Research Article

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# Infinitely many solutions for non-local problems with broken symmetry

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**Abstract:** The aim of this paper is to investigate the existence of solutions of the non-local elliptic problem

$$\begin{cases} (-\Delta)^s u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $n > 2s$, $\Omega$ is an open bounded domain of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, $(-\Delta)^s$ is the non-local Laplacian operator, $2 < p < 2_s^*$ and $h \in L^2(\Omega)$. This problem requires the study of the eigenvalue problem related to the fractional Laplace operator, with or without potential.

**Keywords:** Fractional Laplace operator, variational methods, perturbative method

**MSC 2010:** 35S15, 58E05, 45G05

## 1 Introduction

In the last decade, starting with the works [9, 10, 30], PDE models involving operators whose paradigm is the so-called fractional Laplacian have been widely studied. Operators of this kind act by a global integration with respect to a very singular kernel and not by pointwise differentiation. Needless to say, the interest in these operators comes from very different fields: for instance, in probability theory the fractional Laplace operators of the form $(-\Delta)^s$, $s \in (0,1)$, are infinitesimal generators of a stable Lévy process (cf., e.g., [35]); on the other hand the famous Signorini problem gives a motivation from mechanics and in [11, 21] one can find further applications in fluid mechanics (cf. also [16] for a list of other applications).

In this work we deal with the existence of multiple solutions of inhomogeneous superlinear boundary value problems involving the fractional Laplacian operator. Namely, we deal with the non-local elliptic problem

$$\begin{cases} (-\Delta)^s u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{$P_h^s$}$$

where $s \in (0,1)$, $n > 2s$, $\Omega$ is an open bounded domain of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, $(-\Delta)^s$ is the non-local Laplacian operator, i.e.,

$$(-\Delta)^s u(x) := C(n,s) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\,dy \quad \text{on } \mathbb{R}^n$$

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for a suitable $C(n,s) > 0$ (see for instance [16]), $2 < p < 2_s^*$, with the fractional critical exponent $2_s^* = \frac{2n}{n-2s}$, and $h \in L^2(\Omega)$. By a solution of $(P_h^s)$ we mean a function in the fractional Sobolev space $H^s(\Omega)$ (cf. Section 2) which vanishes almost everywhere outside $\Omega$ and satisfies the first equation of $(P_h^s)$ in the weak sense.

If $h \equiv 0$, i.e., the problem is symmetric, the existence of infinitely many solutions has been proved in [24, Theorem 1.1] by equivariant variational methods, by a proof similar to that performed in [27, Theorem 9.38] for the local case.

We recall that the local case corresponds to $s = 1$ and the problem corresponding to $(P_h^s)$ is

$$\begin{cases} -\Delta u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$P_h$}$$

Therefore, due to the presence of the term $h$, the problem is not symmetric. Problem $(P_h)$ has been firstly studied using some methods devised by [2, 3, 26, 32], commonly referred to as the "perturbation from symmetry" ones. Later on, an improvement of these results was obtained in [34]. More recently, problem $(P_h)$ has been studied in [6, 13] also for non-zero boundary condition.

As far as we know, in the non-local setting the only contribution related to our kind of problem is contained in [28] (cf. also [25, Part II, Chapter 11]). Due to some non-trivial technical difficulties, we limit here ourselves to the case $u = 0$ on $\partial\Omega$, i.e., to the case in which the non-homogeneity originates only in the equation.

More precisely, we can state our main result, the non-local counterpart of [34, Theorem 1] with $u = 0$ on $\mathbb{R}^n \setminus \Omega$.

**Theorem 1.1.** *Let $s \in (0,1)$ with $n > 2s$ and let $\Omega$ be an open bounded domain of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Then, problem $(P_h^s)$ has infinitely many solutions for any $h \in L^2(\Omega)$ provided that*

$$2 < p < \frac{2n-2s}{n-2s}. \tag{1.1}$$

**Remark 1.2.** For the sake of simplicity, in Theorem 1.1 we deal with the model nonlinearity $g(s) = |s|^{p-2}s$, nevertheless our techniques work for any continuous $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following widely used set of assumptions:

- There exist $\mu > 2$ and $\rho \geq 0$ such that
$$(x,s) \in \Omega \times \mathbb{R},\ |s| \geq \rho \;\Rightarrow\; 0 < \mu G(x,s) \leq s g(x,s)$$
with $G(x,s) = \int_0^s g(x,t)\,dt$.
- There exist $p \in (2, 2_s^*)$, $c > 0$ such that
$$|g(x,s)| \leq c(|s|^{p-1} + 1) \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}.$$
- $g(x,-s) = -g(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}$.

**Remark 1.3.** Also in the local case, without additional assumptions (such as radial symmetry, cf., e.g., [12, 33]), it is still an open problem whether there exist infinitely many solutions for $p$ up to $2^* = \frac{2n}{n-2}$: in fact, the result in [34, Theorem 1] holds for $p \in (2, \frac{2n-2}{n-2})$, hence in Theorem 1.1 we have a full extension of this result to the non-local case.

The paper is organized as follows: In Section 2 we recall some basic facts about the fractional Laplacian and fractional Sobolev spaces. In Section 3 we obtain bounds for eigenvalue problems with the fractional Laplacian. In Section 4 we recall Bolle's method. Finally, in Section 5 we prove Theorem 1.1.

## 2 Preliminaries about the fractional Laplacian

It is important to point out that different notions of fractional Laplacian in domains $\Omega \subset \mathbb{R}^n$, with different interpretations of the Dirichlet boundary conditions, have been defined in the literature. Let us highlight the main differences between them.

In this paper we consider equations involving the standard fractional Laplacian on $\mathbb{R}^n$ and require that the solutions vanish outside $\Omega$. In this case the associated *Dirichlet form* is

$$\mathcal{E}_{\mathbb{R}^n}(u,v) = C(n,s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))\cdot(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{2.1}$$

In probabilistic terms, the associated stochastic process is the standard symmetric $\alpha$-stable Lévy process (with $\alpha = 2s$), killed upon living $\Omega$. This operator has been also used for instance in [23], even in a more general form, called there the fractional $p$-Laplacian.

A second option is to consider the so-called *regional fractional Laplacian*, which corresponds to the Dirichlet form

$$\mathcal{E}_{\Omega}(u,v) = C(n,s) \int_{\Omega}\int_{\Omega} \frac{(u(x)-u(y))\cdot(v(x)-v(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

The associated stochastic process is the censored stable process, for which jumps outside $\Omega$ are completely forbidden (cf., e.g., [4]).

Another approach is to consider the fractional powers of the Laplacian in $\Omega$ with Dirichlet conditions, defined from the spectral decomposition. The corresponding operator is called *the spectral fractional Laplacian* and coincides with the one obtained from the Caffarelli–Silvestre extension on a cylinder based in $\Omega$ (see [7]). The associated stochastic process is the subordinate killed Brownian motion studied in [31].

Next we recall some basic definitions. Let us take $s \in (0,1)$ and a measurable function $u : \mathbb{R}^n \to \mathbb{R}$. The Gagliardo seminorm is defined by

$$[u]_{s,2} = (\mathcal{E}_{\mathbb{R}^n}(u,u))^{\frac{1}{2}} = \Big( C(n,s) \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y \Big)^{\frac{1}{2}}.$$

Then, the fractional Sobolev space is defined as

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : [u]_{s,2} < +\infty\}$$

and it is equipped with the norm

$$\|u\|_{s,2} = (|u|_2^2 + [u]_{s,2}^2)^{\frac{1}{2}},$$

where $|\cdot|_2$ denotes the norm on $L^2(\mathbb{R}^n)$.

A fundamental tool is the following fractional Sobolev inequality (cf., e.g., [16, Theorem 6.5]).

**Theorem 2.1** (Fractional Sobolev inequality). *Let $s \in (0,1)$ be such that $n > 2s$. Then there exists a positive constant $K = K(n,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, it results that*

$$\|f\|^2_{L^{2_s^*}(\mathbb{R}^n)} \le K \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

*Consequently, the space $H^s(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ for any $p \in [2, 2_s^*]$.*

Our problem is set in the space

$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : u(x) = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\} \tag{2.2}$$

which is a closed subspace of $H^s(\mathbb{R}^n)$. Since we are working in a bounded Lipschitz domain $\Omega$, the space $C_0^\infty(\Omega)$ is dense in $H_0^s(\Omega)$, i.e., we could alternatively define $H_0^s(\Omega)$ as the completion of $C_0^\infty(\Omega)$ (cf., e.g., [8, Lemma 2.3]).

Moreover, since $\Omega$ is bounded, the following *Poincaré inequality* holds:

$$\|u\|_{L^2} \le C\,[u]_{s,2} \quad \text{for all } u \in C_0^\infty(\Omega)$$

with $C = C(n,s,p,\Omega)$ (cf. [8, Lemma 2.4]).

As a consequence we can use in $H_0^s(\Omega)$ the equivalent norm

$$\|\cdot\| = [\,\cdot\,]_{s,2}.$$

Moreover, $H_0^s(\Omega)$ then becomes a Hilbert space, with the associated inner product given by the Dirichlet form (2.1).

We remark that the following embeddings hold:

$$H_0^s(\Omega) \hookrightarrow L^\mu(\Omega) \quad \text{continuously for } \mu \in [1, 2_s^*]$$

(as a consequence of Theorem 2.1) and

$$H_0^s(\Omega) \hookrightarrow\hookrightarrow L^\mu(\Omega) \quad \text{compactly for } \mu \in [1, 2_s^*) \tag{2.3}$$

(cf. [16, Theorem 7.1]).

# 3 Estimates for eigenvalues of the fractional Laplacian with weights

In this section we prove some estimates for eigenvalues of equations involving the fractional Laplacian in domains; we need such estimates for the proof of our main result, though we believe they have an interest of their own. Indeed, eigenvalues of the fractional Laplacian in domains have recently received some attention in literature. For instance, Z. Q. Chen and R. Song used in [14] probabilistic methods to obtain two-sided estimates for the eigenvalues of the fractional Laplacian in domains, while B. Dyda, A. Kuznetsov and M. Kwaśnicki explicitly computed in [17] the eigenvalues in a ball. Moreover, we refer to [20] and references therein for some Weyl-type laws for $p$-fractional eigenvalue problems.

We obtain new estimates for eigenvalues of equations involving the fractional Laplacian with weights by adapting the technique of P. Li and S. T. Yau in [22]. Indeed the following theorem generalizes [22, Theorem 2].

**Theorem 3.1.** *Let $s \in (0, 1)$, $n > 2s$, $\Omega$ be an open bounded domain of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and $q(x)$ a positive function in $L^r(\Omega)$ with $r > \frac{n}{2s}$. Moreover, let us denote by $\mu_k$ the $k$-th eigenvalue for the problem*

$$\begin{cases} (-\Delta)^s \phi(x) = \mu\, q(x)\phi(x) & \text{in } \Omega, \\ \phi = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.1}$$

*Then there exists a positive constant $\bar{C} = \bar{C}(n, s)$ such that for all $k \in \mathbb{N}$ we have*

$$\mu_k^{\frac{n}{2s}} \int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x \geq \bar{C}k.$$

*Proof.* The arguments in [1, Section 1.3], joined with the compact embeddings in (2.3), guarantee the existence of a sequence $(\mu_k)_k$ of eigenvalues for problem (3.1). Let $(\varphi_k)_k$ be an orthogonal basis of eigenfunctions of $(-\Delta)^s$ in $L^2(\Omega, q(x)\,\mathrm{d}x)$ with corresponding eigenvalues $(\mu_k)_k$. For every $k \in \mathbb{N}$, the first equation of (3.1) in the weak sense is

$$C(n, s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\varphi_k(x) - \varphi_k(y))(\psi(x) - \psi(y))}{|x - y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y = \mu_k \int_\Omega q(x)\,\varphi_k(x)\,\psi(x)\,\mathrm{d}x \tag{3.2}$$

for all test functions $\psi \in H_0^s(\Omega)$. Let us consider (cf., e.g., [19, Propositions 6.4 and 6.5]) the *heat kernel*

$$H(x, y, t) = \sum_{i=1}^{+\infty} e^{-\mu_i t}\varphi_i(x)\varphi_i(y),$$

which is positive by the maximum principle in [30, Proposition 2.17], and the function

$$h(t) = \sum_{i=1}^{+\infty} e^{-2\mu_i t}.$$

Namely, fixing $(y, t) \in \Omega \times \mathbb{R}$, the heat kernel $H(\cdot, y, t)$ is a weak solution of

$$\frac{\partial H}{\partial t}(x, y, t) = -\frac{(-\Delta)_x^s}{q(x)} H(x, y, t),$$

i.e.,

$$C(n, s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(H(x, y, t) - H(z, y, t))(\psi(x) - \psi(z))}{|x - z|^{n+2s}}\, dx\, dz = \int_{\Omega} q(x) \frac{\partial H}{\partial t}(x, y, t)\, \psi(x)\, dx$$

for all $\psi \in H_0^s(\Omega)$: in fact,

$$\begin{aligned}
\int_{\Omega} q(x) \frac{\partial H}{\partial t}(x, y, t)\, \psi(x)\, dx &= -\int_{\Omega} q(x) \sum_{i=1}^{+\infty} \mu_i\, e^{-\mu_i t} \varphi_i(x)\varphi_i(y)\psi(x)\, dx \\
&= -\sum_{i=1}^{+\infty} \mu_i\, e^{-\mu_i t}\varphi_i(y) \int_{\Omega} q(x)\, \varphi_i(x)\psi(x)\, dx \\
&= -C(n, s) \sum_{i=1}^{+\infty} e^{-\mu_i t}\varphi_i(y) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\varphi_i(x) - \varphi_i(z))(\psi(x) - \psi(z))}{|x - z|^{n+2s}}\, dx\, dz \\
&= -C(n, s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(H(x, y, t) - H(z, y, t))(\psi(x) - \psi(z))}{|x - z|^{n+2s}}\, dx\, dz.
\end{aligned}$$

We have that

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} H^2(x, y, t) q(x) q(y)\, dx\, dy &= \int_{\Omega}\int_{\Omega} \Big(\sum_{i=1}^{+\infty} e^{-\mu_i t}\varphi_i(x)\varphi_i(y)\Big)\Big(\sum_{j=1}^{+\infty} e^{-\mu_j t}\varphi_j(x)\varphi_j(y)\Big) q(x)q(y)\, dx\, dy \\
&= \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty} e^{-\mu_i t} e^{-\mu_j t}\Big(\int_{\Omega}\int_{\Omega} \varphi_i(x)\varphi_i(y)\varphi_j(x)\varphi_j(y)\, q(x)q(y)\, dx\, dy\Big) \\
&= \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty} e^{-\mu_i t} e^{-\mu_j t} \delta_{ij} = h(t).
\end{aligned} \tag{3.3}$$

Therefore,

$$\begin{aligned}
h'(t) &= \int_{\Omega}\int_{\Omega} 2H(x, y, t) \frac{\partial H}{\partial t}(x, y, t) q(x) q(y)\, dx\, dy \\
&= \int_{\Omega}\int_{\Omega} 2H(x, y, t)\Big[-\sum_{i=1}^{+\infty} \mu_i\, e^{-\mu_i t}\varphi_i(x)\varphi_i(y)\Big] q(x)q(y)\, dx\, dy \\
&= -2 \sum_{i=1}^{+\infty} \mu_i\, e^{-\mu_i t} \int_{\Omega} q(x)\varphi_i(x)\Big[\int_{\Omega} \varphi_i(y) q(y) H(x, y, t)\, dy\Big]\, dx.
\end{aligned}$$

Using $\psi = H(x, \cdot, t)$ as test function in (3.2), we obtain that

$$\begin{aligned}
h'(t) &= -2C(n, s) \cdot \sum_{i=1}^{+\infty} e^{-\mu_i t} \int_{\Omega} q(x)\varphi_i(x)\Big[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\varphi_i(y) - \varphi_i(z))(H(x, y, t) - H(x, z, t))}{|y - z|^{n+2s}}\, dy\, dz\Big]\, dx \\
&= -2C(n, s) \cdot \int_{\Omega} q(x) \sum_{i=1}^{+\infty} e^{-\mu_i t}\varphi_i(x)\Big[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\varphi_i(y) - \varphi_i(z))(H(x, y, t) - H(x, z, t))}{|y - z|^{n+2s}}\, dy\, dz\Big]\, dx \\
&= -2C(n, s) \cdot \int_{\Omega} q(x)\Big[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(H(x, y, t) - H(x, z, t))(H(x, y, t) - H(x, z, t))}{|y - z|^{n+2s}}\, dy\, dz\Big]\, dx \\
&= -2C(n, s) \int_{\Omega} q(x)\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|H(x, y, t) - H(x, z, t)|^2}{|y - z|^{n+2s}}\, dy\, dz\Big)\, dx.
\end{aligned}$$

Using the fractional Sobolev inequality in Theorem 2.1, it follows that

$$-h'(t) \geq \widetilde{C}(n,s) \int_\Omega q(x)\Big[\int_\Omega H^{\frac{2n}{n-2s}}(x,y,t)\,\mathrm{d}y\Big]^{\frac{n-2s}{n}}\,\mathrm{d}x. \tag{3.4}$$

On the other hand, by (3.3) and using the Hölder inequality, we have that

$$\begin{aligned} h(t) &= \int_\Omega q(x)\Big(\int_\Omega H^2(x,y,t)q(y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &\leq \int_\Omega q(x)\Big(\int_\Omega H^{\alpha p}(x,y,t)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big(\int_\Omega H^{\beta p'}(x,y,t)q^{p'}(y)\,\mathrm{d}y\Big)^{\frac{1}{p'}}\,\mathrm{d}x, \end{aligned} \tag{3.5}$$

where

$$\frac{1}{p}+\frac{1}{p'}=1, \quad \alpha+\beta=2.$$

It must be $\alpha p = 2_s^*$ and $\beta p' = 1$. Hence,

$$p = \frac{2n}{n-2s} - 1 = \frac{n+2s}{n-2s} \;\Rightarrow\; p' = \frac{n+2s}{4s}.$$

By (3.5) it follows that

$$h(t) \leq \int_\Omega q(x)\Big(\int_\Omega H^{\frac{2n}{n-2s}}(x,y,t)\,\mathrm{d}y\Big)^{\frac{n-2s}{n+2s}}\Big(\int_\Omega H(x,y,t)q^{\frac{n+2s}{4s}}(y)\,\mathrm{d}y\Big)^{\frac{4s}{n+2s}}\,\mathrm{d}x.$$

By using again the Hölder inequality with exponents $m = \frac{n+2s}{n}$ and $m' = \frac{n+2s}{2s}$, we get that

$$h(t) \leq \Big(\int_\Omega q(x)\Big(\int_\Omega H^{\frac{2n}{n-2s}}(x,y,t)\,\mathrm{d}y\Big)^{\frac{n-2s}{n}}\,\mathrm{d}x\Big)^{\frac{n}{n+2s}}\Big(\int_\Omega q(x)Q^2(x,t)\,\mathrm{d}x\Big)^{\frac{2s}{n+2s}}, \tag{3.6}$$

where

$$Q(x,t) = \int_\Omega H(x,y,t)q^{\frac{n+2s}{4s}}(y)\,\mathrm{d}y.$$

We claim that function $Q$ satisfies the *heat equation*

$$\frac{\partial Q}{\partial t}(x,t) = -\frac{1}{q(x)}(-\Delta)_x^s Q(x,t)$$

in weak sense, i.e., for any test function $\psi \in H_0^s(\Omega)$ we have

$$\int_\Omega \frac{\partial Q}{\partial t}(x,t)\psi(x)\,q(x)\,\mathrm{d}x = C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(Q(x)-Q(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

and $Q$ is such that

$$Q(x,0) = q^{\frac{n-2s}{4s}}(x),$$

since $H$ is the fundamental solution of the heat equation. Therefore,

$$\begin{aligned} \frac{\partial Q}{\partial t}(x,t) &= \int_\Omega \frac{\partial H}{\partial t}(x,y,t)q^{\frac{n+2s}{4s}}(y)\,\mathrm{d}y \\ &= -\int_\Omega \frac{(-\Delta)_x^s}{q(x)}H(x,y,t)q^{\frac{n+2s}{4s}}(y)\,\mathrm{d}y = -\frac{(-\Delta)_x^s}{q(x)}\int_\Omega H(x,y,t)q^{\frac{n+2s}{4s}}(y)\,\mathrm{d}y \end{aligned}$$

and the claim follows.

Hence, we have that

$$\frac{\partial}{\partial t}\int_\Omega Q^2(x,t)q(x)\,\mathrm{d}x = 2\int_\Omega Q(x,t)\frac{\partial Q}{\partial t}(x,t)q(x)\,\mathrm{d}x$$
$$= -2C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|Q(x,t)-Q(z,t)|^2}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z \le 0.$$

This implies that

$$\int_\Omega Q^2(x,t)q(x)\,\mathrm{d}x \le \int_\Omega Q^2(x,0)q(x)\,\mathrm{d}x = \int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x.$$

Therefore by (3.6), we get that

$$h(t) \le \Big[\int_\Omega q(x)\Big(\int_\Omega H^{\frac{2n}{n-2s}}(x,y,t)\,\mathrm{d}y\Big)^{\frac{n-2s}{n}}\,\mathrm{d}x\Big]^{\frac{n}{n+2s}}\Big[\int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big]^{\frac{2s}{n+2s}}$$

and plainly

$$h^{\frac{n+2s}{n}}(t)\Big(\int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)^{-\frac{2s}{n}} \le \int_\Omega q(x)\Big(\int_\Omega H^{\frac{2n}{n-2s}}(x,y,t)\,\mathrm{d}y\Big)^{\frac{n-2s}{n}}\,\mathrm{d}x.$$

By (3.4) we get that

$$h^{\frac{n+2s}{n}}(t)\Big(\int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)^{-\frac{2s}{n}} \le -\frac{1}{\widetilde{C}(n,s)}h'(t),$$

that is

$$h'(t) \le -\widetilde{C}(n,s)h^{\frac{n+2s}{n}}(t)\Big(\int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)^{-\frac{2s}{n}}.$$

Dividing by $h^{\frac{n+2s}{n}}(t)$, it results that

$$h^{-\frac{n+2s}{n}}(t)h'(t) \le -\widetilde{C}(n,s)\Big(\int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)^{-\frac{2s}{n}},$$

that is

$$\frac{n}{2s}\frac{\mathrm{d}}{\mathrm{d}t}(h^{-\frac{2s}{n}}(t)) \ge \widetilde{C}(n,s)\Big(\int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)^{-\frac{2s}{n}}.$$

Now $h^{-\frac{2s}{n}}(t) \to 0$ as $t \to 0$, since $h(t) \to +\infty$. Hence, integrating in $(0,t)$, we get that

$$\frac{n}{2s}h(t)^{-\frac{2s}{n}} \ge \widetilde{C}(n,s)t\Big(\int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)^{-\frac{2s}{n}}.$$

Finally,

$$h(t) \le \Big(\frac{2s}{n}\widetilde{C}(n,s)\Big)^{-\frac{n}{2s}}\Big(\int_D q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)t^{-\frac{n}{2s}}.$$

By the definition of $h$, we have

$$\sum_{i=1}^{+\infty} e^{-2\mu_i t} \le \Big(\frac{2s}{n}\widetilde{C}(n,s)\Big)^{-\frac{n}{2s}}\Big(\int_D q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)t^{-\frac{n}{2s}}.$$

Setting $t = \frac{n}{4\mu_k}$, we get

$$\Big(\frac{2s}{n}\widetilde{C}(n,s)\Big)^{-\frac{n}{2s}}\Big(\int_\Omega q^{\frac{n}{2s}}(x)\,\mathrm{d}x\Big)\Big(\frac{n}{4\mu_k}\Big)^{-\frac{n}{2s}} \ge \sum_{i=1}^{+\infty} e^{-\frac{n\mu_i}{2\mu_k}} \ge ke^{-\frac{n}{2}}$$

and the proof is complete. □

**Remark 3.2.** We conjecture that Theorem 3.1 holds for a larger class of non-local operators (like those considered in [18] for $p = 2$): in fact, only some structural properties are needed along the computations.

Now, given a potential $V \in L^{\frac{n}{2s}}(\Omega)$, let us consider the eigenvalue problem

$$\begin{cases} (-\Delta)^s u + V(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

and denote by $\mathcal{N}_-^s((-\Delta)^s + V)$ the number of its non-positive eigenvalues and by $V_-(x) := \min\{V(x), 0\}$. Then, by using Theorem 3.1, we can state the analogue of [22, Corollary 2] for the non-local case.

**Corollary 3.3.** *There exists a constant $\underline{C} = \underline{C}(n, s)$ such that*

$$\mathcal{N}_-^s((-\Delta)^s + V) \le \underline{C}\,|V_-|_{\frac{n}{2s}}^{\frac{n}{2s}}.$$

*Proof.* The proof is essentially the same as of [22, Corollary 2]: Indeed, (i) and (ii) of [22] work also in our case, therefore it is enough to consider only non-positive eigenvalues and to work with a strictly negative potential. Step (iii) is not needed since we work on bounded domains of $\mathbb{R}^n$ and moreover the number $\mathcal{N}_-^s((-\Delta)^s + V)$ is equal to the number of eigenvalues less than 1 for (3.1) with $q(x) = -V(x)$ (cf. [22, formula (24)]). Then, denoting by $\mu_k$ the greatest eigenvalue less than or equal to 1, by Theorem 3.1 it follows that

$$|V_-|_{\frac{n}{2s}}^{\frac{n}{2s}} \ge \mu_k^{\frac{n}{2s}} |V_-|_{\frac{n}{2s}}^{\frac{n}{2s}} \ge \bar{C}k \ge \bar{C}\mathcal{N}_-^s((-\Delta)^s + V). \qquad \square$$

## 4 Bolle's perturbation method

In order to apply the method devised by Bolle in [5] for dealing with problems with broken symmetry, let us recall the main theorem stated in [6] and generalized in [15] by considering $C^1$ instead of $C^2$ functionals. The key point is to handle a continuous path of functionals $(I_\theta)_{\theta\in[0,1]}$ starting at a symmetric functional $I_0$ (which corresponds in our case to $h = 0$) and ending at the non-even functional $I_1$ associated to the problem.

Let $X$ be a Banach space equipped with the norm $\|\cdot\|$, let $(X', \|\cdot\|_{X'})$ be its dual and let $I : [0, 1] \times X \to \mathbb{R}$ be a $C^1$ functional; let us set $I_\theta = I(\theta, \cdot) : X \to \mathbb{R}$ and $I'_\theta(\cdot) = \frac{\partial I}{\partial v}(\theta, \cdot)$ for all $\theta \in [0, 1]$.

Let us consider the following assumptions:

($A_1$) $I$ satisfies a variant of the Palais–Smale condition: each sequence $((\theta_n, v_n))_n$ in $[0, 1] \times X$ such that

$$(I(\theta_n, v_n))_n \text{ is bounded} \quad \text{and} \quad \lim_{n\to+\infty} \|I'_{\theta_n}(v_n)\| = 0$$

converges up to subsequences.

($A_2$) For all $b > 0$ there exists $C_b > 0$ such that if $(\theta, v) \in [0, 1] \times X$, then

$$|I_\theta(v)| \le b \implies \left|\frac{\partial I}{\partial \theta}(\theta, v)\right| \le C_b(\|I'_\theta(v)\|_{X'} + 1)(\|v\| + 1).$$

($A_3$) There exist two continuous maps $\eta_1, \eta_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$, with $\eta_1(\theta, \cdot) \le \eta_2(\theta, \cdot)$ for all $\theta \in [0, 1]$, which are Lipschitz continuous with respect to the second variable and such that if $(\theta, v) \in [0, 1] \times X$, then

$$I'_\theta(v) = 0 \implies \eta_1(\theta, I_\theta(v)) \le \frac{\partial I}{\partial \theta}(\theta, v) \le \eta_2(\theta, I_\theta(v)). \tag{4.1}$$

($A_4$) $I_0$ is even and for each finite-dimensional subspace $V$ of $X$ it results that

$$\lim_{\substack{v\in V \\ \|v\|\to+\infty}} \sup_{\theta\in[0,1]} I_\theta(v) = -\infty.$$

For $i \in \{1, 2\}$, let $\psi_i : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be the solution of the problem

$$\begin{cases} \dfrac{\partial \psi_i}{\partial \theta}(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)), \\ \psi_i(0, s) = s. \end{cases}$$

Note that $\psi_i(\theta,\cdot)$ is continuous, non-decreasing on $\mathbb{R}$ and $\psi_1(\theta,\cdot)\le\psi_2(\theta,\cdot)$ for $i\in\{1,2\}$. Set

$$\bar\eta_1(s)=\max_{\theta\in[0,1]}|\eta_1(\theta,s)|,\quad \bar\eta_2(s)=\max_{\theta\in[0,1]}|\eta_2(\theta,s)|.$$

Let us consider a sequence $(e_k)_k\subset X$ of linearly independent unitary vectors and set $E_k=\operatorname{span}\{e_1,\dots,e_k\}$. Moreover, let us define

$$\Gamma=\{\gamma\in C(X,X):\gamma\text{ is odd and there exists }R>0\text{ such that }\gamma(v)=v\text{ if }\|v\|\ge R\}$$

and

$$c_k=\inf_{\gamma\in\Gamma}\sup_{v\in E_k}I_0(\gamma(v)).$$

Then we have $c_k\le c_{k+1}$ for all $k\ge1$.

In this framework, the following abstract result can be proved, cf. [5, Theorem 3] and [6, Theorem 2.2].

**Theorem 4.1.** *Assume that the $C^2$ path of functionals $I:[0,1]\times X\to\mathbb{R}$ satisfies* (A$_1$)–(A$_4$) *above. Then there exists $\overline{K}>0$ such that for all $k\in\mathbb{N}$*
(a) *either $I_1$ has a critical level $\tilde c_k$ with $\psi_2(1,c_k)<\psi_1(1,c_{k+1})\le\tilde c_k$,*
(b) *or $c_{k+1}-c_k\le\overline{K}(\bar\eta_1(c_{k+1})+\bar\eta_2(c_k)+1)$.*

# 5 Proof of Theorem 1.1

Let us consider, as in Theorem 1.1, $s\in(0,1)$, $n>2s$, an open bounded domain $\Omega$ of $\mathbb{R}^n$ with Lipschitz boundary and $h\in L^2(\Omega)$.

The weak solutions of $(P_h^s)$ are the critical points of the functional

$$J_1(v)=\frac12\|v\|^2-\frac1p|v|_p^p-\int_\Omega hv\,dx$$

defined on the space $H_0^s(\Omega)$ introduced in (2.2).

Thus, according to Bolle's perturbation method, we define the path of functionals $J:[0,1]\times H_0^s(\Omega)\to\mathbb{R}$ as follows:

$$J(\theta,v)=\frac12\|v\|^2-\frac1p|v|_p^p-\theta\int_\Omega hv\,dx.$$

For simplicity, we set $J_\theta=J(\theta,\cdot)$, hence the weak solutions of $(P_h^s)$ are the critical points of $J(1,\cdot)=J_1$.

Next we check that, with this choice of the path, the assumptions in Theorem 4.1 are satisfied.

Concerning the Palais–Smale type condition in assumption (A$_1$), we refer to [24, Lemmas 3.2 and 3.3]. Moreover, since for all $(\theta,v)\in[0,1]\times H_0^s(\Omega)$ it results $\frac{\partial J}{\partial\theta}(\theta,v)=-\int_\Omega hv\,dx$, also (A$_2$) holds.

We prove that $J$ satisfies assumption (A$_3$): in fact, the following lemma holds.

**Lemma 5.1.** *If $(\theta,v)\in[0,1]\times H_0^s(\Omega)$ is such that $J'_\theta(v)=0$, then with $\eta_1,\eta_2:[0,1]\times\mathbb{R}\to\mathbb{R}$ defined as*

$$-\eta_1(\theta,s)=\eta_2(\theta,s)=D(s^2+1)^{\frac{1}{2p}},$$

*inequality* (4.1) *holds for a suitable constant $D>0$.*

*Proof.* If $v\in H_0^s(\Omega)$ is a critical point of $J_\theta$, it results that

$$\|v\|^2-|v|_p^p-\theta\int_\Omega hv\,dx=0.$$

Hence,

$$J_\theta(v)=\Big(\frac12-\frac1p\Big)|v|_p^p+\Big(\frac12-1\Big)\theta\int_\Omega hv\,dx,$$

by which

$$|v|_p^p \le D_1(J_\theta(v) + 1) \tag{5.1}$$

follows for a suitable $D_1 > 0$. By (5.1) we get

$$\left|\frac{\partial J}{\partial \theta}(\theta, v)\right| \le D_2(J_\theta^2(v) + 1)^{\frac{1}{2p}}$$

for some $D_2 > 0$, thus the lemma is proved. □

Finally, let us remark that the functional $J_0$ is even on $H_0^s(\Omega)$, hence by the following lemma, also property $(A_4)$ holds.

**Lemma 5.2.** *Let $V$ be a finite-dimensional subspace of $H_0^s(\Omega)$. Then,*

$$\lim_{\substack{v\in V\\ \|v\|\to+\infty}} \sup_{\theta\in[0,1]} J(\theta, v) = -\infty.$$

*Proof.* For all $(\theta, v) \in [0, 1] \times H_0^s(\Omega)$ it results that

$$J_\theta(v) \le \frac{1}{2}\|v\|^2 - \frac{1}{p}|v|_p^p + |h|_2|v|_2,$$

then the assertion follows, since $p > 2$ and on finite-dimensional spaces all norms are equivalent. □

Now we are ready to prove our main result.

*Proof of Theorem 1.1.* As the path of functionals $J$ satisfies $(A_1)$–$(A_4)$, we can apply Theorem 4.1. Let us denote by $(\lambda_k)_k$ the sequence of the eigenvalues of $(-\Delta)^s$ on $H_0^s(\Omega)$ and by $(e_k)_k$ the corresponding sequence of eigenfunctions. According to [29, Section 3], the main features of the spectrum of the fractional Laplace operator are very closed to that of the classical one. Then, we set

$$E_1 = \mathbb{R}e_1, \quad E_{k+1} = E_k \oplus \mathbb{R}e_{k+1} \quad \text{for } k \ge 1$$

and define a corresponding sequence of mini-max levels as

$$c_k = \inf_{\gamma\in\Gamma} \sup_{v\in E_k} J_0(\gamma(v)),$$

where $\Gamma$ is as in Section 4 with $X = H_0^s(\Omega)$. Next we have to estimate the growth of these $c_k$. We claim that condition (b) in Theorem 4.1 can not hold for all $k$ large enough. In fact, taking $\eta_1$, $\eta_2$ as in Lemma 5.1, condition (b) in Theorem 4.1 becomes

$$c_{k+1} - c_k \le \underline{K}((c_k)^{\frac{1}{p}} + (c_{k+1})^{\frac{1}{p}} + 1) \tag{5.2}$$

for a suitable $\underline{K} > 0$; hence, if (5.2) holds for all $k$ large enough, then by [2, Lemma 5.3] it follows that there exist $L > 0$ and $\bar{k} \in \mathbb{N}$ such that

$$c_k \le L\, k^{\frac{p}{p-1}} \quad \text{for all } k \ge \bar{k}. \tag{5.3}$$

On the other hand, [34, Theorem B] implies that for all $k \in \mathbb{N}$ there exists $v_k \in H_0^s(\Omega)$ such that

$$J_0'(v_k) = 0 \quad \text{and} \quad J(v_k) \le c_k, \tag{5.4}$$

with large Morse index greater than or equal to $k$, i.e., the operator

$$J_0''(v_k) = (-\Delta)^s - (p-1)|v_k|^{p-2}$$

has at least $k$ non-positive eigenvalues. Therefore, by Corollary 3.3 with $V(x) = -(p-1)|v_k|^{p-2}$, we infer that for a suitable $C'_{n,s} > 0$ we have

$$k \le \mathcal{N}_-(J_0''(v_k)) \le C'_{n,s}|v_k|_{(p-2)\frac{n}{2s}}^{(p-2)\frac{n}{2s}}.$$

In particular, (5.4) implies $J_0'(v_k)[v_k] = 0$, then

$$\|v_k\|^2 = |v_k|_p^p.$$

Hence for a suitable $C''_{n,s} > 0$ it follows that

$$c_k \geq C''_{n,s} k^{\frac{2sp}{n(p-2)}} \quad \text{for all } k \geq k_0,$$

and by (5.3) this yields a contradiction under assumption (1.1). Therefore condition (a) in Theorem 4.1 holds for infinitely many $k \in \mathbb{N}$. □

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# References

[1] A. Ambrosetti and D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems*, Birkhäuser, Boston, 2011.
[2] A. Bahri and H. Berestycki, A perturbation method in critical point theory and applications, *Trans. Amer. Math. Soc.* **267** (1981), 1–32.
[3] A. Bahri and P. L. Lions, Morse index of some min-max critical points. I: Applications to multiplicity results, *Comm. Pure Appl. Math.* **41** (1988), 1027–1037.
[4] K. Bogdan, K. Burzy and Z. Q. Chen, Censored stable process, *Probab. Theory Related Fields* **127** (2003), 89–152.
[5] P. Bolle, On the Bolza problem, *J. Differential Equations* **152** (1999), 274–288.
[6] P. Bolle, N. Ghoussoub and H. Tehrani, The multiplicity of solutions in non-homogeneous boundary value problems, *Manuscripta Math.* **101** (2000), 325–350.
[7] C. Brändle, E. Colorado and A. De Pablo, A concave-convex elliptic problem involving the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), 39–71.
[8] L. Brasco, E. Lindgren and E. Parini, The fractional Cheeger problem, *Interfaces Free Bound.* **16** (2014), no. 3, 419–458.
[9] L. A. Caffarelli, S. Salsa and L. Silvestre, Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian, *Invent. Math.* **171** (2008), 425–461.
[10] L. A. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), 1245–1260.
[11] L. A. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Ann. of Math. (2)* **171** (2010), 1903–1930.
[12] A. Candela, G. Palmieri and A. Salvatore, Radial solutions of semilinear elliptic equations with broken symmetry, *Topol. Methods Nonlinear Anal.* **27** (2006), 117–132.
[13] A. Candela and A. Salvatore, Multiplicity results of an elliptic equation with non homogeneous boundary conditions, *Topol. Methods Nonlinear Anal.* **11** (1998), 1–18.
[14] Z. Q. Chen and R. Song, Two sided eigenvalue estimates for subordinate Brownian motion in bounded domains, *J. Funct. Anal.* **226** (2005), 90–113.
[15] M. Clapp, Y. Ding and S. Hernández-Linares, Strongly indefinite functionals with perturbed symmetries and multiple solutions of nonsymmetric elliptic systems, *Electron. J. Differential Equations* **2004** (2004), Paper No. 100.
[16] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), 521–573.
[17] B. Dyda, A. Kuznetsov and M. Kwaśnicki, Eigenvalues of the fractional Laplace operator in the unit ball, preprint (2015), http://arxiv.org/abs/1509.08533.
[18] G. Franzina and P. Palatucci, Fractional $p$-eigenvalues, *Riv. Mat. Univ. Parma (N.S.)* **5** (2014), 315–328.
[19] R. K. Getoor, Markov operators and their associated semi-groups, *Pacific J. Math.* **9** (1959), 449–472.
[20] A. Iannizzotto and M. Squassina, Weyl-type laws for fractional $p$-eigenvalue problems, *Asymptot. Anal.* **88** (2014), 233–245.
[21] A. Kiselev, F. Nazarov and A. Volberg, Global well-posedness for the critical 2D dissipative quasi geostrophic equation, *Invent. Math.* **167** (2007), 445–453.

[22] P. Li and S. T. Yau, On the Schrödinger equation and the eigenvalue problem, *Comm. Math. Phys.* **88** (1983), 309–318.
[23] E. Lindgren and P. Lindqvist, Fractional eigenvalues, *Calc. Var. Partial Differential Equations* **49** (2014), 795–826.
[24] G. Molica Bisci, Sequences of weak solutions for fractional equations, *Math. Res. Lett.* **21** (2014), 241–253.
[25] G. Molica Bisci, V. Rădulescu and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia Math. Appl. 162, Cambridge University Press, Cambridge, 2016.
[26] P. H. Rabinowitz, Multiple critical points of perturbed symmetric functionals, *Trans. Amer. Math. Soc.* **272** (1982), 753–769.
[27] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math. 65, American Mathematical Society, Providence, 1986.
[28] R. Servadei, Infinitely many solutions for fractional Laplace equations with subcritical nonlinearity, in: *Recent Trends in Nonlinear Partial Differential Equations. II: Stationary Problems*, Contemp. Math. 595, American Mathematical Society, Providence (2013), 317–340.
[29] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.* **33** (2013), 2105–2137.
[30] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007), 67–112.
[31] R. Song and Z. Vondracek, Potential theory of subordinate killed brownian motion in a domain, *Probab. Theory Related Fields* **125** (2003), 578–592.
[32] M. Struwe, Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems, *Manuscripta Math.* **32** (1980), 335–364.
[33] M. Struwe, Infinitely many solutions of superlinear boundary value problems with rotational symmetry, *Arch. Math. (Basel)* **36** (1981), 360–369.
[34] K. Tanaka, Morse indices at critical points related to the symmetric mountain pass theorem and applications, *Comm. Partial Differential Equations* **14** (1989), 99–128.
[35] E. Valdinoci, From the long jump random walk to the fractional Laplacian, *Bol. Soc. Esp. Mat. Apl.* **49** (2009), 33–44.