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Teaching Mathematics at Distance: A Challenge for Universities

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Abstract: The focus of this research is how Sicilian state university mathematics professors faced the challenge of teaching via distance education during the first wave of the COVID-19 pandemic. Since the pandemic entered our lives suddenly, the professors found themselves having to lecture using an e-learning platform that they had never used before, and for which they could not receive training due to the health emergency. In addition to the emotional aspects related to the particular situation of the pandemic, there are two aspects to consider when teaching mathematics at a distance. The first is related to the fact that at university level, lecturers generally teach mathematics in a formal way, using many symbols and formulas that they are used to writing. The second aspect is that the way mathematics is taught is also related to the students to whom the teaching is addressed. In fact, not only online, but also in face-to-face modality, the teaching of mathematics to students on the mathematics degree course involves a different approach to lessons (as well as to the choice of topics to explain) than teaching mathematics in another degree course. In order to investigate how the Sicilian State university mathematics professors taught mathematics at distance, a questionnaire was prepared and administered one month after the beginning of the lockdown in Italy. Both quantitative and qualitative analyses were made, which allowed us to observe the way that university professors have adapted to the new teaching modality: they started to appropriate new artifacts (writing tablets, mathematical software, e-learning platform) to replicate their face-to-face teaching modality, mostly maintaining their blackboard teacher status. Their answers also reveal their beliefs related to teaching mathematics at university level, noting what has been an advantageous or disadvantageous for them in distance teaching.

Keywords: COVID-19 pandemic; university mathematics professors; teaching mathematics at a distance; blackboard teacher; teacher beliefs

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1. Introduction

Once upon a time, there was a world that believed in technology and connections between people, but one day a viral pandemic stopped everything and compelled the entire world to sit down and think. It was the COVID-19 disease, caused by a new virus that started to infect people in China at the end of 2019 and spread worldwide in 2020. From Wuhan (China), the virus, called SARS-CoV-2, spread across nearby countries in Asia first, and then moved across the world. On 20 February, the first infected person was certified in the north of Italy. Within a few days, the Italian government had decided to close schools and universities all over the country, since the virus was spreading further, and the whole of Italy was "closed" in quarantine: stores, restaurants, and factories as well as schools and universities were closed to the public. On 11 March, 12,462 people had been infected by SARS-CoV-2, and 827 people had died in Italy. This number was fated to grow to over one million people. Schools and universities were still closed in May 2020.

In the first week of March 2020, the Italian Ministry of Education, University, and Research called on schools and universities not to abandon students who were quarantined at home and to start distance teaching.

Figure 1 summarizes the number of infected people in Italy by the end of the first Italian quarantine (June 2020).

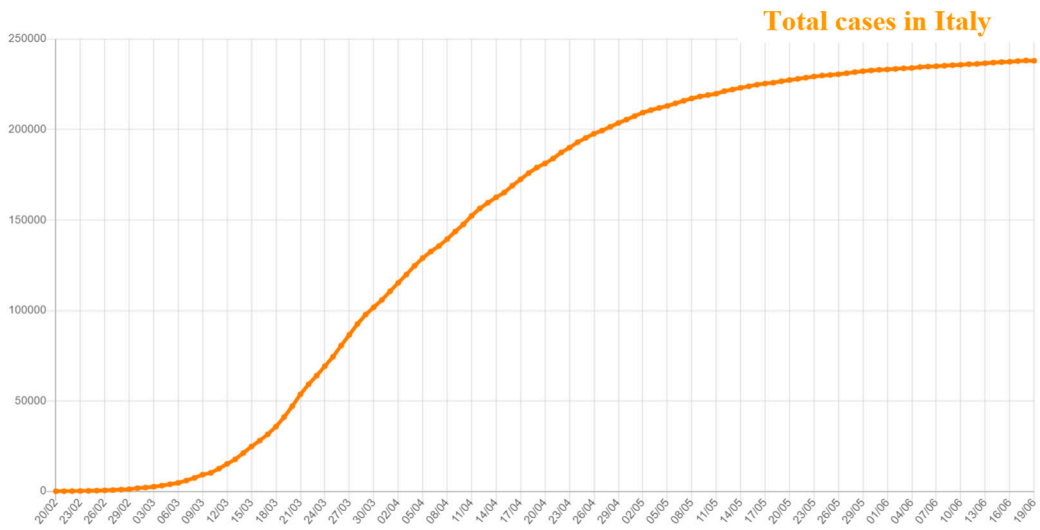

Figure 1. Trend of COVID-19 cases in Italy until June 2020 (taken from https://lab24.ilScheme24.ore.com/coronavirus/ accessed on 30/06/2020).

As shown in Figure 1, there were more than 230,000 infected people out of 60 million inhabitants in Italy in the middle of June 2020 (0.395% of the population). There were not as many cases when distance teaching began, but we thought that the worst was yet to come.

The situation induced by the COVID-19 emergency required us to make an extra effort not only as individuals, but also as educators. We believe that the worldwide emergency requires us to reflect on its consequences for mathematics education and research, in a connected and technological world that suddenly found itself connected only through technology. We used a great deal of technology before the pandemic and believed it was a good mediator of mathematics education between teachers and learners, but we were forced to use only the mediator, and we have to reflect on the importance of the main characters in education: the teachers and learners.

Many studies in mathematics education demonstrate the importance of artifacts, and technological artifacts in particular, becoming instruments to mediate mathematical topics [1–3], but in this situation we were alone with these artifacts and the mediators were far away. Schools in Italy were accustomed to using technological devices (even when teaching face-to-face) and teaching aid platforms. In fact, thanks to law 107/2015 (https://www.gazzettaufficiale.it/eli/id/2015/07/15/15G00122/sg), the Digital Animator was introduced to Italian schools in 2015, which supports headteachers in the design and implementation of digital innovation projects (internal training, involvement of the school community, and the creation of innovative solutions with the use of technology). Nothing similar is planned for Sicilian universities. Moreover, university mathematics courses are traditionally taught by chalk and blackboard, at least in Sicily, where there are universities with ancient traditions (the oldest is in Catania, founded in 1434). The shift to distance learning was therefore a serious challenge for their professors.

The focus of our research is how Sicilian university mathematics professors faced the challenge of teaching via distance education during the COVID-19 pandemic, using a new technology (a distance learning platform) promoted by their universities. In addition to the emotional effects of the pandemic, there are two aspects of teaching mathematics at a

distance to consider. The first, without judging the suitability of this approach, is related to the fact that lecturers at university level generally teach mathematics in a formal way, using many symbols and formulas that they are used to writing for their students to see. The second aspect is that the way mathematics is taught is also related to the students to whom the teaching is addressed. Not only online, but also in face-to-face modality, teaching mathematics to students on the mathematics degree course involves a different approach to lessons (as well as to the choice of topics) than teaching mathematics on another degree course [4].

Our study population comprises all the university professors in the three Sicilian State universities that teach mathematics, on both mathematics and non-mathematics degree courses (e.g., engineering, computer science, etc.). We will specify this in the methodology, but we anticipate that, at least as far as Sicilian universities are concerned, the mathematics or physics degree courses have less students than the other degree courses in which mathematics is taught.

The research questions that guide our study are as follows:

- As they moved to distance teaching, what artifacts did university professors adopt in their didactic transposition of mathematics?
- What effect does the number of students and the university degree course have on how a professor explains mathematics?
- Does distance teaching have disadvantages for teaching mathematics, and if so what?
- Do university mathematics professors gain any advantages in terms of professional skills from a potential crisis? Do opportunities arise from difficulty?

The paper consists of five sections in addition to the introduction. In the next section (Section 2), we present the theoretical background, including features of e-learning environments, the use of artifact in teaching/learning mathematics, the process of objectification, the role of corporeity in the teaching/learning of mathematics, the concept of teacher beliefs, and a teacher's need for virtue when faced with unforeseen situations, such as the COVID-19 pandemic. The methodology that guided the research is presented in Section 3, with a description of the participants involved in the study and an illustration of how the data was collected and analyzed. The results section (Section 4) is dedicated to the exposition of both quantitative and qualitative analyses. The final section (Section 5) is dedicated to discussion and proposes lines for future research.

2. Theoretical Background

With the advent of the pandemic, universities have had to arrange various forms of e-learning in order to complete their teaching activities for the 2019/2020 academic year. E-learning delivers content through electronic information and communications technologies (ICTs). There are several definitions of e-learning. Here we assume the following: ICT is used to support distance teaching/learning processes, based on both e-content delivery and active and/or collaborative learning approaches [5].

To propose and manage e-learning means working within a complex system, involving not only the content to be delivered, but also theoretical models, technological and methodological choices to be made, the use and coordination of human resources, and the integration processes of the organization adopting e-learning as an additional way to acquire new knowledge and skills [5]. There are different kinds of e-learning activities. There are e-learning activities in which materials are provided for self-education (e.g., [6]) and also e-learning activities in virtual spaces able to host collaborative learning communities, organized in real learning groups, or in communities of practice [7], where each member increases their knowledge and skills by building them with the rest of the group, based on the cognitive problems typical of a given profession (e.g., [8,9]).

Issues arise in e-learning in mathematics, regarding the design of effective learning objects, when provided at distance. Borba et al. [10] discuss the use of digital learning objects in mathematics education, in blended teaching, but also in MOOCs and virtual learning environment. "It seems clear that digital technology involves "deconstructing" the

notion of the classroom" ([10], p. 605), and this is more and more true when the classroom does not physically exist anymore.

The management of e-learning activities also involves various figures who are involved with its different aspects: administrative, technological, and didactic. An e-learning system involves the overall management/responsibility of the system, analysis of training needs, path design, teaching management/responsibility for the path, tutoring (network/coaching/mentoring/counselling), knowledge/competence of the content, evaluation of the learning and training process, in-depth knowledge of the e-learning platform, IT administration of the platform, knowledge/competence of the organization's ICT systems, information retrieval, knowledge management, the design and implementation of e-content (graphic story-boarding and multimedia development, didactic communication, etc.), an administrative/operational office, monitoring design and management, quality analysis, and so on [5]. In short, it is a complex system that involves more than simply choosing an e-learning platform in order to achieve the final goal, that is, to create good e-learning paths.

To use technological artifacts (as for other artifacts) in an effective way from an educational point of view means to achieve an *instrumental genesis*, in the sense introduced by Rabarderl [11]. The process can be long and difficult, because it requires two phases, which Rabardel and Samurcay [12] call *instrumentalization* and *instrumentation*. The first phase is concerned with the approach to the artifact, and, for example, with the progressive awareness of its potential and its limits. The second phase, which is deeper, is devoted to rising and developing the artifact's schemes of use, with the appropriation of social utilization schemes for the artifact and/or the arising and development of private schemes. The sudden closure of universities and the switch to distance teaching required efforts to appropriate new artifacts (platforms, devices, etc.) in the senses of both instrumentalization and instrumentation.

We have to consider that "this type of school closure has never happened on this scale before. It will require all stakeholders to rethink how education happens during this emergency scenario and, then beyond. [. . .] Designing the learning experience for students must be differentiated when possible. It is not just as simple as putting your course online. Teachers need to think and choose how they can incorporate a blended learning approach and which tools will best serve their students and pedagogical practice. They need to consider what is accessible and fit for purpose, as well as ways in which to bring connectivity, relationality, and humanity into a distance learning model" ([13], p. 1).

In the particular case of mathematics, we must remember the importance of the embodiment component. Andrà [14] compares the blackboard teacher and the body teacher. The former manages communication through the written word and symbolic language, in which the blackboard is always at the center of attention, because the teacher is writing on it or indicates or underlines a written part on it; for the latter, communication is characterized by an intensive use of iconic and metaphorical gestures, which mainly involve an imaginative and figurative component.

There are several ways to represent mathematical objects: formal mathematical language, iconic representations, etc. The learning process that leads students to recognize the same mathematical object seen from different points of view is called *objectification* [15]. Teaching a discipline like mathematics, made of abstract objects that you cannot touch, often requires the use of metaphors to achieve the objectification of mathematical topics, which students can understand in terms of something already known. According to theorists of the embodied mind [16], people use physical objects or situations to understand complex topics in depth. They specifically use conceptual metaphors: this is not simply a metaphor in the poetic sense of the word, but rather a cognitive mechanism that projects the inferential structure of a source domain into a target domain. It is a map, in the mathematical sense of the word, between the two domains (source and target), preserving all the properties of the two elements corresponding in the map. When using metaphors, it is important that teachers make use of their body and gestures to help students envision

in their mind the new mathematical objects in terms of objects they are familiar with and are able to visualize and manipulate. See [16] for more details. The use of different representations, such as metaphors, in conjunction with formal mathematical language, is useful, because several studies have demonstrated that students have difficulty with the process of objectification [17].

There is a risk of losing the dimension of the body teacher, in favor of the blackboard teacher in the transition from traditional classroom teaching, involving presence, to distance learning, or e-learning. The professors involved in our research were very tied to the blackboard and mostly represented mathematical objects using formal mathematical language. They thus found themselves suddenly deprived of their "safe place".

The ways in which professors are comfortable teaching, their "safe teaching place", are related to their mathematical and pedagogical knowledge and to their beliefs. More than 30 years ago, in Shulman's famous paper "Those who understand" [18], educators pointed out the crucial role of integrating mathematical and pedagogical knowledge. Here we deal only with the Mathematics Teacher's Specialized Knowledge (MTSK) model, introduced by Carrillo et al. [19], inspired by the research of Ball et al. [20] on mathematical knowledge for teaching (MKT). Carrillo and the other authors of the MTSK model discussed the "precise images by which the teacher's practice can be interpreted in the light of those aspects which most influence it, based on the knowledge underlying this practice" ([19], p. 5).

The two domains of the MTSK model are the Mathematical Knowledge and Pedagogical Content Knowledge, divided into various sub-domains as shown in Figure 2. Teacher beliefs are at the center, due to the close relationship between beliefs and the two knowledge domains. By "beliefs" we mean the more or less coherent set of personal truths, mental images, conceptions, meanings, and preferences of teachers introduced by Thompson in 1992 [21], which strongly influence what happens in class and, therefore, student learning. In the MTSK model, beliefs are distinguished (although the boundary is fragile, as shown in Figure 2) as beliefs about math and beliefs about the teaching and learning of math. Beliefs as a mediator between knowledge and practice are discussed in [22], and much work has been published about primary and high school teachers, but the role of beliefs at university level has not yet been fully investigated, as claimed by Mora and Rodriguez [23]. Fukawa-Connely et al. [24] studied teacher beliefs and practices at the tertiary level, and found an interesting contradiction between professors' beliefs about mathematical teaching and their classroom practice: 85% of studied professors (in abstract algebra) used lectures as standard pedagogical practice, 82% answered that lecturing was the best way to teach. However, 56% agreed (plus 26% more who slightly agreed) with the statement "I think students learn better when they do mathematical work (in addition to taking notes and attending to the lecture) in class", showing a mismatch between beliefs about student learning and actual teaching practice. Even when teachers recognize that a constructivist approach is more effective for student learning, they continue to use a transmission-oriented methodology.

If this mismatch is important in a normal situation, then it becomes more serious in an emergency such as that induced by COVID-19, with the sudden immersion in distance teaching.

The new pandemic dimension requires, in the emergency, the even greater involvement by teachers, who must complete their work in new way even without further training. They are called on to activate the creative capacity that Berthoz [25] attributes to our brain, which he calls "vicariance". It is the vicariant component of the human brain that allows people to use multiple and unexpected strategies to achieve a goal, to replace one sense with another (such as when we move in the dark after an accident). Vicariance, says Berthoz, is a simplex principle that refers to the adaptive character of the individual in situations, environments, and interactions with others.

In the following sections, we will see how university professors implemented vicariance for the transposition of knowledge to their students during distance teaching, which

is less structured in an e-learning environment, even when they were usually blackboard teachers, during the COVID-19 pandemic.

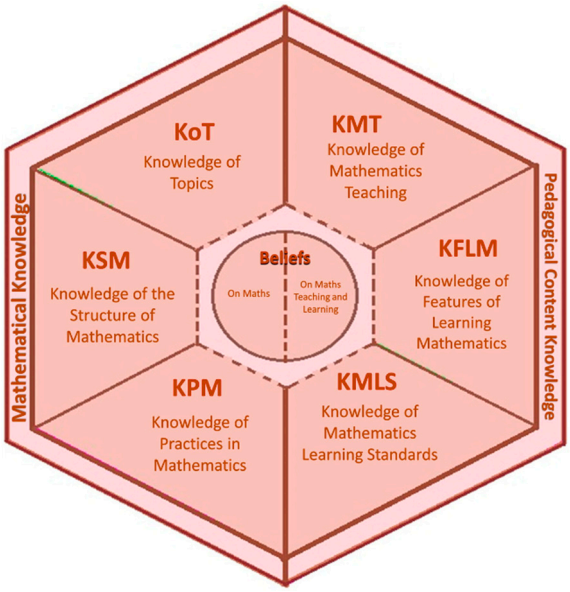

Figure 2. The Mathematics Teacher's Specialized Knowledge model.

3. Methodology

3.1. Participants

The research involved 27 university mathematics professors. They were voluntary respondents to an anonymous questionnaire, addressed to professors who taught mathematics at the three Sicilian state universities (Catania, Messina, and Palermo) in the second semester of the 2019/2020 academic year. The participants represented one-third of the entire population of state university mathematics professors (82) teaching in Sicily, regardless of their semesters of activity [26].

Eighteen of the professors who answered the questionnaire taught only one course, seven taught two courses, and two taught three, for a total of 38 courses. The focus of the questionnaire, as we will specify in more detail in the next section, was to identify strengths or weaknesses highlighted by the shift to distance learning in each course taught. Since those who teach two or three courses teach different subjects to different students, we considered the 38 courses independent of the fact that some were taught by the same person. The data below thus refers to 38 responses.

3.2. Data Collection and Method of Analysis

An anonymous questionnaire to collect data was prepared by the Mathematics Education Research Group (MERG) of the Department of Mathematics and Computer Science at the University of Catania, Italy (which includes Ferrarello, Mammana, Pennisi, and Taranto). It was based on in-depth interviews conducted with experienced professors teaching three different university mathematics courses during the first few weeks of the pandemic (March 2020). This initial step provided data for structuring the different sections and specific contents of the questionnaire.

Three strands of mathematics courses can be identified in Sicilian universities:

(1) Courses as part of mathematics (and possibly physics) degree courses, in which mathematics is quite appreciated by students, generally offered in small classes.
(2) Mathematics courses (analysis, geometry, physical mathematics) in engineering degree courses, in which mathematics is on average appreciated by students, generally offered in large classes (even more than 100 students).

(3) Basic mathematics courses offered for science degree courses (e.g., biology, geology, natural sciences, agriculture, computer science, architecture, economics, etc.), where mathematics is (often) little appreciated by students, generally offered to classes with many students.

The questionnaire was made up of four sections: the first gathered general information; the second investigated teaching habits and emotions related to courses held before the COVID-19 pandemic. The third section investigated new teaching habits and emotions related to the courses held during the COVID-19 pandemic. The fourth and last section included questions about the technologies used for distance learning. Each section contained open-ended, semi-open, closed, and Likert scale questions. We validated the questionnaire by sharing it with a group of six experts in mathematics education and educational psychology. After including their suggestions for modifications to the questionnaire, we conducted a pilot study, administering it to three additional university math professors, who only suggested a few small final changes.

The questionnaire, produced using Google Forms, which is an open source application for online surveys, was administered to the final participants about one month after the start of the lockdown in Italy. The MERG contacted the Directors of the Mathematics Departments of the three Sicilian State universities by email, with the request to distribute the questionnaire to all professors teaching math courses that semester. The data collection lasted one month. The data that we will illustrate in the next section therefore captures a precise moment: April 2020.

The analyses were carried out by MERG, in collaboration with psychologist colleagues (Cassibba, Musso), using Version 24 of the Statistical Package for the Social Sciences (SPSS). The analyses are both quantitative and qualitative. In the quantitative analyses, we generally reported the frequencies of the responses or their cross-tabulation, associated, when appropriate, with nonparametric statistics (i.e., chi-square test, Kruskal-Wallis test, McNemar test, and Median test) given that the data was categorical or ordered categorically. Because of the largely exploratory approach of our study, and to ensure the high sensitivity of statistical tests in the initial phase of our research project, we set the critical p value for significance at 0.10. This a priori choice was also connected to the high sensitivity of the chi-square test to sample size, while the distribution of the variables seemed less problematic considering the use of nonparametric statistics. The responses to the open-ended and semi-open questions were subjected to thematic analysis. The analysis followed the principle of data reduction and the generation of themes, which allowed an in-depth reading of the meaning-making processes related to the closed-ended questions. Specifically, the analysis process was structured in (a) open coding for the generation of the themes; (b) comparison with existing knowledge for the reorganization and grouping of the themes that emerged; and (c) selective coding, to extract illustrative examples of central experiences. The classifications and identification of the themes were carried out by the study authors, reaching a good degree of agreement (k = 0.83); in cases of disagreement, a discussion was initiated, which led, in all cases, to a final shared decision. We used qualitative data in this manuscript to better interpret quantitative results, without their in-depth descriptive analysis.

4. Data and Analysis

The sample of respondents had the following characteristics: 61% were men; 39% were women (Figure 3). Eleven percent were up to 40 years old, 34% were between 41 and 55 years old, and 55% were over 55 years old (Figure 4). This reflects the general state university mathematics professor population in Sicily well in terms of gender and age, with proportional differences under 5% [26].

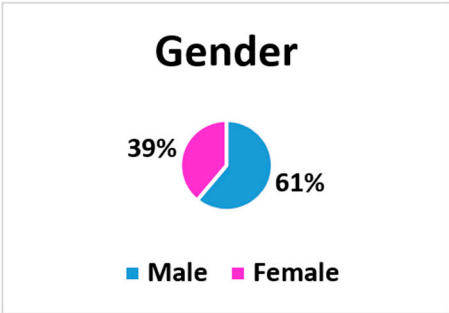

Figure 3. Gender of the sample.

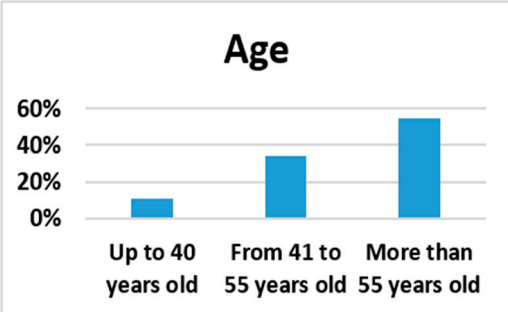

Figure 4. Age of the sample.

Fifty-five percent of respondents taught at the University of Catania, 34% at the University of Messina, and 11% at the University of Palermo (Figure 5). Sixty-eight percent of respondents taught on a Bachelor's degree course and 32% on a Master's degree course. Sixty-six percent taught in Mathematics or Physics, 18% in Engineering, and 16% in "Other" (Figure 6)—the three categories of degree courses specified in the previous section.

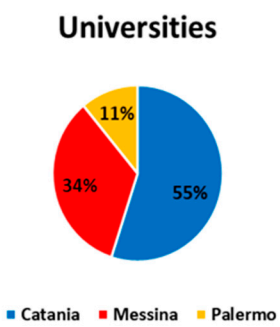

Figure 5. Universities at which the sample teaches.

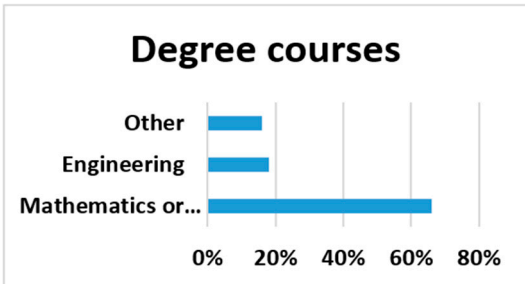

Figure 6. Degree courses of the sample.

Compared to the distribution of mathematics professors across the three state universities in [26], those in the University of Palermo were underrepresented (37% of the population), while those at the University of Catania (41% in the population) and the University of Messina (22% in the population) were overrepresented. These three universities and their professors have very similar characteristics, however, given that the rules for their functioning (e.g., recruitment procedures, number of courses provided, credits per course) are clearly established at both national and regional levels.

Fifty-eight percent of professors had taught their course for less than five years and only 34% had done so for more than 10 years. Before filling in the questionnaire, 37% had taught four to eight distance learning lessons and 63% had taught more than nine distance learning lessons. All respondents gave their lectures at a distance via the Microsoft Teams e-learning platform, as this was chosen by the three universities involved in the study. The didactic offices on Microsoft Teams created as many virtual classrooms (Teams) as courses provided by the university. Table 1 shows the number of students who were members of a Team and the percentage of students regularly attending distance learning, for each degree course. We can see that the majority of degree courses in mathematics or physics (56%) had between 11 and 40 students and 60% of these students attended distance classes assiduously (attendance between 75% and 100%). Most engineering degree courses (57%) had more than 100 students, most of whom (43%) attended distance learning (attendance between 75% and 100%). The other degree courses, including mathematics courses, varied between 41 and 100 students (50%), or over 100 (33%), and 50% of these students had an attendance rate of between 0% and 50%.

Table 1. Cross-tabulation between the number of students who were members of the Teams and the percentage of students who attend distance learning, compared to the degree courses (n = 38).

			% of Distance Learning Students			Total
			From 0 to 50	**From 51 to 75**	**From 76 to 100**	
Mathematics or physics	# students members of Teams	From 1 to 10	0.0%	4.0%	28.0%	32.0%
		From 11 to 40	4.0%	24.0%	28.0%	56.0%
		From 41 to 100	8.0%	0.0%	4.0%	12.0%
		More than 100	0.0%	0.0%	0.0%	0.0%
	Total		12.0%	28.0%	60.0%	100.0%
Engineering	# students members of Teams	From 1 to 10	0.0%	0.0%	0.0%	0.0%
		From 11 to 40	0.0%	0.0%	14.3%	14.3%
		From 41 to 100	0.0%	14.3%	14.3%	28.6%
		More than 100	28.6%	14.3%	14.3%	57.1%
	Total		28.6%	28.6%	42.9%	100.0%
Other	# students members of Teams	From 1 to 10	0.0%	0.0%	0.0%	0.0%
		From 11 to 40	16.7%	0.0%	0.0%	16.7%
		From 41 to 100	0.0%	16.7%	33.3%	50.0%
		More than 100	33.3%	0.0%	0.0%	33.3%
	Total		50.0%	16.7%	33.3%	100.0%
Total	# students members of Teams	From 1 to 10	0.0%	2.6%	18.4%	21.1%
		From 11 to 40	5.3%	15.8%	21.1%	42.1%
		From 41 to 100	5.3%	5.3%	10.5%	21.1%
		More than 100	10.5%	2.6%	2.6%	15.8%
	Total		21.1%	26.3%	52.6%	100.0%

Using a multiple-choice question, we asked the professors how they used to prepare their lessons before the COVID-19 pandemic. The same question was asked in the section "During COVID-19". Table 2 shows that, with the transition to distance learning, the percentage of professors who prepared everything in detail has increased. The chi-square

test, $\chi^2(4) = 22.86$, $p < 0.001$, allows us to confirm that the passage to distance teaching has pushed the professors in a non-random way to better prepare their lessons.

Table 2. Lesson preparation ($n = 38$).

	Before COVID-19 Pandemic	During COVID-19 Pandemic
	Percent	Percent
Everything in detail	39.5	68.4
Basic step ladder	47.4	26.3
No preparation	13.2	5.3
Total	100.0	100.0

Before the COVID-19 pandemic, as Table 3 shows, no professor had ever lectured online. They all lectured at the front of the class using blackboard and chalk. Thirteen percent also used slides (.ppt or .pdf files), and 11% also used mathematical software.

Table 3. How did you previously teach a lesson? (multiple answer options)—($n = 38$).

	Percent
Not online; in front of the class with blackboard and chalk	81.6
Not online; in front of the class with blackboard and chalk; slide	7.9
Not online; in front of the class with blackboard and chalk; slide; mathematical software	5.3
Not online; in front of the class with blackboard and chalk; mathematical software	2.6
Not online; in front of the class with blackboard and chalk; mathematical software; laboratory activities in groups	2.6
Total	100.0

The symbolic and formal writings of mathematics had to be digitized during the COVID-19 pandemic, so we asked the professors how they adapted to this. Table 4 shows that the writing tablet was considered a worthy substitute for the blackboard, and in fact 61% of the professors continued to manually write the symbolic and formal writings of mathematics with a tablet. Thirty = seven percent used mathematical software, 42% showed slides—resources prepared before the lesson (.ppt files or sheets that were handwritten beforehand and then scanned to be shown on screen as .pdf files). In fact, someone wrote:

> *I do not have a writing tablet, blackboard, or anything else, so the only way is to write by hand before the lesson, scan the many sheets and insert them on the PC to share during the lesson: hard work.*

Table 4. What do you use to digitize the formal and symbolic writings of mathematics? (multiple answer options)—($n = 38$).

	Percent
Writing tablet	26.3
Writing tablet; slides	18.4
Mathematical software	18.4
Slides	18.4
Mathematical software; writing tablet	13.2
Mathematical software; slides	2.6
Mathematical software; writing tablet; slides	2.6
Total	100.0

The McNemar test showed that there was a significant increase in the use of slides, from 13% to 42%, $p = 0.01$, and there is also a significant increase in the use of mathematical software, from 11% to 37%, $p = 0.03$.

We investigated how often professors used their handwriting during their distance learning, using a writing tablet and sheets of paper that they then converted into .pdfs for projection. The answers were: *never* (23%), *sometimes* (21%), *often* (16%), and *always* (40%). Table 5 shows that those who chose to use their own writing did so to improve teaching effectiveness and content exposure (58%) or to recreate the traditional atmosphere of class (19%). Some of the answers given by professors, for example, were:

It is easier to show the progress of an exercise.

I believe that the professor's personal writing makes the text "less cold" and more effective from a didactic point of view.

Table 5. Explain how much you use your writing (writing tablet, sheets shown to students, etc.)—(n = 38).

	Percent
To improve teaching effectiveness and content exposure	58.1
To recreate the traditional atmosphere of class	19.4
I prefer to use other modalities	12.9
I cannot use it	9.7
Total	100.0

We were interested in understanding what actions the professors took, and how often they were teaching using a personal computer in distance learning. In particular, we asked if they activated the camera, shared their screen, and asked students to share their screens, and asked them to explain their habits. Sixty percent *always* activated the camera (Table 6). The open-ended responses that explained this answer are related to choices for relational and communication reasons. For example:

To have more contact with students.

Students follow better.

I like the idea of eye contact.

It is important that students see me, both to create a "classroom" relationship and because I often show objects or "help" myself with gestures.

Those who did not use a camera for distance learning explained that they preferred other educational tools or did not do so for privacy reasons. For example:

I do not think it is necessary and I prefer students to focus on the content of the presentation.

[. . .] since my class consists of 180 students, I prefer not to show myself on video to avoid any unpleasant episodes.

Although it was a small percentage of teachers who did not turn the camera on, we wondered if this choice was related to the number of students following their distance teaching, or whether it was related to the degree course. The Kruskal–Wallis Test, in both cases, did not support this hypothesis, $p > 0.50$.

Eighty-seven percent of professors shared their screens frequently (Table 6, *often* + *always*). The open-ended responses showed that professors took this action because they needed to show resources that they had prepared for the lesson (63%) or because they needed to build the lesson in front of the students (37%). For example:

Sharing is necessary to show what I have prepared, but also to show further examples, to do exercises, to focus on key steps . . . using a writing tablet.

I think it is necessary to share the screen in a presentation, especially for mathematical formulas. I couldn't do without it.

During a three-hour lesson in front of the class, I fill four blackboards at least ten times, because I always have to write, now with no other way I show the many papers I write before each lesson, so I always talk and share.

I use Mathematica software. Students see either my notebook where I explain the topic, or my notebook where I solve a problem.

Table 6. Frequency with which a certain action takes place on the PC during the distance lesson (n = 38).

	Activating the Camera	Sharing the Screen
	Percent	Percent
Never	21.1	5.3
Sometimes	18.4	7.9
Often	0.0	10.5
Always	60.5	76.3
Total	100.0	100.0

Using natural language in an informal way, mathematical language and iconic representations (diagrams, function graphs, charts, etc.) did not undergo significant variations during the mathematical lectures taught at a distance. In fact, as Table 7 shows, 87% of professors continued to use these three signs (natural language in an informal way, mathematical language, and iconic representations) in the same way as before the COVID-19 pandemic. This means that the modality of distance learning had not affected the way in which the terminology proper to the discipline and its iconic representations were used in teaching practices. This is independent of the number of students attending that particular course (Median test, $\chi^2(2) = 0.53$, $p > 0.50$). Almost all the professors (95%) used gestures less than before or in the same way as before. In particular, the median test, $\chi^2(2) = 8.67$, $p = 0.013$, shows that professors who had more students (percentage of attendants between 75% and 100%) reported using gestures in the same way as before, while those who have few students (percentage of attendants between 0% and 50%) reported having used fewer gestures.

Table 7. Frequency of formats during distance learning compared to the usual presentation of topics? (n = 38).

	Natural Language in an Informal Way	Mathematical Language	Iconic Representations	Gestures
	Percent	Percent	Percent	Percent
Less than before	5.3	0.0	10.5	63.2
Same as before	86.8	86.8	86.8	31.6
More than before	7.9	13.2	2.6	5.3
Total	100.0	100.0	100.0	100.0

School teachers on some social networks in Italy often noted that during the quarantine: "distance teaching requires twice as long to prepare a lesson and you can explain half the things you have prepared". We asked the university mathematics professors how much they agreed with this, especially the second part. Table 8 shows that 82% (*definitely disagree* + *quite disagree*) disagreed with this statement, and the distribution of the answer options is significantly different, $\chi^2(3) = 27.05$, $p < 0.001$).

The open answers explaining the degree of agreement reveal that 18% found the distance modality ineffective. Twenty-three percent used similar modalities to their previous ones, and 59% used new teaching modalities to try to maintain the same standard as when teaching in person. The fact that the same modalities used in person were adopted even at a distance, and that new "compensatory" tools were used (such as prepared notes, slides,

etc.), allows us to conclude that teaching on Teams had not reduced the number of topics explained by professors. Open answers from the professors included:

> *With distance learning, having already prepared the material/presentation to share during the lesson (which then becomes teaching material available to students), it is possible to explain more quickly and therefore explain more topics than in traditional lessons.*
>
> *The use of the blackboard meant more time to explain the topics. Students don't have to take notes. I provide them at the end of each lesson.*
>
> *Actually, with distance learning you do more with the same amount of time.*
>
> *I spend lot of time handwriting what I would write on the blackboard while I explain, and because I can't see the eyes of the students I'm not inclined to repeat concepts or demonstrations, and so I do everything I set out to do.*

Something more emerges from the open answers: the professors managed to transpose even more knowledge than they were able to transpose in person. Some said that this was probably due to not having to write on the blackboard and wait for the students to transcribe the notes, others said that by sending the (previously prepared) notes to the students, they could afford to go faster. Some also said that more was done because there was a lack of interaction with the students.

Table 8. How much you agree with these statements: "with distance learning you can only explain half the things you have prepared"? (n = 38).

	Percent
Definitely disagree	60.5
Quite disagree	21.1
Quite agree	7.9
Definitely agree	10.5
Total	100.0

We investigated long-distance professor–student relationships through the two following questions. First of all, we used a Likert scale to ask whether the professors could perceive whether the students were keeping up with them in the lesson, and then used an open question to ask what strategies they adopted for perceiving this. Twenty-four percent said *definitely not*, 40% *more no than yes*, 26% *more yes than no*, and 10% *definitely yes*. In most cases, therefore, professors were not able to perceive whether students were keeping up with them in the lesson, and this is also demonstrated by the chi-square test analysis: $\chi^2(3) = 6.42$, $p < 0.10$. A Kruskal–Wallis test showed that this inability depends on neither the number of students attending the course, $\chi^2(2) = 0.699$, $p = 0.70$, nor on the degree course, $\chi^2(2) = 0.960$, $p = 0.62$. The double entry table (Table 9) correlates the strategies adopted by professors with the previous answer. We can see that those who claimed they were able to perceive that the students were keeping up with the lesson did so by asking the students directly whether they were following the explanation (30%), by asking the students questions about the content they had explained (27%), or by deducing it from the interventions made by students (24%).

One comparison was related to concerns about the use of distance learning that had been raised before the COVID-19 pandemic, with respect to concerns about distance learning during the COVID-19 pandemic. Table 10 shows that before starting distance learning due to the pandemic, 74% (*pretty concerned* + *very concerned*) were concerned about the thought of having to give their lessons online. This percentage dropped to 40% during the pandemic wave. This suggests that the professors' concerns were lessened when practicing distance learning.

Table 9. Cross-tabulation between degree of agreement and strategies adopted to perceive whether students are keeping up with the lesson (n = 38).

			Categories of Answers for Professors' Strategies				Total
			Asking the Students If They Are Following the Explanation	**Understanding via Student Interventions**	**Asking Questions about the Lesson Contents**	**Not Verifiable**	
Likert scale items	Definitely not	% of Total	3.1%	0.0%	0.0%	12.1%	15.2%
	More no than yes	% of Total	12.1%	12.1%	12.1%	6.1%	42.4%
	More yes than no	% of Total	12.1%	12.1%	6.1%	0.0%	30.3%
	Definitely yes	% of Total	3.0%	0.0%	9.1%	0.0%	12.1%
Total		% of Total	30.3%	24.2%	27.3%	18.2%	100.0%

Table 10. Likert-scale response percentages about concerns regarding distance learning before and during COVID-19 pandemic (n = 38).

Likert-Item Items	Concern before COVID-19	Concern during COVID-19
Not at all concerned	7.9%	36.8%
Little concerned	18.4%	23.7%
Pretty concerned	42.1%	31.6%
Very concerned	31.6%	7.9%
Total	100.0%	100.0%

As noted in the methodology section, our sample respondents adopted Microsoft Teams as the e-learning platform for distance learning. We emphasize here that none of the professors in the universities at Catania, Messina, or Palermo had received training in the use of this platform. This is certainly due to the fact that the academic community found itself suddenly (within five days) required to provide distance learning in order to continue guaranteeing the right to university education, despite the health emergency and the pandemic situation. Although professors discovered the Teams functions on their own, some expressed their willingness to continue using it. In fact, professors plan to continue using Teams and/or the other platforms at the end of the COVID-19 pandemic (Table 11). Twenty-six percent said they do not want to use any platform, and 5% were unable to respond. Teams was confirmed as the choice of 58% of respondents. Whereas 39% had been using other platforms, there is now increased willingness to use a digital platform, Teams, in the future.

Table 11. Which of these e-learning platforms do you plan to continue using after the end of COVID-19 pandemic? (multiple response options)—(n = 38).

	Percent
Teams	28.9
No platform	26.3
Teams and Studium	18.4
Studium	10.5
Teams and Moodle	10.5
I do not know	5.3
Total	100.0

We wanted to investigate whether the age of the respondents affected this choice. It emerged that 77% of those up to the age of 55 said they wanted to continue using Teams or similar, while 57% of those over 55 said they did not want to continue using it. The older a professor, the less willing they are to continue using distance learning platforms. This assumption is confirmed by the fact that the distribution of the answer is significant, $\chi^2(1) = 4.35$, $p < 0.10$.

Using open questions, we asked the professors to comment on how much they felt they had gained and lost through distance learning. Table 12 shows the cross-tabulation of the thematic categories of answers that emerged for both issues. The chi-square analysis related to this contingency table was not significant, $\chi^2(12) = 15.72$, $p = 0.21$, meaning that the association between the two thematic variables were no different from chance. However, considering each variable individually (see the total column in the Table 12), the frequencies of the identified categories were significantly different for the loss in distance modality, $\chi^2(4) = 22.26$, $p < 0.001$, with the "Human exchange" category more frequent compared to the other categories. No differences were revealed for the gain in distance modality variable.

Table 12. Cross-tabulation between thematic categories of "Gain in distance modality" and "Loss in distance modality" (n = 38).

			Thematic Categories for "Loss in Distance Modality"				Total
			Human Exchange	Interaction for Learning Purposes	Both (the Previous Ones)	No Loss	
Thematic categories for "Gain in distance modality"	No profit	% of Total	13.2%	5.3%	7.9%	0.0%	26.4%
	Better learning/involvement of students	% of Total	13.2%	7.9%	0.0%	2.6%	23.7%
	Improvement of didactical/technological skills	% of Total	10.5%	0.0%	7.9%	0.0%	18.4%
	Less organizational/physical stress	% of Total	5.3%	2.6%	7.9%	0.0%	15.8%
	Improved lesson planning	% of Total	2.6%	10.5%	2.6%	0.0%	15.7%
Total		% of Total	44.8%	26.3%	26.3%	2.6%	100.0%

We note that compared to earnings, 21% of professors believed they benefited from an improvement in terms of their didactical and technological skills:

I discovered other ways of teaching, through the use of technology.

I learned how to use the iPad board and the Microsoft Teams platform.

Twenty-three percent believed that the distance modality had a positive impact on learning/involvement of students.

The ability to produce a PDF file that reproduces exactly what I wrote on the board. In this way students will find faithful notes and avoid transcribing inaccuracies that often cause confusion on important issues such as definitions or various observations.

A greater sense of responsibility on the part of the students, who have understood that to follow well they must "study" punctually and in advance.

The blackboards are published, the lesson is recorded, I have the ability to use software and also to show a video, which in traditional lessons I did not do because it was too laborious.

Fifteen percent said that having to do the lesson at a distance improved the planning of the lesson, and another 15% said they suffered less stress on a physical and organizational level.

The in-depth planning of the lessons.

The lesson is smoother, and the material I have prepared will still help me in the years to come.

Comforts.

Time.

The majority (26%) said they had not benefited from any earnings.

The majority (43%) complained about a loss of human exchange. Twenty-seven percent complained of losing interactions with students for learning purposes, and another 27% claimed to have lost both.

A part of human contact, useful to transmit the passion for discipline.

Eye contact with students.

Human contact with students.

The ability to see positive or negative reactions to what is proposed not explicitly manifested through interventions.

When I saw the empty eyes of so many students I understood that they hadn't understood me and so I repeated it all with other words until I saw their eyes full: I lost the joy of having been able to fill those eyes.

We emphasize that student cameras were generally off and that the Teams platform does not allow the presenter to see more than one student when they share the screen, or more than nine when they do not. The presenter thus cannot see the students' eyes.

5. Discussion and Conclusions

The e-learning system is a complex system that involves several kinds of capabilities (and people). It is not easy to change in a few days from the way one has taught for years to a new way, without any training, while worried about the health, economy, and social perspective of your country. We asked how, in this situation, professors were able to move to distance teaching. The problems of adapting ways of teaching to the new e-learning environment are particularly relevant when teaching mathematics, because of the frequent use of symbols and formulas, as well as gestures and body. Let us review how the university

professors adapted themselves to the new teaching modality, attempting to teach "in the same way" in this new environment. We discovered that professors moved from lecturing with chalk and blackboard (100% used them) to lecturing with writing tablets (61%), which they had never used before, and slides or mathematical software (79%), which they had used to some extent (only 15.8% used slides or software before the pandemic). Most of the professors involved in our research started a process of instrumental genesis, through the appropriation of new artifacts (writing tablets, mathematical software, e-learning platform) in an *instrumentalization* phase: they managed to use them, found their potential and limits, but only to recreate the same teaching they were used to without the artifacts. So far, they had not reached an *instrumentation* phase, with the proper use of the artifacts with appropriation of their schemes of use. The professors we dealt with had a strong belief about teaching math (core in the MTSK model [19], together with math belief): that lecturing is the only way to teach mathematics. For instance, a professor stated: "*I do not have a writing tablet, blackboards, or anything else so the only way is to write by hand before the lesson, scan the sheets that are many and insert them on the PC to share during the lesson: hard work*." They thus continued in the "same way", by using different artifacts, technological ones. Writing tablets and other strategies were used to show their personal handwriting, because almost 60% believed that it was important to improve teaching effectiveness and content exposure (and not only to recreate a sort of classroom environment). No one claimed to have chosen the writing tablet so that they did not have to prepare the lectures in advance and so to save time, suggesting that the use of personal writing was not a comfort choice but rather a didactic one, due to personal beliefs about mathematics, according to this belief which has to be written, and written by hand, because "*I believe that the professor's personal writing makes the text "less cold" and more effective from a didactic point of view*". The use of slides and mathematical software also seems to have been a didactic choice, and not a comfort choice. It required an extra effort, especially in terms of the details to be prepared for the lessons. Nevertheless, professors preparing the lectures in all the details increased from 39.5% to 68.4%. It was probably anxiety and worry that drove them to make detailed preparations, to prevent possible trouble. We want to stress that 81.8% of professors who started to prepare their lessons in detail during the emergency scored their worry about distance teaching as 3 or 4 on a Likert scale from 1 (not worried) to 4 (very worried).

The use of handwriting, slides, and software discussed so far is especially valid for "blackboard teachers" [14], those who, for instance, did not activate their camera because they "*prefer that students focus on the content of the presentation*". As for the "body teachers" [14], those who, for instance think that "*it is important that students see me, both to create a 'classroom' relationship and because I often show object or 'help' myself with gestures*", we find out that only 5.3% used gestures more often than before, regardless of the number of students. Those with a higher percentage of students use them in the same way and no less than before. Therefore, to answer our second research question, the number of students does not apparently affect the use of gestures. Using natural language in an informal way, mathematical language and the iconic representations was also independent of the number of students attending that particular course.

As for the disadvantages of distance teaching, 97.2% of answers reported having lost interaction for human exchange or learning purposes. Today, we are very used to technological devices, and we use them both for personal and didactic reasons: artifacts [12] can mediate mathematical topics, exactly as musical instruments can mediate the composer's feelings when we listen to music. In the situation illustrated in this paper, we had "instruments", but the players and the audience were so far from each other that we wonder whether the concert will be a success or not. We think that face-to-face communication is important in mathematics education, as in musical concerts, from both points of view: that of the students (the audience) and the teachers (the players). Attending a concert is not the same as watching it on the television, even if it is a live concert, and it is not the same thing to play in front of a reactive audience or in an empty theatre. We want to emphasize that there is a big difference between thinking about a course formatted as an on-line course

and adapting a course designed to be a face-to-face course as a distance course. In the musical metaphor, a player who goes to record a new disk is alone in the recording studio, and can play the music until the record is perfect (on-line courses). Alone in the theatre, a player can also execute a perfect performance, but without infusing the audience with the feelings involved in the music (distance courses in the emergency). Looking into the eyes of "audience" enable an understanding of whether teaching is effective or not. Some professors "lost the joy of having been able to fill those eyes". Another said "*I like the idea of eye contact*".

The final and central question to be answered is about the benefits of distance teaching and the opportunities that arise from difficulty. Fifteen point four percent of professors reported that extra time was a benefit (because, they said, you save the time traveling to and from the office) or comfort (because, they said, you are at home). Twenty-five point six percent claimed to have gained nothing. We want to emphasize that the other 59% found benefits in terms of better teaching or learning. In particular, 20.5% found an improvement in their didactical or technological skills. Fifty-eight percent wished to continue using the Teams platform, which no one had used before the emergency and everybody learned without any training. As shown in the previous section, the choice, as expected, is affected by age (the older professors are the least willing to use digital platforms). We hope that those professors who want to use the artifact will develop an instrumentation phase in the instrumental genesis process.

It is said that a crisis is a terrible opportunity to learn and grow up, and we ask whether we, as academics, were able to take this as an opportunity. We think that the professors we analyzed, starting from a worrying situation, showed a competence of vicariance, trying to solve unexpected problems, trying to change their habits, and trying to find something to learn. In fact, a gain took place when they acquired new skills that they will use in the future.

Several professors who helped in the research were available to be contacted again for further research. Our aim is to validate our results with them through interviews and focus groups. This further step will also take into account the fact that the COVID-19 pandemic situation is still ongoing and that they are still teaching at a distance.

We would also like to point out that in the same period during which we administered the questionnaires to university professors, a specular questionnaire was administered to the students of the three Sicilian universities (Catania, Messina, Palermo), with the aim of analyzing the distance teaching from the students' point of view in terms of emotions, learning, and advantages/disadvantages. We therefore intend to engage with the subsequent analysis to show the other side of the coin.

The game is not over!

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