

# Remarkable Classes of Almost 3-Contact Metric Manifolds

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**Abstract:** We introduce a new class of almost 3-contact metric manifolds, called 3-(0,  $\delta$ )-Sasaki manifolds. We show fundamental geometric properties of these manifolds, analyzing analogies and differences with the known classes of 3-( $\alpha, \delta$ )-Sasaki ( $\alpha \neq 0$ ) and 3- $\delta$ -cosymplectic manifolds.

**Keywords:** almost 3-contact metric manifold; 3-Sasaki; 3-cosymplectic; 3-( $\alpha, \delta$ )-Sasaki; 3- $\delta$ -cosymplectic; metric connection with skew torsion

**MSC:** 53C15; 53C25; 53B05

## 1. Introduction

An almost 3-contact metric manifold is a  $(4n + 3)$ -dimensional differentiable manifold  $M$  endowed with three almost contact metric structures  $(\varphi, \xi_i, \eta_i, g)$ ,  $i = 1, 2, 3$ , sharing the same Riemannian metric  $g$  and satisfying suitable compatibility conditions, equivalent to the existence of a sphere of almost contact metric structures. In the recent paper [1], new classes of almost 3-contact metric manifolds were introduced and studied. The first remarkable class is given by 3-( $\alpha, \delta$ )-Sasaki manifolds defined as almost 3-contact metric manifolds  $(M, \varphi_i, \xi_i, \eta_i, g)$  such that

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k, \quad \alpha \in \mathbb{R}^*, \delta \in \mathbb{R}, \quad (1)$$

for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ . This is a generalization of 3-Sasaki manifolds, which correspond to the values  $\alpha = \delta = 1$ . A second class introduced in [1] is given by 3- $\delta$ -cosymplectic manifolds defined by the conditions

$$d\eta_i = -2\delta\eta_j \wedge \eta_k, \quad d\Phi_i = 0, \quad \delta \in \mathbb{R},$$

generalizing 3-cosymplectic manifolds which correspond to the value  $\delta = 0$ .

In the present paper we will introduce a third class of almost 3-contact metric manifolds, which is in fact a second (and alternative) generalization of 3-cosymplectic manifolds. We will consider almost 3-contact metric manifolds whose structure tensor fields satisfy

$$d\eta_i = -2\delta\eta_j \wedge \eta_k, \quad d\Phi_i = -2\delta(\eta_j \wedge \Phi_k - \eta_k \wedge \Phi_j), \quad \delta \in \mathbb{R} \quad (2)$$

for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ . When  $\delta = 0$  we recover a 3-cosymplectic manifold. We will call these manifolds 3-(0,  $\delta$ )-Sasaki manifolds. The choice of name is due to the fact that for a 3-( $\alpha, \delta$ )-Sasaki manifold, Equation (1) implies

$$d\Phi_i = 2(\alpha - \delta)(\eta_j \wedge \Phi_k - \eta_k \wedge \Phi_j), \quad (3)$$

so that the two equations in (2) formally correspond to (1) and (3) with  $\alpha = 0$ , although in this case the second equation is no more a consequence of the first one. In fact the two conditions in (2) are not completely independent (see Remark 1). Examples of 3-(0,  $\delta$ )-Sasaki structures can be defined on the semidirect products  $SO(3) \ltimes \mathbb{R}^{4n}$ . The structure on these Lie groups was introduced in [2] as an example of canonical abelian almost 3-contact



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metric structure. It is also shown in [2] that the Lie group  $SO(3) \ltimes \mathbb{R}^{4n}$  admits co-compact discrete subgroups, so that the corresponding compact quotients admit almost 3-contact metric structures of the same type.

One can show that for all the above three classes of manifolds, 3- $(\alpha, \delta)$ -Sasaki, 3- $\delta$ -cosymplectic, and 3- $(0, \delta)$ -Sasaki manifolds, the structure is hypernormal, the characteristic vector fields  $\xi_i$ ,  $i = 1, 2, 3$ , are Killing and they span an integrable distribution, called vertical, with totally geodesic leaves. Nevertheless, there are remarkable geometric differences between the three classes. In the 3- $(\alpha, \delta)$ -Sasaki case the 1-forms  $\eta_i$  are all contact forms, i.e.,  $\eta_i \wedge (d\eta_i)^n \neq 0$  everywhere on  $M$ , while for the other two classes, the horizontal distribution defined by  $\eta_i = 0$ ,  $i = 1, 2, 3$ , is integrable. Both 3- $\delta$ -cosymplectic manifolds and 3- $(0, \delta)$ -Sasaki manifolds are locally isometric to the Riemannian product of a 3-dimensional Lie group, tangent to the vertical distribution, and a  $4n$ -dimensional manifold tangent to the horizontal distribution. The Lie group is either isomorphic to  $SO(3)$  or flat depending on whether  $\delta \neq 0$  or  $\delta = 0$ . Each horizontal leaf is endowed with a hyper-Kähler structure. The difference between 3- $\delta$ -cosymplectic and 3- $(0, \delta)$ -Sasaki manifolds lies in the projectability of the structure tensor fields  $\varphi_i$ ,  $i = 1, 2, 3$ , with respect to the vertical foliation. They are always projectable for 3- $\delta$ -cosymplectic manifolds, but not for 3- $(0, \delta)$ -Sasaki manifolds with  $\delta \neq 0$ . In this case one can project a transverse quaternionic structure, as it happens for 3- $(\alpha, \delta)$ -Sasaki manifolds. Finally, for the three classes of manifolds, we analyze the existence of a canonical metric connection with totally skew-symmetric torsion.

## 2. Almost Contact and Almost 3-Contact Metric Manifolds

An *almost contact manifold* is a smooth manifold  $M$  of dimension  $2n + 1$ , endowed with a structure  $(\varphi, \xi, \eta)$ , where  $\varphi$  is a  $(1, 1)$ -tensor field,  $\xi$  a vector field, and  $\eta$  a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

implying that  $\varphi\xi = 0$ ,  $\eta \circ \varphi = 0$ , and  $\varphi$  has rank  $2n$ . The tangent bundle of  $M$  splits as  $TM = \mathcal{H} \oplus \langle \xi \rangle$ , where  $\mathcal{H}$  is the  $2n$ -dimensional distribution defined by  $\mathcal{H} = \text{Im}(\varphi) = \text{Ker}(\eta)$ . The vector field  $\xi$  is called the *characteristic* or *Reeb vector field*.

On the product manifold  $M \times \mathbb{R}$  one can define an almost complex structure  $J$  by  $J\left(X, f \frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X) \frac{d}{dt}\right)$ , where  $X$  is a vector field tangent to  $M$ ,  $t$  is the coordinate of  $\mathbb{R}$  and  $f$  is a  $C^\infty$  function on  $M \times \mathbb{R}$ . If  $J$  is integrable, the almost contact structure is said to be *normal* and this is equivalent to the vanishing of the tensor field  $N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi$ , where  $[\varphi, \varphi]$  is the Nijenhuis torsion of  $\varphi$  [3]. More precisely, for any vector fields  $X$  and  $Y$ ,  $N_\varphi$  is given by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + d\eta(X, Y)\xi. \quad (4)$$

It is known that any almost contact manifold admits a compatible metric, that is a Riemannian metric  $g$  such that  $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$  for every  $X, Y \in \mathfrak{X}(M)$ . Then  $\eta = g(\cdot, \xi)$  and  $\mathcal{H} = \langle \xi \rangle^\perp$ . The manifold  $(M, \varphi, \xi, \eta, g)$  is called an *almost contact metric manifold*. The associated fundamental 2-form is defined by  $\Phi(X, Y) = g(X, \varphi Y)$ .

We recall some remarkable classes of almost contact metric manifolds.

- An  $\alpha$ -*contact metric manifold* is defined as an almost contact metric manifold such that

$$d\eta = 2\alpha\Phi, \quad \alpha \in \mathbb{R}^*,$$

When  $\alpha = 1$ , it is called a *contact metric manifold*; the 1-form  $\eta$  is a *contact form*, that is  $\eta \wedge (d\eta)^n \neq 0$  everywhere on  $M$ . An  $\alpha$ -*Sasaki manifold* is a normal  $\alpha$ -contact metric manifold, and again such a manifold with  $\alpha = 1$  is called a *Sasaki manifold*.

- An *almost cosymplectic manifold* is defined as an almost contact metric manifold such that

$$d\eta = 0, \quad d\Phi = 0;$$

if furthermore the structure is normal,  $M$  is called a *cosymplectic manifold*. It is worth remarking that some authors call these manifolds *almost coKähler* and *coKähler*, respectively ([4]).

- A *quasi-Sasaki manifold* is a normal almost contact metric manifold with closed 2-form  $\Phi$ . This class includes both  $\alpha$ -Sasaki and cosymplectic manifolds. The Reeb vector field of a quasi-Sasaki manifold is always Killing.

Both  $\alpha$ -Sasaki manifolds and cosymplectic manifolds can be characterized by means of the Levi-Civita connection  $\nabla^g$ . Indeed, one can show that an almost contact metric manifold  $(M, \varphi, \xi, \eta, g)$  is  $\alpha$ -Sasaki if and only if

$$(\nabla_X^g \varphi)Y = \alpha(g(X, Y)\xi - \eta(X)Y) \quad \forall X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold is cosymplectic if and only if  $\nabla^g \varphi = 0$ ; further, this is equivalent to requiring the manifold to be locally isometric to the Riemannian product of a real line (tangent to the Reeb vector field) and a Kähler manifold.

For a comprehensive introduction to almost contact metric manifolds we refer to [3]. For Sasaki geometry, we also recommend the monograph [5]; the survey [4] covers fundamental properties and recent results on cosymplectic geometry.

An *almost 3-contact manifold* is a differentiable manifold  $M$  of dimension  $4n + 3$  endowed with three almost contact structures  $(\varphi_i, \xi_i, \eta_i)$ ,  $i = 1, 2, 3$ , satisfying the following relations,

$$\begin{aligned} \varphi_k &= \varphi_i \varphi_j - \eta_j \otimes \xi_i = -\varphi_j \varphi_i + \eta_i \otimes \xi_j, \\ \xi_k &= \varphi_i \xi_j = -\varphi_j \xi_i, \quad \eta_k = \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i, \end{aligned}$$

for any even permutation  $(i, j, k)$  of  $(1, 2, 3)$  ([3]). The tangent bundle of  $M$  splits as  $TM = \mathcal{H} \oplus \mathcal{V}$ , where

$$\mathcal{H} := \bigcap_{i=1}^3 \text{Ker}(\eta_i), \quad \mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle.$$

In particular,  $\mathcal{H}$  has rank  $4n$ . We call any vector belonging to the distribution  $\mathcal{H}$  *horizontal* and any vector belonging to the distribution  $\mathcal{V}$  *vertical*. The manifold is said to be *hypernormal* if each almost contact structure  $(\varphi_i, \xi_i, \eta_i)$  is normal. In [6] it was proved that if two of the almost contact structures are normal, then so is the third.

The existence of an almost 3-contact structure is equivalent to the existence of a sphere  $\{(\varphi_x, \xi_x, \eta_x)\}_{x \in S^2}$  of almost contact structures such that

$$\varphi_x \circ \varphi_y - \eta_y \otimes \xi_x = \varphi_{x \times y} - (x \cdot y)I, \quad \varphi_x \xi_y = \xi_{x \times y}, \quad \eta_x \circ \varphi_y = \eta_{x \times y},$$

for every  $x, y \in S^2$ , where  $\cdot$  and  $\times$  denote the standard inner product and cross product on  $\mathbb{R}^3$ . In fact, if the structure is hypernormal, then every structure in the sphere is normal ([7]).

Any almost 3-contact manifold admits a Riemannian metric  $g$  which is compatible with each of the three structures. Then  $M$  is said to be an *almost 3-contact metric manifold* with structure  $(\varphi_i, \xi_i, \eta_i, g)$ ,  $i = 1, 2, 3$ . For ease of notation, we will denote an almost 3-contact metric manifold by  $(M, \varphi_i, \xi_i, \eta_i, g)$ , omitting  $i = 1, 2, 3$ . The subbundles  $\mathcal{H}$  and  $\mathcal{V}$  are orthogonal with respect to  $g$  and the three Reeb vector fields  $\xi_1, \xi_2, \xi_3$  are orthonormal. In fact, the structure group of the tangent bundle is reducible to  $\text{Sp}(n) \times \{1\}$  [8].

Given an almost 3-contact metric structure  $(\varphi_i, \xi_i, \eta_i, g)$ , an  $\mathcal{H}$ -homothetic deformation is defined by

$$\eta'_i = c\eta_i, \quad \xi'_i = \frac{1}{c}\xi_i, \quad \varphi'_i = \varphi_i, \quad g' = ag + b \sum_{i=1}^3 \eta_i \otimes \eta_i, \quad (5)$$

where  $a, b, c$  are real numbers such that  $a > 0$ ,  $c^2 = a + b > 0$ , ensuring that  $(\varphi'_i, \xi'_i, \eta'_i, g')$  is an almost 3-contact metric structure. In particular, the fundamental 2-forms  $\Phi_i$  and  $\Phi'_i$  associated to the structures are related by

$$\Phi'_i = a\Phi_i - b\eta_j \wedge \eta_k, \quad (6)$$

where  $(i, j, k)$  is an even permutation of  $(1, 2, 3)$ .

An almost 3-contact metric manifold is called

- *3- $\alpha$ -Sasaki*, with  $\alpha \in \mathbb{R}^*$ , if  $(\varphi_i, \xi_i, \eta_i, g)$  is  $\alpha$ -Sasaki for all  $i = 1, 2, 3$ , i.e. the structure is hypernormal and

$$d\eta_i = 2\alpha\Phi_i, \quad i = 1, 2, 3; \quad (7)$$

when  $\alpha = 1$ , it is a *3-Sasaki manifold*;

- *3-cosymplectic* if  $(\varphi_i, \xi_i, \eta_i, g)$  is cosymplectic for all  $i = 1, 2, 3$ , i.e. the structure is hypernormal and

$$d\eta_i = 0, \quad d\Phi_i = 0, \quad i = 1, 2, 3; \quad (8)$$

- *3-quasi-Sasaki manifold* if each structure  $(\varphi_i, \xi_i, \eta_i, g)$  is quasi-Sasaki; this class includes both 3- $\alpha$ -Sasaki and 3-cosymplectic manifolds.

These classes were deeply investigated by various authors. See [5,9,10] and references therein for 3-Sasakian geometry, the papers [7,11,12] for 3-cosymplectic manifolds, and [13,14] for 3-quasi-Sasaki manifolds.

In fact, both for 3-Sasaki and 3-cosymplectic manifolds, the hypernormality is consequence of the structure Equations (7) and (8) respectively. This was proved by Kashiwada in [15] for 3-Sasaki manifolds, and in ([16], Theorem 4.13) for 3-cosymplectic manifolds.

In [1] the new classes of 3- $(\alpha, \delta)$ -Sasaki manifolds and 3- $\delta$ -cosymplectic manifolds were introduced, generalizing the classes of 3- $\alpha$ -Sasaki and 3-cosymplectic manifolds, respectively. We will review the definitions and the basic properties of these manifolds in the next section. For both these two classes the hypernormality is a consequence of the defining structure equations for the manifolds, thus generalizing the analogous results for 3-Sasaki and 3-cosymplectic manifolds. This is obtained by using the following Lemma:

**Lemma 1** ([1]). *Let  $(M, \varphi_i, \xi_i, \eta_i, g)$  be an almost 3-contact metric manifold. Then the following formula holds  $\forall X, Y, Z \in \mathfrak{X}(M)$ :*

$$\begin{aligned} g(N_{\varphi_i}(X, Y), Z) = & \\ = & -d\Phi_j(X, Y, \varphi_j Z) + d\Phi_j(\varphi_i X, \varphi_i Y, \varphi_j Z) + d\Phi_k(X, \varphi_i Y, \varphi_j Z) + d\Phi_k(\varphi_i X, Y, \varphi_j Z) \\ & - \eta_i(X)[d\eta_j(\varphi_i Y, \varphi_j Z) + d\eta_k(Y, \varphi_j Z)] + \eta_i(Y)[d\eta_j(\varphi_i X, \varphi_j Z) + d\eta_k(X, \varphi_j Z)] \\ & + \eta_j(Z)[d\eta_j(X, Y) - d\eta_j(\varphi_i X, \varphi_i Y)] - \eta_j(Z)[d\eta_k(X, \varphi_i Y) + d\eta_k(\varphi_i X, Y)]. \end{aligned} \quad (9)$$

In the following we will be concerned with various classes of almost 3-contact metric manifolds where the three Reeb vector fields are all Killing. In this case one can show that there exists a function  $\delta \in C^\infty(M)$  such that

$$\eta_r([\xi_s, \xi_t]) = 2\delta\epsilon_{rst}, \quad r, s, t = 1, 2, 3$$

where  $\epsilon_{rst}$  is the totally skew-symmetric symbol, or equivalently  $d\eta_r(\xi_s, \xi_t) = -2\delta\epsilon_{rst}$ . We call  $\delta$  a *Reeb commutator function*, we refer to [1] for more information on this notion.

### 3. 3- $(\alpha, \delta)$ -Sasaki Manifolds and 3- $\delta$ -Cosymplectic Manifolds

This section is a short review of 3- $(\alpha, \delta)$ -Sasaki manifolds and 3- $\delta$ -cosymplectic manifolds. These were discussed in detail in [1,17].

**Definition 1.** An almost 3-contact metric manifold  $(M, \varphi_i, \xi_i, \eta_i, g)$  is called a  $3-(\alpha, \delta)$ -Sasaki manifold if it satisfies

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k$$

for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ , where  $\alpha \neq 0$  and  $\delta$  are real constants.

When  $\alpha = \delta = 1$ , we have a 3-contact metric manifold, and hence a 3-Sasaki manifold by Kashiwada's theorem [15]. In the following, when considering  $3-(\alpha, \delta)$ -Sasaki manifolds we will always mean  $\alpha \neq 0$ . As an immediate consequences of the definition one obtains the following properties:

1. Each  $\xi_i$  is an infinitesimal automorphism of the distribution  $\mathcal{H}$ , i.e.

$$d\eta_r(X, \xi_s) = 0 \quad X \in \Gamma(\mathcal{H}), \quad r, s = 1, 2, 3;$$

2. The constant  $\delta$  is the Reeb commutator function;
3. The differentials  $d\Phi_i$  are given by

$$d\Phi_i = 2(\delta - \alpha)(\eta_k \wedge \Phi_j - \eta_j \wedge \Phi_k).$$

Applying Lemma 1 one shows the following

**Theorem 1** ([1], Theorem 2.2.1). Any  $3-(\alpha, \delta)$ -Sasaki manifold is hypernormal.

In particular, a  $3-(\alpha, \delta)$ -Sasaki manifold with  $\alpha = \delta$  is  $3-\alpha$ -Sasaki. It can be also shown that the vertical distribution of any  $3-(\alpha, \delta)$ -Sasaki manifold is integrable with totally geodesic leaves and each Reeb vector field  $\xi_i$  is Killing.

We can distinguish three main classes of  $3-(\alpha, \delta)$ -Sasaki manifolds. A  $3-(\alpha, \delta)$ -Sasaki manifold is called *degenerate* if  $\delta = 0$  and *non-degenerate* otherwise. Quaternionic Heisenberg groups are examples of degenerate  $3-(\alpha, \delta)$ -Sasaki manifolds (see ([1], Example 2.3.2)). Considering an  $\mathcal{H}$ -homothetic deformation of a  $3-(\alpha, \delta)$ -Sasaki structure, as in (5), one can verify that the obtained structure  $(\varphi', \xi', \eta', g')$  is a  $3-(\alpha', \delta')$ -Sasaki with

$$\alpha' = \alpha \frac{c}{a}, \quad \delta' = \frac{\delta}{c}.$$

In particular,  $\mathcal{H}$ -homothetic deformations preserve the class of degenerate manifolds. In the nondegenerate case, one sees immediately that  $\alpha' \delta'$  has the same sign as  $\alpha \delta$ . This justifies the distinction between *positive*  $3-(\alpha, \delta)$ -Sasaki manifolds, with  $\alpha \delta > 0$ , and *negative*  $3-(\alpha, \delta)$ -Sasaki manifolds, with  $\alpha \delta < 0$ . In fact, it can be shown that a  $3-(\alpha, \delta)$ -Sasaki manifold is positive if and only if it is  $\mathcal{H}$ -homothetic to a 3-Sasaki manifold, and negative if and only if it is  $\mathcal{H}$ -homothetic to a  $3-(\alpha', \delta')$ -Sasaki manifold with  $\alpha' = -1$ ,  $\delta' = 1$ .

Examples of negative  $3-(\alpha, \delta)$ -Sasaki manifolds can be obtained in the following way. It is known that quaternionic Kähler (not hyper-Kähler) manifolds with negative scalar curvature admit a canonically associated principal  $SO(3)$ -bundle  $P(M)$  which is endowed with a *negative 3-Sasaki structure* [18,19]. This is a 3-structure  $(\varphi_i, \xi_i, \eta_i, \tilde{g})$ ,  $i = 1, 2, 3$ , where  $(\varphi_i, \xi_i, \eta_i)$  is a normal almost 3-contact structure, and  $\tilde{g}$  is a compatible semi-Riemannian metric, with signature  $(3, 4n)$ , where  $4n$  is the dimension of the base space, and  $d\eta_i(X, Y) = 2\tilde{g}(X, \varphi_i Y)$ . Then, one can define the Riemannian metric

$$g = -\tilde{g} + 2 \sum_{i=1}^3 \eta_i \otimes \eta_i,$$

which is compatible with the structure  $(\varphi_i, \xi_i, \eta_i)$ , and satisfies  $d\eta_i = -2\Phi_i - 4\eta_j \wedge \eta_k$ , where  $\Phi_i(X, Y) = g(X, \varphi_i Y)$  (see also [19]). Therefore  $(\varphi_i, \xi_i, \eta_i, g)$  is a  $3-(\alpha, \delta)$ -Sasaki structure with  $\alpha = -1$  and  $\delta = 1$ .

The following Theorem regarding the transverse geometry with respect to the vertical foliation of a  $3-(\alpha, \delta)$ -Sasaki manifold is proved in [17]:

**Theorem 2.** Any  $3-(\alpha, \delta)$ -Sasaki manifold  $M$  admits a locally defined Riemannian submersion  $\pi: M \rightarrow N$  along its horizontal distribution  $\mathcal{H}$  such that  $N$  carries a quaternionic Kähler structure given by

$$\check{\varphi}_i = \pi_* \circ \varphi_i \circ s_*, \quad i = 1, 2, 3,$$

where  $s: U \rightarrow M$  is any local smooth section of the Riemannian submersion. The covariant derivatives of the almost complex structures  $\check{\varphi}_i$  are given by

$$\nabla_X^{\check{\varphi}_i} \check{\varphi}_i = 2\delta(\check{\eta}_k(X)\check{\varphi}_j - \check{\eta}_j(X)\check{\varphi}_k)$$

where  $\check{\eta}_i(X) = \eta_i(s_*X) \circ s$  for  $i = 1, 2, 3$ . The scalar curvature of the base space  $N$  is  $16n(n+2)\alpha\delta$ .

The Riemannian Ricci tensor of any  $3-(\alpha, \delta)$ -Sasaki manifold is computed in [1]:

$$\text{Ric}^g = 2\alpha(2\delta(n+2) - 3\alpha)g + 2(\alpha - \delta)((2n+3)\alpha - \delta) \sum_{i=1}^3 \eta_i \otimes \eta_i. \quad (10)$$

In particular, a  $3-(\alpha, \delta)$ -Sasaki manifold is Riemannian Einstein if and only if  $\delta = \alpha$ , in which case the structure is  $3-\alpha$ -Sasaki, or  $\delta = (2n+3)\alpha$ .

Notice that, by Theorem 2, a non-degenerate  $3-(\alpha, \delta)$ -Sasaki manifold locally fibers over a quaternionic Kähler space of positive or negative scalar curvature, according to  $\alpha\delta > 0$  or  $\alpha\delta < 0$ , respectively. In [17] a systematic study of homogeneous non-degenerate  $3-(\alpha, \delta)$ -Sasaki manifolds has been carried out, obtaining a complete classification in the positive case, where the base space of the homogeneous fibration turns out to be a symmetric Wolf space. In the case  $\alpha\delta < 0$ , one can provide a general construction of homogeneous  $3-(\alpha, \delta)$ -Sasaki manifolds fibering over nonsymmetric Alekseevsky spaces.

We recall now the definition and some basic facts on  $3-\delta$ -cosymplectic manifolds.

**Definition 2.** A  $3-\delta$ -cosymplectic manifold is an almost 3-contact metric manifold satisfying

$$d\eta_i = -2\delta\eta_j \wedge \eta_k, \quad d\Phi_i = 0,$$

for some  $\delta \in \mathbb{R}$  and for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ .

When  $\delta = 0$ , the fact that the forms  $\eta_i$  and  $\Phi_i$  are all closed implies that the structure is hypernormal ([16], Theorem 4.13). In fact this immediately follows from (9). Therefore, a  $3-\delta$ -cosymplectic manifold with  $\delta = 0$  is 3-cosymplectic. In particular, it is 3-quasi-Sasaki and the Reeb vector fields are all Killing. The local structure of these manifolds is described by the following:

**Proposition 1** ([12]). Any 3-cosymplectic manifold of dimension  $4n+3$  is locally the Riemannian product of a hyper-Kähler manifold of dimension  $4n$  and a 3-dimensional flat abelian Lie group.

As a consequence, since every hyper-Kähler manifold is Ricci flat, even the Riemannian Ricci tensor of any 3-cosymplectic manifold vanishes.

As regards  $3-\delta$ -cosymplectic manifolds with  $\delta \neq 0$ , even in this case one can show that the structure is hypernormal, the Reeb vector fields are Killing, and the manifold locally decomposes as a Riemannian product [1]. In particular,

**Proposition 2.** Any  $3-\delta$ -cosymplectic manifold with  $\delta \neq 0$  is locally the Riemannian product of a hyper-Kähler manifold and a 3-dimensional Lie group isomorphic to  $\text{SO}(3)$ , with constant curvature  $\delta^2$ . Consequently, the Riemannian Ricci tensor is  $\text{Ric}^g = 2\delta^2 \sum_{i=1}^3 \eta_i \otimes \eta_i$ .

In both cases, i.e.,  $\delta = 0$  or  $\delta \neq 0$ , the hyper-Kähler manifold is tangent to the horizontal distribution, while the 3-dimensional Lie group is tangent to the vertical distribution. In fact, examples of these manifolds can be obtained taking Riemannian products  $N \times G$ , where  $(N, J_i, h)$ ,  $i = 1, 2, 3$ , is a hyper-Kähler manifold, and  $G$  is a 3-dimensional Lie group, which is either abelian, or isomorphic to  $SO(3)$ . If  $\xi_1, \xi_2, \xi_3$  are generators of the Lie algebra  $\mathfrak{g}$  of  $G$ , satisfying  $[\xi_i, \xi_j] = 2\delta\xi_k$ ,  $\delta \in \mathbb{R}$ , then one can define in a natural manner an almost 3-contact metric structure  $(\varphi_i, \xi_i, \eta_i, g)$  on the product  $N \times G$ , setting

$$\begin{aligned}\varphi_i|_{TN} &= J_i, & \varphi_i\xi_i &= 0, & \varphi_i\xi_j &= \xi_k, & \varphi_i\xi_k &= -\xi_j, \\ \eta_i|_{TN} &= 0, & \eta_i(\xi_i) &= 1, & \eta_i(\xi_j) &= \eta_i(\xi_k) &= 0,\end{aligned}$$

and  $g$  the product metric of  $h$  and the left invariant Riemannian metric on  $G$  with respect to which  $\xi_1, \xi_2, \xi_3$  are an orthonormal basis of  $\mathfrak{g}$ .

For a comparison with the class of  $3-(0, \delta)$ -Sasaki manifolds, which will be introduced in the next section, it is worth remarking that for a  $3-\delta$ -cosymplectic manifold  $(M, \varphi_i, \xi_i, \eta_i, g)$  the Lie derivatives of the structure tensor fields  $\varphi_i$ ,  $i = 1, 2, 3$  with respect to the Reeb vector fields are given by

$$\mathcal{L}_{\xi_i}\varphi_i = 0, \quad \mathcal{L}_{\xi_i}\varphi_j = 2\delta(\eta_i \otimes \xi_j - \eta_j \otimes \xi_i) = -\mathcal{L}_{\xi_j}\varphi_i \quad (11)$$

for every  $i, j = 1, 2, 3$ . Indeed, in a  $3-\delta$ -cosymplectic manifold the Levi-Civita connection satisfies ([1], Proposition 2.1.1):

$$\nabla_{\xi_i}^g \varphi_i = 0,$$

$$(\nabla_{\xi_i}^g \varphi_j)X = \delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) = -(\nabla_{\xi_j}^g \varphi_i)X,$$

$$\nabla_X^g \xi_i = \delta(\eta_k(X)\xi_j - \eta_j(X)\xi_k),$$

where  $(i, j, k)$  is an even permutation of  $(1, 2, 3)$  and  $X \in \mathfrak{X}(M)$ . Therefore,

$$\begin{aligned}(\mathcal{L}_{\xi_i}\varphi_i)X &= [\xi_i, \varphi_i X] - \varphi_i[\xi_i, X] \\ &= \nabla_{\xi_i}^g(\varphi_i X) - \nabla_{\varphi_i X}^g \xi_i - \varphi_i(\nabla_{\xi_i}^g X) + \varphi_i(\nabla_X^g \xi_i) \\ &= (\nabla_{\xi_i}^g \varphi_i)X - \nabla_{\varphi_i X}^g \xi_i + \varphi_i(\nabla_X^g \xi_i) \\ &= -\delta(\eta_k(\varphi_i X)\xi_j - \eta_j(\varphi_i X)\xi_k) + \delta(\eta_k(X)\varphi_i \xi_j - \eta_j(X)\varphi_i \xi_k) = 0.\end{aligned}$$

In the same way,

$$\begin{aligned}(\mathcal{L}_{\xi_i}\varphi_j)X &= (\nabla_{\xi_i}^g \varphi_j)X - \nabla_{\varphi_j X}^g \xi_i + \varphi_j(\nabla_X^g \xi_i) \\ &= \delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) - \delta\eta_k(\varphi_j X)\xi_j - \delta\eta_j(X)\varphi_j \xi_k \\ &= 2\delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) = -(\mathcal{L}_{\xi_j}\varphi_i)X.\end{aligned}$$

#### 4. $3-(0, \delta)$ -Sasaki Manifolds

In this section we introduce the class of  $3-(0, \delta)$ -Sasaki manifolds.

**Definition 3.** An almost 3-contact metric manifold  $(M, \varphi_i, \xi_i, \eta_i, g)$  will be called  $3-(0, \delta)$ -Sasaki manifold if

$$d\eta_i = -2\delta\eta_j \wedge \eta_k, \quad d\Phi_i = -2\delta(\eta_j \wedge \Phi_k - \eta_k \wedge \Phi_j) \quad (12)$$

for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ , and for some real constant  $\delta \in \mathbb{R}$ .

In particular, the structure is not 3-quasi-Sasaki when  $\delta \neq 0$ , and we have the following basic properties for a  $3-(0, \delta)$ -Sasaki manifold:

1. The horizontal distribution  $\mathcal{H}$  is integrable;

2. Each  $\xi_i$  is an infinitesimal automorphism of the distribution  $\mathcal{H}$ , i.e.

$$d\eta_r(X, \xi_s) = 0 \quad X \in \Gamma(\mathcal{H}), \quad r, s = 1, 2, 3;$$

3. The constant  $\delta$  is the Reeb commutator function.

**Remark 1.** In case  $\delta \neq 0$ , the two equations in (12) are not completely independent. Indeed, if one assumes  $d\Phi_i = -2\gamma(\eta_j \wedge \Phi_k - \eta_k \wedge \Phi_j)$ ,  $\gamma \in \mathbb{R}^*$ , differentiating this equation, and combining with  $d\eta_i = -2\delta\eta_j \wedge \eta_k$ , a straightforward computation gives  $\gamma = \delta$ . Thus, there is no freedom for the choice of constant in the second equation.

If  $(\varphi_i, \xi_i, \eta_i, g)$  is a 3- $(0, \delta)$ -Sasaki structure, applying an  $\mathcal{H}$ -homothetic deformation as in (5), an easy computation using (6) shows that the new structure  $(\varphi'_i, \eta'_i, \xi'_i, g')$  is again 3- $(0, \delta')$ -Sasaki, with  $\delta' = \frac{\delta}{c}$ .

**Example 1.** Consider the abelian Lie algebra  $\mathbb{R}^{4n}$  spanned by vectors  $v_r, v_{n+r}, v_{2n+r}, v_{3n+r}$ ,  $r = 1, \dots, n$ , and endowed with the hypercomplex structure  $\{J_1, J_2, J_3\}$  defined by

$$J_i(v_r) = v_{in+r}, \quad J_i(v_{in+r}) = -v_r, \quad J_i(v_{jn+r}) = v_{kn+r}, \quad J_i(v_{kn+r}) = -v_{jn+r},$$

for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ . Let us consider also the Lie algebra  $\mathfrak{so}(3)$  spanned by  $\xi_1, \xi_2, \xi_3$  with Lie brackets given by  $[\xi_i, \xi_j] = 2\delta\xi_k$ ,  $\delta \neq 0$ . Let  $\rho$  be the representation of  $\mathfrak{so}(3)$  on  $\mathbb{R}^{4n}$  given by

$$\rho : \mathfrak{so}(3) \rightarrow \mathfrak{gl}(4n, \mathbb{R}), \quad \rho(\xi_i) = \delta J_i, \quad i = 1, 2, 3.$$

On the Lie algebra  $\mathfrak{g} = \mathfrak{so}(3) \ltimes_{\rho} \mathbb{R}^{4n}$  on can define in a natural way an almost 3-contact metric structure  $(\varphi_i, \xi_i, \eta_i, g)$ , with

$$\varphi_i|_{\mathbb{R}^{4n}} = J_i, \quad \varphi_i(\xi_i) = 0, \quad \varphi_i(\xi_j) = \xi_k = -\varphi_j(\xi_k),$$

$$\eta_i|_{\mathbb{R}^{4n}} = 0, \quad \eta_i(\xi_i) = 1, \quad \eta_i(\xi_j) = \eta_i(\xi_k) = 0,$$

and where  $g$  is the inner product such that the vectors  $\xi_i, v_l$ ,  $i = 1, 2, 3$ ,  $l = 1, \dots, 4n$  are orthonormal. In particular, the non zero brackets on  $\mathfrak{g}$  are given by

$$[\xi_i, \xi_j] = 2\delta\xi_k, \quad [\xi_i, X] = \delta\varphi_i(X), \quad X \in \mathbb{R}^{4n}.$$

The representation  $\rho : \mathfrak{so}(3) \rightarrow \mathfrak{gl}(4n, \mathbb{R})$  can be integrated to a representation  $\tilde{\rho} : \mathrm{SO}(3) \rightarrow \mathrm{GL}(4n, \mathbb{R})$ . Therefore, identifying  $\mathbb{R}^{4n}$  with  $\mathbb{H}^n$  in a natural way, the simply connected Lie group  $G = \mathrm{SO}(3) \ltimes_{\tilde{\rho}} \mathbb{H}^n$ , with Lie algebra  $\mathfrak{g}$ , admits a left invariant almost 3-contact metric structure  $(\varphi_i, \xi_i, \eta_i, g)$ . One can easily verify that this structure satisfies (12).

**Remark 2.** For more details on the above example we refer to [2], where  $\mathfrak{g}$  is described as a remarkable example of a Lie algebra endowed with an abelian almost 3-contact metric structure. In fact, the structure defined on  $\mathfrak{g}$  belongs to the class of canonical abelian structures, so that the Lie group  $G$  admits a unique metric connection with totally skew symmetric torsion  $\nabla$  such that

$$\nabla_X \varphi_i = 2\delta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k)$$

for every vector field  $X$  and for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ . The torsion of the canonical connection  $\nabla$  is  $T = 2\delta\eta_1 \wedge \eta_2 \wedge \eta_3$ , which satisfies  $\nabla T = 0$ .

It is also shown in [2] that the Lie group  $G$  admits co-compact discrete subgroups, so that the corresponding compact quotients admit almost 3-contact metric structures of the same type.

**Proposition 3.** Let  $(M, \varphi_i, \xi_i, \eta_i, g)$  be a 3- $(0, \delta)$ -Sasaki manifold. Then the structure is hypernormal.

**Proof.** In order to compute the tensor fields  $N_{\varphi_i}$ , we apply Lemma 1. We always denote by  $X, Y, Z$  horizontal vector fields and by  $(i, j, k)$  an even permutation of  $(1, 2, 3)$ .

Being  $d\Phi_i(X, Y, Z) = 0$ , then  $N_{\varphi_i}(X, Y, Z) = 0$  for every  $i = 1, 2, 3$ . Furthermore, since the horizontal distribution is integrable, by the definition of the tensor field  $N_{\varphi_i}$  (see (4)), one has  $N_{\varphi_i}(X, Y, \xi_r) = 0$  for all  $r = 1, 2, 3$ . Notice that, since

$$\xi_i \lrcorner \Phi_i = 0, \quad \xi_j \lrcorner \Phi_i = -\eta_k, \quad \xi_k \lrcorner \Phi_i = \eta_j,$$

from the second equation in (12), we have,

$$\xi_i \lrcorner d\Phi_i = 0, \quad \xi_j \lrcorner d\Phi_i = -2\delta(\Phi_k + \eta_{ij}), \quad \xi_k \lrcorner d\Phi_i = 2\delta(\Phi_j + \eta_{ki}). \quad (13)$$

Therefore, from Lemma 1, applying (12) and (13), we have

$$\begin{aligned} N_{\varphi_i}(X, \xi_i, Z) &= -d\Phi_j(X, \xi_i, \varphi_j Z) + d\Phi_k(\varphi_i X, \xi_i, \varphi_j Z) + d\eta_j(\varphi_i X, \varphi_j Z) + d\eta_k(X, \varphi_j Z) \\ &= -2\delta\Phi_k(\varphi_j Z, X) - 2\delta\Phi_j(\varphi_j Z, \varphi_i X) \\ &= 2\delta\Phi_j(\varphi_i X, \varphi_j Z) + 2\delta\Phi_k(X, \varphi_j Z) = -2\delta g(\varphi_i X, Z) - 2\delta g(X, \varphi_i Z) = 0, \\ N_{\varphi_i}(X, \xi_j, Z) &= d\Phi_j(\varphi_i X, \xi_k, \varphi_j Z) + d\Phi_k(\varphi_i X, \xi_j, \varphi_j Z) \\ &= -2\delta\Phi_i(\varphi_j Z, \varphi_i X) + 2\delta\Phi_i(\varphi_j Z, \varphi_i X) = 0, \\ N_{\varphi_i}(X, \xi_k, Z) &= -d\Phi_j(X, \xi_k, \varphi_j Z) - d\Phi_k(X, \xi_j, \varphi_j Z) \\ &= 2\delta\Phi_i(\varphi_j Z, X) - 2\delta\Phi_i(\varphi_j Z, X) = 0. \end{aligned}$$

Equations (13) implies  $d\Phi_r(X, \xi_s, \xi_t) = 0$  for every  $r, s, t = 1, 2, 3$  and  $X \in \Gamma(\mathcal{H})$ . Taking also into account that  $d\eta_r(X, \xi_s) = 0$ , we deduce from (9) that

$$N_{\varphi_r}(X, \xi_s, \xi_t) = N_{\varphi_r}(\xi_s, \xi_t, X) = 0.$$

Finally, (9) implies together with  $d\eta_r(\xi_s, \xi_t) = -2\delta\epsilon_{rst}$  that

$$N_{\varphi_i}(\xi_i, \xi_j, \xi_k) = N_{\varphi_i}(\xi_i, \xi_k, \xi_j) = N_{\varphi_i}(\xi_j, \xi_k, \xi_i) = 0,$$

completing the proof that  $M$  is hypernormal.  $\square$

**Proposition 4.** Let  $(M, \varphi_i, \xi_i, \eta_i, g)$  be a 3-(0,  $\delta$ )-Sasaki manifold. Then the Levi-Civita connection satisfies for all  $X, Y \in \mathfrak{X}(M)$  and any cyclic permutation  $(i, j, k)$  of  $(1, 2, 3)$ :

$$\begin{aligned} (\nabla_X^g \varphi_i)Y &= 2\delta [\eta_k(X)\varphi_j Y - \eta_j(X)\varphi_k Y] \\ &\quad - \delta [\eta_j(X)\eta_j(Y) + \eta_k(X)\eta_k(Y)]\xi_i + \delta \eta_i(Y) [\eta_j(X)\xi_j + \eta_k(X)\xi_k] \end{aligned} \quad (14)$$

and

$$\nabla_X^g \xi_i = \delta (\eta_k(X)\xi_j - \eta_j(X)\xi_k), \quad (15)$$

$$\nabla_{\xi_i}^g \xi_i = 0, \quad \nabla_{\xi_i}^g \xi_j = -\nabla_{\xi_j}^g \xi_i = \delta \xi_k. \quad (16)$$

In particular, each  $\xi_i$  is a Killing vector field.

**Proof.** Since the structure is hypernormal, by ([3], Lemma 6.1), the Levi-Civita connection satisfies

$$\begin{aligned} 2g((\nabla_X^g \varphi_i)Y, Z) &= d\Phi_i(X, \varphi_i Y, \varphi_i Z) - d\Phi_i(X, Y, Z) \\ &\quad + d\eta_i(\varphi_i Y, X)\eta_i(Z) - d\eta_i(\varphi_i Z, X)\eta_i(Y). \end{aligned} \quad (17)$$

Further, an easy computation (see [1]) shows that for every cyclic permutation  $(i, j, k)$  of  $(1, 2, 3)$ ,

$$\begin{aligned}\Phi_j(\varphi_i X, \varphi_i Y) &= -\Phi_j(X, Y) - (\eta_k \wedge \eta_i)(X, Y), \\ \Phi_k(\varphi_i X, \varphi_i Y) &= -\Phi_k(X, Y) - (\eta_i \wedge \eta_j)(X, Y), \\ \Phi_j(\varphi_i X, Y) &= -\Phi_k(X, Y) - \eta_i(X)\eta_j(Y), \\ \Phi_k(\varphi_i X, Y) &= \Phi_j(X, Y) - \eta_i(X)\eta_k(Y).\end{aligned}$$

Then, using the second equation in (12) and the above equations, we have

$$\begin{aligned}d\Phi_i(X, \varphi_i Y, \varphi_i Z) &= \\ &= -2\delta[\eta_j(X)\Phi_k(\varphi_i Y, \varphi_i Z) + \eta_j(\varphi_i Y)\Phi_k(\varphi_i Z, X) + \eta_j(\varphi_i Z)\Phi_k(X, \varphi_i Y) \\ &\quad - \eta_k(X)\Phi_j(\varphi_i Y, \varphi_i Z) - \eta_k(\varphi_i Y)\Phi_j(\varphi_i Z, X) - \eta_k(\varphi_i Z)\Phi_j(X, \varphi_i Y)] \\ &= -2\delta[-\eta_j(X)\Phi_k(Y, Z) - \eta_j(X)(\eta_i \wedge \eta_j)(Y, Z) \\ &\quad - \eta_k(Y)\Phi_j(Z, X) + \eta_k(Y)\eta_i(Z)\eta_k(X) + \eta_k(Z)\Phi_j(Y, X) - \eta_k(Z)\eta_i(Y)\eta_k(X) \\ &\quad + \eta_k(X)\Phi_j(Y, Z) + \eta_k(X)(\eta_k \wedge \eta_i)(Y, Z) \\ &\quad + \eta_j(Y)\Phi_k(Z, X) + \eta_j(Y)\eta_i(Z)\eta_j(X) - \eta_j(Z)\Phi_k(Y, X) - \eta_j(Z)\eta_i(Y)\eta_j(X)] \\ &= d\Phi_i(X, Y, Z) + 4\delta[\eta_j(X)\Phi_k(Y, Z) - \eta_k(X)\Phi_j(Y, Z)] \\ &\quad + 4\delta\eta_j(X)[\eta_i(Y)\eta_j(Z) - \eta_j(Y)\eta_i(Z)] \\ &\quad + 4\delta\eta_k(X)[\eta_i(Y)\eta_k(Z) - \eta_k(Y)\eta_i(Z)].\end{aligned}$$

On the other hand, again using the first equation in (12), we obtain

$$\begin{aligned}d\eta_i(\varphi_i Y, X)\eta_i(Z) - d\eta_i(\varphi_i Z, X)\eta_i(Y) &= \\ &= -2\delta(\eta_j \wedge \eta_k)(\varphi_i Y, X)\eta_i(Z) + 2\delta(\eta_j \wedge \eta_k)(\varphi_i Z, X)\eta_i(Y) \\ &= -2\delta\eta_i(Z)[- \eta_k(Y)\eta_k(X) - \eta_j(X)\eta_j(Y)] + 2\delta\eta_i(Y)[- \eta_k(Z)\eta_k(X) - \eta_j(X)\eta_j(Z)].\end{aligned}$$

Inserting the above computations in (17), we conclude that

$$\begin{aligned}g((\nabla_X^g \varphi_i)Y, Z) &= 2\delta[\eta_k(X)g(\varphi_j Y, Z) - \eta_j(X)g(\varphi_k Y, Z)] \\ &\quad - \delta\eta_i(Z)[\eta_k(Y)\eta_k(X) + \eta_j(X)\eta_j(Y)] + \delta\eta_i(Y)[\eta_k(Z)\eta_k(X) + \eta_j(X)\eta_j(Z)]\end{aligned}$$

which implies (14). As regards the proof (15), applying (14) for  $Y = \xi_i$ , we get

$$(\nabla_X^g \varphi_i)\xi_i = -\delta(\eta_j(X)\xi_j + \eta_k(X)\xi_k).$$

Applying  $\varphi_i$  on both hand-sides, we obtain (15). Equations (16) are immediate consequences of (15). Furthermore, we also get

$$g(\nabla_X^g \xi_i, Y) = -\delta(\eta_j \wedge \eta_k)(X, Y)$$

for every  $X, Y \in \mathfrak{X}(M)$ . Since  $\nabla^g \xi_i$  is skew-symmetric,  $\xi_i$  is Killing.  $\square$

**Corollary 1.** Let  $(M, \varphi_i, \xi_i, \eta_i, g)$  be a 3-(0,  $\delta$ )-Sasaki manifold. Then for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$  we have

$$\mathcal{L}_{\xi_i} \varphi_i = 0, \quad \mathcal{L}_{\xi_i} \varphi_j = -\mathcal{L}_{\xi_j} \varphi_i = 2\delta\varphi_k. \quad (18)$$

**Proof.** For the first Lie derivative, notice that by (14) we have  $\nabla_{\xi_i}^g \varphi_i = 0$ . Then, applying also (15), for every vector field  $X$  we have

$$\begin{aligned}(\mathcal{L}_{\xi_i} \varphi_i)X &= (\nabla_{\xi_i}^g \varphi_i)X - \nabla_{\varphi_i X}^g \xi_i + \varphi_i(\nabla_X^g \xi_i) \\ &= -\delta(\eta_k(\varphi_i X)\xi_j - \eta_j(\varphi_i X)\xi_k) + \delta(\eta_k(X)\varphi_i \xi_j - \eta_j(X)\varphi_i \xi_k) = 0.\end{aligned}$$

Now, using (14) for the covariant derivative  $\nabla^g \varphi_j$ , for every vector field  $Y$ , we have

$$(\nabla_{\xi_i}^g \varphi_j)Y = 2\delta \varphi_k Y - \delta(\eta_i(Y)\xi_j - \eta_j(Y)\xi_i).$$

Therefore, applying also (15), we get

$$\begin{aligned} (\mathcal{L}_{\xi_i} \varphi_j)X &= (\nabla_{\xi_i}^g \varphi_j)X - \nabla_{\varphi_j X}^g \xi_i + \varphi_j(\nabla_X^g \xi_i) \\ &= 2\delta \varphi_k X - \delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) - \delta\eta_k(\varphi_j X)\xi_j - \delta\eta_j(X)\varphi_j \xi_k \\ &= 2\delta \varphi_k X. \end{aligned}$$

Analogously,  $\mathcal{L}_{\xi_j} \varphi_i = -2\delta \varphi_k$ .  $\square$

**Theorem 3.** Let  $(M, \varphi_i, \xi_i, \eta_i, g)$  be a  $3-(0, \delta)$ -Sasaki manifold. Then both the horizontal and the vertical distribution are integrable with totally geodesic leaves. Each leaf of the vertical distribution is locally isomorphic to the Lie group  $SO(3)$ , with constant sectional curvature  $\delta^2$ ; each leaf of the horizontal distribution is endowed with a hyper-Kähler structure. Consequently, the Riemannian Ricci tensor of  $M$  is given by

$$\text{Ric}^g = 2\delta^2 \sum_{i=1}^3 \eta_i \otimes \eta_i. \quad (19)$$

**Proof.** We already know that the horizontal distribution  $\mathcal{H}$  is integrable. From (15), for every  $X, Y \in \Gamma(\mathcal{H})$  and  $i = 1, 2, 3$ , we have

$$g(\nabla_X^g Y, \xi_i) = -g(\nabla_X^g \xi_i, Y) = 0,$$

so that the distribution  $\mathcal{H}$  has totally geodesic leaves. Furthermore, Equation (16) implies that the vertical distribution  $\mathcal{V}$  is also integrable with totally geodesic leaves. In particular  $[\xi_i, \xi_j] = 2\delta \xi_k$  for an even permutation  $(i, j, k)$  of  $(1, 2, 3)$ , so that the leaves of  $\mathcal{V}$  are locally isomorphic to the Lie group  $SO(3)$  and have constant sectional curvature  $\delta^2$ . On each leaf of the horizontal distribution  $\mathcal{H}$  one can consider the almost hyper-Hermitian structure defined by  $(J_i := \varphi_i|_{\mathcal{H}}, g)$ , which turns out to be hyper-Kähler due to (14). Consequently,  $M$  is locally the Riemannian product of 3-dimensional sphere of curvature  $\delta^2$  and a  $4n$ -dimensional manifold  $M'$ , which is endowed with a hyper-Kähler structure. Since any hyper-Kähler manifold is Ricci flat, we obtain that the Riemannian Ricci tensor of  $M$  is given by (19).  $\square$

**Remark 3.** From Theorem 3 it follows that any  $3-(0, \delta)$ -Sasaki manifold is locally isometric to the Riemannian product of 3-dimensional sphere and a  $4n$ -dimensional manifold  $M'$ , which is endowed with a hyper-Kähler structure. We recall that  $3-\delta$ -cosymplectic manifolds are also locally isometric to the Riemannian product of a 3-dimensional sphere of constant curvature  $\delta^2$  and a hyper-Kähler manifold. Nevertheless, there is a difference between the two geometries. Looking at the transverse geometry of the foliation defined by the vertical distribution  $\mathcal{V}$ , in both cases the Riemannian metric  $g$  is projectable, being the vector fields  $\xi_i$ ,  $i = 1, 2, 3$ , all Killing. In the case of  $3-\delta$ -cosymplectic manifolds, each tensor field  $\varphi_i$  is also projectable, as by (11), the Lie derivatives with respect to the Reeb vector fields satisfy  $(\mathcal{L}_{\xi_i} \varphi_j)X = 0$  for every  $i, j = 1, 2, 3$  and for every horizontal vector field  $X$ . In the case of  $3-(0, \delta)$ -Sasaki manifolds, owing to (18), the tensor fields are not projectable. Nevertheless, taking into account the horizontal parts  $\Phi_i^{\mathcal{H}} := \Phi_i + \eta_j \wedge \eta_k$  of the fundamental 2-forms  $\Phi_i$ , one can verify that horizontal 4-form

$$\Phi_1^{\mathcal{H}} \wedge \Phi_1^{\mathcal{H}} + \Phi_2^{\mathcal{H}} \wedge \Phi_2^{\mathcal{H}} + \Phi_3^{\mathcal{H}} \wedge \Phi_3^{\mathcal{H}}$$

is projectable and defines a transversal quaternionic structure, which turns out to be locally hyper-Kähler.

### 5. Connections with Totally Skew-Symmetric Torsion

In this section we will show that every 3-(0,  $\delta$ )-Sasaki manifold is *canonical* in the sense of the definition given in [1], thus admitting a special metric connection with totally skew-symmetric torsion, called canonical. Recall that a metric connection  $\nabla$  with torsion  $T$  on a Riemannian manifold  $(M, g)$  is said to have *totally skew-symmetric torsion*, or *skew torsion* for short, if the  $(0, 3)$ -tensor field  $T$  defined by  $T(X, Y, Z) := g(T(X, Y), Z)$  is a 3-form. The relation between  $\nabla$  and the Levi-Civita connection  $\nabla^g$  is then given by

$$\nabla_X Y = \nabla_X^g Y + \frac{1}{2} T(X, Y).$$

For more details we refer to [20]. We recall now the definition and the characterization of canonical almost 3-contact metric manifolds.

**Definition 4** ([1]). *An almost 3-contact metric manifold  $(M, \varphi_i, \xi_i, \eta_i, g)$  is called canonical if the following conditions are satisfied:*

- (i) *each  $N_{\varphi_i}$  is totally skew-symmetric on  $\mathcal{H}$ ,*
- (ii) *each  $\xi_i$  is a Killing vector field,*
- (iii) *for any  $X, Y, Z \in \Gamma(\mathcal{H})$  and any  $i, j = 1, 2, 3$ ,*

$$N_{\varphi_i}(X, Y, Z) - d\Phi_i(\varphi_i X, \varphi_i Y, \varphi_i Z) = N_{\varphi_j}(X, Y, Z) - d\Phi_j(\varphi_j X, \varphi_j Y, \varphi_j Z),$$

- (iv)  *$M$  admits a Reeb Killing function  $\beta \in C^\infty(M)$ , that is the tensor fields  $A_{ij}$  defined on  $\mathcal{H}$  by*

$$A_{ij}(X, Y) := g((\mathcal{L}_{\xi_j} \varphi_i)X, Y) + d\eta_j(X, \varphi_i Y) + d\eta_j(\varphi_i X, Y),$$

*satisfy*

$$A_{ii}(X, Y) = 0, \quad A_{ij}(X, Y) = -A_{ji}(X, Y) = \beta \Phi_k(X, Y),$$

*for every  $X, Y \in \Gamma(\mathcal{H})$  and every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ .*

Here  $N_{\varphi_i}$  also denotes the  $(0, 3)$ -tensor field defined by  $N_{\varphi_i}(X, Y, Z) := g(N_{\varphi_i}(X, Y), Z)$  and we say that  $N_{\varphi_i}$  is totally skew-symmetric on  $\mathcal{H}$  if the  $(0, 3)$ -tensor is a 3-form on  $\mathcal{H}$ .

**Theorem 4** ([1]). *An almost 3-contact metric manifold  $(M, \varphi_i, \xi_i, \eta_i, g)$  is canonical, with Reeb Killing function  $\beta$ , if and only if it admits a metric connection  $\nabla$  with skew torsion such that*

$$\nabla_X \varphi_i = \beta(\eta_k(X) \varphi_j - \eta_j(X) \varphi_k)$$

*for every vector field  $X$  on  $M$  and for every even permutation  $(i, j, k)$  of  $(1, 2, 3)$ . If such a connection  $\nabla$  exists, it is unique and its torsion is given by*

$$\begin{aligned} T(X, Y, Z) &= N_{\varphi_i}(X, Y, Z) - d\Phi_i(\varphi_i X, \varphi_i Y, \varphi_i Z), \\ T(X, Y, \xi_i) &= d\eta_i(X, Y), \\ T(X, \xi_i, \xi_j) &= -g([\xi_i, \xi_j], X), \\ T(\xi_1, \xi_2, \xi_3) &= 2(\beta - \delta), \end{aligned}$$

*for every  $X, Y, Z \in \Gamma(\mathcal{H})$ , and  $i, j = 1, 2, 3$ , and where  $\delta$  is the Reeb commutator function.*

The connection  $\nabla$  is called the *canonical connection* of  $M$ , and also satisfies

$$\nabla_X \xi_i = \beta(\eta_k(X) \xi_j - \eta_j(X) \xi_k), \quad \nabla_X \eta_i = \beta(\eta_k(X) \eta_j - \eta_j(X) \eta_k) \quad (20)$$

for every vector field  $X$  on  $M$ . Therefore, when  $\beta = 0$  the canonical connection parallelizes all the structure tensor fields, in which case we call the almost 3-contact metric manifold *parallel*.

Both  $3-(\alpha, \delta)$ -Sasaki manifolds and  $3-\delta$ -cosymplectic manifolds turn out to be canonical. In particular,

**Theorem 5 ([1]).** *Every  $3-(\alpha, \delta)$ -Sasaki manifold is a canonical almost 3-contact metric manifold, with constant Reeb Killing function  $\beta = 2(\delta - 2\alpha)$ . The torsion  $T$  of the canonical connection  $\nabla$  is given by*

$$T = \sum_{i=1}^3 \eta_i \wedge d\eta_i + 8(\delta - \alpha) \eta_{123} = 2\alpha \sum_{i=1}^3 \eta_i \wedge \Phi_i^{\mathcal{H}} + 2(\delta - 4\alpha) \eta_{123}$$

and satisfies  $\nabla T = 0$ .

We denote by  $\eta_{123}$  the 3-form  $\eta_1 \wedge \eta_2 \wedge \eta_3$ . From the above theorem, it follows that any  $3-(\alpha, \delta)$ -Sasaki manifold is a parallel canonical manifold if and only if  $\delta = 2\alpha$ , in which case the  $3-(\alpha, \delta)$ -Sasaki structure is positive ( $\alpha\delta > 0$ ).

Regarding  $3-\delta$ -cosymplectic manifolds, we have:

**Proposition 5 ([1]).** *Any  $3-\delta$ -cosymplectic manifold is a parallel canonical almost 3-contact metric manifold. The torsion of the canonical connection is given by*

$$T = -2\delta \eta_{123}.$$

For the class of  $3-(0, \delta)$ -Sasaki manifolds, we have the following

**Proposition 6.** *Every  $3-(0, \delta)$ -Sasaki manifold is a canonical almost 3-contact metric manifold, with constant Reeb Killing function  $\beta = 2\delta$ . The torsion  $T$  of the canonical connection  $\nabla$  is given by*

$$T = 2\delta \eta_{123},$$

which satisfies  $\nabla T = 0$ .

**Proof.** Let  $(M, \varphi_i, \xi_i, \eta_i, g)$  be a  $3-(0, \delta)$ -Sasaki manifold. We showed that the structure is hypernormal and the Reeb vector fields are Killing. Furthermore, by the second equation in (12),  $d\Phi_i(X, Y, Z) = 0$  for every  $X, Y, Z \in \Gamma(\mathcal{H})$ . Therefore, conditions (i), (ii) and (iii) in Definition 4 are easily verified. As regards condition (iv), applying the first equation in (4) and Corollary 1, for every  $X, Y \in \Gamma(\mathcal{H})$  we have

$$A_{ii}(X, Y) = 0, \quad A_{ij}(X, Y) = -A_{ji}(X, Y) = 2\delta \Phi_k(X, Y).$$

Hence, the structure is canonical with Reeb commutator function  $\beta = 2\delta$ . Now, by Theorem 4, taking also into account the fact that the vertical distribution is integrable, the only non-vanishing term of the canonical connection is  $T(\xi_1, \xi_2, \xi_3) = 2\delta$ , so that  $T = 2\delta \eta_{123}$ . Although the structure is not parallel when  $\delta \neq 0$ , the torsion satisfies  $\nabla T = 0$ , as by (20), the 3-form  $\eta_{123}$  is parallel with respect to  $\nabla$ .  $\square$

The above result generalizes the result obtained in [2] for the Lie group described in Example 1 (see also Remark 2).

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