\alpha_k\mu_k}$ and using (1.8) and Proposition 2.2, we see that

$$\int_{-Rr_k}^{Rr_k} \lambda_k \mu_k^i u_k^{2-i} e^{\alpha_k u_k^2}\,dx = \underbrace{r_k\lambda_k\mu_k^2 e^{\alpha_k\mu_k^2}}_{=\frac{1}{\alpha_k}} \int_{-R}^{R} \Big(1 + \frac{\eta_k}{2\alpha_k\mu_k^2}\Big)^{2-i} e^{\eta_k + \frac{\eta_k^2}{4\alpha_k\mu_k^2}}\,d\xi \to \frac{1}{\pi}\int_{-R}^{R} e^{\eta_\infty}\,d\xi$$

as $k \to \infty$, as claimed in (2.9).

In order to prove the last statement, recalling that $\|(-\Delta)^{1/4}u_k\|_{L^2} = 1$, we write

$$1 = \int_{-Rr_k}^{Rr_k} \lambda_k u_k^2 e^{\alpha_k u_k^2}\,dx + \int_{I\setminus(-Rr_k,Rr_k)} \lambda_k u_k^2 e^{\alpha_k u_k^2}\,dx =: (\mathrm{I})_k + (\mathrm{II})_k.$$

By (2.7) and (2.9), we get

$$\lim_{k\to\infty} (I)_k = \frac{1}{\pi}\int_{-R}^{R} e^{\eta_\infty}\,dx = 1 + o(1),$$

with $o(1) \to 0$ as $R \to \infty$. This in turn implies that

$$\lim_{R\to\infty}\lim_{k\to\infty} (\mathrm{II})_k = 0,$$

which is possible only if $u_\infty \equiv 0$ or $\lambda_\infty = 0$ (by Fatou's lemma). But on account of (2.5), also in the latter case we have $u_\infty \equiv 0$. $\square$

Lemma 2.5. *For $A > 1$, set $u_k^A := \min\{u_k, \frac{\mu_k}{A}\}$. Then we have*

$$\limsup_{k\to\infty} \|(-\Delta)^{\frac14} u_k^A\|^2_{L^2(\mathbb{R})} \le \frac{1}{A}.$$

Proof. We set $\bar u_k^A := \min\{\tilde u_k, \frac{\mu_k}{A}\}$. Since $\bar u_k^A$ is an extension (in general not harmonic) of u_k^A, we have

$$\|(-\Delta)^{\frac14} u_k^A\|^2_{L^2(\mathbb{R})} \le \int_{\mathbb{R}^2_+} |\nabla \bar u_k^A|^2\,dx\,dy. \tag{2.10}$$

Using integration by parts and the harmonicity of $\tilde u_k$, we get

$$\begin{aligned}\int_{\mathbb{R}^2_+} |\nabla \bar u_k^A|^2\,dx\,dy &= \int_{\mathbb{R}^2_+} \nabla\bar u_k^A \cdot \nabla\tilde u_k\,dx\,dy\\ &= -\int_{\mathbb{R}} u_k^A(x)\frac{\partial\tilde u_k(x,0)}{\partial y}\,dx\\ &= \int_{\mathbb{R}} (-\Delta)^{\frac12} u_k u_k^A\,dx.\end{aligned} \tag{2.11}$$

Note that, even if $\mathbb{R}^2_+$ is unbounded, the integration by parts above holds since $|\tilde{u}(x,y)| = O(|(x,y)|^{-1})$ and $|\nabla\tilde{u}(x,y)| = O(|(x,y)|^{-2})$ for $|(x,y)|$ large (see Lemma A.4). Proposition 2.2 implies that $u_k^A(r_k x) = \frac{\mu_k}{A}$ for $|x| \le R$ and $k \ge k_0(R)$. Then, with (2.7) and (2.9), we obtain

$$\begin{aligned}\int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A \, dx &\ge \int_{-Rr_k}^{Rr_k} \lambda_k u_k e^{\alpha_k u_k^2} u_k^A \, dx \\ &\xrightarrow{k\to\infty} \frac{1}{\pi A}\int_{-R}^{R} e^{\eta_\infty}\, d\xi \\ &\xrightarrow{R\to\infty} \frac{1}{A}.\end{aligned}$$

Set now $v_k^A := (u_k - \frac{\mu_k}{A})^+ = u_k - u_k^A$. With similar computations, we get

$$\begin{aligned}\int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k v_k^A \, dx &\ge \int_{-Rr_k}^{Rr_k} \lambda_k u_k v_k^A e^{\alpha_k u_k^2} \, dx \\ &\xrightarrow{k\to\infty} \frac{1}{\pi}\Big(1 - \frac{1}{A}\Big)\int_{-R}^{R} e^{\eta_\infty}\, d\xi \\ &\xrightarrow{R\to\infty} \frac{A-1}{A}.\end{aligned}$$

Since

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A \, dx + \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k v_k^A \, dx = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k \, dx = 1,$$

we get that

$$\lim_{k\to\infty} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A \, dx = \frac{1}{A}.$$

Then we conclude using (2.10) and (2.11). □

Proposition 2.6. *We have*

$$C_\pi = \lim_{k\to\infty} \frac{1}{\lambda_k \mu_k^2}. \tag{2.12}$$

Moreover,

$$\lim_{k\to\infty} \mu_k \lambda_k = 0. \tag{2.13}$$

Proof. Fix $A > 1$ and let u_k^A be defined as in Lemma 2.5. We split

$$\int_I (e^{\alpha_k u_k^2} - 1)\, dx = \int_{I\cap\{u_k \le \frac{\mu_k}{A}\}} (e^{\alpha_k (u_k^A)^2} - 1)\, dx + \int_{I\cap\{u_k > \frac{\mu_k}{A}\}} (e^{\alpha_k u_k^2} - 1)\, dx =: \text{(I)} + \text{(II)}.$$

Using Corollary 2.4 and Vitali's theorem, we see that

$$\text{(I)} \le \int_I (e^{\alpha_k (u_k^A)^2} - 1)\, dx \to 0 \quad \text{as } k \to \infty$$

since $e^{\alpha_k (u_k^A)^2}$ is uniformly bounded in $L^A(I)$ by Lemma 2.5 together with Theorem A.

By (2.6) and Corollary 2.4, we now estimate

$$\text{(II)} \le \frac{A^2}{\lambda_k \mu_k^2} \int_{I\cap\{u_k > \frac{\mu_k}{A}\}} \lambda_k u_k^2 (e^{\alpha_k u_k^2} - 1)\, dx \le \frac{A^2}{\lambda_k \mu_k^2}(1 + o(1)),$$

with $o(1) \to 0$ as $k \to \infty$. Together with (2.2), and by letting $A \downarrow 1$, this gives

$$C_\pi \le \lim_{k\to\infty} \frac{1}{\lambda_k \mu_k^2}.$$

The converse inequality follows from (2.9) as follows:

$$\int_I (e^{\alpha_k u_k^2} - 1)\,dx \geq \int_{-Rr_k}^{Rr_k} e^{\alpha_k u_k^2}\,dx + o(1) = \frac{1}{\lambda_k \mu_k^2}\Big(\frac{1}{\pi}\int_{-R}^{R} e^{\eta_\infty}\,dx + o(1)\Big) + o(1),$$

with $o(1) \to 0$ as $k \to \infty$. Letting $R \to \infty$ and recalling (2.7), we obtain (2.12).

Finally, (2.13) follows at once from (2.12) because otherwise we would have $C_\pi = 0$, which is clearly impossible. □

Proposition 2.7. *Let us set $f_k := \lambda_k \mu_k u_k e^{\alpha_k u_k^2}$. Then we have*

$$\int_I f_k \varphi\,dx \to \varphi(0)$$

as $k \to \infty$ for any $\varphi \in C(\bar{I})$. In particular, $f_k \rightharpoonup \delta_0$ in the sense of Radon measures in I.

Proof. Take $\varphi \in C(\bar{I})$. For given $R > 0$ and $A > 1$, we split

$$\int_I \varphi f_k\,dx = \int_{-Rr_k}^{Rr_k} \varphi f_k\,dx + \int_{\{u_k > \frac{\mu_k}{A}\}\setminus(-Rr_k, Rr_k)} \varphi f_k\,dx + \int_{\{u_k \leq \frac{\mu_k}{A}\}} \varphi f_k\,dx =: I_1 + I_2 + I_3.$$

On $\{u_k \leq \frac{\mu_k}{A}\}$ we have $u_k = u_k^A$, and Lemma 2.5 and Theorem A imply that $u_k e^{\alpha_k u_k^2}$ is uniformly bounded in L^1 (depending on A). Thus using (2.13), we get $I_3 \to 0$.

With (2.6) and (2.9) we also get

$$\begin{aligned} I_2 &\leq A\|\varphi\|_{L^\infty(I)} \int_{\{u_k > \frac{\mu_k}{A}\}\setminus(-Rr_k, Rr_k)} \lambda_k u_k^2 e^{\alpha_k u_k^2}\,dx \\ &\leq A\|\varphi\|_{L^\infty(I)}\Big(1 - \int_{-Rr_k}^{Rr_k} \lambda_k u_k^2 e^{\alpha_k u_k^2}\,dx\Big) \\ &= A\|\varphi\|_{L^\infty(I)}\Big(1 - \frac{1}{\pi}\int_{-R}^{R} e^{\eta_\infty}\,dx + o(1)\Big), \end{aligned}$$

with $o(1) \to 0$ as $k \to \infty$. Thanks to (2.7), we conclude that $I_2 \to 0$ as $k \to \infty$ and $R \to \infty$.

As for I_1, again with (2.9) we compute

$$I_1 = (\varphi(0) + o(1))\Big(\frac{1}{\pi}\int_{-R}^{R} e^{\eta_\infty}\,dx + o(1)\Big),$$

so that $I_1 \to \varphi(0)$ as $k \to \infty$ and $R \to \infty$. □

Given $x \in I$, let $G_x : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the Green's function of $(-\Delta)^{1/2}$ on I with singularity at x. We recall that we have the explicit formula (see, e.g., [3])

$$G_x(y) := \begin{cases} \frac{1}{\pi}\log\Big(\frac{1 - xy + \sqrt{(1-x^2)(1-y^2)}}{|x-y|}\Big), & y \in I, \\ 0, & y \in \mathbb{R} \setminus I. \end{cases} \tag{2.14}$$

In the following, we further denote

$$S(x,y) := G_x(y) - \frac{1}{\pi}\log\frac{1}{|x-y|}. \tag{2.15}$$

Lemma 2.8. *We have $\mu_k u_k \to G := G_0$ in $L^\infty_{\mathrm{loc}}(\bar{I} \setminus \{0\}) \cap L^1(I)$ as $k \to +\infty$.*

Proof. Let us set

$$v_k := \mu_k u_k - G \quad \text{and} \quad f_k = \mu_k \lambda_k u_k e^{\alpha_k u_k^2}.$$

Arguing as in Proposition 2.7, we show that $\|f_k\|_{L^1(\mathbb{R})} \to 1$ as $k \to \infty$. Moreover, since u_k is decreasing with respect to $|x|$, we get that $u_k \to 0$ and $f_k \to 0$ locally uniformly in $\bar{I} \setminus \{0\}$ as $k \to \infty$. By Green's representation formula, we have

$$\begin{aligned} |v_k(x)| &= \Big| \int_I G_x(y) f_k(y)\,dy - G(x) \Big| \\ &\le \int_I |G_x(y) - G(x)| f_k(y)\,dy + |\|f_k\|_{L^1(I)} - 1|\,|G(x)|, \quad x \in I. \end{aligned} \tag{2.16}$$

Fix $\sigma \in (0,1)$. If we assume $|x| \ge \sigma$ and $|y| \le \frac{\sigma}{2}$, then we have

$$\begin{aligned} |G_x(y) - G(x)| &\le \frac{1}{\pi}\Big|\log \frac{|x|}{|x-y|}\Big| + |S(x,y) - S(x,0)| \\ &\le \frac{1}{\pi}\Big|\log\Big|\frac{x}{|x|} - \frac{y}{|x|}\Big|\Big| + \sup_{|x|\ge\sigma,\,|y|\le\frac{\sigma}{2}} |\nabla_y S(x,y)|\,|y| \\ &\le C|y|, \end{aligned}$$

where C is a constant depending only on σ. Then, for any $\varepsilon \in (0, \frac{\sigma}{2})$, we can write

$$\begin{aligned} |v_k(x)| &\le \int_I |G_x(y) - G(x)| f_k(y)\,dy + o(1) \\ &= \int_{-\varepsilon}^{\varepsilon} |G_x(y) - G(x)| f_k(y)\,dy + \int_{I\setminus(-\varepsilon,\varepsilon)} |G_x(y) - G(x)| f_k(y)\,dy + o(1) \\ &\le C\varepsilon \|f_k\|_{L^1(-\varepsilon,\varepsilon)} + \big(\sup_{z\in I}\|G_z\|_{L^1(I)} + |G(x)|\big)\|f_k\|_{L^\infty(I\setminus(-\varepsilon,\varepsilon))} + o(1) \\ &\le C\varepsilon + o(1), \end{aligned} \tag{2.17}$$

where $o(1) \to 0$ uniformly in $I \setminus (-\sigma, \sigma)$ as $k \to \infty$. Clearly, (2.17) implies

$$\limsup_{k\to\infty} \|v_k\|_{L^\infty(I\setminus(-\sigma,\sigma))} \le C\varepsilon.$$

Since ε and σ can be arbitrarily small, this shows that $v_k \to 0$ in $L^\infty_{\mathrm{loc}}(\bar{I} \setminus \{0\})$. With a similar argument, we prove the L^1 convergence. Indeed, integrating (2.16), for $\varepsilon \in (0,1)$ we get

$$\begin{aligned} \|v_k\|_{L^1(I)} &\le \int_I \int_I |G_x(y) - G(x)| f_k(y)\,dy\,dx + |\|f_k\|_{L^1(I)} - 1|\|G\|_{L^1(I)} \\ &\le \int_I f_k(y) \int_I |G_x(y) - G(x)|\,dx\,dy + o(1) \\ &\le \int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I |G_x(y) - G(x)|\,dx\,dy + 2\sup_{z\in I}\|G_z\|_{L^1(I)}\|f_k\|_{L^\infty(I\setminus(-\varepsilon,\varepsilon))} + o(1) \\ &= \int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I |G_x(y) - G(x)|\,dx\,dy + o(1). \end{aligned} \tag{2.18}$$

Since

$$\sup_{y\in(-\varepsilon,\varepsilon)} \sup_{x\in I} |S(x,y) - S(x,0)| = O(\varepsilon),$$

we get

$$\int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I |G_y(x) - G(x)|\,dx\,dy = \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I \Big|\log \frac{|x|}{|x-y|}\Big|\,dx\,dy + O(\varepsilon).$$

Moreover, using the change of variables $x = yz$, we obtain

$$\int_I \left|\log \frac{|x|}{|x-y|}\right| dx = |y| \int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \left|\log \frac{|z|}{|z-1|}\right| dz = O\Big(|y| \log \frac{1}{|y|}\Big).$$

Then we have

$$\int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I |G_y(x) - G_0(x)|\, dx\, dy = \int_{-\varepsilon}^{\varepsilon} f_k(y) O\Big(|y| \log \frac{1}{|y|}\Big)\, dy + O(\varepsilon) = O\Big(\varepsilon \log \frac{1}{\varepsilon}\Big). \tag{2.19}$$

Clearly, (2.18) and (2.19) yield $\limsup_{k\to+\infty} \|v_k - G\|_{L^1(I)} = O(\varepsilon \log \frac{1}{\varepsilon})$. Since ε can be arbitrarily small, we get the conclusion. □

Proposition 2.9. *We have $\mu_k \tilde{u}_k \to \tilde{G}$ in*

$$C^0_{\mathrm{loc}}(\overline{\mathbb{R}^2_+} \setminus \{(0,0)\}) \cap C^1_{\mathrm{loc}}(\mathbb{R}^2_+),$$

where $\tilde{G}$ is the Poisson extension of G.

Proof. As in the proof of Lemma 2.8, we denote $v_k := \mu_k u_k - G$. Let us consider the Poisson extension $\widetilde{v}_k = \mu_k \widetilde{u}_k - \widetilde{G}$. For any fixed $\varepsilon > 0$, we can split

$$\widetilde{v}_k(x,y) = \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{y v_k(\xi)}{(x-\xi)^2 + y^2}\, d\xi + \frac{1}{\pi} \int_{I\setminus(-\varepsilon,\varepsilon)} \frac{y v_k(\xi)}{(x-\xi)^2 + y^2}\, d\xi.$$

By Lemma 2.8, we have

$$\begin{aligned} \left|\frac{1}{\pi} \int_{I\setminus(-\varepsilon,\varepsilon)} \frac{y v_k(\xi)}{(x-\xi)^2 + y^2}\, d\xi\right| &\le \frac{1}{\pi} \|v_k\|_{L^\infty(I\setminus(-\varepsilon,\varepsilon))} \int_{\mathbb{R}} \frac{y}{(x-\xi)^2 + y^2}\, d\xi \\ &= \|v_k\|_{L^\infty(I\setminus(-\varepsilon,\varepsilon))} \to 0 \end{aligned}$$

as $k \to \infty$. Moreover, assuming $(x,y) \in \mathbb{R}^2_+ \setminus B_{2\varepsilon}(0,0)$, we get

$$\left|\frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{y v_k(\xi)}{(x-\xi)^2 + y^2}\, d\xi\right| \le \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{y |v_k(\xi)|}{|(x,y) - (\xi,0)|^2}\, d\xi \le \frac{y}{\pi \varepsilon^2} \|v_k\|_{L^1(I)} \to 0.$$

Hence $\widetilde{v}_k \to 0$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2_+ \setminus B_{2\varepsilon}(0,0))$. Finally, since can ε be arbitrarily small and $\tilde{v}_k$ is harmonic in $\mathbb{R}^2_+$, we get $\widetilde{v}_k \to 0$ in

$$C^0_{\mathrm{loc}}(\overline{\mathbb{R}^2_+} \setminus \{(0,0)\}) \cap C^1_{\mathrm{loc}}(\mathbb{R}^2_+).$$ □

2.3 The Two Main Estimates and Completion of the Proof

We shall now conclude our contradiction argument by showing the incompatibility of (2.3) with (2.2) and the definition of C_π. In this final part of the proof, we will use the precise asymptotic of $\widetilde{G}$ near $(0,0)$. Since $\log|(x,y)|$ is the Poisson integral of $\log|x|$ (see Proposition A.3), and since $S(0,\cdot) \in C(\mathbb{R})$, equation (2.15) guarantees the existence of the limit

$$S_0 := \lim_{(x,y)\to(0,0)} \tilde{G}(x,y) + \frac{1}{\pi} \log|(x,y)| = \lim_{x\to 0} G(x) + \frac{1}{\pi} \log|x|.$$

In fact, using (2.14), we get $S_0 = \frac{\log 2}{\pi}$. More precisely, noting that $S(0,\cdot) \in C^\infty(I)$, we can write

$$\tilde{G}(x,y) = \frac{1}{\pi} \log \frac{1}{|(x,y)|} + S_0 + h(x,y), \tag{2.20}$$

with

$$h \in C^\infty(\overline{\mathbb{R}^2_+} \cap B_1(0,0)) \cap C(\overline{\mathbb{R}^2_+}) \quad \text{and} \quad h(0,0) = 0.$$

Proposition 2.10. *If* (2.3) *holds, then* $C_\pi \le 2\pi e^{\pi S_0} = 4\pi$.

Proof. For a fixed large $L > 0$ and a fixed and small $\delta > 0$, set

$$a_k := \inf_{\partial B_{Lr_k}\cap\mathbb{R}^2_+} \tilde u_k, \quad b_k := \sup_{\partial B_\delta\cap\mathbb{R}^2_+} \tilde u_k, \quad \tilde v_k := (\tilde u_k \wedge a_k) \vee b_k.$$

Recalling that $\|\nabla \tilde u_k\|^2_{L^2} = 1$, we have

$$\int_{(B_\delta\setminus B_{Lr_k})\cap\mathbb{R}^2_+} |\nabla \tilde v_k|^2\,dx\,dy \le 1 - \int_{\mathbb{R}^2_+\setminus B_\delta} |\nabla \tilde u_k|^2\,dx\,dy - \int_{\mathbb{R}^2_+\cap B_{Lr_k}} |\nabla \tilde u_k|^2\,dx\,dy.$$

Clearly, the left-hand side bounds

$$\inf_{\substack{\tilde u|_{\mathbb{R}^2_+\cap\partial B_{Lr_k}}=a_k\\ \tilde u|_{\mathbb{R}^2_+\cap\partial B_\delta}=b_k}} \int_{(B_\delta\setminus B_{Lr_k})\cap\mathbb{R}^2_+} |\nabla \tilde u|^2\,dx\,dy = \int_{(B_\delta\setminus B_{Lr_k})\cap\mathbb{R}^2_+} |\nabla \tilde\Phi_k|^2\,dx\,dy = \pi\frac{(a_k-b_k)^2}{\log\delta - \log(Lr_k)},$$

where the function $\tilde\Phi_k$ is the unique solution to

$$\begin{cases} \Delta\tilde\Phi_k = 0 & \text{in } \mathbb{R}^2_+\cap(B_\delta\setminus B_{Lr_k}),\\ \tilde\Phi_k = a_k & \text{on } \mathbb{R}^2_+\cap\partial B_{Lr_k},\\ \tilde\Phi_k = b_k & \text{on } \mathbb{R}^2_+\cap\partial B_\delta,\\ \dfrac{\partial\tilde\Phi_k}{\partial y} = 0 & \text{on } \partial\mathbb{R}^2_+\cap(B_\delta\setminus B_{Lr_k}), \end{cases}$$

given explicitly by

$$\tilde\Phi_k = \frac{b_k - a_k}{\log\delta - \log(Lr_k)}\log|(x,y)| + \frac{a_k\log\delta - b_k\log Lr_k}{\log\delta - \log(Lr_k)}.$$

Using Proposition 2.2, we obtain

$$a_k = \mu_k + \frac{-\frac{1}{\pi}\log L + O(L^{-1}) + o(1)}{\mu_k},$$

where for fixed $L > 0$ we have $o(1) \to 0$ as $k \to \infty$, and $|O(L^{-1})| \le \frac{C}{L}$ uniformly for L and k large. Moreover, using Proposition 2.9 and (2.20), we obtain

$$b_k = \frac{-\frac{1}{\pi}\log\delta + S_0 + O(\delta) + o(1)}{\mu_k},$$

where for fixed $\delta > 0$ we have $o(1) \to 0$ as $k \to \infty$, and $|O(\delta)| \le C\delta$ uniformly for δ small and k large.

Still with Proposition 2.2, we get

$$\lim_{k\to\infty} \mu_k^2 \int_{\mathbb{R}^2_+\cap B_{Lr_k}} |\nabla \tilde u_k|^2\,dx\,dy = \frac{1}{4\pi^2}\int_{\mathbb{R}^2_+\cap B_L} |\nabla\tilde\eta_\infty|^2\,dx\,dy = \frac{1}{\pi}\log\frac{L}{2} + O\Big(\frac{\log L}{L}\Big).$$

Similarly with Proposition 2.9 we get

$$\begin{aligned}
\liminf_{k\to\infty} \mu_k^2 \int_{\mathbb{R}^2_+\setminus B_\delta} |\nabla\tilde u_k|^2\,dx\,dy &\ge \int_{\mathbb{R}^2_+\setminus B_\delta} |\nabla\tilde G|^2\,dx\,dy\\
&= \int_{\mathbb{R}^2_+\cap\partial B_\delta} -\frac{\partial\tilde G}{\partial r}\tilde G\,d\sigma + \int_{\mathbb{R}\setminus(-\delta,\delta)} -\frac{\partial\tilde G(x,0)}{\partial y}G(x)\,dx\\
&= \int_{\mathbb{R}^2_+\cap\partial B_\delta} \Big(\frac{1}{\pi\delta} + O(1)\Big)\Big(-\frac{1}{\pi}\log\delta + S_0 + O(\delta)\Big)\,d\sigma\\
&= -\frac{1}{\pi}\log\delta + S_0 + O(\delta\log\delta),
\end{aligned}$$

where we used Lemma A.4, the expansion in (2.20) and the boundary conditions

$$\begin{cases} \widetilde{G}(x,0) = G(x) = 0 & \text{for } x \in \mathbb{R} \setminus I, \\ -\dfrac{\partial \tilde{G}(x,0)}{\partial y} = (-\Delta)^{\frac{1}{2}} G(x) = 0 & \text{for } x \in I \setminus \{0\}. \end{cases}$$

We then get

$$\frac{\pi(a_k - b_k)^2}{\log \delta - \log(Lr_k)} \le 1 - \frac{-\frac{1}{\pi}\log \delta + S_0 + O(\delta \log \delta) + \frac{1}{\pi}\log \frac{L}{2} + O(\frac{\log L}{L})}{\mu_k^2}$$

or

$$\begin{aligned} \pi(a_k - b_k)^2 &= \pi\mu_k^2 - 2\log L + O(L^{-1}) + 2\log \delta - 2\pi S_0 + O(\delta) + o(1) + \frac{O(\log^2 L + \log^2 \delta)}{\mu_k^2} \\ &\le (\log \delta - \log L + \log(\lambda_k \mu_k^2) + \alpha_k \mu_k^2 + \log \alpha_k) \\ &\quad \times \Big(1 - \frac{-\frac{1}{\pi}\log \delta + S_0 + O(\delta \log \delta) + \frac{1}{\pi}\log \frac{L}{2} + O(\frac{\log L}{L})}{\mu_k^2}\Big) \\ &= \log \frac{\delta}{L} + \log(\lambda_k \mu_k^2) + \alpha_k \mu_k^2 + \log \alpha_k + \alpha_k\Big(\frac{1}{\pi}\log \frac{2\delta}{L} - S_0\Big) \\ &\quad + O(\delta \log \delta) + O\Big(\frac{\log L}{L}\Big) + \frac{O(\log^2 \delta) + O(\log^2 L) + O(1)}{\mu_k^2}. \end{aligned}$$

Rearranging gives

$$\log \frac{1}{\lambda_k \mu_k^2} \le \Big(1 - \frac{\alpha_k}{\pi}\Big)\log \frac{L}{\delta} + (\alpha_k - \pi)\mu_k^2 + (2\pi - \alpha_k)S_0 + \frac{\alpha_k}{\pi}\log 2 + \log \alpha_k + O(\delta \log \delta) + O\Big(\frac{\log L}{L}\Big) + o(1),$$

with $o(1) \to 0$ as $k \to \infty$. Then, recalling that $\alpha_k \uparrow \pi$ and letting first $k \to \infty$ and then $L \to \infty$ and $\delta \to 0$, we obtain

$$\limsup_{k\to\infty} \log \frac{1}{\lambda_k \mu_k^2} \le \pi S_0 + \log(2\pi) = \log(4\pi).$$

Using Proposition 2.6, we conclude. □

Proposition 2.11. *There exists a function $u \in \tilde{H}^{1/2,2}(I)$ with $\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \le 1$ such that*

$$\int_I (e^{\pi u^2} - 1)\,dx > 2\pi e^{\pi S_0} = 4\pi.$$

Proof. For $\varepsilon > 0$ choose $L = L(\varepsilon) > 0$ such that, as $\varepsilon \to 0$, we have $L \to \infty$ and $L\varepsilon \to 0$. Fix

$$\Gamma_{L\varepsilon} := \{(x,y) \in \mathbb{R}^2_+ : \tilde{G}(x,y) = \gamma_{L\varepsilon} := \min_{\mathbb{R}^2_+ \cap \partial B_{L\varepsilon}} \tilde{G}\}$$

and

$$\Omega_{L\varepsilon} := \{(x,y) \in \mathbb{R}^2_+ : \tilde{G}(x,y) > \gamma_{L\varepsilon}\}.$$

By the maximum principle, we have

$$\mathbb{R}^2_+ \cap B_{L\varepsilon} \subset \Omega_{L\varepsilon}.$$

Indeed, $\tilde{G}$ is harmonic in $\mathbb{R}^2_+$, $\tilde{G} \ge \gamma_{L\varepsilon}$ on $\partial(\mathbb{R}^2_+ \cap B_{L\varepsilon}) \setminus \{(0,0)\}$, and $\tilde{G} \to +\infty$ as $(x,y) \to (0,0)$. Notice also that (2.20) gives

$$\gamma_{L\varepsilon} = -\frac{1}{\pi}\log(L\varepsilon) + S_0 + O(L\varepsilon). \tag{2.21}$$

For some constants B and c to be fixed, we set

$$U_\varepsilon(x,y) := \begin{cases} c - \dfrac{\log(\frac{x^2}{\varepsilon^2} + (1 + \frac{y}{\varepsilon})^2) + 2B}{2\pi c} & \text{for } (x,y) \in \mathbb{R}^2_+ \cap B_{L\varepsilon}(0,-\varepsilon), \\ \dfrac{\gamma_{L\varepsilon}}{c} & \text{for } (x,y) \in \Omega_{L\varepsilon} \setminus B_{L\varepsilon}(0,-\varepsilon), \\ \dfrac{\tilde{G}(x,y)}{c} & \text{for } (x,y) \in \mathbb{R}^2_+ \setminus \Omega_{L\varepsilon}. \end{cases}$$

Observe that $\mathbb{R}^2_+ \cap B_{L\varepsilon}(0,-\varepsilon) \subseteq \mathbb{R}^2_+ \cap B_{L\varepsilon} \subseteq \Omega_{L\varepsilon}$. To have continuity on $\mathbb{R}^2_+ \cap \partial B_{L\varepsilon}(0,-\varepsilon)$ we impose

$$\frac{-\log L^2 - 2B}{2\pi c} + c = \frac{\gamma_{L\varepsilon}}{c},$$

which, together with (2.21), gives the relation

$$B = \pi c^2 + \log\varepsilon - \pi S_0 + O(L\varepsilon). \tag{2.22}$$

Moreover,

$$\int\limits_{\mathbb{R}^2_+\cap B_{L\varepsilon}(0,-\varepsilon)} |\nabla U_\varepsilon|^2\,dx\,dy = \frac{1}{4\pi^2 c^2}\int\limits_{\mathbb{R}^2_+\cap B_L(0,-1)} |\nabla \log(x^2+(1+y)^2)|^2\,dx\,dy = \frac{\frac{1}{\pi}\log(\frac{L}{2}) + O(\frac{\log L}{L})}{c^2}$$

and

$$\begin{aligned}\int\limits_{\mathbb{R}^2_+\setminus\Omega_{L\varepsilon}} |\nabla U_\varepsilon|^2\,dx\,dy &= \frac{1}{c^2}\int\limits_{\mathbb{R}^2_+\setminus\Omega_{L\varepsilon}} |\nabla\tilde G|^2\,dx\,dy\\ &= \frac{1}{c^2}\int\limits_{\mathbb{R}^2_+\cap\partial\Omega_{L\varepsilon}} \frac{\partial\tilde G}{\partial\nu}\tilde G\,d\sigma - \underbrace{\frac{1}{c^2}\int\limits_{(\mathbb{R}\times\{0\})\setminus\bar\Omega_{L\varepsilon}} \frac{\partial\tilde G}{\partial y}\tilde G\,dx}_{=0}\\ &= \frac{\frac{1}{\pi}\log(\frac{1}{L\varepsilon}) + S_0 + O(L\varepsilon\log(L\varepsilon))}{c^2},\end{aligned}$$

where the last equality follows from (2.20). We now impose $\|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^2_+)} = 1$, obtaining

$$-\log\varepsilon - \log 2 + \pi S_0 + O(L\varepsilon\log(L\varepsilon)) + O\Big(\frac{\log L}{L}\Big) = \pi c^2, \tag{2.23}$$

which, together with (2.22), implies

$$B = -\log 2 + O(L\varepsilon\log(L\varepsilon)) + O\Big(\frac{\log L}{L}\Big). \tag{2.24}$$

Let now

$$I^1_{L,\varepsilon} = (-\varepsilon\sqrt{L^2-1}, \varepsilon\sqrt{L^2-1})$$

and $I^2_{L\varepsilon}$ be the disjoint sub-intervals of I obtained by intersecting $I\times\{0\}$ respectively with

$$B_{L\varepsilon}(0,-\varepsilon) \quad\text{and}\quad \overline{\mathbb{R}^2_+\setminus\Omega_{L\varepsilon}}.$$

Then, for $u_\varepsilon(x) := U_\varepsilon(x,0)$, using a change of variables and (2.23)–(2.24), we get

$$\begin{aligned}\int\limits_{I^1_{L,\varepsilon}} e^{\pi u_\varepsilon^2}\,dx &= \varepsilon\int\limits_{-\sqrt{L^2-1}}^{\sqrt{L^2-1}} \exp\Big(\pi\Big(c - \frac{\log(1+x^2)+2B}{2\pi c}\Big)^2\Big)\,dx\\ &> \varepsilon e^{\pi c^2 - 2B}\int\limits_{-\sqrt{L^2-1}}^{\sqrt{L^2-1}} \frac{1}{1+x^2}\,dx\\ &= 2e^{\pi S_0 + O(L\varepsilon\log(L\varepsilon)) + O(\frac{\log L}{L})}\pi\Big(1 + O\Big(\frac{1}{L}\Big)\Big)\\ &= 2\pi e^{\pi S_0} + O(L\varepsilon\log(L\varepsilon)) + O\Big(\frac{\log L}{L}\Big).\end{aligned}$$

Moreover,

$$\int\limits_{I^2_{L\varepsilon}} (e^{\pi u_\varepsilon^2} - 1)\,dx \ge \int\limits_{I^2_{L\varepsilon}} \pi u_\varepsilon^2\,dx = \frac{1}{c^2}\int\limits_{I^2_{L\varepsilon}} \pi G^2\,dx =: \frac{\nu_{L\varepsilon}}{c^2},$$

with

$$\nu_{L\varepsilon} > \nu_{\frac{1}{2}} > 0 \quad \text{for } L\varepsilon < \frac{1}{2}.$$

Now observe that $c^2 = -\frac{\log \varepsilon}{\pi} + O(1)$ by (2.23), and choose $L = \log^2 \varepsilon$ to obtain

$$O(L\varepsilon \log(L\varepsilon)) + O\Big(\frac{\log L}{L}\Big) = O\Big(\frac{\log\log\varepsilon}{\log^2 \varepsilon}\Big) = o\Big(\frac{1}{c^2}\Big),$$

so that

$$\int_I (e^{\pi u_\varepsilon^2} - 1)\,dx \geq 2\pi e^{\pi S_0} + \frac{\nu_{\frac{1}{2}}}{c^2} + o\Big(\frac{1}{c^2}\Big) > 2\pi e^{\pi S_0}$$

for ε small enough.

Finally, notice that

$$\|(-\Delta)^{\frac{1}{4}} u_\varepsilon\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}^2_+} |\nabla \tilde{u}_\varepsilon|^2\,dx\,dy \leq \int_{\mathbb{R}^2_+} |\nabla U_\varepsilon|^2\,dx\,dy \leq 1$$

since the Poisson extension $\tilde{u}_\varepsilon$ minimizes the Dirichlet energy among extensions with finite energy. □

3 Proof of Theorem 1.3

Let $u_k \in H \cap C^\infty(\mathbb{R})$ be a sequence of positive even and decreasing solutions to (1.5) satisfying the energy bound (1.6) and with $\lambda_k \to \lambda_\infty \geq 0$ as $k \to \infty$.

First we show that case (i) holds when $\mu_k \leq C$.

Lemma 3.1. *If $\mu_k \leq C$, then* (i) *holds.*

Proof. By assumption, we know that u_k and

$$f_k := (-\Delta)^{\frac{1}{2}} u_k = \lambda_k u_k e^{\alpha_k u_k^2} - u_k$$

are uniformly bounded in $L^\infty(\mathbb{R})$. Then, by elliptic estimates and a bootstrap argument (see [11, Theorem 1.5] and [22, Corollary 25]), we can find $u_\infty \in C^\infty(\mathbb{R})$ such that, up to a subsequence, $u_k \to u_\infty$ in $C^\ell_{\mathrm{loc}}(\mathbb{R})$ for every $\ell \geq 0$. To prove that u_∞ satisfies (1.7), note that $f_k \to f_\infty := \lambda_\infty u_\infty e^{\pi u_\infty^2} - u_\infty$ locally uniformly on $\mathbb{R}$ and set $M = \sup_k(\|f_k\|_{L^\infty(\mathbb{R})} + \mu_k)$. For any $\varphi \in \mathcal{S}(\mathbb{R})$ (the Schwarz space of rapidly decreasing functions) and any $R > 0$, we have that

$$\begin{aligned}\int_{\mathbb{R}} |f_k - f_\infty|\,|\varphi|\,dx &\leq \|f_k - f_\infty\|_{L^\infty((-R,R))} \int_{-R}^{R} |\varphi| + 2M\|\varphi\|_{L^1((-R,R)^c)} \\ &\xrightarrow{k\to+\infty} M\|\varphi\|_{L^1((-R,R)^c)} \\ &\xrightarrow{R\to+\infty} 0.\end{aligned}$$

Similarly, recalling that $(-\Delta)^{\frac{1}{2}}\varphi$ has quadratic decay at infinity (see, e.g., [18, Proposition 2.1]), we get

$$\begin{aligned}\int_{\mathbb{R}} |u_k - u_\infty|\,|(-\Delta)^{\frac{1}{2}}\varphi|\,dx &\leq \|(-\Delta)^{\frac{1}{2}}\varphi\|_{L^\infty((-R,R))}\|u_k - u_\infty\|_{L^1((-R,R))} + C\int_{(-R,R)^c} \frac{|u_k(x) - u_\infty(x)|}{|x|^2}\,dx \\ &\leq \|(-\Delta)^{\frac{1}{2}}\varphi\|_{L^\infty((-R,R))}\|u_k - u_\infty\|_{L^1((-R,R))} + 2CM\int_{(-R,R)^c} \frac{dx}{x^2}\,dx \\ &\xrightarrow{k,R\to+\infty} 0.\end{aligned}$$

Hence u is a weak solution of (1.7). □

From now on we will assume that $\mu_k \to +\infty$ and prove that Theorem 1.3 (ii) holds.

Lemma 3.2. *Let η_k be defined as in Theorem 1.3. Then η_k is bounded in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R})$ for $\alpha \in (0,1)$.*

Proof. Note that

$$\begin{aligned} r_k\mu_k^2 &= \frac{1}{\alpha_k\lambda_k e^{\alpha_k\mu_k^2}} = \frac{1}{\alpha_k\|u_k\|_H^2 e^{\alpha_k\mu_k^2}}\int_{\mathbb{R}} u_k^2 e^{\alpha_k u_k^2}\,dx \\ &\le C\frac{1}{\alpha_k\|u_k\|_H^2 e^{\frac{\alpha_k}{2}\mu_k^2}}\int_{\mathbb{R}} u_k^2 e^{\frac{\alpha_k}{2}u_k^2}\,dx \\ &\le C\frac{\|u_k\|_{L^4}^2\sqrt{D_{\alpha_k}}}{\alpha_k\|u_k\|_H^2 e^{\frac{\alpha_k}{2}\mu_k^2}} \\ &\le C\frac{\sqrt{D_\pi}}{\alpha_k e^{\frac{\alpha_k}{2}\mu_k^2}} \to 0. \end{aligned}$$

Moreover, we have that

$$(-\Delta)^{\frac{1}{2}}\eta_k = 2\frac{u_k(r_k\cdot)}{\mu_k}e^{\alpha_k u_k^2(r_k\cdot)-\alpha_k\mu_k^2} - 2\alpha_k r_k\mu_k^2\frac{u_k(r_k\cdot)}{\mu_k}$$

is bounded in L^∞. Since $\eta_k \le 0$ and $\eta_k(0) = 0$, this implies that η_k is bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R})$ and then in $C^\alpha_{\mathrm{loc}}(\mathbb{R})$ for any $\alpha \in (0,1)$. ☐

The bound of Lemma 3.2 implies that, up to a subsequence, $\eta_k \to \eta_\infty$ in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R})$ for some function η_∞. However, it does not provide a limit equation for η_∞. In order to prove that η_∞ solves

$$(-\Delta)^{\frac{1}{2}}\eta_\infty = 2e^{\eta_\infty},$$

we will prove that η_k is bounded in $L_s(\mathbb{R})$ for any $s > 0$. This bound can be obtained thanks to the commutator estimates proved in [35]. Part of the argument must be modified since the u_k's are not compactly supported. We start by recalling the following technical lemma, which is a consequence of the estimates in [35].

Lemma 3.3. *For any $s \in (0,1)$, there exists a constant $C = C(s)$ such that, for any $\varphi, \psi \in C^\infty_c(\mathbb{R}^n)$ and $\rho \in \mathbb{R}^+$, we have*

$$\|\varphi(-\Delta)^{\frac{s}{2}}\psi\|_{L^{(\frac{1}{s},\infty)}((-\rho,\rho))} \le C(E_1(\varphi,\psi) + E_{2,2\rho}(\varphi,\psi)),$$

where

$$\begin{aligned} E_1(\varphi,\psi) &= \|(-\Delta)^{\frac{1}{4}}\varphi\|_{L^2(\mathbb{R})}\|(-\Delta)^{\frac{1}{4}}\psi\|_{L^2(\mathbb{R})} \\ E_{2,\rho}(\varphi,\psi) &= \|(-\Delta)^{\frac{1}{4}}\varphi\|_{L^2(\mathbb{R})}\|(-\Delta)^{\frac{1}{2}}\psi\|_{L\log^{\frac{1}{2}}L(-\rho,\rho)}. \end{aligned}$$

Proof. Let $\theta \in C^\infty_c((-2,2))$ be a cut-off function such that $\theta \equiv 1$ on $(-1,1)$ and $0 \le \theta \le 1$. Let us denote $\theta_\rho = \theta(\frac{\cdot}{\rho})$. Let us also introduce the Riesz operators

$$I_{1-s}u := \kappa_s|\cdot|^{-s} * u \quad \text{for } s \in (0,1),$$

where the constant κ_s is defined by the identity

$$\widehat{\kappa_s|\cdot|^{-s}} = |\cdot|^{s-1}.$$

With this definition, I_{1-s} is the inverse of $(-\Delta)^{\frac{1-s}{2}}$. Then we can split

$$\begin{aligned} \varphi(-\Delta)^{\frac{s}{2}}\psi &= \varphi I_{1-s}(-\Delta)^{\frac{1}{2}}\psi \\ &= \varphi I_{1-s}(\theta_{2\rho}(-\Delta)^{\frac{1}{2}}\psi) + \varphi I_{1-s}((1-\theta_{2\rho})(-\Delta)^{\frac{1}{2}}\psi) \\ &= \varphi I_{1-s}(\theta_{2\rho}(-\Delta)^{\frac{1}{2}}\psi) + [\varphi, I_{1-s}]((1-\theta_{2\rho})(-\Delta)^{\frac{1}{2}}\psi) + I_{1-s}((1-\theta_{2\rho})\varphi(-\Delta)^{\frac{1}{2}}\psi) \\ &=: J_1 + J_2 + J_3, \end{aligned}$$

where we use the commutator notation $[u, I_{1-s}](v) = uI_{1-s}v - I_{1-s}(uv)$ for any $u, v \in C_c^\infty(\mathbb{R})$. Applying respectively [35, Proposition 3.2, Proposition 3.4 and Proposition A.3], we get that

$$\begin{aligned}\|J_1\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)} &= \big\|I_{\frac{1}{2}}((-\Delta)^{\frac{1}{4}}\varphi)I_{1-s}(\theta_{2\rho}(-\Delta)^{\frac{1}{2}}\psi)\big\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)}\\ &\le C\|(-\Delta)^{\frac{1}{4}}\varphi\|_{L^2(\mathbb{R})}\|(-\Delta)^{\frac{1}{2}}\psi\|_{L\log^{\frac{1}{2}}L(-2\rho,2\rho)}\\ &= CE_{2,2\rho}(\varphi,\psi),\end{aligned}$$

that

$$\begin{aligned}\|J_2\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)} &= \big\|[\varphi, I_{1-s}]((1-\theta_{2\rho})(-\Delta)^{\frac{1}{4}}(-\Delta)^{\frac{1}{4}}\psi)\big\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)}\\ &\le C\|(-\Delta)^{\frac{1}{4}}\varphi\|_{L^2(\mathbb{R})}\|(-\Delta)^{\frac{1}{s}}\psi\|_{L^2(\mathbb{R})}\\ &= CE_1(\varphi,\psi),\end{aligned}$$

and that

$$\begin{aligned}\|J_3\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)} &\le \|I_{1-s}(\varphi(-\Delta)^{\frac{1}{2}}\psi)\|_{L^{(\frac{1}{s},\infty)}(\mathbb{R})}\\ &\le C\|\varphi(-\Delta)^{\frac{1}{2}}\psi\|_{L^1(\mathbb{R})}\\ &= C\|(-\Delta)^{\frac{1}{4}}\varphi(-\Delta)^{\frac{1}{4}}\psi\|_{L^1(\mathbb{R})}\\ &\le CE_1(\varphi,\psi),\end{aligned}$$

as desired. □

As a consequence of Lemma 3.3, we obtain the following crucial estimate.

Lemma 3.4. *For any $s \in (0,1)$ there exists a constant $C = C(s)$ such that*

$$\int_{(-\rho,\rho)} |u(-\Delta)^{\frac{s}{2}}u|\,dx \le C\rho^{1-s}(E_1(u,u) + E_{2,2\rho}(u,u))$$

for any $\rho > 0$ and $u \in H \cap C^\infty(\mathbb{R})$. Here E_1 and $E_{2,2\rho}$ are defined as in Lemma 3.3.

Proof. By the Hölder inequality for Lorentz spaces (see, e.g., [43, Theorem 3.5]), we have

$$\begin{aligned}\|u(-\Delta)^{\frac{s}{2}}u\|_{L^1(-\rho,\rho)} &\le \|\chi_{(-\rho,\rho)}\|_{L^{(\frac{1}{1-s},1)}(\mathbb{R})}\|u(-\Delta)^{\frac{s}{2}}u\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)}\\ &\le C\rho^{1-s}\|u(-\Delta)^{\frac{s}{2}}u\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)}.\end{aligned} \tag{3.1}$$

We shall bound the right-hand side of (3.1) by approximating u with compactly supported functions and applying Lemma 3.3. To this purpose, we take a sequence of cut-off functions $(\tau_j)_{j\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R})$ such that $\tau_j(x) = 1$ for $|x| \le j$, $\tau_j(x) = 0$ for $|x| \ge j+1$, $0 \le \tau_j \le 1$, and $|\tau_j'| \le 2$. We define $u_j := \tau_j u$. We claim that

$$u_j \to u \quad \text{in } H^{\frac{1}{2},2}(\mathbb{R}) \cap L^q(\mathbb{R}),\ q \in (2,\infty), \tag{3.2}$$

and

$$(-\Delta)^{\frac{s}{2}}u_j \to (-\Delta)^{\frac{s}{2}}u \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}), \tag{3.3}$$

The first claim is proved in [14, Lemma 12]. We shall prove the second claim. Set $v_j = u_j - u$. Then, for any fixed $R_0 > 0$ and $x \in (-R_0, R_0)$, if $j > 2R_0$, we have

$$|(-\Delta)^{\frac{s}{2}}v_j| \le K_s \int_{\mathbb{R}\setminus(-j,j)} \frac{|v_j(y)|}{|x-y|^{1+s}}\,dy \le 2^{1+s}K_s \int_{\mathbb{R}\setminus(-j,j)} \frac{|u(y)|}{|y|^{1+s}}\,dy \le \frac{C\|u\|_{L^2(\mathbb{R})}}{j^{1+2s}}.$$

with C depending only on s. As $j \to \infty$, we get (3.3).

Now, By Lemma 3.3, we know that, for any j,

$$\|u_j(-\Delta)^{\frac{s}{2}}u_j\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)} \le C(E_1(u_j,u_j) + E_{2,2\rho}(u_j,u_j)), \tag{3.4}$$

where C depends only on s. Clearly, (3.2) yields

$$E_1(u_j, u_j) \to E_1(u, u).$$

Moreover,

$$E_{2,2\rho}(u_j, u_j) = E_{2,2\rho}(u, u) \quad \text{for } j \ge 2\rho.$$

Finally, (3.2) and (3.3) imply that $u_j(-\Delta)^{s/2}u_j \to u(-\Delta)^{s/2}u$ in $L^q_{\mathrm{loc}}(\mathbb{R})$ for every $q \in [1, \infty)$, and therefore in $L^{(1/s,\infty)}(-\rho, \rho)$. Then, passing to the limit in (3.4), we get

$$\|u(-\Delta)^{\frac{s}{2}}u\|_{L^{(\frac{1}{s},\infty)}(-\rho,\rho)} \le C(E_1(u, u) + E_{2,2\rho}(u, u)),$$

and together with (3.1) we conclude. □

We can now apply Lemma 3.4 to u_k. After scaling, we get the following bound on η_k.

Lemma 3.5. *For any $s \in (0, 1)$, there exists a constant $C = C(s) > 0$ such that*

$$\int_{-R}^{R} |(-\Delta)^{\frac{s}{2}}\eta_k|\,dx \le CR^{1-s} \quad \textit{for any } R > 0 \textit{ and } k \ge k_0(R).$$

Proof. First we observe that

$$f_k := (-\Delta)^{\frac{1}{2}}u_k = \lambda_k u_k e^{\alpha_k u_k^2} - u_k$$

is bounded in $L\log^{\frac{1}{2}} L_{\mathrm{loc}}(\mathbb{R})$. Indeed, we have

$$\log^{\frac{1}{2}}(2 + |f_k|) \le C(1 + u_k),$$

so that

$$|f_k|\log^{\frac{1}{2}}(2 + |f_k|) \le C|f_k|(1 + u_k) = O(|f_k|u_k + 1).$$

Since $|f_k|u_k$ is bounded in $L^1(\mathbb{R})$ by (1.5) and (1.6), we get that f_k is bounded in $L\log^{\frac{1}{2}} L_{\mathrm{loc}}(\mathbb{R})$.

Then Lemma 3.4 and (1.6) imply the existence of $C = C(s)$ such that

$$\int_{-\rho}^{\rho} |u_k(-\Delta)^{\frac{s}{2}}u_k|\,dx \le C\rho^{1-s}, \quad \rho \in (0, 1).$$

For any $R > 0$, we can apply this with $\rho = Rr_k$ and rewrite it in terms of η_k. Then we obtain

$$\int_{-R}^{R} \Big(1 + \frac{\eta_k}{\mu_k^2}\Big)|(-\Delta)^{\frac{s}{2}}\eta_k|dx \le CR^{1-s}.$$

Since, by Lemma 3.2, η_k is locally bounded, if k is sufficiently large, we get

$$1 + \frac{\eta_k}{\mu_k^2} \ge \frac{1}{2}$$

and the proof is complete. □

Lemma 3.6. *The sequence (η_k) is bounded in $L_s(\mathbb{R})$ for any $s > 0$.*

Proof. It is sufficient to prove the statement for $s \in (0, \frac{1}{2})$. Since $\eta_k \le 0$, Lemma 3.5 gives

$$\begin{aligned}
C &\ge \frac{1}{K_s}\int_{-1}^{1} |(-\Delta)^s\eta_k|\,dx \\
&\ge \Big|\int_{-1}^{1}\int_{\mathbb{R}} \frac{\eta_k(x) - \eta_k(y)}{|x - y|^{1+2s}}\,dy\,dx\Big| \\
&\ge \underbrace{\int_{-1}^{1}\int_{-2}^{2} \frac{\eta_k(x) - \eta_k(y)}{|x - y|^{1+2s}}\,dy\,dx}_{=:I_1} + \underbrace{\int_{-1}^{1}\int_{(-2,2)^c} \frac{\eta_k(x)\,dy\,dx}{|x - y|^{1+2s}}}_{=:I_2} + \underbrace{\int_{-1}^{1}\int_{(-2,2)^c} \frac{-\eta_k(y)\,dy\,dx}{|x - y|^{1+2s}}}_{=:I_3}.
\end{aligned}$$

Take $2s < \alpha < 1$. Since η_k is bounded in $C^\alpha_{\mathrm{loc}}(\mathbb{R})$ by Lemma 3.2, we have that

$$|I_1| \le C \int_{-1}^{1} \int_{-2}^{2} \frac{dy\,dx}{|x-y|^{1+2s-\alpha}} \le C \int_{-3}^{3} \frac{dz}{|z|^{1+2s-\alpha}} = C.$$

Similarly,

$$|I_2| \le \int_{-1}^{1} |\eta_k(x)| \int_{(x-1,x+1)^c} \frac{1}{|x-y|^{1+2s}}\,dy\,dx \le C.$$

Therefore, we obtain that

$$I_3 = \int_{-1}^{1} \int_{(-2,2)^c} \frac{|\eta_k(y)|}{|x-y|^{1+2s}}\,dy\,dx \le C.$$

But for $x \in (-1,1)$ and $y \notin (-2,2)$ we have

$$|x-y| \le |y| + |x| \le 2|y| \le 2(1+|y|^{1+2s})^{\frac{1}{1+2s}}.$$

Hence

$$I_3 = \int_{-1}^{1} \int_{(-2,2)^c} \frac{|\eta_k(y)|}{|x-y|^{1+2s}}\,dy\,dx \ge \frac{1}{2^{2s}} \int_{(-2,2)^c} \frac{|\eta_k(y)|}{1+|y|^{1+2s}}\,dy.$$

This and Lemma 3.2 imply that η_k is bounded in $L_s(\mathbb{R})$. □

Proof of Theorem 1.3 (completed). By Lemma 3.2, up to a subsequence, we can assume that $\eta_k \to \eta_\infty$ in $C^\alpha_{\mathrm{loc}}(\mathbb{R})$ for any $\alpha \in (0,1)$, with $\eta_\infty \in C^\alpha_{\mathrm{loc}}(\mathbb{R})$. Let us set

$$f_k := (-\Delta)^{\frac{1}{2}}\eta_k = 2\Big(1 + \frac{\eta_k}{2\alpha_k\mu_k^2}\Big)e^{\eta_k + \frac{\eta_k^2}{4\alpha_k\mu_k^2}} - 2r_k\alpha_k\mu_k^2\Big(1 + \frac{\eta_k}{2\alpha_k\mu_k^2}\Big).$$

As observed in the proof of Lemma 3.2, we have $r_k\mu_k^2 \to 0$ as $k \to \infty$, and thus $f_k \to 2e^{\eta_\infty}$ locally uniformly on $\mathbb{R}$. Moreover, f_k is bounded in $L^\infty(\mathbb{R})$. Then, for any Schwarz function $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |f_k - 2e^{\eta_\infty}|\,|\varphi|\,dx \le o(1) \int_{(-R,R)} |\varphi|\,dx + (\|f_k\|_{L^\infty(\mathbb{R})} + \|2e^{\eta_\infty}\|_{L^\infty(\mathbb{R})}) \int_{(-R,R)^c} |\varphi|\,dx \to 0$$

as $k, R \to +\infty$. On the other hand, we know by Lemma 3.6 that η_k is bounded in $L_s(\mathbb{R})$ and, consequently, $\eta_\infty \in L_s(\mathbb{R})$, $s > 0$. In particular, for $s \in (0, \frac{1}{2})$, letting first $k \to \infty$ and then $R \to \infty$, we get

$$\begin{aligned}\int_{\mathbb{R}} |\eta_k - \eta_\infty|\,|(-\Delta)^{\frac{1}{2}}\varphi|\,dx &\le \|(-\Delta)^{\frac{1}{2}}\varphi\|_{L^\infty(-R,R)}\|\eta_k - \eta_\infty\|_{L^1(-R,R)} + C\int_{(-R,R)^c} \frac{|\eta_k(x) - \eta_\infty(x)|}{|x|^2}\,dx \\ &\le C\|\eta_k - \eta_\infty\|_{L^1(-R,R)} + CR^{2s-1}(\|\eta_k\|_{L_s(\mathbb{R})} + \|\eta_\infty\|_{L_s(\mathbb{R})}) \to 0.\end{aligned}$$

Then η_∞ is a weak solution $(-\Delta)^{\frac{1}{2}}\eta_\infty = 2e^{\eta_\infty}$ and $\eta_\infty \in L_s(\mathbb{R})$ for any s. Moreover, repeating the argument of Corollary 2.4 and using (1.6), we get

$$\frac{1}{\pi}\int_{-R}^{R} e^{\eta_\infty}\,d\xi = \lim_{k\to\infty} \int_{-Rr_k}^{Rrk} \lambda_k u_k^2 e^{\alpha_k u_k^2}\,dx \le \limsup_{k\to\infty} \|u_k\|_H^2 = \Lambda,$$

which implies $e^{\eta_\infty} \in L^1(\mathbb{R})$. Then $\eta_\infty(x) = -\log(1+x^2)$; see, e.g., [7, Theorem 1.8].

To complete the proof, we shall study the properties of the weak limit u_∞ of u_k in H. First, we show that u_∞ is a weak solution of (1.7). Let us denote

$$g_k := \lambda_k u_k e^{\alpha_k u_k^2}, \quad g_\infty := \lambda_\infty u_\infty e^{\pi u_\infty^2}.$$

Take any function $\varphi \in \mathcal{S}(\mathbb{R})$. On the one hand, since $(-\Delta)^{\frac{1}{2}}\varphi \in L^2(\mathbb{R})$ and $u_k \rightharpoonup u_\infty$ weakly in $L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} (u_k - u_\infty)(-\Delta)^{\frac{1}{2}}\varphi\,dx + \int_{\mathbb{R}} (u_k - u_\infty)\varphi\,dx \to 0$$

as $k \to \infty$. On the other hand, for any large $t > 0$ we get

$$\int_{\mathbb{R}} |g_k - g_\infty|\,|\varphi|\,dx \le \int_{\{u_k \le t\}} |g_k - g_\infty|\,|\varphi|\,dx + \frac{\|\varphi\|_{L^\infty(\mathbb{R})}}{t} \int_{\mathbb{R}} u_k(g_k + g_\infty)\,dx = o(1) + O(t^{-1}) \to 0$$

as $k, t \to \infty$, where we used that $g_\infty \in L^2(\mathbb{R})$ by Theorem A (see, e.g., [21, Lemma 2.3]) together with the dominated convergence theorem and the bounds $\|u_k g_k\|_{L^1(\mathbb{R})} \le \Lambda$ and $\|u_k\|_{L^2(\mathbb{R})} \le \Lambda$. Then u_∞ is a weak solution of (1.7).

Now, observe that

$$\|u_k\|_H^2 = \int_{\mathbb{R}} g_k u_k\,dx = \int_{-Rr_k}^{Rr_k} g_k u_k\,dx + \int_{\mathbb{R}\setminus(-Rr_k, Rr_k)} g_k u_k\,dx$$

with

$$\lim_{k\to\infty} \int_{-Rr_k}^{Rr_k} u_k g_k\,dx = \frac{1}{\pi} \int_{-R}^{R} e^{\eta_\infty}\,dx \to 1$$

as $R \to \infty$, and

$$\liminf_{k\to\infty} \int_{\mathbb{R}\setminus(-Rr_k, Rr_k)} g_k u_k\,dx = \int_{\mathbb{R}} g_\infty u_\infty\,dx = \|u_\infty\|_H^2$$

for any $R > 1$, by Fatou's lemma. Thus we conclude that

$$\|u_k\|_H^2 \ge \|u_\infty\|_H^2 + 1.$$

Finally, to prove that $u_k \to u_\infty$ in $C^\ell_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$ for every $\ell \ge 0$, we use the monotonicity of u_k, which implies that u_k is locally bounded away from 0. Hence we can conclude by elliptic estimates, as in Lemma 3.1. □

4 Proof of Theorem 1.2

Let us set

$$E_\alpha(u) = \int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,dx, \quad D_\alpha := \sup_{u \in H: \|u\|_H \le 1} E_\alpha(u).$$

The proof of Theorem 1.2 is organized as follows. First, we prove that D_α is attained for $\alpha \in (0, \pi)$ sufficiently close to π. Then we fix a sequence $(\alpha_k)_{k\in N}$ such that $\alpha_k \nearrow \pi$ as $k \to +\infty$, and for any large k we take a positive extremal $u_k \in H$ for D_{α_k}. With a contradiction argument similar to the one of Section 2, we show that $\mu_k := \sup_{\mathbb{R}} u_k \le C$. Finally, we show that $u_k \to u_\infty$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}) \cap L^2(\mathbb{R})$, where u_∞ is a maximizer for D_π.

4.1 Subcritical Extremals: Ruling out Vanishing

The following lemma describes the effect of the lack of compactness of the embedding $H \subseteq L^2(\mathbb{R})$ on E_α, and holds uniformly for $\alpha \in [0, \pi]$.

Lemma 4.1. *Let $(\alpha_k) \subseteq [0, \pi]$ and $(u_k) \subseteq H$ be two sequences such that the following conditions hold:*
(i) $\alpha_k \to \alpha_\infty \in [0, \pi]$ *as* $k \to \infty$.
(ii) $\|u_k\|_H \le 1$, $u_k \rightharpoonup u_\infty$ *weakly in* H, $u_k \to u_\infty$ *a.e. in* $\mathbb{R}$, *and* $e^{\alpha_k u_k^2} \to e^{\alpha_\infty u_\infty^2}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R})$ *as* $k \to \infty$.
(iii) *The* u_k*'s are even and monotone decreasing, i.e.* $u_k(-x) = u_k(x) \ge u_k(y)$ *for* $0 \le x \le y$.
Then we have

$$E_{\alpha_k}(u_k) = E_{\alpha_\infty}(u_\infty) + \alpha_\infty\big(\|u_k\|^2_{L^2(\mathbb{R})} - \|u_\infty\|^2_{L^2(\mathbb{R})}\big) + o(1)$$

as $k \to \infty$.

Proof. Since u_k is even and decreasing, we know that

$$u_k(x)^2 \le \frac{\|u_k\|^2_{L^2(\mathbb{R})}}{2|x|} \le \frac{1}{2|x|} \tag{4.1}$$

for any $x \in \mathbb{R} \setminus \{0\}$. In particular, there exists a constant $C > 0$, such that

$$e^{\alpha_k u_k^2(x)} - 1 - \alpha_k u_k^2(x) \le C|x|^{-4}$$

for $|x| \ge 1$. Applying the dominated convergence theorem for $|x| \ge 1$, using the assumption that

$$e^{\alpha_k u_k^2} \to e^{\alpha_\infty u_\infty^2} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}),$$

and recalling that (u_k) is precompact in $L^1_{\mathrm{loc}}(\mathbb{R})$, we find that

$$\int_{\mathbb{R}} (e^{\alpha_k u_k^2} - 1 - \alpha_k u_k^2)\, dx \to \int_{\mathbb{R}} (e^{\alpha_\infty u_\infty^2} - 1 - \alpha_\infty u_\infty^2)\, dx,$$

and the lemma follows. □

Lemma 4.2. *Take $\alpha \in (0, \pi)$. If $D_\alpha > \alpha$, then D_α is attained by an even and decreasing function, i.e. there exists $u_\alpha \in H$ even and decreasing such that $\|u_\alpha\|_H = 1$ and $E_\alpha(u_\alpha) = D_\alpha$.*

Proof. Let $(u_k) \subset H$ be a maximizing sequence for E_α. Without loss of generality, we can assume $u_k \to u_\infty \in H$ weakly in H and a.e. on $\mathbb{R}$. Moreover, up to replacing u_k with its symmetric decreasing rearrangement, we can assume that u_k is even and decreasing (see [44]). Since $\alpha \in (0, \pi)$, the sequence $e^{\alpha u_k^2} - 1$ is bounded in $L^{\frac{\pi}{\alpha}}(\mathbb{R})$, with $\frac{\pi}{\alpha} > 1$. Then, by Vitali's theorem, we get $e^{\alpha u_k^2} \to e^{\alpha u_\infty^2}$ in $L^1_{\mathrm{loc}}(\mathbb{R})$, and Lemma 4.1 yields

$$E_\alpha(u_k) = E_\alpha(u_\infty) + \alpha\big(\|u_k\|^2_{L^2(\mathbb{R})} - \|u_\infty\|^2_{L^2(\mathbb{R})}\big) + o(1). \tag{4.2}$$

This implies that $u_\infty \not\equiv 0$ since otherwise we have $E_\alpha(u_k) = \alpha\|u_k\|^2_{L^2(\mathbb{R})} + o(1) \le \alpha + o(1)$, which contradicts the assumption $D_\alpha > \alpha$. Let us denote

$$L := \limsup_{k\to\infty} \|u_k\|^2_{L^2(\mathbb{R})}, \qquad \tau := \frac{\|u_\infty\|^2_{L^2(\mathbb{R})}}{L}.$$

Observe that $L, \tau \in (0, 1]$. Let us consider the sequence $v_k(x) = u_k(\tau x)$. Clearly, we have $v_k \rightharpoonup v_\infty$ weakly in H, where $v_\infty(x) := u_\infty(\tau x)$. Moreover, since $\|v_\infty\|^2_{L^2} = L$ and

$$\|(-\Delta)^{\frac14} v_\infty\|^2_{L^2} \le \liminf_{k\to\infty} \|(-\Delta)^{\frac14} v_k\|^2_{L^2} = \liminf_{k\to\infty} \|(-\Delta)^{\frac14} u_k\|^2_{L^2} \le 1 - L,$$

we get $\|v_\infty\|_H \le 1$. By (4.2), we have

$$D_\alpha \le E_\alpha(u_\infty) + \alpha L(1-\tau) = \tau E_\alpha(v_\infty) + \alpha L(1-\tau) \le \tau D_\alpha + \alpha L(1-\tau). \tag{4.3}$$

If $\tau < 1$, this implies $D_\alpha \le \alpha L \le \alpha$, contradicting the assumptions. Hence $\tau = 1$ and (4.3) gives $D_\alpha = E_\alpha(u_\infty)$. Finally, we have $\|u_\infty\|_H = 1$ since otherwise

$$E_\alpha\Big(\frac{u_\infty}{\|u_\infty\|_H}\Big) > E_\alpha(u_\infty) = D_\alpha.$$ □

Lemma 4.3. *There exists $\alpha^* \in (0, \pi)$ such that $D_\alpha > \alpha$ for any $\alpha \in (\alpha^*, \pi]$. In particular, D_α is attained by an even and decreasing function u_α for any $\alpha \in (\alpha^*, \pi)$ by Lemma 4.2.*

Proof. This follows from Proposition 4.14 by continuity. Indeed Proposition 4.14 gives $D_\pi > 2\pi e^{-\gamma} > \pi$. □

4.2 The Critical Case

Next, we take a sequence α_k such that $\alpha_k \nearrow \pi$ as $k \to \infty$. For any large k, Lemma 4.3 yields the existence of $u_k \in H$ even and decreasing such that $D_{\alpha_k} = E_{\alpha_k}(u_k)$. Each u_k satisfies

$$(-\Delta)^{\frac12} u_k + u_k = \lambda_k u_k e^{\alpha_k u_k^2}$$

and $\|u_k\|_H = 1$. Note that $u_k \in C^\infty(\mathbb{R})$ by elliptic estimates (see [28, Theorem 13], [11, Theorem 1.5] and

[22, Corollary 25]). Multiplying the equation by u_k and using the basic inequality $te^t \geq e^t - 1$ for $t \geq 0$, we infer

$$\frac{1}{\lambda_k} = \int_{\mathbb{R}} u_k^2 e^{\alpha_k u_k^2}\,dx \geq \frac{1}{\alpha_k} E_{\alpha_k}(u_k) = \frac{1}{\alpha_k} D_{\alpha_k}.$$

Since $D_{\alpha_k} \to D_\pi > 0$, we get that λ_k is uniformly bounded.

Then the sequence u_k satisfies the alternative of Theorem 1.3. If case (i) holds, then we can argue as in Lemma 4.2 and Lemma 4.3 and prove that D_π is attained. Therefore, we shall assume by contradiction that case (ii) occurs.

Let r_k and η_k be as in Theorem 1.3. Let $\tilde{\eta}_k$ denote the Poisson integral of η_k.

Proposition 4.4. *We have $\tilde{\eta}_k \to \tilde{\eta}_\infty$ in $C^\ell_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ for every $\ell \geq 0$, where*

$$\tilde{\eta}_\infty(x, y) = -\log((1 + y)^2 + x^2)$$

is the Poisson integral (compare to (A.5)*) of $\eta_\infty := -\log(1 + x^2)$.*

Proof. By Theorem 1.3, we know that $\eta_k \to \eta_\infty$ in $C^\ell_{\mathrm{loc}}(\mathbb{R})$ and that η_k is bounded in $L_{1/2}$. Then we can repeat the argument of the proof of Proposition 2.2. □

Remark 4.5. As in (2.9), the convergence $\eta_k \to \eta_\infty$ in $L^\infty_{\mathrm{loc}}(\mathbb{R})$ implies

$$\lim_{k\to\infty} \int_{-r_k R}^{r_k R} \lambda_k \mu_k^i u_k^{2-i} e^{\alpha_k u_k^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{\eta_\infty}\,dx$$

for $i = 0, 1, 2$ and for any $R > 0$.

Lemma 4.6. *We have $u_k \to 0$ in $L^2(\mathbb{R})$.*

Proof. Indeed, otherwise up to a subsequence we would have

$$\|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2(\mathbb{R})} \leq \frac{1}{A}$$

for some $A > 1$. Consider the function $v_k = (u_k - u_k(1))^+$. Then

$$v_k \in \tilde{H}^{\frac{1}{2},2}(I) \quad \text{and} \quad \|(-\Delta)^{\frac{1}{4}} v_k\|_{L^2(\mathbb{R})} \leq \frac{1}{A}.$$

The Moser–Trudinger inequality (1.3) gives that $e^{\alpha_k v_k^2}$ is bounded in $L^A(\mathbb{R})$. Since

$$u_k^2 \leq (1 + \varepsilon) v_k^2 + \frac{1}{\varepsilon}(u_k - v_k)^2$$

and $|v_k - u_k| \leq u_k(1) \to 0$ as $k \to \infty$, we get that $e^{\alpha_k u_k^2}$ is uniformly bounded in $L^p(\mathbb{R})$ for every $1 < p < A$. Therefore, we have

$$\int_{(-1,1)} (e^{\alpha_k u_k^2} - 1)\,dx \to 0$$

as $k \to \infty$. But then, by Lemma 4.1, we find $D_\pi \leq \pi$, which contradicts Lemma 4.3. □

Lemma 4.7. *For $A > 1$, set $u_k^A := \min\{u_k, \frac{\mu_k}{A}\}$. Then we have*

$$\limsup_{k\to\infty} \|(-\Delta)^{\frac{1}{4}} u_k^A\|^2_{L^2(\mathbb{R})} \leq \frac{1}{A}.$$

Proof. The proof is similar to the one of Lemma 2.5. We set

$$\bar{u}_k^A := \min\Big\{\tilde{u}_k, \frac{\mu_k}{A}\Big\}.$$

Note that $\bar{u}_k^A = \tilde{u}_k$ for $|(x, y)| \to \infty$ by (4.1) and Lemma A.6. Then, since $(-\Delta)^{1/2} u_k \in L^2(\mathbb{R})$, we get (see (A.9))

$$\lim_{R\to+\infty} \int_{\partial B_R \cap \mathbb{R}^2_+} \bar{u}_k^A \frac{\partial \tilde{u}_k}{\partial \nu}\,d\sigma = \lim_{R\to+\infty} \int_{\partial B_R \cap \mathbb{R}^2_+} \tilde{u}_k \frac{\partial \tilde{u}_k}{\partial \nu} = 0.$$

Since $\bar{u}_k^A$ is an extension of u_k^A, using integration by parts and the harmonicity of $\tilde{u}_k$, we get

$$\begin{aligned}\|(-\Delta)^{\frac{1}{4}} u_k^A\|_{L^2(\mathbb{R})}^2 \le \int_{\mathbb{R}^2_+} |\nabla \bar{u}_k^A|^2 \,dx\,dy &= \int_{\mathbb{R}^2_+} \nabla \bar{u}_k^A \cdot \nabla \tilde{u}_k \,dx\,dy \\ &= -\int_{\mathbb{R}} u_k^A(x) \frac{\partial \tilde{u}_k(x,0)}{\partial y}\,dx \\ &= \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A \,dx. \end{aligned} \tag{4.4}$$

Proposition 4.4 implies that $u_k^A(r_k x) = \frac{\mu_k}{A}$ for $|x| \le R$ and $k \ge k_0(R)$. Noting that $u_k^A \le u_k$ and using Lemma 4.6 and Remark 4.5, we get

$$\begin{aligned}\int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A \,dx &\ge \int_{-Rr_k}^{Rr_k} \lambda_k u_k e^{\alpha_k u_k^2} u_k^A \,dx - \int_{\mathbb{R}} u_k u_k^A \,dx \\ &= \frac{1}{A} \int_{-Rr_k}^{Rr_k} \lambda_k \mu_k u_k e^{\alpha_k u_k^2} u_k^A \,dx + O(\|u_k\|_{L^2(\mathbb{R})}^2) \\ &\xrightarrow{k\to\infty} \frac{1}{\pi A} \int_{-R}^{R} e^{\eta_\infty} \,d\xi \\ &\xrightarrow{R\to\infty} \frac{1}{A}.\end{aligned}$$

Set now $v_k^A := (u_k - \frac{\mu_k}{A})^+$. With similar computations, we get

$$\begin{aligned}\int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k v_k^A \,dx &\ge \int_{-Rr_k}^{Rr_k} \lambda_k u_k v_k^A e^{\alpha_k u_k^2} \,dx + O(\|u_k\|_{L^2(\mathbb{R})}^2) \\ &\xrightarrow{k\to\infty} \frac{1}{\pi}\Big(1 - \frac{1}{A}\Big) \int_{-R}^{R} e^{\eta_\infty} \,d\xi \\ &\xrightarrow{R\to\infty} \frac{A-1}{A}.\end{aligned}$$

Since

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A \,dx + \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k v_k^A \,dx = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k \,dx = 1 - \|u_k\|_{L^2(\mathbb{R})}^2 \to 1$$

as $k \to \infty$, we get that

$$\lim_{n\to\infty} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A \,dx = \frac{1}{A}.$$

Then we conclude using (4.4). □

Proposition 4.8. *We have*

$$D_\pi = \lim_{k\to\infty} \frac{1}{\lambda_k \mu_k^2}. \tag{4.5}$$

Moreover,

$$\lim_{k\to\infty} \mu_k \lambda_k = 0. \tag{4.6}$$

Proof. Fix $A > 1$ and write

$$\int_{\mathbb{R}} (e^{\alpha_k u_k^2} - 1) \,dx = \text{(I)} + \text{(II)} + \text{(III)},$$

where (I), (II) and (III) denote respectively the integrals over the regions

$$\Big\{u_k \le \frac{\mu_k}{A}\Big\} \cap (-1,1), \quad \Big\{u_k \le \frac{\mu_k}{A}\Big\} \cap (-1,1)^c, \quad \Big\{u_k > \frac{\mu_k}{A}\Big\}.$$

Using Lemmas 4.11 and 4.7 together with Theorem A, we see that

$$(\mathrm{I}) \le \int_{-1}^{1} (e^{\alpha_k (u_k^A)^2} - 1) \to 0 \quad \text{as } k \to \infty$$

since $e^{\alpha_k (u_k^A)^2} - 1$ is uniformly bounded in L^p for any $1 \le p < A$. By (4.1) and Lemma 4.6, we find

$$(\mathrm{II}) \le \int_{(-1,1)^c} (e^{\alpha_k u_k^2} - 1)\,dx \le C \int_{\mathbb{R}} u_k^2\,dx \to 0 \quad \text{as } k \to \infty.$$

We now estimate

$$(\mathrm{III}) \le \frac{A^2}{\lambda_k \mu_k^2} \int_{\{u_k > \frac{\mu_k}{A}\}} \lambda_k u_k^2 e^{\alpha_k u_k^2}\,dx \le \frac{A^2}{\lambda_k \mu_k^2}(1 + o(1)),$$

with $o(1) \to 0$ as $k \to \infty$, where we used that

$$\int_{I \cap \{u_k > \frac{\mu_k}{A}\}} \lambda_k u_k^2 e^{\alpha_k u_k^2}\,dx \le \|u_k\|_H^2 = 1.$$

Letting $A \downarrow 1$ gives

$$\sup_H E_\pi \le \lim_{k\to\infty} \frac{1}{\lambda_k \mu_k^2}.$$

The converse inequality follows from Remark 4.5:

$$\int_{\mathbb{R}} (e^{\alpha_k u_k^2} - 1)\,dx \ge \int_{-Rr_k}^{Rr_k} e^{\alpha_k u_k^2}\,dx + o(1) = \frac{1}{\lambda_k \mu_k^2}\Big(\int_{-R}^{R} e^{\eta_\infty}\,dx + o(1)\Big) + o(1).$$

with $o(1) \to 0$ as $k \to \infty$. Letting $R \to \infty$, we obtain (4.5).

Finally, (4.6) follows at once from (4.5), because otherwise we would have $D_\pi = 0$, which is clearly impossible. □

Lemma 4.9. *We have*

$$f_k := \lambda_k \mu_k u_k e^{\alpha_k u_k^2} \rightharpoonup \delta_0$$

as $k \to \infty$, *in the sense of Radon measures in* $\mathbb{R}$.

Proof. The proof follows step by step the proof of Proposition 2.7, with (1.4), Proposition 4.4, Remark 4.5 Lemma 4.6 and Lemma 4.7 used in place of (1.3), Proposition 2.2, equality (2.9), Lemma 2.4, and Lemma 2.5. We omit the details. □

For $x \in \mathbb{R}$, let G_x be the Green function of $(-\Delta)^{\frac{1}{2}} + \mathrm{Id}$ on $\mathbb{R}$ with singularity at x. In the following, we set $G := G_0$. By translation invariance, we get $G_x(y) = G(y - x)$ for any $x, y \in \mathbb{R}$, $x \neq y$. Moreover, the inversion formula for the Fourier transform implies that

$$G(x) = \frac{1}{2}\sin|x| - \frac{1}{\pi}\sin(|x|)\mathrm{Si}(|x|) - \frac{1}{\pi}\cos(|x|)\mathrm{Ci}(|x|), \tag{4.7}$$

where

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt \quad \text{and} \quad \mathrm{Ci}(x) = -\int_x^{+\infty} \frac{\cos t}{t}\,dt.$$

We recall that the identity

$$\mathrm{Ci}(x) = \log x + \gamma + \int_0^x \frac{\cos t - 1}{t}\,dt \tag{4.8}$$

holds for any $x \in \mathbb{R} \setminus \{0\}$, where γ denotes the Euler–Mascheroni constant; see, e.g., [17, Chapter 12.2].

Proposition 4.10. *The function G satisfies the following properties:*
(i) *We have $G \in C^\infty(\mathbb{R} \setminus \{0\})$ and*

$$G(x) = -\frac{1}{\pi}\log|x| - \frac{\gamma}{\pi} + O(|x|), \quad G'(x) = -\frac{1}{\pi x} + O(1) \qquad \text{as } x \to 0.$$

(ii) *We have $G(x) = O(|x|^{-2})$, $G'(x) = O(|x|^{-3})$ and $G''(x) = O(|x|^{-4})$ as $|x| \to \infty$.*
(iii) *Let $\tilde{G}$ be the Poisson extension of G. There exists a function*

$$f \in C^1(\overline{\mathbb{R}^2_+})$$

such that $f(0,0) = 0$ and

$$\tilde{G}(x,y) = -\frac{1}{\pi}\ln|(x,y)| - \frac{\gamma}{\pi} + \frac{x}{\pi}\arctan\frac{x}{y} - \frac{y}{2\pi}\log(x^2+y^2) + f(x,y) \quad \text{in } \mathbb{R}^2_+.$$

Proof. Property (i) follows directly by formula (4.7) and the identity in (4.8). Similarly, since

$$Si(t) = \frac{\pi}{2} - \frac{\cos t}{t} - \frac{\sin t}{t^2} + O(t^{-3}), \quad Ci(t) = \frac{\sin t}{t} - \frac{\cos t}{t^2} + O(t^{-3})$$

as $t \to +\infty$, we get (ii).

Given $R > 0$, let $\psi \in C_c^\infty(\mathbb{R})$ be a cut-off function with $\psi \equiv 1$ on $(-R,R)$. Let us set $g_0 := -\frac{1}{\pi}\log|\cdot| - \frac{\gamma}{\pi}$, $g_1 := \frac{1}{2}|\cdot|\psi$ and $g_2 := G - g_0 - g_1$. By Proposition A.3, we have

$$\tilde{g}_0(x,y) = -\frac{1}{\pi}\log|(x,y)| - \frac{\gamma}{\pi}, \quad (x,y) \in \mathbb{R}^2_+.$$

Let us set $\theta(x,y) := \arctan\frac{x}{y}$, the angle between the y-axis and the segment connecting the origin to (x,y). A direct computation shows that

$$\Delta(x\theta(x,y)) = \frac{2y}{x^2+y^2} = \frac{1}{2}\Delta(y\log(x^2+y^2)).$$

Then the function

$$h(x,y) := \tilde{g}_1(x,y) - \frac{1}{\pi}x\theta(x,y) + \frac{1}{2\pi}y\log(x^2+y^2)$$

is harmonic in $\mathbb{R}^2_+$, continuous on $\overline{\mathbb{R}}^2_+$, and identically 0 on $(-R,R)\times\{0\}$. By [45, Theorem C], we get that

$$h \in C^\infty(\overline{\mathbb{R}^2_+} \cap B_R(0,0)).$$

Finally, note that formula (4.7) implies $g_2(0) = 0$ and $g_2 \in C^{1,\alpha}(\mathbb{R})$ for any $\alpha \in (0,1)$. Hence, standard elliptic regularity yields

$$\tilde{g}_2 \in C^{1,\alpha}(\overline{\mathbb{R}^2_+} \cap B_R(0,0))$$

for any $\alpha \in (0,1)$. In particular, $\tilde{g}_2(0,0) = g_2(0) = 0$. □

Lemma 4.11. *We have $\mu_k u_k \to G$ in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R} \setminus (-\varepsilon,\varepsilon))$ for any $\varepsilon > 0$.*

Proof. Let us set $v_k := \mu_k u_k - G$ and $f_k = \mu_k \lambda_k u_k e^{\alpha_k u_k^2}$. By Lemma 4.9, we have $\|f_k\|_{L^1(I)} \to 1$ as $k \to +\infty$, $I = (-1,1)$. Then, arguing as in Lemma 2.9, we get

$$\begin{aligned} |v_k(x)| &= \Big|\int_{\mathbb{R}} G(y-x)f_k(y)\,dy - G(x)\Big| \\ &\le \int_I |G(x-y) - G(x)|f_k(y)\,dy + \underbrace{\big|\|f_k\|_{L^1(I)} - 1\big|}_{=o(1)}|G(x)| + \underbrace{\int_{\mathbb{R}\setminus I} G(x-y)f_k(y)\,dy}_{=:\,w_k(x)}. \end{aligned} \tag{4.9}$$

Using (4.1), Lemma 4.6 and (4.6), we get that $f_k \to 0$ in $L^2(\mathbb{R} \setminus I)$. In particular,

$$|w_k(x)| \le \|f_k\|_{L^2(\mathbb{R}\setminus I)}\|G\|_{L^2(\mathbb{R})} \to 0.$$

Fix $\sigma \in (0,1)$ and assume $|x| \geq \sigma$. If we further take $|y| \leq \frac{\sigma}{2}$, then Proposition 4.10 implies

$$|G(x-y) - G(x)| \leq C|y|,$$

where C is a constant depending only on σ. Thus, for any $\varepsilon \in (0, \frac{\sigma}{2})$, setting $I_\varepsilon = \mathbb{R} \setminus (-\varepsilon, \varepsilon)$, we can write

$$\begin{aligned}
|v_k(x)| &\leq \int_I |G(x-y) - G(x)| f_k(y)\,dy + o(1)\|G\|_{L^\infty(\mathbb{R}\setminus(-\sigma,\sigma))} + o(1)\\
&\leq C\int_{-\varepsilon}^{\varepsilon} |y| f_k(y)\,dy + \int_{I_\varepsilon} |G(x-y)| f_k(y)\,dy + |G(x)|\int_{I_\varepsilon} f_k(y)\,dy + o(1)\\
&\leq C\varepsilon\|f_k\|_{L^1(I)} + \|f_k\|_{L^\infty(I_\varepsilon)}\big(\|G\|_{L^1(\mathbb{R})} + \|G\|_{L^\infty(\mathbb{R}\setminus(-\sigma,\sigma))}\big) + o(1)\\
&\leq C\varepsilon + o(1), \qquad (4.10)
\end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$ (depending on ε and σ). Here, we used that $f_k \to 0$ in $L^\infty(\mathbb{R} \setminus (-\varepsilon, \varepsilon))$ by (4.1) and (4.6). Since ε is arbitrarily small, (4.10) shows that $v_k \to 0$ in $L^\infty(\mathbb{R} \setminus (-\sigma, \sigma))$.

Next, we prove the L^2 convergence. First, Hölder's inequality and Fubini's theorem give

$$\|w_k\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}\setminus I} G(x-y) f_k(y)\,dy \Big)^2 dx \leq \|G\|_{L^1(\mathbb{R})}^2 \|f_k\|_{L^2(\mathbb{R}\setminus I)}^2 \to 0$$

as $k \to \infty$. With a similar argument, after integrating (4.9) and using the triangular inequality in L^2, we find

$$\begin{aligned}
\|v_k\|_{L^2(\mathbb{R})} &\leq \Big(\int_{\mathbb{R}} \Big(\int_I |G(x-y) - G(x)| f_k(y)\,dy \Big)^2 dx \Big)^{\frac{1}{2}} + |\|f_k\|_{L^1(I)} - 1| \|G\|_{L^2(\mathbb{R})} + \|w_k\|_{L^2(\mathbb{R})}\\
&\leq \underbrace{\Big(\int_I f_k(y)\,dy \Big)^{\frac{1}{2}}}_{=1+o(1)} \Big(\int_I f_k(y) \int_{\mathbb{R}} |G(x-y) - G(x)|^2\,dx\,dy \Big)^{\frac{1}{2}} + o(1).
\end{aligned}$$

Since $G \in L^2(\mathbb{R})$, the function $\psi(y) := \int_{\mathbb{R}} |G(x-y) - G(x)|^2\,dx$ is continuous on $\mathbb{R}$ and $\psi(0) = 0$. Let $\varphi \in C(\mathbb{R})$ be a compactly supported function such that $\varphi \equiv \psi$ on I. Then Lemma 4.9 implies

$$\int_I f_k(y) \int_{\mathbb{R}} |G(x-y) - G(x)|^2\,dx\,dy = \int_{\mathbb{R}} f_k(y)\varphi(y)\,dy + o(1) \to 0$$

as $k \to \infty$, and the conclusion follows. □

Repeating the argument of Proposition 2.9, we get the following lemma.

Lemma 4.12. *We have $\mu_k \tilde{u}_k \to \tilde{G}$ in*

$$C^0_{\mathrm{loc}}(\overline{\mathbb{R}^2_+} \setminus \{(0,0)\}) \cap C^1_{\mathrm{loc}}(\mathbb{R}^2_+),$$

where $\tilde{G}$ is the Poisson extension of G.

With Proposition 4.4 and Lemma 4.12, we can give an upper bound on D_π.

Proposition 4.13. *Under the assumption that $\mu_k \to \infty$ as $k \to \infty$, we have $D_\pi \leq 2\pi e^{-\gamma}$.*

Proof. For a fixed and small $\delta > 0$, set

$$a_k := \inf_{\partial B_{Lr_k} \cap \mathbb{R}^2_+} \tilde{u}_k, \quad b_k := \sup_{\partial B_\delta \cap \mathbb{R}^2_+} \tilde{u}_k, \quad \tilde{v}_k := (\tilde{u}_k \wedge a_k) \vee b_k.$$

Recalling that

$$\|\nabla \tilde{u}_k\|_{L^2(\mathbb{R}^2_+)}^2 = \|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2(\mathbb{R})}^2 = 1 - \|u_k\|_{L^2(\mathbb{R})}^2,$$

we have

$$\int_{(B_\delta \setminus B_{Lr_k}) \cap \mathbb{R}^2_+} |\nabla \tilde{v}_k|^2\,dx\,dy \leq 1 - \|u_k\|_{L^2}^2 - \int_{\mathbb{R}^2_+ \setminus B_\delta} |\nabla \tilde{u}_k|^2\,dx - \int_{\mathbb{R}^2_+ \cap B_{Lr_k}} |\nabla \tilde{u}_k|^2\,dx.$$

Clearly, the left-hand side bounds

$$\inf_{\substack{\tilde u|_{\mathbb{R}^2_+\cap\partial B_{Lr_k}}=a_k\\ \tilde u|_{\mathbb{R}^2_+\cap\partial B_\delta}=b_k}} \int_{(B_\delta\setminus B_{Lr_k})\cap\mathbb{R}^2_+} |\nabla\tilde u|^2\,dx\,dy = \pi\frac{(a_k-b_k)^2}{\log\delta-\log(Lr_k)}.$$

Using Proposition 4.4, Proposition 4.10 and Lemma 4.12, we obtain

$$\begin{cases} a_k = \mu_k + \dfrac{-\frac{1}{\pi}\log L + O(L^{-1}) + o(1)}{\mu_k},\\ b_k = \dfrac{-\frac{1}{\pi}\log\delta - \frac{\gamma}{\pi} + O(\delta|\log\delta|) + o(1)}{\mu_k}, \end{cases} \tag{4.11}$$

where $o(1)\to 0$ as $k\to\infty$ for fixed $L>0$, $\delta>0$, and $|O(L^{-1})|\le CL^{-1}$, $|O(\delta|\log\delta|)|\le C\delta|\log\delta|$, uniformly for δ small and L, k large. Still by Proposition 4.4, we get

$$\lim_{k\to\infty}\mu_k^2\int_{B^+_{Lr_k}}|\nabla\tilde u_k|^2\,dx\,dy = \frac{1}{4\pi^2}\int_{B^+_L}|\nabla\tilde\eta_\infty|^2\,dx\,dy = \frac{1}{\pi}\log\frac{L}{2} + O\Big(\frac{\log L}{L}\Big).$$

Similarly Lemma 4.12 and Proposition 4.10 yield

$$\liminf_{k\to\infty}\mu_k^2\int_{\mathbb{R}^2_+\setminus B_\delta}|\nabla\tilde u_k|^2\,dx\,dy \ge \int_{\mathbb{R}^2_+\setminus B_\delta}|\nabla\tilde G|^2\,dx\,dy$$

with

$$\begin{aligned}\int_{\mathbb{R}^2_+\setminus B_\delta}|\nabla\tilde G|^2\,dx\,dy &= \int_{\mathbb{R}^2_+\cap\partial B_\delta} -\frac{\partial\tilde G}{\partial r}\tilde G\,d\sigma + \int_{\mathbb{R}\setminus(-\delta,\delta)} -\frac{\partial\tilde G(x,0)}{\partial y}G(x)\,dx\\ &= \int_{\mathbb{R}^2_+\cap\partial B_\delta}\Big(\frac{1}{\pi\delta}+O(|\log\delta|)\Big)\Big(-\frac{1}{\pi}\log\delta - \frac{\gamma}{\pi} + O(\delta|\log\delta|)\Big)\,d\sigma - \int_{\mathbb{R}\setminus(-\delta,\delta)} G(x)^2\,dx\\ &= -\frac{1}{\pi}\log\delta - \frac{\gamma}{\pi} - \|G\|^2_{L^2(\mathbb{R})} + O(\delta\log^2\delta),\end{aligned}$$

where we used Lemma A.5 and

$$-\frac{\partial\tilde G(x,0)}{\partial y} = (-\Delta)^{\frac12}G(x) = -G(x)\quad\text{for } x\in\mathbb{R}\setminus\{0\}.$$

From Lemma 4.11 we get that $\mu_k u_k\to G$ in $L^2(\mathbb{R})$, and hence

$$\|u_k\|^2_{L^2(\mathbb{R})} = \frac{\|G\|^2_{L^2(\mathbb{R})}+o(1)}{\mu_k^2}$$

as $k\to+\infty$. We then get

$$\frac{\pi(a_k-b_k)^2}{\log\delta-\log(Lr_k)} \le 1 - \frac{\frac{1}{\pi}\log\frac{L}{2\delta} - \frac{\gamma}{\pi} + O(\delta\log^2\delta) + O(\frac{\log L}{L}) + o(1)}{\mu_k^2}.$$

Using (4.11) and rearranging as in the proof of Proposition 2.10, we find

$$\begin{aligned}\log\frac{1}{\lambda_k\mu_k^2} &\le \Big(1-\frac{\alpha_k}{\pi}\Big)\log\frac{L}{\delta} + (\alpha_k-\pi)\mu_k^2 + \Big(\frac{\alpha_k}{\pi}-2\Big)\gamma + \frac{\alpha_k}{\pi}\log 2 + \log\alpha_k\\ &\quad + O(\delta\log^2\delta) + O\Big(\frac{\log L}{L}\Big) + o(1),\end{aligned}$$

with $o(1)\to 0$ as $k\to\infty$. Then, recalling that $\alpha_k\uparrow\pi$ and letting first $k\to\infty$ and then $L\to\infty$ and $\delta\to 0$, we obtain

$$\limsup_{k\to\infty}\log\frac{1}{\lambda_k\mu_k^2} \le -\gamma + \log(2\pi).$$

Using Proposition 4.8 we conclude. □

Proposition 4.14. *There exists a function $u \in H^{1/2,2}(\mathbb{R})$ such that $\|u\|_H \le 1$ and $E_\pi(u) > 2\pi e^{-\gamma}$.*

Proof. For $\varepsilon > 0$ choose $L = L(\varepsilon) > 0$ such that, as $\varepsilon \to 0$, we have $L \to \infty$ and $L\varepsilon \to 0$. Fix

$$\Gamma_{L\varepsilon} := \{(x,y) \in \mathbb{R}^2_+ : \tilde{G}(x,y) = \gamma_{L\varepsilon} := \min_{\mathbb{R}^2_+ \cap \partial B_{L\varepsilon}} \tilde{G}\}$$

and

$$\Omega_{L\varepsilon} := \{(x,y) \in \mathbb{R}^2_+ : \tilde{G}(x,y) > \gamma_{L\varepsilon}\}.$$

By the maximum principle, we have $\mathbb{R}^2_+ \cap B_{L\varepsilon} \subset \Omega_{L\varepsilon}$. Notice also that Proposition 4.10 gives

$$\gamma_{L\varepsilon} = -\frac{1}{\pi}\log(L\varepsilon) - \frac{\gamma}{\pi} + O(L\varepsilon|\log(L\varepsilon)|)$$

and $\Omega_{L\varepsilon} \subseteq \mathbb{R}^2_+ \cap B_{2L_\varepsilon}$. For suitable constants $B, c \in \mathbb{R}$ to be fixed, we set

$$U_\varepsilon(x,y) := \begin{cases} c - \dfrac{\log(\frac{x^2}{\varepsilon^2} + (1+\frac{y}{\varepsilon})^2) + 2B}{2\pi c} & \text{for } (x,y) \in B_{L\varepsilon}(0,-\varepsilon) \cap \mathbb{R}^2_+, \\ \dfrac{\gamma_{L\varepsilon}}{c} & \text{for } (x,y) \in \Omega_{L\varepsilon} \setminus B_{L\varepsilon}(0,-\varepsilon), \\ \dfrac{\tilde{G}(x,y)}{c} & \text{for } (x,y) \in \mathbb{R}^2_+ \setminus \Omega_{L\varepsilon}. \end{cases}$$

Observe that $\mathbb{R}^2_+ \cap B_{L\varepsilon}(0,-\varepsilon) \subseteq \mathbb{R}^2 \cap B_{L\varepsilon} \subseteq \Omega_{L\varepsilon}$. We choose B in order to have continuity on $\mathbb{R}^2_+ \cap \partial B_{L\varepsilon}(0,-\varepsilon)$, i.e. we impose

$$\frac{-\log L^2 - 2B}{2\pi c} + c = \frac{\gamma_{L\varepsilon}}{c},$$

which gives the relation

$$B = \pi c^2 + \log\varepsilon + \gamma + O(L\varepsilon|\log(L\varepsilon)|). \tag{4.12}$$

This choice of B also implies that the function cU_ε does not depend on the value of c. Then we can choose c by imposing

$$\|\nabla U_\varepsilon\|^2_{L^2(\mathbb{R}^2_+)} + \|u_\varepsilon\|^2_{L^2(\mathbb{R})} = 1, \tag{4.13}$$

where we set $u_\varepsilon(x) = U_\varepsilon(x,0)$. Since the harmonic extension $\tilde{u}_\varepsilon$ minimizes the Dirichlet energy among extensions with finite energy, we have

$$\|(-\Delta)^{\frac{1}{4}} u_\varepsilon\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}^2_+} |\nabla \tilde{u}_\varepsilon|^2\,dx\,dy \le \int_{\mathbb{R}^2_+} |\nabla U_\varepsilon|^2\,dx\,dy,$$

and (4.13) implies

$$\|u_\varepsilon\|^2_{H^{\frac{1}{2},2}(\mathbb{R})} \le 1.$$

In order to obtain a more precise expansion of B and c, we compute

$$\begin{aligned} \int_{B_{L\varepsilon}(0,-\varepsilon)\cap\mathbb{R}^2_+} |\nabla U_\varepsilon|^2\,dx\,dy &= \frac{1}{4\pi^2 c^2} \int_{B_L(0,-1)\cap\mathbb{R}^2_+} |\nabla \log(x^2 + (1+y)^2)|^2\,dx\,dy \\ &= \frac{\frac{1}{\pi}\log(\frac{L}{2}) + O(\frac{\log L}{L})}{c^2} \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \int_{\mathbb{R}^2_+\setminus\Omega_{L\varepsilon}} |\nabla U_\varepsilon|^2\,dx\,dy &= \frac{1}{c^2}\int_{\mathbb{R}^2_+\setminus\Omega_{L\varepsilon}} |\nabla\tilde{G}|^2\,dx\,dy \\ &= -\frac{1}{c^2}\int_{\mathbb{R}^2_+\cap\partial\Omega_{L\varepsilon}} \frac{\partial\tilde{G}}{\partial\nu}\tilde{G}\,d\sigma - \frac{1}{c^2}\int_{(\mathbb{R}\times\{0\})\setminus\bar{\Omega}_{L\varepsilon}} \frac{\partial\tilde{G}}{\partial y}\tilde{G}\,d\sigma \\ &= (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

By the divergence theorem, for $\tau < L\varepsilon$ and letting $\tau \to 0$, we have

$$
\begin{aligned}
(\mathrm{I}) &= -\frac{\gamma_{L\varepsilon}}{c^2} \int\limits_{(\mathbb{R}\times\{0\})\cap(\bar{\Omega}_{L\varepsilon}\setminus B_\tau)} \frac{\partial \tilde{G}}{\partial \nu}\, d\sigma - \frac{\gamma_{L\varepsilon}}{c^2} \int\limits_{\mathbb{R}^2_+\cap\partial B_\tau} \frac{\partial \tilde{G}}{\partial \nu}\, d\sigma \\
&= \frac{\gamma_{L\varepsilon}}{c^2}\Big(\int\limits_{(\mathbb{R}\times\{0\})\cap\bar{\Omega}_{L\varepsilon}} G\, d\sigma + 1\Big) \\
&= \frac{\gamma_{L\varepsilon}}{c^2}(1 + O(L\varepsilon \log(L\varepsilon))) \\
&= \frac{\frac{1}{\pi}\log(\frac{1}{L\varepsilon}) - \frac{\gamma}{\pi} + O(L\varepsilon\log^2(L\varepsilon))}{c^2},
\end{aligned} \tag{4.15}
$$

where in the third identity we used that $\Omega_{L\varepsilon} \subset B_{2L\varepsilon}$ for $L\varepsilon$ small enough. Observe also that

$$
\|u_\varepsilon\|^2_{L^2(\mathbb{R})} = \frac{1}{c^2} \int\limits_{(\mathbb{R}\times\{0\})\setminus\bar{\Omega}_{L\varepsilon}} G^2\, dx + \frac{O(L\varepsilon\log^2(L\varepsilon))}{c^2} = -(\mathrm{II}) + \frac{O(L\varepsilon\log^2(L\varepsilon))}{c^2}.
$$

Together with (4.13)–(4.15) this gives

$$
-\log\varepsilon - \log 2 - \gamma + O(L\varepsilon\log^2(L\varepsilon)) + O\Big(\frac{\log L}{L}\Big) = \pi c^2,
$$

which, together with (4.12), implies

$$
B = -\log 2 + O(L\varepsilon\log^2(L\varepsilon)) + O\Big(\frac{\log L}{L}\Big).
$$

Now, observe that

$$
B_{L\varepsilon}(0, -\varepsilon) \cap (\mathbb{R}\times\{0\}) = (-\varepsilon\sqrt{L^2-1}, \varepsilon\sqrt{L^2-1})
$$

and that

$$
\begin{aligned}
\int\limits_{-\varepsilon\sqrt{L^2-1}}^{\varepsilon\sqrt{L^2-1}} e^{\pi u_\varepsilon^2}\, dx &= \varepsilon \int\limits_{-\sqrt{L^2-1}}^{\sqrt{L^2-1}} \exp\Big(\pi\Big(c - \frac{\log(1+x^2) + 2B}{2\pi c}\Big)^2\Big)\, dx \\
&> \varepsilon e^{\pi c^2 - 2B} \int\limits_{-\sqrt{L^2-1}}^{\sqrt{L^2-1}} \frac{1}{1+x^2}\, dx \\
&= 2e^{-\gamma + O(L\varepsilon\log^2(L\varepsilon)) + O(\frac{\log L}{L})}\pi\Big(1 + O\Big(\frac{1}{L}\Big)\Big) \\
&= 2\pi e^{-\gamma} + O(L\varepsilon\log^2(L\varepsilon)) + O\Big(\frac{\log L}{L}\Big).
\end{aligned}
$$

Moreover,

$$
\int\limits_{(\mathbb{R}\times\{0\})\setminus\bar{\Omega}_{L\varepsilon}} (e^{\pi u_\varepsilon^2} - 1)\, dx \geq \int\limits_{(\mathbb{R}\times\{0\})\setminus\bar{\Omega}_{L\varepsilon}} \pi u_\varepsilon^2\, dx = \frac{1}{c^2} \int\limits_{(\mathbb{R}\times\{0\})\setminus\bar{\Omega}_{L\varepsilon}} \pi G^2\, dx =: \frac{\nu_{L\varepsilon}}{c^2},
$$

with

$$
\nu_{L\varepsilon} > \nu_{\frac{1}{2}} > 0 \quad \text{for } L\varepsilon < \frac{1}{2}.
$$

Now choose $L = \log^2\varepsilon$ to obtain

$$
O(L\varepsilon\log^2(L\varepsilon)) + O\Big(\frac{\log L}{L}\Big) = O\Big(\frac{\log\log\varepsilon}{\log^2\varepsilon}\Big) = o\Big(\frac{1}{c^2}\Big),
$$

so that

$$
E_\pi(u_\varepsilon) = \int\limits_{\mathbb{R}} (e^{\pi u_\varepsilon^2} - 1)\, dx \geq 2\pi e^{-\gamma} + \frac{\nu_{\frac{1}{2}}}{c^2} + o\Big(\frac{1}{c^2}\Big) > 2\pi e^{-\gamma}
$$

for ε small enough. □

Proof of Theorem 1.2 (completed). By Propositions 2.11 and 4.14, we know that $\mu_k \le C$. By the dominated convergence theorem, we have $e^{\alpha_k u_k^2} \to e^{\pi u_\infty^2}$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Then, by Lemma 4.2, we infer

$$E_{\alpha_k}(u_k) = E_\pi(u_\infty) + \pi(\|u_k\|^2_{L^2(\mathbb{R})} - \|u_\infty\|^2_{L^2(\mathbb{R})}) + o(1). \tag{4.16}$$

This implies that $u_\infty \not\equiv 0$; otherwise we would have

$$E_{\alpha_k}(u_k) \le \pi\|u_k\|^2_{L^2(\mathbb{R})} + o(1) \le \pi + o(1),$$

which contradicts the strict inequality $D_\pi > 2\pi e^{-\gamma} > \pi$ since $E_{\alpha_k}(u_k) = D_{\alpha_k} \to D_\pi$ as $k \to \infty$.

Let us set

$$L := \limsup_{k\to\infty}\|u_k\|_2^2 \quad \text{and} \quad \tau = \frac{\|u_\infty\|^2_{L^2(\mathbb{R})}}{L}$$

and observe that $L, \tau \in (0, 1]$. Let us consider the sequence $v_k(x) = u_k(\tau x)$. Clearly, we have $v_k \rightharpoonup v_\infty$ in H, where $v_\infty(x) := u_\infty(\tau x)$. Since $\|v_\infty\|^2_{L^2} = L$ and

$$\|(-\Delta)^{\frac14} v_\infty\|^2_{L^2} \le \liminf_{k\to\infty}\|(-\Delta)^{\frac14} v_k\|^2_{L^2} = \liminf_{k\to\infty}\|(-\Delta)^{\frac14} u_k\|^2_{L^2} \le 1 - L,$$

we get $\|v_\infty\|_{H^{1/2,2}} \le 1$. By (4.16), we have

$$D_\pi = E_\pi(u_\infty) + \pi L(1-\tau) = \tau E_\pi(v_\infty) + \pi L(1-\tau) \le \tau D_\pi + \pi L(1-\tau).$$

If $\tau < 1$, this implies $D_\pi \le \pi L$, which is not possible. Hence, we must have $\tau = 1$ and $E_\pi(u_\infty) = D_\pi$. □

A Appendix: The Half-Laplacian on $\mathbb{R}$

For $u \in \mathcal{S}$ (the Schwarz space of rapidly decaying functions), we set

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\hat{u}(\xi), \quad \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx.$$

One can prove that it holds (see, e.g., [10, Section 3])

$$(-\Delta)^s u(x) = K_s \mathrm{P.V.}\int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\,dy := K_s \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\,dy, \tag{A.1}$$

from which it follows that

$$\sup_{x\in\mathbb{R}}|(1+x^{1+2s})(-\Delta)^s\varphi(x)| < \infty \quad \text{for every } \varphi \in \mathcal{S}.$$

Then one can set

$$L_s(\mathbb{R}) := \Big\{u \in L^1_{\mathrm{loc}}(\mathbb{R}) : \|u\|_{L_s} := \int_{\mathbb{R}} \frac{|u(x)|}{1+|x|^{1+2s}}\,dx < \infty\Big\}, \tag{A.2}$$

and for every $u \in L_s(\mathbb{R})$ one defines the tempered distribution $(-\Delta)^s u$ by

$$\langle(-\Delta)^s u, \varphi\rangle := \int_{\mathbb{R}} u(-\Delta)^s\varphi\,dx = \int_{\mathbb{R}} u\mathcal{F}^{-1}(|\xi|\hat{\varphi}(\xi))\,dx \quad \text{for every } \varphi \in \mathcal{S}. \tag{A.3}$$

Moreover, for $p \ge 1$ and $s \in (0, 1)$ we will define

$$H^{s,p}(\mathbb{R}) := \{u \in L^p(\mathbb{R}) : (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R})\}. \tag{A.4}$$

In the case $s = \frac12$, we have $K_{1/2} = \frac1\pi$ in (A.1) and a simple alternative definition of $(-\Delta)^{\frac12}$ can be given via the Poisson integral. For $u \in L_{1/2}(\mathbb{R})$, define the Poisson integral

$$\tilde{u}(x, y) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{yu(\xi)}{(y^2+(x-\xi)^2)}\,d\xi, \quad y > 0, \tag{A.5}$$

which is harmonic in $\mathbb{R}^2_+ = \mathbb{R}\times(0,\infty)$ and satisfies the boundary condition $\tilde{u}|_{\mathbb{R}\times\{0\}} = u$ in the following sense.

Proposition A.1. *If $u \in L^{1/2}(\mathbb{R})$, then $\tilde{u}(\cdot, y) \in L^1_{\mathrm{loc}}(\mathbb{R})$ for $y \in (0, \infty)$ and $\tilde{u}(\cdot, y) \to u$ in the sense of distributions as $y \to 0^+$. If $u \in L^{1/2}(\mathbb{R}) \cap C((a, b))$ for some interval $(a, b) \subseteq \mathbb{R}$, then $\tilde{u}$ extends continuously to $(a, b) \times \{0\}$ and $\tilde{u}(x, 0) = u(x)$ for any $x \in (a, b)$. If $u \in H^{1/2}(\mathbb{R})$, then $\tilde{u} \in H^1(\mathbb{R}^2_+)$, the identity*

$$\|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2_+)} = \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})}$$

holds, and $\tilde{u}|_{\mathbb{R}\times\{0\}} = u$ in the sense of traces.

Then we have (see, e.g., [4])

$$(-\Delta)^{\frac{1}{2}} u = -\frac{\partial \tilde{u}}{\partial y}\Big|_{y=0}, \tag{A.6}$$

where the identity is pointwise if u is regular enough (for instance $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R})$), and has to be read in the sense of tempered distributions in general, with

$$\Big\langle -\frac{\partial \tilde{u}}{\partial y}\Big|_{y=0}, \varphi \Big\rangle := \Big\langle u, -\frac{\partial \tilde{\varphi}}{\partial y}\Big|_{y=0} \Big\rangle, \quad \varphi \in \mathcal{S},\ \tilde{\varphi} \text{ as in (A.5)}. \tag{A.7}$$

More precisely, we obtain the following proposition.

Proposition A.2. *If $u \in L_{1/2}(\mathbb{R}) \cap C^{1,\alpha}_{\mathrm{loc}}((a, b))$ for some interval $(a, b) \subset \mathbb{R}$ and some $\alpha \in (0, 1)$, then the tempered distribution $(-\Delta)^{1/2} u$ defined in* (A.3) *coincides on the interval (a, b) with the functions given by* (A.1) *and* (A.6)*. For general $u \in L_{1/2}(\mathbb{R})$, the definitions* (A.3) *and* (A.6) *are equivalent, where the right-hand side of* (A.6) *is defined by* (A.7)*.*

It is known that the Poisson integral of a function $u \in L_{1/2}(\mathbb{R})$ is the unique harmonic extension of u under some growth constraints at infinity. In fact, combining [48, Theorem 2.1 and Corollary 3.1] and [45, Theorem C], we get the following proposition.

Proposition A.3. *For any $u \in L_{1/2}(\mathbb{R})$, the Poisson extension $\tilde{u}$ satisfies*

$$\tilde{u}(x, y) = o(y^{-1}(x^2 + y^2)) \quad \text{as } |(x, y)| \to \infty.$$

Moreover, if U is a harmonic function in $\mathbb{R}^2_+$ which satisfies $U(x, y) = o(y^{-1}(x^2 + y^2))$ as $|(x, y)| \to \infty$ and $U(\cdot, y) \to u$ as $y \to 0^+$ in the sense of distributions, then $U = \tilde{u}$ in $\mathbb{R}^2_+$.

Assume that $u \in H^{1/2,2}(\mathbb{R})$ solves $(-\Delta)^{1/2} u = f$ in $\mathbb{R}$ with $f \in L^2(\mathbb{R})$. The identity

$$\|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2_+)} = \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} uf\, dx \tag{A.8}$$

of Proposition A.1 can be seen as an integration-by-parts formula on $\mathbb{R}^2_+$ for $\tilde{u}$, since

$$\frac{\partial \tilde{u}}{\partial y}\Big|_{y=0} = -f$$

in the sense of distributions. On the other hand, the standard integration-by-parts formula for $\tilde{u}$ on $\mathbb{R}^2_+ \cap B_R$ implies

$$\|\nabla \tilde{u}\|_{L^2(\mathbb{R}^2_+\cap B_R)} = \int_{-R}^{R} uf\, dx + \int_{\partial B_R \cap \mathbb{R}^2_+} \tilde{u} \frac{\partial \tilde{u}}{\partial \nu}\, d\sigma,$$

where

$$\frac{\partial \tilde{u}}{\partial \nu}(x, y) = \nabla \tilde{u}(x, y) \cdot \frac{(x, y)}{|(x, y)|}$$

is the normal derivative of $\tilde{u}$ on ∂B_R. Then the validity of the integration-by-parts formula (A.8) is equivalent to

$$\lim_{\mathbb{R}\to+\infty} \int_{\partial B_R \cap \mathbb{R}^2_+} \tilde{u} \frac{\partial \tilde{u}}{\partial \nu}\, d\sigma = 0. \tag{A.9}$$

We shall prove that (A.9) holds even when $u \notin H^{1/2,2}(\mathbb{R})$, provided u has a good decay at infinity. This provides integration by parts formulas for the Poisson extension of Green's functions, which are not in $H^{1/2,2}(\mathbb{R})$. In the following, we set

$$P(x,y) := \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

Lemma A.4. *Assume that $u \in L^1(\mathbb{R})$ and $u \equiv 0$ in $\mathbb{R}\setminus I$. Then $\exists R_0 > 0$ such that*

$$|\tilde{u}(x,y)| \le \frac{C}{\sqrt{x^2+y^2}} \quad \text{and} \quad |\nabla\tilde{u}(x,y)| \le \frac{C}{x^2+y^2} \tag{A.10}$$

for any $(x,y) \in \mathbb{R}^2_+$ with $|(x,y)| \ge R_0$. In particular, (A.9) *holds.*

Proof. Note that we have $P(x-\xi,y) \le \frac{1}{\pi y}$ and $|\nabla P(x-\xi,y)| \le \frac{1}{y}P(x-\xi,y)$ for any $(x,y) \in \mathbb{R}^2_+$. In particular, we have

$$|\tilde{u}(x,y)| \le \frac{1}{\pi y}\int_{\mathbb{R}}|u(\xi)|\,d\xi \quad \text{and} \quad |\nabla\tilde{u}(x,y)| \le \frac{1}{\pi y^2}\int_{\mathbb{R}}|u(\xi)|\,d\xi.$$

Moreover, for $|x| \ge 2$ we have $|x-\xi| \ge \frac{|x|}{2}$ for any $\xi \in I$, and hence

$$|\tilde{u}(x,y)| \le \frac{y}{\frac{x^2}{4}+y^2}\|u\|_{L^1(\mathbb{R})} \le \frac{4\|u\|_{L^1(\mathbb{R})}}{\sqrt{x^2+y^2}}.$$

Similarly, using again that $|\nabla P(x-\xi)| \le \frac{1}{y}P(x,y)$, we get

$$|\nabla\tilde{u}(x,y)| \le \frac{4\|u\|_{L^1(\mathbb{R})}}{x^2+y^2}.$$ □

Lemma A.5. *Let $u \in L^1(\mathbb{R}) \cap C^2(\mathbb{R}\setminus(-R,R))$ for some $R > 0$. If $|u''(x)| = O(|x|^{-4})$ as $|x| \to \infty$, then* (A.10) *and* (A.9) *hold.*

Proof. It is sufficient to prove the lemma for $u \in C^2(\mathbb{R})$. Indeed, we can write $u = u_1 + u_2$, with $u_1 = u\varphi$ and $u_2 = u(1-\varphi)$, where $\varphi \in C_c^\infty(\mathbb{R})$ is a cut-off function which equals 1 in $(-R,R)$. Note that $\tilde{u}_2 \in C^2(\mathbb{R})$ and that it coincides with u when $|x|$ is large. By lemma A.4, inequalities (A.10) hold for $\tilde{u}_1$. Hence, (A.10) holds for $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$ if it holds for $\tilde{u}_2$.

Thus, we assume that $u \in C^2(\mathbb{R})$. Let us consider the complex map $f(z) = \frac{iz+1}{z+i}$, $z \in \mathbb{C}$, which defines a biholomorphic map between the half plane and the complex disk. In Euclidean coordinates, f and its inverse correspond respectively to the conformal diffeomorphisms $\Phi : \mathbb{R}^2_+ \to B_1 \subseteq \mathbb{R}^2$, $\Psi : B_1 \to \mathbb{R}^2_+$ given by

$$\Phi(x,y) := \Big(\frac{2x}{x^2+(y+1)^2}, \frac{x^2+y^2-1}{x^2+(y+1)^2}\Big),$$

$$\Psi(x,y) := \Big(\frac{2x}{x^2+(y-1)^2}, \frac{1-x^2-y^2}{x^2+(y-1)^2}\Big).$$

Note that Φ and Ψ extend continuously to diffeomorphisms

$$\Phi : \overline{\mathbb{R}^2_+} \to \overline{B_1}\setminus\{(0,1)\}, \quad \Psi : \overline{B_1}\setminus\{(0,1)\} \to \overline{\mathbb{R}^2_+}.$$

In fact, we have $\Psi|_{\partial B_1}(z) = \pi_N(z)$ and $\Phi(x,0) = \pi_N^{-1}(x)$, where $\pi_N : \partial B_1 \to \mathbb{R}$, $\pi(z_1,z_2) = \frac{z_1}{1-z_2}$, is the stereographic projection from the north pole. Let us consider the function

$$v(x) := \begin{cases} u(\pi_N(z)), & z \in \partial B_1 \setminus (0,1), \\ 0, & z = (0,1). \end{cases}$$

Near the point $(0,1)$, in local coordinates given by the stereographic projection

$$\pi_S^{-1}(t) = \Big(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\Big), \quad t \in \mathbb{R},$$

we get that $v(\pi_S^{-1}(t)) = u(t^{-1})$ for any $t \neq 0$. The decay assumption on u'' implies that $|\frac{d^k}{dt^k}u(t^{-t})| = O(t^{2+k})$ for $k = 0,1,2$ as $t \to 0$. In particular, we get

$$\lim_{t\to 0} u(t^{-1}) = 0, \quad \lim_{t\to 0}\frac{d}{dt}u(t^{-1}) = 0, \quad u''(t^{-1}) = O(1) \text{ as } t \to 0.$$

This implies that v is of class $C^{1,1}(\partial B_1)$. Let then V be the unique harmonic extension of v to B_1, which can be defined by the Poisson formula for B_1. The regularity of v implies that $V \in C^{1,\alpha}(\overline{B_1})$ for any $\alpha \in (0,1)$. In particular, V and the elements of its Jacobian matrix DV are bounded on $\overline{B_1}$. Since Φ is a conformal map and V is bounded, Proposition A.3 implies that $\tilde{u}(x,y) = V(\Phi(x,y))$. Moreover, we have $|V(z)| \le C|z-(0,1)|$ for any $z \in B_1$. Then we get

$$|\tilde{u}(x,y)| \le C|\Phi(x,y)-(0,1)| = \frac{2}{\sqrt{x^2+(1+y)^2}} = O\Big(\frac{1}{\sqrt{x^2+y^2}}\Big).$$

Similarly, since $|DV| = O(1)$, we find

$$|\nabla\tilde{u}(x,y)| \le C\Big|\frac{\partial\Phi}{\partial y}(x,y)\Big| = \frac{2C}{x^2+(1+y)^2} = O\Big(\frac{1}{x^2+y^2}\Big),$$

as desired. □

With no regularity assumptions, one can still prove that $\tilde{u}$ decays at infinity if u does.

Lemma A.6. *Assume that $u \in L^p(\mathbb{R})$ for some $p \ge 1$. If $\lim_{|x|\to\infty} u(x) = 0$, then $\lim_{|(x,y)|\to 0} \tilde{u}(x,y) = 0$.*

Proof. By Hölder's inequality, we have

$$|\tilde{u}(x,y)| \le \frac{1}{\pi}\Big(\int_{\mathbb{R}} P(x-\xi,y)^{\frac{p}{p-1}}\,d\xi\Big)^{\frac{p-1}{p}} \|u\|_{L^p(\mathbb{R})} \le \frac{C}{y^{1/p}} \to 0$$

as $y \to +\infty$. Then it is sufficient to prove that $\lim_{|x|\to+\infty}|\tilde{u}(x,y)| = 0$, uniformly with respect to $y \in (0,\infty)$. To see this, we write the integral in the definition of $\tilde{u}$ as the sum of integrals on $I(x) := (-\frac{|x|}{2}, \frac{|2|}{2})$ and on $\mathbb{R} \setminus I(x)$. For $\xi \in I(x)$, we have $|x-\xi| \ge \frac{|x|}{2}$ and

$$\int_{I(x)} P(x-\xi,y)u(\xi)\,d\xi \le \frac{4y}{x^2+y^2}\int_{-\frac{|x|}{2}}^{\frac{|x|}{2}} |u(\xi)|\,d\xi \le \frac{4y|x|^{1-\frac{1}{p}}\|u\|_{L^p(\mathbb{R})}}{x^2+y^2} \le \frac{2\|u\|_{L^p(\mathbb{R})}}{|x|^{\frac{1}{p}}}.$$

Instead, for $|\xi| \ge \frac{|x|}{2}$, we have

$$\int_{-\frac{|x|}{2}}^{\frac{|x|}{2}} P(x-\xi,y)u(\xi)\,d\xi \le \sup_{|\xi|\ge\frac{|x|}{2}} |u(\xi)| \int_{\mathbb{R}} P(x-\xi,y)\,d\xi = \sup_{|\xi|\ge\frac{|x|}{2}} |u(\xi)| \to 0$$

as $|x| \to +\infty$. This gives the conclusion. □

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