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Spectral properties of the singular Friedrichs–Lee Hamiltonian

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ABSTRACT

We show that the Friedrichs–Lee model, which describes the one-excitation sector of a two-level atom interacting with a structured boson field, can be generalized to singular atom–field couplings. We provide a characterization of its spectrum and resonances and discuss the inverse spectral problem.

I. INTRODUCTION

The Friedrichs–Lee Hamiltonian is a self-adjoint operator on a Hilbert space, which describes the behavior of an eigenvalue coupled to a continuous spectrum, and it is a rare example of a solvable model with a rich mathematical structure.[15] It was originally introduced by Lee[18] as a solvable quantum-field theoretical model suitable for the investigation of the renormalization procedure. Lee's Hamiltonian has a conserved quantum number labeling reducing subspaces (excitation sectors). Its reduction to the first nontrivial excitation sector, which we will refer to as the Friedrichs–Lee Hamiltonian, is the quantum-mechanical model used by Friedrichs in his seminal study of the perturbation of continuous spectra;[11] similar models for unstable relativistic boson fields have been studied in the literature.[3]

Since its inception, the Friedrichs–Lee Hamiltonian has proven to be a very useful model in many applications, ranging from quantum field theory of unstable particles[4] to non-relativistic quantum electrodynamics,[5] to quantum field theory on manifolds,[16] to quantum optics,[12] and to quantum probability.[30]

In this paper, we aim at a complete study of the mathematical properties of the Friedrichs–Lee operator by extending it to a larger class of possibly singular couplings (thus providing rigorous foundations to many formal computations usually carried out in the physical literature) and providing a characterization of its spectrum with respect to the spectrum of the uncoupled operator. This paper is organized as follows:

- In Sec. II, we derive the expression of the Friedrichs–Lee Hamiltonian as the restriction to the one-excitation sector of Lee's field-theory model.
- In Sec. III, we introduce the singular Friedrichs–Lee Hamiltonian, proving that it includes the case of a regular coupling (Theorem III.2) and showing that a singularly coupled model can always be obtained as the norm resolvent limit of a proper sequence of regular models (Theorem III.4).
- In Sec. IV, we characterize the spectrum of the Friedrichs–Lee Hamiltonian: in Theorem IV.1, we find its essential and discrete components, and in Theorem IV.3, we find its absolutely continuous, singular continuous, and pure point components, the latter being strictly dependent on a Herglotz function known as the self-energy of the model.
- In Sec. V, we apply the results of Secs. III and IV to some examples of Friedrichs–Lee Hamiltonians.
- In Sec. VI, we discuss the resonances of the model, showing that they can be characterized as complex eigenvalues of a deformation of the Hamiltonian (Theorem VI.7).
- In Sec. VII, we introduce the inverse spectral problem, i.e., the choice of a coupling yielding the desired dynamics for a model with a given field structure.

Future developments may include the generalization of the singular coupling and spectral characterization to the n-atom Friedrichs–Lee Hamiltonian or to higher excitation sectors, as well as applications to physically interesting systems.

II. PHYSICAL MODEL

Let (X, μ) be a σ-finite measure space and $\mathscr{H} = L^2(X,\mu)$ be the space of square-integrable complex-valued functions with respect to μ, with $\langle\cdot|\cdot\rangle$ denoting the inner product, conjugate-linear in the first entry, and $\|\cdot\|$ denoting the induced norm. Let H_{field} be a Hamiltonian operator on the Bose–Fock space $\mathscr{F}$ of $\mathscr{H}$ with formal expression

$$H_{\text{field}} = \int_X \omega(k)\, a^*(k)\, a(k)\, \mathrm{d}\mu(k), \tag{1}$$

where $\omega : X \to \mathbb{R}$ is a measurable function and $a(k)$ and $a^*(k)$ are the operator-valued distributions associated with a family of annihilation and creation operators, respectively, satisfying the formal commutation relations $[a(k), a^*(k')] = \delta(k - k')$. Physically, H_{field} is the operator associated with the energy of a bosonic field, (X, μ) is the momentum space of the bosons, and $\omega(k)$ is the dispersion relation, i.e., the energy of a quantum with momentum k. For example, the choices $X = \mathbb{R}^3$, $\omega(k) = (|k|^2 + m^2)^{1/2}$, and μ being the Lebesgue measure on $\mathbb{R}^3$ describe a relativistic bosonic field associated with particles of mass m. At the mathematical level, H_{field} is the second quantization of the multiplication operator associated with the function ω and is a densely defined self-adjoint operator on $\mathscr{F}$.[24]

Let us consider a nondegenerate two-level atom, with ground state (in Dirac's notation) $|{\downarrow}\rangle$ and excited state $|{\uparrow}\rangle$ in $\mathbb{C}^2$. Let

$$H_{\text{atom}} = \varepsilon_{\text{a}} |{\uparrow}\rangle\langle{\uparrow}| \tag{2}$$

be its Hamiltonian, where $\varepsilon_{\text{a}} \in \mathbb{R}$ is the energy of the excited state and the ground state energy is set to zero. The operator $H_{\text{atom}} \otimes I + I \otimes H_{\text{field}}$, defined on a dense subspace of $\mathbb{C}^2 \otimes \mathscr{F}$, represents the system atom–field in the absence of mutual interaction. A physically meaningful coupling between the atom and the field can be introduced as follows: given $g \in \mathscr{H}$, let

$$V_g = \int_X \Big(\sigma^+ \otimes \overline{g(k)}\, a(k) + \sigma^- \otimes g(k)\, a^*(k)\Big)\, \mathrm{d}\mu(k), \tag{3}$$

where $\sigma^+ = |{\uparrow}\rangle\langle{\downarrow}|$ and $\sigma^- = |{\downarrow}\rangle\langle{\uparrow}|$ are the ladder operators, that is, σ^+ raises the ground to the excited state and σ^- lowers the excited to the ground state.[5] The total Hamiltonian H_{Lee} associated with the atom–field system is formally given by

$$H_{\text{Lee}} = H_{\text{atom}} \otimes I + I \otimes H_{\text{field}} + V_g, \tag{4}$$

where we use the same notation for identity operators acting on different Hilbert spaces. This is a generalization of the standard Lee model.[18] Physically, μ controls and weighs the values of momenta available to the bosons and must be chosen according to the physical setting: for instance, for an electromagnetic field in free space, μ is the Lebesgue measure on $X = \mathbb{R}^3$, while, for a field confined in an optical cavity, at least one component of the momenta will be discrete.

Summing up, the analytic features of our model will depend on three physically important quantities:

- The space (X, μ) of all possible momenta of the field quanta,
- The *dispersion relation* $\omega(k)$ that gives the energy of a quantum with momentum k, and
- The *form factor* $g(k)$ that controls the coupling between the atom and a field quantum with momentum k.

The operator H_{Lee} does not conserve the total number of bosons in the theory: the number operator, formally defined as

$$N_{\text{field}} = \int_X a^*(k) a(k)\, \mathrm{d}\mu(k), \tag{5}$$

does not commute with H_{Lee} for any nonzero form factor g. However, the operator

$$N_{\text{tot}} = |{\uparrow}\rangle\langle{\uparrow}| \otimes I + I \otimes N_{\text{field}}, \tag{6}$$

representing the total number of excitations in the system, commutes with H_{Lee} for every choice of g; since the operator N_{tot} has spectrum $\sigma(N_{\text{tot}}) = \mathbb{N}$, one can study the evolution of the system generated by the restriction of H_{Lee} to each eigenspace of N_{tot}.

The simplest nontrivial choice is the one-excitation sector that is isomorphic to $\mathbb{C} \oplus \mathscr{H}$. The generic normalized element $\Psi \in \mathbb{C} \oplus \mathscr{H}$ may be expressed as

$$\Psi = \begin{pmatrix} x \\ \xi \end{pmatrix}, \qquad x \in \mathbb{C}, \quad \xi \in \mathscr{H}, \tag{7}$$

the normalization being

$$|x|^2 + \int_X |\xi(k)|^2\, \mathrm{d}\mu(k) = 1, \tag{8}$$

where $|x|^2$ is the probability that the atom is in its excited state and ξ is the wave function of the boson in the field. In particular, the state

$$\Psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

represents the excited atom interacting with the vacuum. The restriction of H_{Lee} to the one-excitation sector $\mathbb{C} \oplus \mathscr{H}$, which we will denote as H_{FL}, is the *Friedrichs–Lee Hamiltonian.*[11,15] Its domain is

$$D(H_{\text{FL}}) = \mathbb{C} \oplus D(\Omega) = \left\{ \begin{pmatrix} x \\ \xi \end{pmatrix} \middle| \, x \in \mathbb{C}, \, \xi \in D(\Omega) \right\}, \tag{10}$$

and it acts on a generic vector of the domain as

$$H_{\text{FL}} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \varepsilon_{\text{a}} x + \langle g|\xi\rangle \\ xg + \Omega\xi \end{pmatrix}, \tag{11}$$

where Ω is the multiplication operator associated with the dispersion relation ω, that is, $(\Omega\xi)(k) = \omega(k)\xi(k)$; we will refer to Ω as the *inner Hamiltonian* of the model. The action of the Hamiltonian H_{FL} in (11) can be obtained by using a formal matrix representation,

$$H_{\text{FL}} = \begin{pmatrix} \varepsilon_{\text{a}} & \langle g| \\ g & \Omega \end{pmatrix}, \tag{12}$$

where $\langle g|$ is, in Dirac notation, the linear functional on $\mathscr{H}$ associated with g.

III. THE SINGULAR FRIEDRICHS–LEE HAMILTONIAN

The Hamiltonian (11) with matrix representation (12) cannot include a *singular coupling* between the field and atom, i.e., a form factor $g \notin \mathscr{H} = L_2(X,\mu)$. This obstruction is relevant at a physical level: for instance, a flat form factor between the field and atom [i.e., $g(k) =$ constant] cannot be generally included (if μ is not a finite measure), thus preventing the description of interesting phenomena [e.g., exponential decay of the state Ψ_0 in Eq. (9)].

To extend the Friedrichs–Lee model to a (possibly) singular coupling between the atom and field,[19] the formalism of Hilbert scales will be extensively used (see Ref. 1).

Definition III.1. Given a Hilbert space $\mathscr{K}$ and a self-adjoint operator T on it, the space $\mathscr{K}_s$, for any $s \geq 0$, is the domain of $|T|^{s/2}$ endowed with the norm

$$\|\varphi\|_s := \| |T - i|^{s/2} \varphi \| \tag{13}$$

and $\mathscr{K}_{-s}$ is its dual space, i.e., the space of continuous functionals on it, endowed with the norm $\|\varphi\|_{-s} := \| |T - i|^{-s/2} \varphi \|$.

With an abuse of notation, the pairing between $\varphi \in \mathscr{K}_{-s}$ and $\zeta \in \mathscr{K}_s$ will still be denoted as $\langle \varphi|\zeta\rangle$, i.e., with the notation of the scalar product on $\mathscr{K}$. Besides, $\mathscr{K}_2 = D(T)$ and the following properties hold:

- $\mathscr{K}_s \subset \mathscr{K}_{s'}$ for any $s > s'$, with dense inclusion.
- $\mathscr{K}_0$, $\mathscr{K}_1$, and $\mathscr{K}_2$ are the original Hilbert space $\mathscr{K}$, the form domain of T, and the domain of T, respectively.
- T maps $\mathscr{K}_s$ into $\mathscr{K}_{s-2}$ and $\frac{1}{T-i}$ maps $\mathscr{K}_s$ into $\mathscr{K}_{s+2}$.

In the following, we will consider the Hilbert space $\mathscr{K} = \mathscr{H}$ and the self-adjoint operator $T = \Omega$ as the multiplication operator by a real-valued measurable function $\omega : X \to \mathbb{R}$. In this case, we have, for every $s \geq 0$,

$$g \in \mathscr{H}_s \qquad \text{iff} \qquad \int_X |\omega(k) - i|^s |g(k)|^2 \, \mathrm{d}\mu(k) < \infty, \tag{14}$$

$$g \in \mathscr{H}_{-s} \qquad \text{iff} \qquad \int_X \frac{|g(k)|^2}{|\omega(k) - i|^s} \, \mathrm{d}\mu(k) < \infty. \tag{15}$$

Our first result is the following theorem, which extends the Friedrichs–Lee Hamiltonian to the case $g \in \mathscr{H}_{-2}$.

Theorem III.2. *Let $\varepsilon \in \mathbb{R}$, Ω be a self-adjoint operator on the Hilbert space $\mathscr{H} = L^2(X,\mu)$ acting as the multiplication operator by a real-valued measurable function $\omega : X \to \mathbb{R}$, and $g \in \mathscr{H}_{-2}$. Consider the operator $H_{g,\varepsilon}$ on the Hilbert space $\mathbb{C} \oplus \mathscr{H}$ with domain*

$$D(H_{g,\varepsilon}) = \left\{ \begin{pmatrix} x \\ \xi - x\frac{\Omega}{\Omega^2+1}g \end{pmatrix} \middle| \, x \in \mathbb{C}, \ \xi \in \mathscr{H}_2 \right\} \tag{16}$$

such that

$$H_{g,\varepsilon}\begin{pmatrix} x \\ \xi - x\frac{\Omega}{\Omega^2+1}g \end{pmatrix} = \begin{pmatrix} \varepsilon x + \langle g|\xi\rangle \\ \Omega\xi + x\frac{1}{\Omega^2+1}g \end{pmatrix}. \tag{17}$$

Then, we have the following:

(i) *If $g \in \mathscr{H}$, then $H_{g,\varepsilon}$ reduces to the Friedrichs–Lee Hamiltonian H_{FL} in (11) with an atom excitation energy*

$$\varepsilon_{\mathrm{a}} = \varepsilon + \left\langle g \middle| \frac{\Omega}{\Omega^2+1}g \right\rangle. \tag{18}$$

(ii) *$H_{g,\varepsilon}$ is self-adjoint and, for all $z \in \mathbb{C}\backslash\mathbb{R}$, its resolvent operator is*

$$\frac{1}{H_{g,\varepsilon}-z}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \dfrac{x - \langle g|\frac{1}{\Omega-z}\xi\rangle}{\varepsilon - z - \Sigma_g(z)} \\ \dfrac{1}{\Omega-z}\xi - \dfrac{x - \langle g|\frac{1}{\Omega-z}\xi\rangle}{\varepsilon - z - \Sigma_g(z)}\dfrac{1}{\Omega-z}g \end{pmatrix}, \qquad \begin{pmatrix} x \\ \xi \end{pmatrix} \in \mathbb{C}\oplus\mathscr{H}, \tag{19}$$

where

$$\Sigma_g(z) := \left\langle g \middle| \left(\frac{1}{\Omega-z} - \frac{\Omega}{\Omega^2+1}\right)g \right\rangle \tag{20}$$

is the self-energy function of $H_{g,\varepsilon}$.

(iii) *The evolution group generated by $H_{g,\varepsilon}$ is given by*

$$U_{H_{g,\varepsilon}}(t) = \frac{1}{2\pi i}\mathrm{PV}\int_{i\delta-\infty}^{i\delta+\infty} e^{-izt}\,\frac{1}{H_{g,\varepsilon}-z}\,\mathrm{d}z \tag{21}$$

for all $t > 0$, where $\delta > 0$ is an arbitrary constant and the principal-value integral must be understood in the strong sense. Moreover, if

$$\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = U_{H_{g,\varepsilon}}(t)\Psi_0 \tag{22}$$

is the evolution at time $t > 0$ of the initial state Ψ_0 in (9), we have

$$x(t) = \frac{1}{2\pi i}\mathrm{PV}\int_{i\delta-\infty}^{i\delta+\infty} \frac{e^{-izt}}{\varepsilon - z - \Sigma_g(z)}\,\mathrm{d}z, \tag{23}$$

$$\xi(t,k) = -\frac{1}{2\pi i}\mathrm{PV}\int_{i\delta-\infty}^{i\delta+\infty} \frac{g(k)\,e^{-izt}}{[\varepsilon - z - \Sigma_g(z)][\omega(k) - z]}\,\mathrm{d}z. \tag{24}$$

Proof. (i) If $g \in \mathscr{H}$, then $\frac{\Omega}{\Omega^2+1}g \in \mathscr{H}_2 = D(\Omega)$, and hence, the domains in Eqs. (10) and (16) coincide. Applying H_{FL} to any vector of the form (16) yields the same result as in Eq. (17), and hence, the two operators coincide.

(ii) Since $D(\Omega)$ is dense in $\mathscr{H}$, then $D(H_{g,\varepsilon})$ is dense in $\mathbb{C}\oplus\mathscr{H}$. Moreover, the self-energy function Σ_g in Eq. (20) is well defined and a direct calculation shows that $H_{g,\varepsilon}$ is symmetric. Finally, the bounded operator acting on $\mathbb{C}\oplus\mathscr{H}$ as in Eq. (19) is the inverse of $H_{g,\varepsilon} - z$ for any $z \in \mathbb{C}\backslash\mathbb{R}$ and the self-adjointess of $H_{g,\varepsilon}$ easily follows from that.

(iii) Equation (21) follows from Eq. (19) and from the link between the resolvent and the evolution group associated with any self-adjoint operator;[9] particularly, Eq. (23) follows by substituting $x = 1$ and $\xi = 0$ in Eq. (19) and applying Eq. (21). □

Remark III.3. We can distinguish three separate cases:

1. $g \in \mathscr{H}$: the domain $D(H_{g,\varepsilon})$ does not depend on g, and both $\varepsilon_{\rm a}$ and ε are finite quantities, representing the "bare" and "dressed" (coupling-dependent) excitation energies of the atom, respectively. The formal matrix expression (12) of the Hamiltonian holds.
2. $g \in \mathscr{H}_{-1}\backslash\mathscr{H}$: the domain $D(H_{g,\varepsilon})$ depends on g, but again, both $\varepsilon_{\rm a}$ and ε are finite quantities with the same physical meaning as above, since $\langle g|\frac{\Omega}{\Omega^2+1}g\rangle$ is finite. Again the Hamiltonian can be written as in Eq. (12).
3. $g \in \mathscr{H}_{-2}\backslash\mathscr{H}_{-1}$: the domain $D(H_{g,\varepsilon})$ depends on g, and the bare excitation energy $\varepsilon_{\rm a}$ is not defined, since $\langle g|\frac{\Omega}{\Omega^2+1}g\rangle$ is not finite; because of that, Eq. (12) is ill-defined.

The latter situation is related to the renormalization procedure of quantum field theory in which the bare (hence unobservable) value of a parameter, e.g., the electron charge, diverges in such a way to obtain a finite value of the measurable dressed one. Besides, in the first two cases, we may equivalently write

$$\varepsilon - \Sigma_g(z) = \varepsilon_{\rm a} - \tilde{\Sigma}_g(z), \tag{25}$$

where $\tilde{\Sigma}_g(z) = \langle g|\frac{1}{\Omega-z}g\rangle$ is the "bare" self-energy. In this sense, the extension of the model to the case $g \in \mathscr{H}_{-1}$ is straightforward up to an algebraic technicality, i.e., the choice of a convenient representation of the domain, while the further extension to the case $g \in \mathscr{H}_{-2}$ requires an "infinite" term to be added to both the bare excitation energy and the bare self-energy.

Finally, the three cases reflect the possible situations in which Ψ_0 has

1. finite mean value and variance of energy $H_{g,\varepsilon}$,
2. finite mean value but infinite variance of $H_{g,\varepsilon}$, and
3. infinite mean value and variance of $H_{g,\varepsilon}$.

When $g \notin \mathscr{H}$, we will say that the atom–field coupling is *singular*, as opposed to the regular case $g \in \mathscr{H}$. We would like to obtain a singular Friedrichs–Lee Hamiltonian as the limit, in a suitable sense, of a sequence of regular models.[8,31] The usual notions of norm and strong operator convergence cannot apply because of the unboundedness of the operators and the fact that the domain of the singular Hamiltonian is coupling-dependent; however, we can resort to the notions of *resolvent* and *dynamical* convergence.

Recall that, given a family $(T_n)_{n\in\mathbb{N}}$ of self-adjoint operators on a Hilbert space $\mathscr{K}$ and a self-adjoint operator T, the sequence $(T_n)_{n\in\mathbb{N}}$ is said to converge to T:

- in the *norm* (*strong*, respectively) *resolvent sense* if, for all $z \in \mathbb{C}\backslash\mathbb{R}$, $\frac{1}{T_n-z} \to \frac{1}{T-z}$ in the norm (strong, respectively) sense, as $n \to \infty$, and
- in the *norm* (*strong*, respectively) *dynamical sense* if, for all $t \in \mathbb{R}$, $e^{-itT_n} \to e^{-itT}$ in the norm (resp. strong) sense, as $n \to \infty$.

The following result holds:

Theorem III.4 (Singular coupling limit). *Let $\varepsilon \in \mathbb{R}$.*

(i) *If $(g_n)_{n\in\mathbb{N}} \subset \mathscr{H}$ is a convergent sequence in the* norm *of $\mathscr{H}_{-2}$ with limit $g \in \mathscr{H}_{-2}$, then $H_{g_n,\varepsilon} \to H_{g,\varepsilon}$ in the* norm *resolvent sense and in the strong dynamical sense as $n \to \infty$.*
(ii) *Conversely, for every singular Friedrichs–Lee Hamiltonian $H_{g,\varepsilon}$, $g \in \mathscr{H}_{-2}$, there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset \mathscr{H}$ such that $H_{g_n,\varepsilon} \to H_{g,\varepsilon}$ in the* norm *resolvent sense and in the strong dynamical sense as $n \to \infty$.*

Proof. First of all, notice that norm resolvent convergence obviously implies strong resolvent convergence and the latter is *equivalent* to strong dynamical convergence (see Ref. 20), and hence, we only need to prove the results about norm resolvent convergence.

(i) A direct calculation shows that the resolvent operator in Eq. (19), with form factor $g_n \in \mathscr{H}$ and $g \in \mathscr{H}_{-2}$, respectively, can be written as the sum of the resolvent of the uncoupled operator $H_{0,\varepsilon}$ plus a finite-rank term, namely,

$$\frac{1}{H_{g_n,\varepsilon}-z} = \frac{1}{H_{0,\varepsilon}-z} + K_n, \tag{26}$$

$$\frac{1}{H_{g,\varepsilon}-z} = \frac{1}{H_{0,\varepsilon}-z} + K, \tag{27}$$

where for all $\Psi = \begin{pmatrix} x \\ \xi \end{pmatrix} \in \mathbb{C}\oplus\mathscr{H}$,

$$K_n\Psi = \frac{1}{\varepsilon - z - \Sigma_{g_n}(z)}\begin{pmatrix} x\dfrac{\Sigma_{g_n}(z)}{\varepsilon - z} - \langle \dfrac{1}{\Omega-\bar{z}}g_n|\xi\rangle \\ x\dfrac{1}{\Omega-z}g_n + \langle\dfrac{1}{\Omega-\bar{z}}g_n|\xi\rangle\dfrac{1}{\Omega-z}g_n \end{pmatrix} \tag{28}$$

and

$$K\Psi = \frac{1}{\varepsilon - z - \Sigma_g(z)}\begin{pmatrix} x\frac{\Sigma_g(z)}{\varepsilon - z} - \langle \frac{1}{\Omega - \bar{z}} g|\xi\rangle \\ x\frac{1}{\Omega - z}g + \langle \frac{1}{\Omega - \bar{z}}g|\xi\rangle \frac{1}{\Omega - z}g \end{pmatrix}. \tag{29}$$

Hence, the claim is proven if we show that for all $z \in \mathbb{C}\backslash\mathbb{R}$,

- $\left\| \frac{1}{\Omega - z} g_n - \frac{1}{\Omega - z} g \right\| \to 0$, as $n \to \infty$, and
- $\Sigma_{g_n}(z) \to \Sigma_g(z)$, as $n \to \infty$.

We recall that $g_n \to g$, as $n \to \infty$, in the norm of $\mathscr{H}_{-2}$, means that $\left\| \frac{1}{\Omega - i} g_n - g \right\| \to 0$, and equivalently, by the first resolvent formula, $\left\| \frac{1}{\Omega - z} g_n - \frac{1}{\Omega - z} g \right\| \to 0$, for every $z \in \mathbb{C}\backslash\mathbb{R}$; moreover, by continuity of the pairing between $\mathscr{H}_2$ and $\mathscr{H}_{-2}$, this also proves $\Sigma_{g_n}(z) \to \Sigma_g(z)$.

(ii) Let $g \in \mathscr{H}_{-2}$. Since $\mathscr{H}$ is dense in $\mathscr{H}_{-2}$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $g_n \to g$, as $n \to \infty$, in the norm of $\mathscr{H}_{-2}$, and hence, as in (i), a sequence of regular Friedrichs–Lee Hamiltonians converges in the norm resolvent sense to the singular one.

□

Remark III.5. If $g \in \mathscr{H}_{-2}$, the approximating sequence of Hamiltonians $H_{g_n,\varepsilon}$ is characterized by a diverging bare excitation energy $\varepsilon_{a,n} = \varepsilon + \langle g_n | \frac{\Omega}{\Omega^2+1} g_n \rangle$ *and* a diverging bare self-energy $\tilde{\Sigma}_{g_n}(z) = \langle g_n | \frac{1}{\Omega - z} g_n \rangle$; their difference converges to a finite limit that depends on the value of the dressed excitation energy ε; this is clearly a renormalization procedure, as discussed in Remark III.3.

Also notice that Theorem III.4 holds even if the dressed energy ε of the approximating sequence, instead of being kept fixed, is replaced with a converging sequence $\varepsilon_n \to \varepsilon$.

Remark III.6. There is an interesting connection between the Friedrichs–Lee model and rank-one perturbations of self-adjoint operators. Given a Hilbert space $\mathscr{K}$, a self-adjoint operator T on $\mathscr{K}$, and a functional $\varphi \in \mathscr{K}_{-2}$, consider the formal operator

$$T + \alpha|\varphi\rangle\langle\varphi|, \qquad \alpha \in \mathbb{R}. \tag{30}$$

If $\varphi \in \mathscr{K}_{-2}\backslash\mathscr{K}$, this is only a formal expression, with which one can associate a well-defined self-adjoint operator through a restriction–extension procedure[1,22,23,27] that we briefly recall in the following. Consider the densely defined symmetric operator T_φ obtained by restricting T to $\ker(\langle\varphi|)$; one can prove that its adjoint T_φ^* is the operator with domain

$$D(T_\varphi^*) = \left\{ \xi - x\frac{T}{T^2+1}\varphi \,\middle|\, \xi \in D(T),\, x \in \mathbb{C} \right\} \tag{31}$$

acting as

$$T_\varphi^*\left(\xi - x\frac{T}{T^2+1}\varphi\right) = T\xi + x\frac{1}{T^2+1}\varphi. \tag{32}$$

Then, for $\varphi \in \mathscr{K}_{-1}\backslash\mathscr{K}$, the restriction $T_{\varphi,\alpha}$ of T_φ^* to the domain

$$D(T_{\varphi,\alpha}) = \left\{ \xi - x\frac{T}{T^2+1}\varphi \,\middle|\, \xi \in D(T),\, x \in \mathbb{C},\, \langle\varphi|\xi\rangle = -x\left(\frac{1}{\alpha} + c_\varphi\right) \right\}, \tag{33}$$

with $c_\varphi = \langle\varphi|\frac{T}{T^2+1}\varphi\rangle$, is a self-adjoint operator that corresponds to a singular rank-one perturbation of T: indeed, if one applies the formal expression (30) on vectors in the above domain, all terms outside the Hilbert space $\mathscr{K}$ cancel out because of the constraint in the domain definition.

If, instead, $\varphi \in \mathscr{K}_{-2}\backslash\mathscr{K}_{-1}$, there is an issue: the action of $\langle\varphi|$ on $\frac{T}{T^2+1}\varphi$ is not defined. However, we can still fix some arbitrary $c \in \mathbb{R}$ and define $T_{\varphi,\alpha}^c$ as the restriction of T_φ^* to the domain

$$D(T_{\varphi,\alpha}^c) = \left\{ \xi - x\frac{T}{T^2+1}\varphi \,\middle|\, \xi \in D(T),\, x \in \mathbb{C},\, \langle\varphi|\xi\rangle = -x\left(\frac{1}{\alpha} + c\right) \right\}. \tag{34}$$

Interestingly, in both cases, the dependence on the parameter α is in the domain of the operator and not in its action. Besides, in the case $\varphi \in \mathscr{K}_{-2}\backslash\mathscr{K}_{-1}$, the operator depends on $1/\alpha + c$ rather than on α and c separately.

Now we present an alternative procedure obtained by introducing an extra degree of freedom in the system, i.e., by searching for self-adjoint realizations on the larger Hilbert space $\mathbb{C} \oplus \mathscr{K}$ rather than on $\mathscr{K}$. Let us define the operator $\tilde{T}_{\varphi,\alpha}^c$ with the domain

$$D(\tilde{T}_{\varphi,\alpha}^c) = \left\{ \begin{pmatrix} x \\ \xi - x\frac{T}{T^2+1}\varphi \end{pmatrix} \,\middle|\, \xi \in D(T),\, x \in \mathbb{C} \right\} \tag{35}$$

acting as

$$\tilde{T}^{c}_{\varphi,\alpha}\begin{pmatrix} x \\ \xi - x\frac{T}{T^2+1}\varphi \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{\alpha}+c\right)x + \langle\varphi|\xi\rangle \\ T\xi + x\frac{1}{T^2+1}\varphi \end{pmatrix}. \tag{36}$$

In practice, we are handling the "diverging" terms by enlarging the Hilbert space instead of imposing a constraint: the dependence on α (also on c) is therefore moved to the action of the operator rather than to its domain.

Now, by choosing $\mathscr{K} = \mathscr{H}$, Ω as the multiplication operator by ω, and $\varphi = g$, the operator $\tilde{T}^{c}_{\varphi,\alpha}$ corresponds to a singular Friedrichs–Lee Hamiltonian $H_{g,\varepsilon}$ with the dressed excitation energy

$$\varepsilon = \frac{1}{\alpha} + c. \tag{37}$$

Notice that the freedom in the choice of c reflects the fact that, in the Friedrichs–Lee model, a bare excitation energy is not defined for $g \notin \mathscr{H}_{-1}$: the operator really depends only on $1/\alpha + c$, in the same way as the Friedrichs–Lee Hamiltonian ultimately depends on the dressed energy ε alone.

IV. SPECTRAL PROPERTIES

After having introduced the model, let us characterize its spectral properties with respect to two common decompositions of the spectrum of a self-adjoint operator:

- absolutely continuous, singular continuous, and pure point spectrum and
- essential and discrete spectrum

(the discrete spectrum being the set of all isolated eigenvalues of finite multiplicity[20]).

In the absence of coupling (i.e., for $g = 0$), the spectrum of the Friedrichs–Lee Hamiltonian $H_{0,\varepsilon}$ is obviously

$$\sigma(H_{0,\varepsilon}) = \{\varepsilon\} \cup \sigma(\Omega), \tag{38}$$

with $\sigma(\Omega)$ being the spectrum of Ω, i.e., the closure of the μ-essential range of ω,

$$\mu\text{- Ran}(\omega) = \left\{\lambda \in \mathbb{R} \,|\, \mu\left(\omega^{-1}((\lambda-\delta,\lambda+\delta))\right) > 0, \quad \forall\delta > 0\right\}, \tag{39}$$

which coincides with the support of the induced measure ν_Ω defined as

$$\nu_\Omega(B) = \int_{\omega^{-1}(B)} \mathrm{d}\mu(k) \tag{40}$$

for every Borel set $B \subset \mathbb{R}$.

Here and henceforth, the *support of a Borel measure* ν on $\mathbb{R}$ is the set

$$\operatorname{supp}(\nu) = \{\lambda \in \mathbb{R} \,|\, \nu((\lambda-\delta,\lambda+\delta)) > 0, \quad \forall\delta > 0\}. \tag{41}$$

This is the smallest closed set C such that $\nu(\mathbb{R}\backslash C) = 0$ and is also known as the set of growth points of ν or the spectrum of ν.

Moreover, by abusing the English language with the use of an adjective to modify rather than limiting a noun, we define a *minimal support* (or an essential support) of the measure ν a set M such that $\nu(\mathbb{R}\backslash M) = 0$ and that every Borel subset $M_0 \subset M$ with $\nu(M_0) = 0$ has also zero Lebesgue measure. Notice that a minimal support is not unique: minimal supports may differ by sets of zero Lebesgue and ν measure, and it can happen either that $\operatorname{supp}(\nu)$ is a minimal support or that it is not—there exists always a minimal support of ν whose closure coincides with $\operatorname{supp}(\nu)$, but, in general, the closure of a minimal support may differ from $\operatorname{supp}(\nu)$ by a set of nonzero Lebesgue measure.

The absolutely continuous (ac), singular continuous (sc), and pure point (pp) components of $\sigma(\Omega)$ coincide with the supports of the ac, sc, and pp components of ν_Ω, whence

- $\sigma_{\mathrm{ac}}(H_{0,\varepsilon}) = \sigma_{\mathrm{ac}}(\Omega)$,
- $\sigma_{\mathrm{sc}}(H_{0,\varepsilon}) = \sigma_{\mathrm{sc}}(\Omega)$, and
- $\sigma_{\mathrm{pp}}(H_{0,\varepsilon}) = \{\varepsilon\} \cup \sigma_{\mathrm{pp}}(\Omega)$,
  with Ψ_0 being the eigenvector in Eq. (9) belonging to ε. At the physical level, $\operatorname{supp}(\nu_\Omega) = \sigma(\Omega)$ is the energy space of the boson. As for the distinction between the essential and discrete spectrum, in the most general case, we have
- $\sigma_{\mathrm{ess}}(H_{0,\varepsilon}) = \sigma_{\mathrm{ess}}(\Omega)$ and
- $\sigma_{\mathrm{dis}}(H_{0,\varepsilon})\backslash\{\varepsilon\} = \sigma_{\mathrm{dis}}(\Omega)$,
  with ε belonging to the discrete spectrum $\sigma_{\mathrm{dis}}(H_{0,\varepsilon})$ if and only if ε is isolated from the spectrum of Ω.

We want to find a complete characterization of the spectral properties of $H_{g,\varepsilon}$ with respect to the spectrum of Ω for a generic form factor $g \in \mathscr{H}_{-2}$. First of all, let us examine the behavior of the discrete/essential decomposition of the spectrum.

Theorem IV.1. *Let $\varepsilon \in \mathbb{R}$ and $g \in \mathscr{H}_{-2}$. The essential spectrum of $H_{g,\varepsilon}$ in (16) and (17) coincides with the essential spectrum of Ω, with the possible exception of the accumulation points of the eigenvalues of Ω.*

Proof. Suppose $g \in \mathscr{H}$, i.e., consider the regular model. Then, using the matrix representation for $H_{g,\varepsilon}$, we can write

$$H_{g,\varepsilon} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \Omega \end{pmatrix} + \begin{pmatrix} \varepsilon_{\mathrm{a}} - \varepsilon & \langle g| \\ g & 0 \end{pmatrix} \equiv H_{0,\varepsilon} + K_{g,\varepsilon}, \tag{42}$$

with ε_{a} as in Eq. (18). $K_{g,\varepsilon}$ is finite-rank and hence leaves the essential spectrum unchanged, and thus, $\sigma_{\mathrm{ess}}(H_{g,\varepsilon}) = \sigma_{\mathrm{ess}}(H_{0,\varepsilon}) = \sigma_{\mathrm{ess}}(\Omega)$.

If $g \in \mathscr{H}_{-2} \backslash \mathscr{H}$, Theorem III.4 ensures that $H_{g,\varepsilon}$ will be the norm resolvent limit of a sequence of regular models sharing the same essential spectrum. Under these conditions, the norm resolvent limit preserves the essential spectrum with the possible exception of accumulation points of the eigenvalues of Ω.[20] This proves the claim. □

Remark IV.2. As a consequence of Theorem IV.1, when $\sigma_{\mathrm{ess}}(\Omega)$ is entirely continuous or dense pure point, it will coincide with the essential spectrum of the corresponding Friedrichs–Lee operator. However, there may be conversion of the dense pure point spectrum into continuous spectrum or vice versa.

Now, let us study the decomposition of the spectrum $\sigma(H_{g,\varepsilon})$ into its absolutely continuous, singular continuous, and pure point components. We define the *coupling measure* associated with Ω and g as

$$\kappa_g(B) = \int_{\omega^{-1}(B)} |g(k)|^2 \, \mathrm{d}\mu(k) \tag{43}$$

for all Borel sets $B \subset \mathbb{R}$.

We have the following result:

Theorem IV.3. *Let $\varepsilon \in \mathbb{R}$ and $g \in \mathscr{H}_{-2}$. Let $\mathscr{H}_g = L^2(X_g, \mu)$, with*

$$X_g = \omega^{-1}(\operatorname{supp}(\kappa_g)), \tag{44}$$

and let $\mathscr{H}_g^{\perp}$ be its orthogonal complement. Let $G_g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the function

$$G_g(\lambda) = \int_{\mathbb{R}} \frac{1}{(\lambda - \lambda')^2} \, \mathrm{d}\kappa_g(\lambda'), \tag{45}$$

and let Σ_g^+ be the boundary value on the real axis of the self-energy function Σ_g in (20), defined almost everywhere by

$$\Sigma_g^+(\lambda) = \lim_{\delta \downarrow 0} \Sigma_g(\lambda + i\delta). \tag{46}$$

Then,

- $\sigma_{\mathrm{ac}}(H_{g,\varepsilon}) = \sigma_{\mathrm{ac}}(\Omega)$,
- $\sigma_{\mathrm{pp}}(H_{g,\varepsilon}) = \sigma_{\mathrm{pp}}\left(\Omega|_{\mathscr{H}_g^{\perp}}\right) \cup \overline{\left\{\lambda \in \mathbb{R} \,\middle|\, \varepsilon - \lambda = \Sigma_g^+(\lambda),\ G_g(\lambda) < \infty\right\}}$,
- $\sigma_{\mathrm{sc}}\left(H_{g,\varepsilon}|_{\mathscr{H}_g^{\perp}}\right) = \sigma_{\mathrm{sc}}\left(\Omega|_{\mathscr{H}_g^{\perp}}\right)$,
- *the set $\left\{\lambda \in \mathbb{R} \,\middle|\, \varepsilon - \lambda = \Sigma_g^+(\lambda),\ G_g(\lambda) = \infty\right\}$ is a minimal support for a maximal singular continuous measure of $H_{g,\varepsilon}$, and*
- *the restrictions of Ω and $H_{g,\varepsilon}$ to their absolutely continuous subspaces are unitarily equivalent.*

Remark IV.4. This result can be explained as follows: First of all, the space X of field momenta can be split into a subset X_g of momenta, which are effectively coupled to the atom (i.e., on which the form factor g is μ-supported) and a complementary subset of uncoupled momenta; this subdivision induces a correspondent subdivision of the energy space (i.e., the support of ν_Ω) into coupled and uncoupled energies, i.e., the support of κ_g and its complement.

As expected, the uncoupled part of the spectrum is independent of g and hence, particularly, is the same as in the case $g = 0$. As for the coupled one, it turns out that the absolutely continuous spectrum is still unchanged, but the singular (i.e., pure point and singular continuous) spectrum will be minimally supported by the set of solutions of the equation

$$\varepsilon - \lambda = \Sigma_g^+(\lambda). \tag{47}$$

Finally, notice that the pole equation (47) admits the unique solution $\lambda = \varepsilon$ only when $g = 0$ and, in this case, necessarily $G(\lambda) = 0$, and hence, ε is in the pure point spectrum; our result is in full agreement with the uncoupled case discussed above.

Remark IV.5. If one substitutes g with βg for some $\beta \in \mathbb{R}$ and hence $\Sigma_{\beta g}(z) = \beta^2 \Sigma_g(z)$, the singular spectrum becomes supported by the set of solutions of the equation $\Sigma_g^+(\lambda) = \frac{\varepsilon - \lambda}{\beta^2}$, which will be a different set for every value of β. Therefore, the singular spectrum of the Friedrichs–Lee Hamiltonian $H_{g,\varepsilon}$ is highly coupling-dependent.

The Proof of Theorem IV.3 will be given in Sec. IV C. First, we need some mathematical preliminaries.

A. The self-energy as a Borel transform

Definition IV.6. Let ν be a Borel measure on $\mathbb{R}$ satisfying the growth condition

$$\int_{\mathbb{R}} \frac{1}{1+\lambda^2} \, \mathrm{d}\nu(\lambda) < \infty. \tag{48}$$

Its (regularized) *Borel transform* is the function $B_\nu : \mathbb{C}\backslash\mathrm{supp}(\nu) \to \mathbb{C}$ acting as

$$B_\nu(z) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) \mathrm{d}\nu(\lambda). \tag{49}$$

Remark IV.7. In the literature, the (standard) Borel transform of a Borel measure ν on $\mathbb{R}$ is often defined as follows:

$$\tilde{B}_\nu(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, \mathrm{d}\nu(\lambda).$$

However, this definition makes sense for a smaller class of measure, i.e., measures ν satisfying the growth condition

$$\int_{\mathbb{R}} \frac{1}{1+|\lambda|} \, \mathrm{d}\nu(\lambda) < \infty. \tag{50}$$

For such measures, $\tilde{B}_\nu(z)$ and $B_\nu(z)$ only differ by a finite real constant; this difference is, in fact, immaterial for our purposes, since, as we will show in the next proposition, ν depends only on the imaginary part of the boundary values of $B_\nu(z)$ on the real line, but the choice (49) is more convenient since it is well-defined for a larger class of measures. Indeed, for a Friedrichs–Lee Hamiltonian with form factor g, we have

- $g \in \mathscr{H}$ iff $\int_{\mathbb{R}} \mathrm{d}\kappa_g(\lambda) < \infty$,
- $g \in \mathscr{H}_{-1}$ iff $\int_{\mathbb{R}} \frac{1}{1+|\lambda|} \, \mathrm{d}\kappa_g(\lambda) < \infty$, and
- $g \in \mathscr{H}_{-2}$ iff $\int_{\mathbb{R}} \frac{1}{1+\lambda^2} \, \mathrm{d}\kappa_g(\lambda) < \infty$,

and hence, in particular, the case $g \in \mathscr{H}_{-2}\backslash\mathscr{H}_{-1}$ corresponds to the case in which κ_g does not admit a standard Borel transform [i.e., the bare self-energy $\tilde{\Sigma}_g(z)$] but does have a regularized Borel transform.

Proposition IV.8. Let $g \in \mathscr{H}_{-2}$. The self-energy Σ_g of a Friedrichs–Lee Hamiltonian defined in Eq. (20) has the following properties:

(i) *Σ_g is analytic in $\mathbb{C}\backslash\mathrm{supp}(\kappa_g)$ and $\mathrm{Im}\,\Sigma_g(z) > 0$ for all $z \in \mathbb{C}^+$, i.e., it is a Herglotz function.*[13]
(ii) *The coupling measure κ_g in (43) can be uniquely reconstructed from the imaginary part of Σ_g by the Stieltjes inversion formula,*

$$\frac{1}{2}\left(\kappa_g((\lambda_0, \lambda)) + \kappa_g([\lambda_0, \lambda])\right) = \frac{1}{\pi} \lim_{\delta \downarrow 0} \int_{\lambda_0}^{\lambda} \mathrm{Im}\Sigma_g(\lambda + i\delta) \, \mathrm{d}\lambda. \tag{51}$$

(iii) *The boundary values of Σ_g along $\mathrm{supp}(\kappa_g)$ are linked as follows to the Lebesgue decomposition of κ_g:*

1. $\mathrm{supp}(\kappa_g) = \overline{\{\lambda \in \mathbb{R} \,|\, \mathrm{Im}\Sigma_g^+(\lambda) > 0\}}$,

2. $M_{\mathrm{ac}} = \{\lambda \in \operatorname{supp}(\kappa_g) \,|\, 0 < \operatorname{Im}\Sigma_g^+(\lambda) < \infty\}$ *is a minimal support of* κ_g^{ac}*, and the density of* κ_g^{ac} *is* $\rho_g(\lambda) = \frac{1}{\pi}\operatorname{Im}\Sigma_g^+(\lambda)$,
3. $M_{\mathrm{sc}} = \{\lambda \in \operatorname{supp}(\kappa_g) \,|\, \operatorname{Im}\Sigma_g^+(\lambda) = \infty, \lim_{\delta\downarrow 0}\delta\operatorname{Im}\Sigma_g(\lambda + i\delta) = 0\}$ *is a minimal support of* κ_g^{sc}*, and*
4. $\operatorname{supp}(\kappa_g^{\mathrm{pp}}) = \overline{M}_{\mathrm{pp}}$*, where* $M_{\mathrm{pp}} = \{\lambda \in \operatorname{supp}(\kappa_g) \,|\, \operatorname{Im}\Sigma_g^+(\lambda) = \infty,\ \lim_{\delta\downarrow 0}\delta\operatorname{Im}\Sigma_g(\lambda + i\delta) > 0\}$*, and* $\kappa_g(\{\lambda\}) = \lim_{\delta\downarrow 0}\delta\operatorname{Im}\Sigma_g(\lambda + i\delta)$,

(iv) *Let* $G_g(\lambda)$ *be the function defined in* (45)*, and then,*

1. $\lim_{\delta\downarrow 0}\frac{1}{\delta}\operatorname{Im}\Sigma_g(\lambda + i\delta) = G_g(\lambda)$*, and*
2. *in addition, if* $G_g(\lambda) < \infty$*, then* $\Sigma_g^+(\lambda)$ *is finite and real and* $\Sigma_g(\lambda + i\delta) = \Sigma_g^+(\lambda) + i\delta G_g(\lambda) + o(\delta)$*, as* $\delta\downarrow 0$*, i.e.,* $G_g(\lambda)$ *is the upper derivative of* $\Sigma_g(z)$ *in the direction of the imaginary axis.*

Proof. By (43), the self-energy of the Friedrichs–Lee Hamiltonian $H_{g,\varepsilon}$ in Eq. (20) can be written as

$$\Sigma_g(z) = \int_{\mathbb{R}}\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right)\mathrm{d}\kappa_g(\lambda). \tag{52}$$

Therefore, $\Sigma_g(z) = B_{\kappa_g}(z)$, i.e., the self-energy is the regularized Borel transform of κ_g, and the properties (i)–(iv) follow from the general theory of Borel transform (see Ref. 13 and references therein).

□

Remark IV.9. A commonly shared name for analytic maps of the upper complex half-plane into itself is lacking in the mathematical literature, as discussed in Ref. 13: what is here called "Herglotz function" is elsewhere referred to as Nevanlinna, Pick, Nevanlinna–Pick, or R-function. This is a good example where Arnold's principle (see the Preface of Ref. 28) definitely applies. In this paper, we will use the name Herglotz, which seems to be more common in the mathematical physics literature.

B. Cyclic subspaces and spectral properties

In Subsection IV A, we have shown the link between the self-energy Σ_g and the properties of κ_g; now, we will link the latter with the spectral properties of the Friedrichs–Lee Hamiltonian. We will start from some basic definitions.

Definition IV.10. Let T be a self-adjoint operator on a Hilbert space $\mathscr{K}$ and let $\varphi \in \mathscr{K}_{-2}$. The cyclic subspace $\mathscr{K}_\varphi$ of $\mathscr{K}$ spanned by φ is defined as follows:

$$\mathscr{K}_\varphi = \overline{\operatorname{Span}\left\{\frac{1}{T-z}\varphi \,\middle|\, z \in \mathbb{C}\backslash\mathbb{R}\right\}}. \tag{53}$$

In particular, if $\mathscr{K}_\varphi = \mathscr{K}$, φ is called a cyclic vector and T is said to have a simple spectrum.

Proposition IV.11 (Refs. 20 and 27). Let T be a self-adjoint operator on a Hilbert space $\mathscr{K}$, and let $\varphi \in \mathscr{K}_{-2}$. The following properties hold:

- $\mathscr{K}_\varphi$ *is a reducing subspace for* T*, and hence,* $T = T|_{\mathscr{K}_\varphi} \oplus T|_{\mathscr{K}_\varphi^\perp}$.
- $T|_{\mathscr{K}_\varphi}$ *is unitarily equivalent to the position operator* (i.e., *the multiplication operator by the identity function*) *on the Hilbert space* $L^2(\mathbb{R}, \nu_\varphi)$*, where* ν_φ *is the spectral measure of* T *at* φ*, defined by* $\nu_\varphi(B) = \langle\varphi|E_T(B)\varphi\rangle$*, for all bounded Borel sets* $B \subset \mathbb{R}$*, with* $E_T(\cdot)$ *being the spectral projection of* T.
- $\sigma\left(T|_{\mathscr{K}_\varphi}\right) = \operatorname{supp}(\nu_\varphi)$ *and* $\sigma_j\left(T|_{\mathscr{K}_\varphi}\right) = \operatorname{supp}(\nu_\varphi^j)$ *for* $j \in \{\mathrm{ac}, \mathrm{sc}, \mathrm{pp}\}$.

Remark IV.12. In our case, $\mathscr{K} = \mathscr{H} = L^2(X,\mu)$, $T = \Omega$, and $\varphi = g$. The cyclic subspace $\mathscr{H}_g$ spanned by g, as a consequence of Stone–Weierstrass theorem, is equal to $L^2(X_g, \mu)$, with X_g given by Eq. (44); this justifies the use of the notation $\mathscr{H}_g = L^2(X_g,\mu)$ in Theorem IV.3. In particular, $\Omega|_{\mathscr{H}_g}$ and $\Omega|_{\mathscr{H}_g^\perp}$ are the multiplication operators by the restriction of ω to X_g and $X\backslash X_g$, respectively. Finally, g is cyclic iff its support is a set of full measure μ.

Remark IV.13. As a consequence of Propositions IV.8 and IV.11, the spectrum of $\Omega|_{\mathscr{H}_g}$ and its decomposition can be studied by analyzing the boundary values of the (regularized) Borel transform (49) of the coupling measure κ_g.

Proposition IV.14. Let $\varepsilon \in \mathbb{R}$, $g \in \mathscr{H}_{-2}$, and $g \neq 0$, and let $H_{g,\varepsilon}$ be the corresponding Friedrichs–Lee Hamiltonian (16) *and* (17)*. Then,*

(i) *the cyclic subspace of* $\mathbb{C} \oplus \mathscr{H}$ *spanned by* Ψ_0 *in Eq.* (9) *is* $\mathbb{C}\oplus\mathscr{H}_g$*, with* $\mathscr{H}_g$ *being the cyclic subspace of* $\mathscr{H}$ *spanned by* g*. In particular, if* g *is cyclic in* $\mathscr{H}$*,* Ψ_0 *is cyclic in* $\mathbb{C} \oplus \mathscr{H}$*, and*

(ii) *$H_{g,\varepsilon}|_{(\mathbb{C}\oplus\mathscr{H}_g)^\perp} = H_{0,\varepsilon}|_{(\mathbb{C}\oplus\mathscr{H}_g)^\perp}$, and hence, the spectrum of $H_{g,\varepsilon}|_{(\mathbb{C}\oplus\mathscr{H}_g)^\perp}$ is the same as that of the uncoupled case.*

Proof. Let $\mathscr{K} = \mathbb{C} \oplus \mathscr{H}$. By definition, the cyclic subspace $\mathscr{K}_{\Psi_0}$ spanned by Ψ_0 is

$$\mathscr{K}_{\Psi_0} = \overline{\operatorname{Span}\left\{ \frac{1}{H_{g,\varepsilon} - z}\Psi_0 \,\middle|\, z \in \mathbb{C}\backslash\mathbb{R} \right\}}, \tag{54}$$

where, by Eq. (19), for any $z \in \mathbb{C}\backslash\mathbb{R}$, we have

$$\frac{1}{H_{g,\varepsilon} - z}\Psi_0 = \frac{1}{\varepsilon - z - \Sigma_g(z)}\begin{pmatrix} 1 \\ -\frac{1}{\Omega - z}g \end{pmatrix}. \tag{55}$$

To prove that $\mathscr{K}_{\Psi_0} = \mathbb{C}\oplus\mathscr{H}_g$, we will prove the equivalent equality $\mathscr{K}_{\Psi_0}^\perp = (\mathbb{C}\oplus\mathscr{H}_g)^\perp = \{0\}\oplus\mathscr{H}_g^\perp$.

Let $\Psi = \begin{pmatrix} x \\ \xi \end{pmatrix}$ be orthogonal to vectors of the form (55) such that

$$\frac{1}{\varepsilon - z - \Sigma_g(z)}\left[x - \left\langle \xi \,\middle|\, \frac{1}{\Omega - z}g \right\rangle\right] = 0, \qquad \forall z \in \mathbb{C}\backslash\mathbb{R}. \tag{56}$$

This implies that $x = \left\langle \xi \,\middle|\, \frac{1}{\Omega - z}g \right\rangle$ for all $z \in \mathbb{C}\backslash\mathbb{R}$ and hence necessarily that

$$x = 0, \qquad \left\langle \xi \,\middle|\, \frac{1}{\Omega - z}g \right\rangle = 0 \quad \forall z \in \mathbb{C}\backslash\mathbb{R}, \tag{57}$$

with the second equality meaning that $\xi \in \mathscr{H}_g^\perp$. This proves (i). Besides, by the expression of the Hamiltonian in Eq. (17), for any $\xi \in \mathscr{H}_g^\perp \cap D(\Omega)$, we have

$$H_{g,\varepsilon}\begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ \Omega\xi \end{pmatrix} \tag{58}$$

so that $\{0\}\oplus\mathscr{H}_g^\perp$ is invariant under $H_{g,\varepsilon}$ (which is true, in general, for any cyclic subspace) and, particularly, the action of $H_{g,\varepsilon}$ on it is independent of g. This proves (ii). □

C. Proof of Theorem IV.3

We are now ready to prove Theorem IV.3.

Proof. By Proposition IV.14, we know that $H_{g,\varepsilon}|_{(\mathbb{C}\oplus\mathscr{H}_g)^\perp} = H_{0,\varepsilon}|_{(\mathbb{C}\oplus\mathscr{H}_g)^\perp}$, so we must only find the spectrum of the restriction of $H_{g,\varepsilon}$ to the cyclic subspace $\mathbb{C}\oplus\mathscr{H}_g$. To simplify the notation, without loss of generality, let us suppose g cyclic for Ω; then, Proposition IV.14 implies that Ψ_0 is cyclic for $H_{g,\varepsilon}$. Since both operators, Ω and $H_{g,\varepsilon}$, are self-adjoint on their Hilbert spaces, by Proposition IV.11, the two operators are equivalent to the position operators on $L^2(\mathbb{R}, \kappa_g)$ and $L^2(\mathbb{R}, \nu_{\Psi_0})$, respectively, with ν_{Ψ_0} being the spectral measure of $H_{g,\varepsilon}$ at Ψ_0; finally, by Proposition IV.8, the ac, sc, and pp components of the maximal spectral measures κ_g and ν_{Ψ_0} of both operators can be inferred by the boundary values of the imaginary parts of their Borel transforms.

Now, the (regularized) Borel transform of κ_g is the self-energy $\Sigma_g(z)$. The (standard) Borel transform of ν_{Ψ_0} is defined as follows:

$$\Pi_g(z) = \left\langle \Psi_0 \,\middle|\, \frac{1}{H_{g,\varepsilon} - z}\Psi_0 \right\rangle, \tag{59}$$

where we dropped the regularizing term, which is not needed because $\Psi_0 \in \mathbb{C} \oplus \mathscr{H}$, and hence, ν_{Ψ_0} is a finite measure. A straightforward calculation yields

$$\Pi_g(z) = \frac{1}{\varepsilon - z - \Sigma_g(z)}, \tag{60}$$

and hence, for any $\lambda \in \mathbb{R}$ and $\delta > 0$,

$$\operatorname{Im}\Pi_g(\lambda + i\delta) \sim \frac{\operatorname{Im}\Sigma_g(\lambda + i\delta)}{|\varepsilon - \lambda - \Sigma_g(\lambda + i\delta)|^2}, \quad \text{as } \delta \downarrow 0. \tag{61}$$

First of all, let us examine the absolutely continuous components. By Eq. (61), $\operatorname{Im}\Sigma_g(\lambda + i\delta)$ has a finite nonzero limit if and only if $\operatorname{Im}\Pi_g(\lambda + i\delta)$ has a finite nonzero limit; by Proposition IV.8, this proves that κ_g^{ac} and $\nu_{\Psi_0}^{\mathrm{ac}}$ are minimally supported on (Lebesgue-almost everywhere) equal sets; since they are absolutely continuous measures, this means that their densities are supported on (Lebesgue-almost everywhere) equal sets, and hence, the position operators on the corresponding L^2 spaces are unitarily equivalent. This proves the equality between the absolutely continuous spectra of Ω and $H_{g,\varepsilon}$.

Now let λ be in a minimal support of the singular spectrum of $H_{g,\varepsilon}$; by Proposition IV.8 and Eq. (61), this happens iff $\varepsilon - \lambda = \Sigma_g^+(\lambda)$ [since, if $\operatorname{Im}\Sigma_g(\lambda + i\delta)$ diverges as $\delta \downarrow 0$, the denominator diverges faster], also implying that $\Sigma_g^+(\lambda)$ is real. To distinguish between the pure point and singular continuous components, we must examine the limiting value of $\delta\Pi_g(\lambda + i\delta)$ as $\delta \downarrow 0$. We have

$$\delta \operatorname{Im} \Pi_g(\lambda + i\delta) \sim \frac{\frac{1}{\delta}\operatorname{Im}\Sigma_g(\lambda + i\delta)}{\frac{1}{\delta^2}\left(\varepsilon - \lambda - \operatorname{Re}\Sigma_g(\lambda + i\delta)\right)^2 + \frac{1}{\delta^2}\operatorname{Im}\Sigma_g(\lambda + i\delta)^2}. \tag{62}$$

Now, when $G_g(\lambda) = \infty$, the denominator in (62) diverges faster than the numerator, and hence, $\delta\Pi_g(\lambda + i\delta) \to 0$ as $\delta \downarrow 0$. Besides, when $G_g(\lambda) < \infty$, the first term in the denominator in (62) vanishes since $\operatorname{Re}\Sigma_g(\lambda + i\delta) \sim \varepsilon - \lambda + o(\delta)$, and hence, $\left(\varepsilon - \lambda - \operatorname{Re}\Sigma_g(\lambda + i\delta)\right)^2 \sim o(\delta^2)$; hence, we are left with

$$\delta \operatorname{Im} \Pi_g(\lambda + i\delta) \sim \frac{\delta}{\operatorname{Im}\Sigma_g(\lambda + i\delta)} \to \frac{1}{G_g(\lambda)} > 0, \quad \text{as } \delta \downarrow 0. \tag{63}$$

As a result, the singular continuous component of ν_{Ψ_0} is minimally supported on the set

$$\left\{\lambda \in \mathbb{R} \,\middle|\, \varepsilon - \lambda = \Sigma_g^+(\lambda),\ G_g(\lambda) = \infty\right\}, \tag{64}$$

and the support $\operatorname{supp}(\nu_{\Psi_0}^{\mathrm{pp}})$ of the pure point component of ν_{Ψ_0} is

$$\operatorname{supp}(\nu_{\Psi_0}^{\mathrm{pp}}) = \overline{\left\{\lambda \in \mathbb{R} \,\middle|\, \varepsilon - \lambda = \Sigma_g^+(\lambda),\ G_g(\lambda) < \infty\right\}}. \tag{65}$$

□

Remark IV.15. Notice that $\operatorname{Im}\Sigma_g^+(\lambda) = \infty$ iff $\lim_{\delta\downarrow 0} \operatorname{Im}\Pi_g(\lambda + i\delta) = 0$. In the case in which the singular spectra of both Ω and $H_{g,\varepsilon}$ are purely discrete, this means that inside the support of the spectral measure, the eigenvalues of Ω and $H_{g,\varepsilon}$ are completely disjoint: physically, no stable state of the field with energy coupled to the atom preserves its stability.

As for the energy ε of the excited atom, which is an eigenvalue of the uncoupled operator,

- if $\varepsilon \notin \operatorname{supp}(\kappa_g)$, it will also be an eigenvalue of the coupled operator and
- if $\varepsilon \in \operatorname{supp}(\kappa_g)$, then it is an eigenvalue of the coupled operator if it satisfies the following equation:

$$\Sigma_g^+(\varepsilon) = 0. \tag{66}$$

In particular, if the spectral density of the absolutely continuous part of κ_g is continuous, it must vanish at ε.

Remark IV.16. The eigensystem for the Friedrichs–Lee Hamiltonian can be solved explicitly. Let $\lambda \in \mathbb{R}$, $x \in \mathbb{C}$, and $\xi \in D(\Omega)$ such that

$$H_{g,\varepsilon}\begin{pmatrix} x \\ \xi - x\dfrac{\Omega}{\Omega^2 + 1}g \end{pmatrix} = \lambda\begin{pmatrix} x \\ \xi - x\dfrac{\Omega}{\Omega^2 + 1}g \end{pmatrix}; \tag{67}$$

a direct calculation shows that λ must solve Eq. (47) and

$$\xi(k) = -x\left(\frac{1}{\omega(k) - \lambda} - \frac{\omega(k)}{\omega(k)^2 + 1}\right)g(k), \tag{68}$$

where the condition on λ ensures ξ to be square-integrable.

Remark IV.17. An important consequence of Theorem IV.3 is that, modulo a possible uncoupled part, the spectrum of the Friedrichs–Lee Hamiltonian with a given ε depends entirely on the coupling measure κ_g or equivalently on the self-energy $\Sigma_g(z)$, which can always be reconstructed from the coupling measure through the inversion formula (51). Different choices of the momentum space (X, μ), of the dispersion relation ω and of the form factor g, but yielding the same κ_g and the same uncoupled part are fully equivalent at the spectral level.

To this respect, it is worth noticing that *every* Borel measure on $\mathbb{R}$ satisfying the growth condition (48) can be obtained as the coupling measure of a Friedrichs–Lee Hamiltonian. Indeed, if ν is the desired coupling measure, just choose $(X, \mu) = (\mathbb{R}, \nu)$, $\omega(k) = k$, and $g(k) = 1$ for all $k \in \mathbb{R}$. Then, one immediately obtains

$$\kappa_g(B) = \int_{\omega^{-1}(B)} |g(k)|^2 \, \mathrm{d}\mu(k) = \int_B \mathrm{d}\nu(\lambda) = \nu(B) \tag{69}$$

for any Borel set $B \subset \mathbb{R}$.

V. SOME EXAMPLES

In this section, we discuss the spectral properties of some interesting examples of Friedrichs–Lee models. The section is organized as follows:

- Examples V.1–V.3 concern the case in which $\sigma(\Omega)$ is purely absolutely continuous.
- Example V.4 explores a purely discrete $\sigma(\Omega)$.
- Finally, in Example V.5, we investigate a pure point $\sigma(\Omega)$, with dense eigenvalues in $[0, 1]$, which becomes singular continuous when the coupling is switched on.

Some considerations are now in order. Suppose that $\sigma(\Omega)$ is purely absolutely continuous in some (possibly unbounded) closed interval $J \subset \mathbb{R}$, and hence,

- $\sigma_{\mathrm{ac}}(H_{0,\varepsilon}) = J$ and
- $\sigma_{\mathrm{pp}}(H_{0,\varepsilon}) = \{\varepsilon\}$,

where ε can also be in J. Now, switching on a coupling g, we will still have $\sigma_{\mathrm{ac}}(H_{g,\varepsilon}) = J$. In particular, ε is again in the absolutely continuous spectrum of $H_{g,\varepsilon}$ if and only if $\varepsilon \in J$, but in general, it will not be in $\sigma_{\mathrm{pp}}(H_{g,\varepsilon})$. However, $\varepsilon \in \sigma_{\mathrm{pp}}(H_{g,\varepsilon})$ if and only if $\Sigma_g^+(\varepsilon) = 0$, i.e., if ε is a zero for the density (assumed to be continuous) of κ_g and, in addition, $\mathrm{Re}\Sigma_g^+(\varepsilon) = 0$. Physically, the eigenvalue becomes unstable whenever it lies in a set of coupled values of energy, except when the coupling density vanishes at that point. Of course, depending on the choice of κ_g, the pure point spectrum may contain other elements. In particular, if the coupling density is nonzero on the whole real line, the singular spectrum is empty and the eigenstate Ψ_0 in Eq. (9) of the uncoupled Hamiltonian $H_{0,\varepsilon}$ becomes unstable.

More generally, if some $\lambda \in \mathbb{R}$ is a zero of the coupling density [hence $\mathrm{Im}\Sigma_g^+(\lambda) = 0$], a necessary and sufficient condition for it to be in the singular spectrum is that the equation $\varepsilon - \lambda = \mathrm{Re}\Sigma_g^+(\lambda)$ is fulfilled.

Example V.1 (Lebesgue coupling measure on $\mathbb{R}$). Suppose that $J = \mathbb{R}$ and the coupling measure (43) reads

$$\mathrm{d}\kappa_g(\lambda) = \frac{\beta}{2\pi} \, \mathrm{d}\lambda \tag{70}$$

for some $\beta > 0$. This is obtained from a Friedrichs–Lee model with dispersion relation $\omega(k) = k$ on $L^2(\mathbb{R})$ and a flat form factor $g(k) = \sqrt{\beta/2\pi}$ (see Remark IV.17). Then, a straightforward calculation shows that

$$\Sigma_g(z) = \begin{cases} \dfrac{i\beta}{2}, & \mathrm{Im}\, z > 0, \\ -\dfrac{i\beta}{2}, & \mathrm{Im}\, z < 0. \end{cases} \tag{71}$$

This implies that the pole equation (47) does not have any solution, and hence, $\sigma(H_{g,\varepsilon})$ is purely absolutely continuous. The uncoupled eigenvalue ε "dissolves" in the continuum for any nonzero value of the coupling β. Physically, the bound state with energy ε becomes unstable. Indeed, one can show that the evolution of the state Ψ_0 in Eq. (9),

$$\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = U_{H_{g,\varepsilon}}(t)\Psi_0, \tag{72}$$

yields $x(t) = e^{-\left(\frac{\beta}{2} + i\varepsilon\right)t}$, and hence, $|x(t)|^2 = e^{-\beta t}$: an exponential decay takes place. Notice that a purely exponential decay law at both short and large times is possible since

- the initial state Ψ_0 is not in the domain of $H_{g,\varepsilon}$, but it is in the domain of $H_{0,\varepsilon}$, since, being κ_g Lebesgue, $g \in \mathscr{H}_{-2} \setminus \mathscr{H}$,[10] and
- Ω is unbounded both from below and from above, since $\sigma(\Omega) = \mathrm{supp}(\kappa_g) = \mathbb{R}$, and hence, Paley–Wiener's theorem, which prohibits an exponential decay at large times, does not apply.[9,17,21]

In Sec. VI, we will interpret this result in the framework of the theory of resonances.

Example V.2 (Lebesgue coupling measure on $[0,\infty)$*).* As a second example, consider now a Friedrichs–Lee Hamiltonian with dispersion relation $\omega(k) = k$ on $\mathscr{H} = L^2(0,\infty)$ and a flat form factor $g(k) = \sqrt{\beta}$ for some $\beta > 0$ so that the coupling measure (43) reads

$$\mathrm{d}\kappa_g(\lambda) = \beta\chi_{[0,\infty)}(\lambda)\,\mathrm{d}\lambda. \tag{73}$$

In this case, the uncoupled spectrum is composed of the eigenvalue $\{\varepsilon\}$ and an absolutely continuous part in $J = [0,\infty)$. Again, the self-energy can be evaluated exactly,

$$\Sigma_g(z) = -\beta\,\log(-z), \tag{74}$$

with log being the principal value of the complex logarithm, i.e., $\log(-z) = \log|z| + i\mathrm{Arg}(-z)$, with $Arg(z) \in (-\frac{\pi}{2},\frac{\pi}{2})$. One can check that this function is indeed analytic in $\mathbb{C}\backslash[0,\infty)$ and has a branch cut along the support of the measure. Let us search for the singular spectrum when the coupling is switched on. Solutions of the pole equation (47) must be searched in $(-\infty,0)$ since the coupling density is nonzero in $[0,\infty)$; we have

$$\beta\,\log(-\lambda) = -\varepsilon + \lambda, \tag{75}$$

and hence,

$$-\frac{\lambda}{\beta}e^{-\lambda/\beta} = \frac{1}{\beta}e^{-\varepsilon/\beta}, \tag{76}$$

which has a unique real solution expressed through the principal branch W_0 of Lambert's W-function (or product-log function),[7]

$$E(\varepsilon,\beta) = -\beta\,W_0\left(\frac{1}{\beta}e^{-\varepsilon/\beta}\right). \tag{77}$$

Hence, we have a unique eigenvalue $E(\varepsilon,\beta)$ for $H_{g,\varepsilon}$. It is interesting to study the asymptotic behavior of $E(\varepsilon,\beta)$ as a function of the excitation energy ε of the atom. From the properties of the Lambert function,

$$W_0(x) \sim \log x, \quad \text{as } x \to \infty, \qquad W_0(x) \sim x, \quad \text{as } x \to 0, \tag{78}$$

we have

$$E(\varepsilon,\beta) \sim \varepsilon, \quad \text{as } \varepsilon \to -\infty, \qquad E(\varepsilon,\beta) \sim -e^{-\varepsilon/\beta}, \quad \text{as } \varepsilon \to \infty. \tag{79}$$

This means that, when ε is far away from the lowest energy level of the inner Hamiltonian Ω, the coupled eigenvalue $E(\varepsilon,\beta)$ is close to ε, and hence, the spectrum is nearly unchanged. When ε approaches and eventually reaches $\sigma(\Omega)$, the approximation $E(\varepsilon,\beta) \sim \varepsilon$ is no longer valid and, as $\varepsilon \to \infty$, $E(\varepsilon,\beta)$ approaches the boundary of $\sigma(\Omega)$.

Example V.3 (Sinusoidal coupling measure). Now, we consider a Friedrichs–Lee Hamiltonian in which the coupling density has support on the whole real line but admits some isolated zeros. Let the coupling measure (43) be

$$\mathrm{d}\kappa_g(\lambda) = \frac{\beta}{2\pi}(1-\cos(\tau\lambda))\,\mathrm{d}\lambda \tag{80}$$

for some $\beta > 0$ and $\tau \in \mathbb{R}$, i.e., an absolutely continuous measure with sinusoidal density. This can be obtained from a Friedrichs–Lee model with the dispersion relation $\omega(k) = k$ on $L^2(\mathbb{R})$ and the form factor $g(k) = (\beta(1-\cos\tau k)/2\pi)^{1/2}$. As in Example V.1, here, the uncoupled operator $H_{0,\varepsilon}$ has a spectrum composed of the eigenvalue ε embedded in an absolutely continuous spectrum covering the whole real line; however, here, the coupling density vanishes at $\lambda_j = 2j\pi/\tau$, with $j \in \mathbb{Z}$.

The self-energy is

$$\Sigma_g(z) = \begin{cases} \dfrac{i\beta}{2}\left(1-e^{i\tau z}\right), & \text{Im } z > 0, \\ \dfrac{-i\beta}{2}\left(1-e^{-i\tau z}\right), & \text{Im } z < 0, \end{cases} \tag{81}$$

which is indeed discontinuous on the whole real line except for the zeros λ_j; moreover, it is a periodic function. The pole equation (47) reads

$$\begin{cases} \varepsilon - \lambda = \dfrac{\beta}{2} \sin \tau\lambda, \\ \cos \tau\lambda = 1. \end{cases} \tag{82}$$

The second equation is satisfied only when, as anticipated, λ is one of the zeros of the (continuous) coupling density; if so, the first equation simply becomes $\varepsilon - \lambda = 0$. The following phenomenon occurs: the singular spectrum of $H_{g,\varepsilon}$ is empty except for some "resonant" values of the parameter ε, namely, $\varepsilon = \frac{2\pi j}{\tau}$ for some $j \in \mathbb{Z}$—when this happens, $\sigma_{\rm pp}(H_{g,\varepsilon}) = \{\varepsilon\}$.

Example V.4 (Periodic discrete coupling measure). Consider a Friedrichs–Lee model with the dispersion relation $\omega(k) = k$ and the flat form factor $g(k) = \sqrt{\beta/2\pi}$ ($\beta > 0$) on $L^2(\mathbb{R}, \mu)$, where $\mu = \sum_{j\in\mathbb{Z}} \delta_{j\tau}$ ($\tau > 0$) with δ_k being the Dirac measure at $k \in \mathbb{R}$. It is easy to show that the coupling measure (43) is

$$\kappa_g = \frac{\beta}{2\pi} \sum_{j\in\mathbb{Z}} \delta_{j\tau}, \tag{83}$$

i.e., κ_g is supported on $\tau\mathbb{Z}$. A direct calculation shows that

$$\Sigma_g(z) = -\frac{\beta}{2} \cot\left(\frac{\pi z}{\tau}\right),$$

which can be extended to the real line except for the poles at $\tau\mathbb{Z}$. The spectrum of the uncoupled operator will be purely singular and consists of the solutions of the equation

$$\cot\left(\frac{\pi\lambda}{\tau}\right) = \frac{2}{\beta}(\lambda - \varepsilon),$$

which admits a countable set of isolated solutions $\{E_j(\varepsilon, \beta, \tau)\}_{j\in\mathbb{Z}}$ (hence, the spectrum of $H_{g,\varepsilon}$ is again pure point), where $E_j(\varepsilon, \beta, \tau) \in (j\tau, (j+1)\tau)$. In particular, each $E_j(\varepsilon, \beta)$ varies smoothly with β, and

- $E_j(\varepsilon, \beta, \tau) \to \tau j$, as $\beta \to 0$, i.e., in the limit of small coupling, we recover the uncoupled spectrum of Ω, and
- $E_j(\varepsilon, \beta, \tau) \to \tau\left(j + \frac{1}{2}\right)$, as $\beta \to \infty$, i.e., in the limit of large coupling, the spectrum is rigidly shifted by $\frac{\tau}{2}$.

Different from the previous cases, the singular spectrum (which is again pure point) is nonempty for every value of the parameters and indeed contains a countable number of points.

Example V.5 (Generation of a singular continuous spectrum). This example is an adaptation of Example 2 in Ref. 29 to our framework. Recall that a real number in $[0, 1]$ is said to be a dyadic rational of order $n \in \mathbb{N}$ if it can be written in the form $j/2^n$ for some integer j. Dyadic rationals of any order are the numbers whose expansion in base 2 is finite; such numbers are dense in $[0, 1]$.

For any integer $n \geq 1$, let us consider the normalized Borel measure

$$\nu_n = \frac{1}{2^n} \sum_{j=1}^{2^n} \delta_{j/2^n} \tag{84}$$

and a sequence of positive numbers $(a_n)_{n\in\mathbb{N}}$. We define the measure

$$\nu = \sum_{n=1}^{\infty} a_n \nu_n. \tag{85}$$

ν is therefore a pure point measure with points on the dense set of dyadic rationals between 0 and 1; besides, it is a finite measure iff $\sum_n a_n < \infty$, the latter sum being $\nu(\mathbb{R}) = \nu([0, 1])$.

Consider a Friedrichs–Lee model $H_{g,\varepsilon}$ with field momentum space $(X, \mu) = (\mathbb{R}, \nu)$, dispersion relation $\omega(k) = k$, and flat form factor $g(k) = \beta^{1/2}$, with $\beta > 0$, so that the coupling measure reads $\kappa_g = \beta\nu$. Hence, the spectrum of the uncoupled Hamiltonian $H_{0,\varepsilon}$ is entirely pure point, consisting of the dyadic rationals in $[0, 1]$ and (if not already dyadic) the atom excitation energy ε. By Theorem IV.3, we know that, by switching on the atom–field interaction, the new spectrum will be entirely singular (since creation of an absolutely continuous spectrum is prohibited) and will be the closure of set of all solutions of the pole equation (47). The discriminant between singular continuous and pure point spectrum is given by the value of the function $G_g(\lambda)$ in (45). In our case,

$$G_g(\lambda) = \int_{\mathbb{R}} \frac{1}{(\lambda-\lambda')^2}\, \mathrm{d}\kappa_g(\lambda') = \beta \sum_{n=1}^{\infty} a_n L_n(\lambda), \tag{86}$$

with

$$L_n(\lambda) = \int_{\mathbb{R}} \frac{1}{(\lambda-\lambda')^2}\, \mathrm{d}\nu_n(\lambda') = \frac{1}{2^n}\sum_{j=1}^{2^n} \frac{1}{\left(\lambda - \frac{j}{2^n}\right)^2}. \tag{87}$$

If λ is dyadic, then $G_g(\lambda) = \infty$. Besides, for any n, there will be some integer $j_0 \in \{1,\ldots,n\}$ for which the quantity $|\lambda - j_0 2^{-n}|$ is the smallest among the others, i.e., $j_0 2^{-n}$ is the dyadic of order n that is closest to λ. Therefore, $|\lambda - j_0 2^{-n}|$ is smaller than the spacing between any two consecutive dyadic rationals of order n, which equals 2^{-n}. This means that

$$\left|\lambda - \frac{j_0}{2^n}\right| \le 2^{-n}, \tag{88}$$

and thus,

$$\frac{1}{2^n}\frac{1}{\left(\lambda - \frac{j_0}{2^n}\right)^2} \ge \frac{1}{2^n} 2^{2n} = 2^n. \tag{89}$$

In other words, for each n, the sum in the definition of $L_n(\lambda)$ contains one term, which is larger than 2^n; hence, $L_n(\lambda) > 2^n$, and thus,

$$G_g(\lambda) \ge \beta \sum_{n=1}^{\infty} a_n 2^n. \tag{90}$$

This means that if we choose $(a_n)_{n\in\mathbb{N}}$ in such a way that

$$\sum_{n=1}^{\infty} a_n 2^n = \infty, \tag{91}$$

then, necessarily, $G_g(\lambda) = \infty$ for *every* $\lambda \in [0,1]$, meaning that the spectrum of $H_{g,\varepsilon}$ will be fully singular continuous. Interestingly, the same phenomenon happens for every value of $\beta > 0$, however small; this is an example of *instability* of the dense set of eigenvalues under perturbations.

VI. RESONANCES

We now complete the discussion on the spectral properties of the Friedrichs–Lee operator by studying the resonances of the model. In the following, we will denote by $\mathbb{C}^\pm := \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}$ the open upper and lower half-planes, and for any subset $A \subset \mathbb{C}$, we will use the notation $A^\pm := A \cap \mathbb{C}^\pm$ and $A^0 := A \cap \mathbb{R}$ for its components in $\mathbb{C}^\pm$ and $\mathbb{R}$, respectively, so that A is the disjoint union $A = A^+ \cup A^- \cup A^0$.

Let us recall the definition of a resonance for a self-adjoint operator T.

Definition VI.1. Let T be a self-adjoint operator on a Hilbert space $\mathscr{K}$, and let $z_0 \in \mathbb{C}$ with $\operatorname{Im} z_0 < 0$. Then, z_0 is a *resonance* for T if there is some $\psi \in \mathscr{K}$ such that the function

$$z \in \mathbb{C}^+ \mapsto R_\psi(z) = \left\langle \psi \,\middle|\, \frac{1}{T-z}\psi \right\rangle \in \mathbb{C} \tag{92}$$

admits a meromorphic continuation from the upper to the lower half-plane having a pole at z_0.

Remark VI.2. The function (92) is a Herglotz function; indeed, it is the (standard) Borel transform of the (finite) spectral measure of T at the vector ψ, and hence, the properties listed in Proposition IV.8 hold true; in particular, every singularity of R_ψ lies on the real line. In particular, if $\sigma(T)$ has an absolutely continuous component along some interval $J \subset \mathbb{R}$, then R_ψ has a branch cut along J with finite boundary values and thus will admit an analytic continuation "through the cut" from the upper to the lower plane.[13]

A pole at z_0 of the meromorphic continuation is identified as a resonance, since it yields a contribution proportional to $e^{-iz_0 t}$ to the survival amplitude of ψ; if z_0 is close to the real line, this contribution may dominate at large times and make ψ a metastable state with energy $\operatorname{Re} z_0$ and decay rate $|\operatorname{Im} z_0|$.

Remark VI.3. The definition of resonance presented above is not the only possible one (see the discussions in Ref. 26 and in p. 165 of Ref. 14). In scattering situations, quantum resonances are most commonly defined with respect to a Hamiltonian pair (T, T_0), with T_0 playing the role of a free Hamiltonian and T being the perturbed one: in this framework, they are associated with poles of the meromorphic continuation of the S-matrix, and it must be ensured (see p. 55 of Ref. 25) that the function

$$z \in \mathbb{C}^+ \mapsto R^0_\psi(z) = \left\langle \psi \,\middle|\, \frac{1}{T_0 - z}\psi \right\rangle \in \mathbb{C} \tag{93}$$

admits a meromorphic continuation from the upper to the lower half-plane, which is analytic at the resonance z_0. In our case, the role of the Hamiltonian pair (T, T_0) can be played by the coupled and uncoupled Friedrichs–Lee Hamiltonians, for which it is not difficult to show that the aforementioned condition holds.

Now, for any $\psi \in \mathscr{K}$, *real* poles of R_ψ, i.e., real resonances, are obviously simple poles of the resolvent of T, i.e., *eigenvalues* of T. Conversely, an eigenvalue λ_0 of T is *not* necessarily a pole of R_ψ for all nonzero $\psi \in \mathscr{K}$, since the spectral measure of T at ψ may not be supported in a neighborhood of λ_0 (e.g., if ψ is an eigenvector of T with eigenvalue $\lambda \neq \lambda_0$); however, if we consider any dense subset $\mathscr{D}$ of $\mathscr{K}$, necessarily, there will be some $\psi \in \mathscr{D}$ such that R_ψ has a pole at λ_0.

It would be useful to characterize *non-real* resonances in a similar way. Aguilar–Balslev–Combes–Simon theory of resonances[2,6,26] (see also Ref. 14 and 31) allows us to identify resonances of a self-adjoint operator T as the (complex) eigenvalues of a "deformed" Hamiltonian $T(w)$, with w being a complex parameter.

We will apply this formalism to the Friedrichs–Lee Hamiltonian $H_{g,\varepsilon}$; for this purpose, we will need some assumptions on the structure of the inner Hamiltonian Ω and of the form factor g. First of all, we will make an assumption about the spectral properties of Ω.

Hypothesis 1. There is an interval $J \subset \mathbb{R}$ such that $J \subset \sigma_{\mathrm{ac}}(\Omega)$ and $J \cap (\sigma_{\mathrm{sc}}(\Omega) \cup \sigma_{\mathrm{pp}}(\Omega)) = \emptyset$.

Remark VI.4. Notice that, by Theorem IV.1 and Hypothesis 1, $J \subset \sigma_{\mathrm{ess}}(\Omega)$ and $\sigma_{\mathrm{dis}}(\Omega) \cap J = \emptyset$.

Before introducing other hypotheses, we need the following definition:

Definition VI.5. A *spectral deformation family* is a family $(U(w))_{w \in W^0}$ of linear operators on $\mathscr{H}$ with the following properties:

- $W^0 \subset \mathbb{R}$ is an open and connected neighborhood of 0.
- $U(w)$ is unitary for all $w \in W^0$, and $U(0) = I$.
- $(U(w))_{w \in W^0}$ admits a dense set $\mathscr{A}$ of *analytic vectors* such that for any $\psi \in \mathscr{A}$, the map $w \in W^0 \mapsto \psi(w) = U(w)\psi$ has an analytic continuation in an open and connected set $W \subset \mathbb{C}$, with $W^0 = W \cap \mathbb{R}$; moreover, $\mathscr{A}(w) = \{\psi(w) \mid \psi \in \mathscr{A}\}$ is dense in $\mathscr{H}$ for all $w \in W$.

Given a spectral deformation family $(U(w))_{w \in W^0}$, we assume that for all $w \in W^0$,

$$U(w)D(\Omega) = D(\Omega) \tag{94}$$

and define the deformation of the inner Hamiltonian Ω as

$$w \in W^0 \mapsto \Omega(w)\psi = U(w)\Omega U(w)^*\psi \tag{95}$$

for all $\psi \in D(\Omega)$.

More generally, we will need that the operators $\Omega(w)$ are embedded in a *type-A analytic* family, as follows:

Hypothesis 2. There is a family of closed operators $(\Omega(w))_{w \in W}$, defined on the common domain $D(\Omega)$, such that, for all $\psi \in D(\Omega)$, the map

$$w \in W \mapsto \Omega(w)\psi \in \mathscr{H} \tag{96}$$

is the analytic continuation from W^0 into W of the map (95).

Notice that for $w \in W^0$, $\Omega(w)$ is unitarily equivalent to Ω and, particularly, the two operators share the same spectrum; for non-real w, the spectrum of $\Omega(w)$ in general differs from that of Ω. For our purposes, we must require that the essential spectrum of $\Omega(w)$ in (96) changes "nicely" as a function of $w \in W$, in the sense that it must be always possible, for any non-real w, to "move continuously" through J from the upper to the lower complex half-plane so that the matrix elements of the resolvent of $\Omega(w)$ are analytic continuation of those of Ω through J. In order to accomplish these properties, we will make the following assumption:

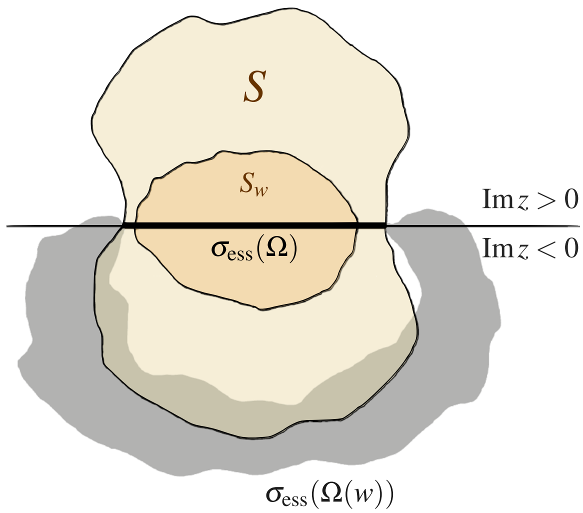

FIG. 1. Graphical representation of Hypothesis 3. For any $w \in W^+$, the set S^+ does not intersect the essential spectrum of the deformed Hamiltonian $\Omega(w)$ (gray region). Moreover, there exists a set $S_w \subset S$ such that $S_w^- \cup S_w^0$ does not intersect the essential spectrum of $\Omega(w)$.

Hypothesis 3. There exists an open connected subset $S \supset J$ of the complex plane such that, for every $w \in W^+$, the following properties hold (see Fig. 1):

- $\sigma_{\mathrm{ess}}(\Omega(w)) \cap S^+ = \emptyset$.
- *There exists some open connected $S_w \subset S$, with $S_w^0 \neq \emptyset$, such that $\sigma_{\mathrm{ess}}(\Omega(w)) \cap (S_w^- \cup S_w^0) = \emptyset$.*

In other words, we require that the region S is such that, for any $w \in W^+$,

- the set S^+ does not contain elements of the essential spectrum of the deformed Hamiltonian $\Omega(w)$ and
- there is a subset $S_w \subset S$ whose component with the nonpositive imaginary part, i.e., $S_w^- \cup S_w^0$, does not intersect the essential spectrum of $\Omega(w)$.

Hypotheses 1–3 would be sufficient to characterize all resonances of Ω as eigenvalues of its deformations $\Omega(w)$ (see Ref. 14); since we are interested in characterizing the resonances of $H_{g,\varepsilon}$, we will need additional hypotheses on the form factor $g \in \mathscr{H}_{-2}$.

First of all, we define the deformation of the form factor g,

$$w \in W^0 \mapsto \langle g(w)|\psi\rangle = \langle g|U(w)^*\psi\rangle \tag{97}$$

for all $\psi \in D(\Omega)$ and require that the functionals $g(w)$ are embedded in an analytic family.

Hypothesis 4. There exists a family $(g(w))_{w\in W} \subset \mathscr{H}_{-2}$ such that for all $\psi \in D(\Omega)$, the map

$$w \in W \mapsto \langle g(w)|\psi\rangle \in \mathbb{C} \tag{98}$$

is the anti-analytic continuation from W^0 into W of the map in (97).

Finally, we will need the following assumption on the form factor g.

Hypothesis 5. For all $\lambda \in J$, the boundary value Σ_g^+ in (46) of the self-energy has a positive imaginary part, namely,

$$\mathrm{Im}\Sigma_g^+(\lambda) > 0. \tag{99}$$

Remark VI.6. Notice that, by Theorem IV.3, Hypothesis 1, and Hypothesis 5, it follows that $J \subset \sigma_{\mathrm{ac}}(H_{g,\varepsilon})$ and $J \cap (\sigma_{\mathrm{sc}}(H_{g,\varepsilon}) \cup \sigma_{\mathrm{pp}}(H_{g,\varepsilon})) = \emptyset$.

Hypotheses 4 and 5 on the form factor g, together with Hypotheses 1–3 on the inner Hamiltonian Ω, allow us to characterize all resonances of the Friedrichs–Lee Hamiltonian $H_{g,\varepsilon}$.

Theorem VI.7. *Let $H_{g,\varepsilon}$ a Friedrichs–Lee Hamiltonian (16) and (17) with the excitation energy $\varepsilon \in \mathbb{R}$, inner Hamiltonian Ω, and form factor $g \in \mathscr{H}_{-2}$. Suppose that Hypotheses 1–5 hold true, and define, for all $w \in W$, the operator $H_{g,\varepsilon}(w)$ as follows:*

$$D(H_{g,\varepsilon}(w)) = \left\{ \left(\begin{matrix} x \\ \xi - x\dfrac{\Omega(w)}{\Omega(w)^2+1} g(w) \end{matrix} \right) \middle| \, x \in \mathbb{C}, \ \xi \in D(\Omega) \right\}, \tag{100}$$

$$H_{g,\varepsilon}(w) \left(\begin{matrix} x \\ \xi - x\dfrac{\Omega(w)}{\Omega(w)^2+1} g(w) \end{matrix} \right) = \left(\begin{matrix} \varepsilon x + \langle g(w) | \xi \rangle \\ \Omega(w)\xi + x\dfrac{1}{\Omega(w)^2+1} g(w) \end{matrix} \right). \tag{101}$$

Let $S_W^- = \bigcup_{w \in W^+} S_w^-$. Then, the following holds:

(i) *There is a dense set $\tilde{\mathscr{A}} \subset \mathbb{C} \oplus \mathscr{H}$ of analytic vectors such that, for every $\Psi \in \tilde{\mathscr{A}}$, the function*

$$z \in \mathbb{C}^+ \mapsto \left\langle \Psi \middle| \frac{1}{H_{g,\varepsilon} - z} \Psi \right\rangle \in \mathbb{C} \tag{102}$$

has a meromorphic continuation across J from $\mathbb{C}^+$ to S_W^-.

(ii) *$z_0 \in S_W^-$ is a resonance of $H_{g,\varepsilon}$ if and only if z_0 is an eigenvalue of $H_{g,\varepsilon}(w_0)$ for some $w_0 \in W^+$.*

Proof. Our proof will be organized as follows: First of all, we will prove that $H_{g,\varepsilon}$ satisfies properties analogous to the ones fulfilled by Ω:

(a) $H_{g,\varepsilon}$ has a purely absolutely continuous spectrum on J.

(b) For all $w \in W^0$, we define
$$\tilde{U}(w) = I \oplus U(w), \qquad \tilde{\mathscr{A}} = \mathbb{C} \oplus \mathscr{A}; \tag{103}$$
then, it results that $(\tilde{U}(w))_{w \in W^0}$ is a spectral deformation family with a dense set of analytic vectors $\tilde{\mathscr{A}}$. Moreover, for all $w \in W^0$, the Hamiltonian $H_{g,\varepsilon}(w)$ in Eq. (101) coincides with $\tilde{U}(w) H_{g,\varepsilon} \tilde{U}(w)^*$ and the map $w \in W^0 \mapsto H_{g,\varepsilon}(w) = \tilde{U}(w) H_{g,\varepsilon} \tilde{U}(w)^*$ admits an analytic continuation from W^0 into W.

(c) For every $w \in W^+$, $\sigma_{\mathrm{ess}}(H_{g,\varepsilon}(w)) \cap S^+ = \emptyset$ and $\sigma_{\mathrm{ess}}(H_{g,\varepsilon}(w)) \cap (S_w^- \cup S_w^0) = \emptyset$; then, following an analogous argument as in Ref. 14, we will deduce (i) and (ii) from (a)–(c).

Property (a) follows directly from Theorems IV.1 and IV.3 using Hypotheses 1 and 5, as explained in Remarks VI.4 and VI.6.

Let us prove property (b). It is straightforward that $\tilde{\mathscr{A}}$ and $(\tilde{U}(w))_{w \in W}$ in (103) have the desired properties. To prove the equality $H_{g,\varepsilon}(w) = \tilde{U}(w) H_{g,\varepsilon} \tilde{U}(w)^*$ for all $w \in W^0$, notice that, for any $w \in W^0$, the domain of $\tilde{U}(w) H_{g,\varepsilon} \tilde{U}(w)^*$ is

$$\begin{aligned} \tilde{U}(w) D(H_{g,\varepsilon}) &= \left\{ \left(\begin{matrix} x \\ U(w)\xi' - xU(w)\dfrac{\Omega}{\Omega^2+1} g \end{matrix} \right) \middle| \, x \in \mathbb{C}, \ \xi' \in D(\Omega) \right\} \\ &= \left\{ \left(\begin{matrix} x \\ \xi - xU(w)\dfrac{\Omega}{\Omega^2+1} g \end{matrix} \right) \middle| \, x \in \mathbb{C}, \ \xi \in D(\Omega) \right\}, \end{aligned} \tag{104}$$

where we have used the assumption $U(w)D(\Omega) = D(\Omega)$ in (94); besides,

$$U(w)\frac{\Omega}{\Omega^2+1} g = \frac{\Omega(w)}{\Omega(w)^2+1} g(w), \tag{105}$$

easily following from the equality $\Omega(w) = U(w)\Omega U(w)^*$ for $w \in W^0$, and hence, the domains (100) and (104) coincide. The equality between the actions of $H_{g,\varepsilon}(w)$ and $\tilde{U}(w)H_{g,\varepsilon}\tilde{U}(w)^*$ on the domain can be shown in a similar way.

Now, we prove (c) and (d). By Hypothesis 1, for every $w \in W^+$, $\Omega(w)$ has no essential spectrum in S^+, and hence, by Hypotheses 2–4, then the function

$$(z,w) \mapsto \Sigma_g(z,w) = \left\langle g(w) \,\middle|\, \left(\frac{1}{\Omega(w)-z} - \frac{\Omega(w)}{\Omega(w)^2+1}\right) g(w) \right\rangle \tag{106}$$

is meromorphic in $S^+ \times (W^+ \cup W^0)$. Moreover, since $U(w)$ is unitary for all $w \in W^0$, it results that

$$\forall z \in S^+,\ \forall w \in W^0:\ \Sigma_g(z,w) = \Sigma_g(z). \tag{107}$$

Hence, by the identity principle for meromorphic functions, we have

$$\forall z \in S^+,\ \forall w \in (W^+ \cup W^0):\ \Sigma_g(z,w) = \Sigma_g(z), \tag{108}$$

and hence, the function $z \in S^+ \times (W^+ \cup W^0) \mapsto \Sigma_g(z,w)$ is analytic because the self-energy $z \in S^+ \mapsto \Sigma_g(z)$ is analytic. Now, fix $w \in W^+$. By Hypothesis 3, there is $S_w \subset S$ such that $\sigma_{\mathrm{ess}}(\Omega(w))$ has no intersection with the set $S_w^- \cup S_w^0$. Hence, for this value of w, the map $z \in S^+ \mapsto \Sigma_g(z,w)$ can be meromorphically continued from S^+ to S_w^- across $S_w^0 \subset J$. Repeating the process for every $w \in W^+$, the function $z \in S^+ \mapsto \Sigma_g(z,w)$ is meromorphically continued from S^+ to $S_W^- \cup S_W^0$, obtaining the function

$$z \in S^+ \cup S_W^- \cup S_W^0 \mapsto \Sigma_g(z,w) \in \mathbb{C}; \tag{109}$$

this continuation is obviously unique and independent of w.

Now, we compute the resolvent of $H_{g,\varepsilon}(w)$ with $w \in W^+$. It can be easily proven, in the same way as for the non-deformed Friedrichs–Lee Hamiltonian, that for every $z \in S^+ \cup S_W^- \cup S_W^0$, with $z \notin \sigma(H_{g,\varepsilon}(w))$,

$$\frac{1}{H_{g,\varepsilon}(w)-z}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \dfrac{x - \left\langle g(w) \,\middle|\, \frac{1}{\Omega(w)-z}\xi \right\rangle}{\varepsilon - z - \Sigma_g(z,w)} \\ \dfrac{1}{\Omega(w)-z}\xi - \dfrac{x - \left\langle g(w) \,\middle|\, \frac{1}{\Omega(w)-z}\xi \right\rangle}{\varepsilon - z - \Sigma_g(z,w)} \dfrac{1}{\Omega(w)-z} g(w) \end{pmatrix}. \tag{110}$$

Therefore, the elements of the spectrum of $H_{g,\varepsilon}(w)$ will be either

- elements of the spectrum of $\Omega(w)$ or
- solutions of the equation $\varepsilon - z = \Sigma_g(z,w)$.

By Hypothesis (3), fixing $w \in W^+$, then $\sigma_{\mathrm{ess}}(\Omega(w)) \cap S^+ = \emptyset$ and $\sigma_{\mathrm{ess}}(\Omega(w)) \cap (S_w^0 \cup S_w^-) = \emptyset$, i.e., the region $S^+ \cup S_w^0 \cup S_w^-$ does not contain the essential spectrum of $\Omega(w)$. Moreover, since the function in (109) is meromorphic, the equation $\varepsilon - z = \Sigma_g(z,w)$ can only admit isolated solutions, i.e., the spectrum of $H_{g,\varepsilon}(w)$ can only have isolated eigenvalues in $S_w^- \cup S_w^0$.

Now, we can proceed with the proof of (i) and (ii).

(i) Let $\Psi \in \tilde{\mathscr{A}}$ and consider the following analytic function:

$$z \in \mathbb{C}^+ \mapsto R_\Psi(z) = \left\langle \Psi \,\middle|\, \frac{1}{H_{g,\varepsilon}-z}\Psi \right\rangle \in \mathbb{C}. \tag{111}$$

Since $(\tilde{U}(w))_{w\in W^0}$ is a spectral deformation family with a dense set of analytic vectors $\tilde{\mathscr{A}}$, the map $w \in W^0 \mapsto \Psi(w) = \tilde{U}(w)\Psi \in \mathbb{C} \oplus \mathscr{H}$ admits an analytic continuation in W. Now, we consider the function

$$(z,w) \in S^+ \times (W^+ \cup W^0) \mapsto R_\Psi(z,w) = \left\langle \Psi(w) \,\middle|\, \frac{1}{H_{g,\varepsilon}(w)-z}\Psi(w) \right\rangle \in \mathbb{C}; \tag{112}$$

using the expression for the resolvent in Eq. (110), Hypotheses 2–4, and (b) and (c), we obtain that the function in (112) is meromorphic. Moreover, it is easy to observe that, for all $w \in W^0$,

$$\left\langle \Psi(w) \,\middle|\, \frac{1}{H_{g,\varepsilon}(w)-z}\Psi(w) \right\rangle = \left\langle \tilde{U}(w)\Psi \,\middle|\, \tilde{U}(w)\frac{1}{H_{g,\varepsilon}-z}\Psi \right\rangle, \tag{113}$$

and thus, since $\tilde{U}(w)$ is unitary,

$$\forall z \in S^+,\ \forall w \in W^0:\ R_\Psi(z,w) = R_\Psi(z). \tag{114}$$

Hence, by the identity principle for meromorphic functions, we have

$$\forall z \in S^+, \ \forall w \in W^+ \cup W^0 : \ R_\Psi(z, w) = R_\Psi(z), \tag{115}$$

and so, the function $z \in S^+ \times (W^+ \cup W^0) \mapsto R_\Psi(z, w)$ is analytic because $z \in S^+ \mapsto R_\Psi(z)$ is analytic. Now, fix $w \in W^+$. By (c), we have $\sigma_{\mathrm{ess}}(H_{g,\varepsilon}(w)) \cap (S_w^- \cup S_w^0) = \emptyset$; hence, for this value of w, the map $z \in S^+ \mapsto R_\Psi(z, w)$ can be meromorphically continued from S^+ to S_w^- across $S_w^0 \subset J$. Repeating the process for every $w \in W^+$, the function $z \in S^+ \mapsto R_\Psi(z, w)$ can be meromorphically continued from S^+ to $S_W^- \cup S_W^0$, obtaining the function

$$z \in S^+ \cup S_W^- \cup S_W^0 \mapsto R_\Psi(z, w) \in \mathbb{C}; \tag{116}$$

this continuation is obviously unique and independent of w.

(ii) Let $z_0 \in S_W^-$ be a resonance for $H_{g,\varepsilon}$, i.e., $z_0 \in S_{w_0}^-$ for some $w_0 \in W^+$. By definition, there is some $\Psi \in \mathbb{C} \oplus \mathscr{H}$ such that the function

$$z \in \mathbb{C}^+ \mapsto R_\psi(z) = \left\langle \Psi \middle| \frac{1}{H_{g,\varepsilon} - z} \Psi \right\rangle \in \mathbb{C} \tag{117}$$

admits a meromorphic continuation from the upper to the lower half-plane having a pole at z_0. Since $\tilde{\mathscr{A}}$ is dense in $\mathbb{C} \oplus \mathscr{H}$, we can assume that $\Psi \in \tilde{\mathscr{A}}$. Then, by (116), the meromorphic continuation from the upper to the lower half-plane of (117) is the expectation value of the resolvent of $H_{g,\varepsilon}(w_0)$ at $\Psi(w_0)$, i.e., $\left\langle \Psi(w_0) \middle| \frac{1}{H_{g,\varepsilon}(w_0) - z} \Psi(w_0) \right\rangle$, and the poles of the latter object are necessarily eigenvalues of $H_{g,\varepsilon}(w_0)$; hence, z_0 is an eigenvalue of $H_{g,\varepsilon}(w_0)$.

Conversely, suppose that $z_0 \in S_W^-$ is an eigenvalue of $H_{g,\varepsilon}(w_0)$ for some $w_0 \in W^+$. Since $\tilde{\mathscr{A}}(w_0) = \{\Psi(w_0) \,|\, \Psi \in \tilde{\mathscr{A}}\}$ is dense in $\mathbb{C} \oplus \mathscr{H}$, there must be necessarily some $\Psi \in \tilde{\mathscr{A}}$ such that the map in (116) has a pole in z_0, and hence, z_0 is a resonance of $H_{g,\varepsilon}$. □

Example VI.8. Let Ω be the multiplication operator by a continuous dispersion relation ω on a momentum space (X, μ) with an absolutely continuous measure μ so that Hypothesis 1 is trivially satisfied. A spectral deformation family may be constructed by considering an *isometric global flow* R_w for $w \in W^0$, i.e., an operator acting on the momentum space X as follows:

$$(U(w)\psi)(k) = \mathscr{J}_w(k)^{1/2} \psi(R_w(k)) \tag{118}$$

for $\psi \in \mathscr{H}$, with $\mathscr{J}_w(k)$ being the Jacobian of the transformation. We must then find some open and connected $W \subset \mathbb{C}$, with $W^0 = W \cap \mathbb{R}$, such that

- there exists a dense set $\mathscr{A}$ of analytic vectors such that $w \in W^0 \mapsto \psi(w) = U(w)\psi \in \mathscr{H}$ has an analytic extension to the whole set W,
- the operator $\Omega(w) = U(w)\Omega U(w)^*$ defined, for $w \in W^0$, as

$$(\Omega(w)\psi)(k) = \omega(R_w(k))\psi(k) \tag{119}$$

  for all $\psi \in D(\Omega)$ admits a strongly analytic extension $(\Omega(w))_{w \in W}$ to W (Hypothesis 2), and
- the form factor $g(w)$ defined, for $w \in W^0$, as $g(w) = U(w)g$ admits an analytic extension to W (Hypothesis 4).

This means that the dispersion relation ω, the form factor g, and every function $\psi \in \mathscr{A}$, which are functions of the *real* variable k, must admit an extension as a function of a *complex* variable ranging in some open subset of the complex plane. Besides, the flow must be chosen in such a way that Hypothesis 3 holds, i.e., the spectrum of $\Omega(w)$ (i.e., its essential range) is deformed in the desired way to unearth the resonances.

As a concrete example, consider a Friedrichs–Lee operator on $L^2(\mathbb{R})$ with the dispersion relation $\omega(k) = k$ [hence, with $\sigma(\Omega) = \mathbb{R}$] and flat singular form factor $g(k) = \sqrt{\beta/2\pi}$, $\beta > 0$; this is a realization of a Friedrichs–Lee model with flat coupling measure studied in Example V.1. Define, for any $w \in \mathbb{R}$,

$$R_w(k) = k - w, \tag{120}$$

whose Jacobian for real w is 1; hence, we obtain

$$(U(w)\psi)(k) = \psi(k-w), \qquad (\Omega(w)\psi)(k) = (k-w)\psi(k). \tag{121}$$

The extension of $\Omega(w)$ to complex values of w, with $D(\Omega(w)) = D(\Omega)$, is immediate; besides,

$$\sigma(\Omega(w)) = \sigma_{\mathrm{ess}}(\Omega(w)) = \mathbb{R} - i\mathrm{Im}\, w, \tag{122}$$

and hence, for $w \in W^+$, the (essential) spectrum of Ω shifts rigidly into the lower half-plane—Hypothesis 2 holds with $S^+ = \mathbb{C}^+$ and $S_w^- = \{w' \in \mathbb{C} \,|\, \mathrm{Im} w' < -\mathrm{Im} w < 0\}$.

We must also find a set of analytic vectors $\mathscr{A}$ for $U(w)$. Let $\mathscr{A}$ be the set of functions $\psi \in L^2(\mathbb{R})$ with the form

$$\psi(k) = P(k)e^{-ak^2}, \tag{123}$$

where P is a polynomial and $a > 0$; such functions are dense in $L^2(\mathbb{R})$, for any $w \in \mathbb{C}$, the function $\psi(w,\cdot) = \psi(\,\cdot - w) \in L^2(\mathbb{R})$ is naturally defined, and hence, $\mathscr{A}$ is a dense set of vectors on which the action of $U(w)$ can be defined for all $w \in \mathbb{C}$. Finally, in our simple case, we have $g(w,k) = \sqrt{\beta/2\pi}$ for all $w \in \mathbb{C}$, i.e., $g(w,\cdot)$ is again a constant function.

We have thus found a dense set of analytic vectors $\tilde{\mathscr{A}} = \mathbb{C} \oplus \mathscr{A}$ such that, for every $\Psi \in \tilde{\mathscr{A}}$, the function $z \in \mathbb{C}^+ \mapsto R_\Psi(z) = \left\langle \Psi \middle| \frac{1}{H_{g,\varepsilon}-z} \Psi \right\rangle \in \mathbb{C}$ has an analytic continuation in $S_W^- = \mathbb{C}^-$. By Theorem VI.7, all poles of those functions are solutions of the equation

$$\varepsilon - z = \Sigma_g(z, w) \tag{124}$$

for some $w \in \mathbb{C}^+$. In our case, taking $\mathrm{Im}\, w$ large enough, this equation reads

$$\varepsilon - z = \frac{i\beta}{2}, \tag{125}$$

and hence, we have a single resonance in $z = \varepsilon - \frac{i\beta}{2}$. The associated eigenvector Ψ of $H_{g,\varepsilon}(w)$ is characterized by a boson wavefunction $\xi(w,\cdot)$ defined by

$$\xi(w,k) = \sqrt{\frac{\beta}{2\pi}} \frac{1}{k - w - \varepsilon + \frac{i\beta}{2}}. \tag{126}$$

By construction, the Fourier–Laplace transform of $R_\Psi(z)$, i.e., its survival amplitude, will be an exponential function with decay rate $\beta/2$. In particular, in the limit $\beta \to 0$, its decay rate and boson component vanish, and we recover the (real) eigenvalue $\lambda = \varepsilon$, with eigenvector Ψ_0, of the uncoupled Friedrichs–Lee Hamiltonian.

Summing up, as anticipated in the discussion in Example V.2, when switching on a flat coupling $g(k) = \sqrt{\beta/2\pi}$ in the Friedrichs–Lee Hamiltonian with $X = L^2(\mathbb{R})$ and $\omega(k) = k$, the uncoupled eigenvalue $\lambda = \varepsilon$ is converted into a *resonance* associated with an unstable state whose survival probability decays exponentially with rate β.

VII. INVERSE PROBLEM: THE CASE OF EXPONENTIAL DECAY

Theorem IV.3 allows us to fully characterize the spectrum and hence the dynamics of a Friedrichs–Lee Hamiltonian with some given momentum space (X, μ), dispersion relation ω, and form factor g. On the other hand, we may want to solve the *inverse problem*, i.e., finding a Friedrichs–Lee model with some *given* spectrum, and hence some given dynamics.

As pointed out in Remark IV.17, the spectrum of the Friedrichs–Lee Hamiltonian with a fixed ε depends entirely on the coupling measure κ_g in Eq. (43) or, equivalently, on the self-energy Σ_g in Eq. (20), and different choices of the momentum space (X, μ), the dispersion relation ω, and the form factor g, yielding the same κ_g, are fully equivalent at the spectral level.

A trivial solution (although not necessarily physically meaningful) always exists: just choose $\omega(k) = k$, $(X,\mu) = (\mathbb{R}, \kappa_g)$, and $g(k) = 1$. However, in the applications, the momentum space (X, μ) and the dispersion relation ω are fixed by the nature and the structure of the bosonic bath, while the form factor g may be suitably engineered. Thus, we are led to study the following inverse problem: *for a given choice of (X, μ) and ω, what choices of g correspond to a Friedrichs–Lee Hamiltonian whose coupling measure κ_g yields our desired dynamics?*

As an example, let us analyze the following problem: let us construct a Friedrichs–Lee operator such that the square modulus of $x(t)$, defined as in Eq. (22), decays with a purely exponential law. By Eq. (23), an exponential law is obtained if κ_g is, up to a multiplicative constant, the Lebesgue measure on the whole real line. Indeed, if $\mathrm{d}\kappa_g(\lambda) = \frac{\beta}{2\pi}\mathrm{d}\lambda$, for some $\beta > 0$, then by Eq. (20), one readily obtains (71), namely,

$$\Sigma_g(z) = \begin{cases} +\dfrac{i\beta}{2}, & \mathrm{Im}\, z > 0, \\ -\dfrac{i\beta}{2}, & \mathrm{Im}\, z < 0, \end{cases} \tag{127}$$

and hence, by Eq. (23), for $t > 0$,

$$x(t) = e^{-(\beta/2+i\varepsilon)t}, \tag{128}$$

implying $|x(t)|^2 = e^{-\beta t}$ for any value of ε. A Friedrichs–Lee model on a measure space (X, μ), with dispersion ω and form factor g, will thus yield an exponential decay if and only if

$$\frac{\beta}{2\pi}(\lambda - \lambda_0) = \int_{\omega^{-1}([\lambda_0,\lambda])} |g(k)|^2 \, \mathrm{d}\mu(k), \quad \forall \lambda_0, \lambda \in \mathbb{R}, \ \lambda \ge \lambda_0. \tag{129}$$

Let us find some form factors g satisfying Eq. (129), with space $X = \mathbb{R}^d$ and μ as the Lebesgue measure on $\mathbb{R}^d$, for different choices of the dispersion relation ω.

Example VII.1. Consider a dispersion relation ω depending only on the projection of the momentum in some direction; without loss of generality, we fix a reference frame such that

$$\omega(k_1, \dots, k_d) = w(k_1), \tag{130}$$

with $w : \mathbb{R} \to \mathbb{R}$ being a differentiable and strictly increasing function. The latter hypothesis ensures that

$$\omega^{-1}([\lambda_0, \lambda]) = \left[w^{-1}(\lambda_0), w^{-1}(\lambda)\right] \times \mathbb{R}^{d-1}, \tag{131}$$

and hence, Eq. (129) becomes

$$\frac{\beta}{2\pi}(\lambda - \lambda_0) = \int_{w^{-1}(\lambda_0)}^{w^{-1}(\lambda)} f(k_1) \, \mathrm{d}k_1, \tag{132}$$

with

$$f(k_1) = \int_{\mathbb{R}^{d-1}} |g(k_1, k_2, \dots, k_d)|^2 \, \mathrm{d}k_2 \dots \mathrm{d}k_d. \tag{133}$$

Equation (132) is satisfied iff $f(k_1) = \frac{\beta}{2\pi} w'(k_1)$, and hence,

$$g(k_1, k_2, \dots, k_d) = e^{i\phi(k_1)} \sqrt{\frac{\beta}{2\pi} w'(k_1)} \, h(k_2, \dots, k_d), \tag{134}$$

with $\phi : \mathbb{R} \to \mathbb{R}$ being an arbitrary phase term and $h : \mathbb{R}^{d-1} \to \mathbb{C}$ being a function satisfying

$$\int_{\mathbb{R}^{d-1}} |h(k_2, \dots, k_d)|^2 \, \mathrm{d}k_2 \dots \mathrm{d}k_d = 1. \tag{135}$$

This result can be readily generalized to the case in which w is piecewise monotonically increasing or decreasing: in this case, the form factor is

$$g(k_1, k_2, \dots, k_d) = e^{i\phi(k_1)} \sqrt{\frac{\beta}{2\pi} |w'(k_1)|} \, h(k_2, \dots, k_d). \tag{136}$$

Example VII.2. As a second example, consider a dispersion relation ω depending only on the modulus of the momentum. For any $k \in \mathbb{R}^d$, we can write $k = rn$, where $r = |k|$ and $n \in \mathbb{S}^1$, with $\mathbb{S}^1$ being the unit sphere. We assume that

$$\omega(k) = w(r), \tag{137}$$

with $w : \mathbb{R}^+ \to \mathbb{R}$ being a differentiable and strictly increasing function. Again, we obtain an equation analogous to Eq. (132),

$$\frac{\beta}{2\pi}(\lambda - \lambda_0) = \int_{w^{-1}(\lambda_0)}^{w^{-1}(\lambda)} f(r) \, r^{d-1} \, \mathrm{d}r, \tag{138}$$

where

$$f(r) = \int_{S^1} |g(k)|^2 \, \mathrm{d}S(n). \tag{139}$$

Equation (138) is satisfied when $f(r) = \frac{\beta}{2\pi} w'(r)/r^{d-1}$, and hence, the form factor is

$$g(k) = e^{i\phi(r)} \sqrt{\frac{\beta}{2\pi} \frac{w'(r)}{r^{d-1}}} h(n), \tag{140}$$

again with $\phi : \mathbb{R} \to \mathbb{R}$ being an arbitrary phase term and $h : \mathbb{S}^1 \to \mathbb{C}$ satisfying

$$\int_{S^1} |h(n)|^2 \, \mathrm{d}S(n) = 1. \tag{141}$$

This procedure can be generalized for a function w piecewise monotonically increasing or decreasing, and in this case, we obtain the form factor,

$$g(k) = e^{i\phi(r)} \sqrt{\frac{\beta}{2\pi} \frac{|w'(r)|}{r^{d-1}}} h(n). \tag{142}$$

VIII. CONCLUDING REMARKS

In this article, we have introduced a generalization of the Friedrichs–Lee Hamiltonian, a model of single-excitation interaction between a two-level atom and a boson field, which provides a reasonable description of a vast class of physical models.

Our contribution is twofold. First of all, we have shown that the model can rigorously accommodate, via proper choice of the domain and by exploiting the formalism of Hilbert scales, "singular" (i.e., not square-integrable) form factors, which are often encountered in the physical literature; the possibility of approximating singular models via regular ones in the norm resolvent sense, for instance, via a cutoff procedure, is also explored. Our technique represents a rigorous implementation of a renormalization procedure of the atom excitation energy.

Second, we have provided a general discussion of the spectral properties of the model. The spectrum of Friedrichs–Lee Hamiltonians has been studied with respect to the spectrum of the uncoupled operator: a full characterization of the pure point and absolutely continuous spectra, as well as a minimal support for a maximal singular continuous measure, has been obtained. Besides, we have studied the insurgence of resonances and their algebraic characterization as eigenvalues of a deformed Friedrichs–Lee Hamiltonian. Some insightful examples have been discussed; finally, a rigorous description of exponential decay phenomena in this framework has been addressed.

Future investigation may be devoted to generalizing our results to the many-atom case, where novel phenomena, absent in the one-atom case, may occur, such as the insurgence of bound states in the continuum, corresponding to the eigenvalues embedded in a continuous spectrum.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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