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A note on $L^p - L^q$ estimates for semilinear critical dissipative Klein-Gordon equations

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Abstract In this note, we consider a semilinear wave equation with scale-invariant mass and dissipation. The scale of the dissipation is the same of the mass term and this creates an interplay in which a relation between the coefficients comes into play to determine if the dissipation is dominant with respect to the mass, or viceversa. We prove global existence of small data solutions for supercritical power nonlinearities when the mass term is dominant with respect to the dissipative term ("Klein-Gordon type" case). The critical exponent is a modified Strauss exponent for global small data solutions to semilinear Klein-Gordon equations.

Keywords Semilinear scale-invariant Klein-Gordon equation · Time-dependent coefficients · Global existence small data solutions

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1 Introduction

We consider a semilinear wave equation with scale-invariant mass and dissipation

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = |u|^p, & t > 0,\, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $\mu \geq 0$ and $\nu \geq 0$. The equation in (1) may be interpreted as a Klein-Gordon equation, in which the mass decays in time with critical speed, to which a weak

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dissipation is applied. The scale of the dissipation is the same of the mass term and this creates an interplay in which a relation between the constants μ and ν comes into play to determine if the dissipation is dominant with respect to the mass, or viceversa.

In this note, we prove global existence of small data energy solutions for supercritical power nonlinearities, in the "Klein-Gordon type" case. By "Klein-Gordon type" case, we mean that the mass term is dominant with respect to the dissipative term. That is, the discriminant of the relation between the dissipative and the mass terms, defined by

$$\delta := (\mu-1)^2 - 4\nu^2 \tag{2}$$

is nonpositive. The case of large, positive discriminant, for which the dissipation term is dominant on the mass term, has been studied in [14,15,19] with classical tools for damped waves.

The definition of "Klein-Gordon type" case is motivated by the fact that, after the change of unknown $v(t,x) = (1+t)^{\frac{\mu}{2}} u(t,x)$, the linear problem corresponding to (1), i.e.

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = 0, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x\in\mathbb{R}^n. \end{cases} \tag{3}$$

is transformed into a non dissipative problem

$$\begin{cases} v_{tt} - \Delta v + \frac{m}{(1+t)^2} v = 0, & t>0,\ x\in\mathbb{R}^n, \\ v(0,x) = u_0(x), & x\in\mathbb{R}^n, \\ v_t(0,x) = u_1(x) + (\mu/2)u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{4}$$

where $m = (1-\delta)/4$. The mass term in (4) is *effective* [1,2] whenever $m \geq 1/4$, i.e. for $\delta \leq 0$ if (4) is obtained by (1) via the above change of unknown. Here the threshold $m = 1/4$ of effectiveness means that, roughly speaking, the profile of the solution to (4) is independent of m for $m \geq 1/4$. On the other hand, its profile quickly changes as $m \to 0$, approaching the profile of the solution to the wave problem with zero mass. In other words, a mass term of type $m(1+t)^{-2}u$ is critical, in the sense that it describes the transition from a Klein-Gordon equation with mass to a wave model with zero mass.

By the same transformation $v(t,x) = (1+t)^{\frac{\mu}{2}} u(t,x)$, problem (1) is transformed into

$$\begin{cases} v_{tt} - \Delta v + \frac{m}{(1+t)^2} v = (1+t)^{-\frac{\mu}{2}(p-1)} |v|^p, & t>0,\ x\in\mathbb{R}^n, \\ v(0,x) = u_0(x), & x\in\mathbb{R}^n, \\ v_t(0,x) = u_1(x) + (\mu/2)u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{5}$$

The presence of the decreasing coefficient $(1+t)^{-\frac{\mu}{2}(p-1)}$ in (5) suggests that the critical exponent for which global small data solutions exist in the supercritical case, will be decreasing with respect to μ.

The interplay between this transition case between the wave model and the Klein-Gordon model, together with the presence of the coefficient $(1+t)^{-\frac{\mu}{2}(p-1)}$ in front of the nonlinearity determines our critical exponent, obtained by using a classical contraction argument and suitable $L^p - L^q$ estimates on the conjugate line for the corresponding linear problem (3).

We emphasize that the presence of time-dependent coefficients in our problems makes ineffective several tools from classical theory of wave equations and classical Klein-Gordon equations.

In the best case scenario, described by the assumption $\mu \geq \mu^*(n)$, where $\mu^*(n) = n + 1/(n+1)$, we can prove global small data solutions to (1) (or, equivalently, (5)) for supercritical powers $p > p_0(n,\mu)$ (and $p \leq 1 + 2/(n-1)$ if $n \geq 2$), where

$$p_0(n,\mu) := \frac{n + 2 + \sqrt{(n+\mu-3)^2 + 8(2n+\mu-1)}}{2n+\mu-1}, \tag{6}$$

that is, $p_0(n,\mu)$ is the solution (with $p > 1$) to

$$\left(n + \tfrac{\mu-1}{2}\right)(p-1)^2 + (n+\mu-3)(p-1) - 4 = 0. \tag{7}$$

As we expect, $p_0(n,\mu)$ is nonincreasing with respect to n and μ (see later, Section 1.2). Moreover, $p_0(n,\mu) \to 1$ as $\mu \to \infty$, for any fixed n. On the other hand, it holds $p_0(n,\mu^*(n)) = 1 + 2/n$, the Fujita exponent for global small data solutions to the classical damped wave equation [13,23,26]. More in general, for $p > 1 + 2/n$ (and $p \leq 1 + 2/(n-2)$ if $n \geq 3$), there exist global small data solutions to

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = |u|^p, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{8}$$

for a large class of "effective" dissipative terms $b(t) > 0$ (see [7]), including $b(t) = (1+t)^\beta$ for $\beta \in (-1,1)$ (see [12]). The exponent $1 + 2/n$ is optimal, in the sense that no global solution to (8) may exist for $1 < p \leq 1 + 2/n$ under a sign assumption on the data [5]. In particular, this effectiveness holds for $b(t) = \mu(1+t)^{-1}$, i.e. for

$$\begin{cases} u_{tt} - \Delta u + \mu(1+t)^{-1}u_t = |u|^p, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{9}$$

when $\mu \geq n+2$ (see [3,25]). Clearly, problem (9) is obtained by setting $\nu = 0$ in (1). The interplay between effective damping as in [7], and time-dependent mass, has been shown to lead to a modified scale of Fujita critical exponents [4,9].

Problem (9) recently attracted a lot of interest. Indeed, this case is critical and it represents a transition case between the case of an effectively damped wave equation and a wave equation without damping, as μ becomes smaller. In particular (see [6, 8]), the critical exponent is given by $p_S(n+2)$ when $\mu = 2$ and $n \geq 3$, where $p_S(n)$ denotes the critical exponent for the semilinear wave equation, conjectured by W. A. Strauss [22]. Namely,

$$p_S(n) = \frac{n + 1 + \sqrt{(n+1)^2 + 8(n-1)}}{2(n-1)},$$

i.e. p_S is the solution to

$$(n-1)(p-1)^2 + (n-3)(p-1) - 4 = 0.$$

Remark 1 We notice that, formally setting $\mu = 1$, we find $p_0(n,1) = p_S(n+1)$. Here $p_S(n+1)$ is the critical exponent for global small data solutions to the semilinear Klein-Gordon equation found by W. A. Strauss in [22]. On the other hand, formally setting $\mu = -1$, we remark that $p_0(n,-1)$ is the critical exponent for global small data solutions to semilinear wave equations found in Theorem 14 in [22], by using $L^p - L^q$ estimates on the conjugate line.

Recently, it has been proved [10] (see also [24]) that solutions to (9) blows up in finite time if $1+2/n < p \leq p_S(n+\mu)$ for suitable data, whenever $p_S(n+\mu) \geq 1+2/n$, i.e. $\mu < (n^2+n+2)/(n+2)$. This result proved one side of the conjecture formulated in [8] that the critical exponent for (9) is given by $\max\{p_S(n+\mu), 1+2/n\}$, and extended a previous blow-up result [11] valid for $p \leq p_S(n+2\mu)$.

For our problem (1), also a competition between different critical exponents appears. In the range $\mu \in (\bar\mu(n), \mu^*(n))$, where $\bar\mu(n) = (n-1)^2/n$, we obtain global existence of small data solutions for $p > p_1(n,\mu)$, where now

$$p_1(n,\mu) := \frac{1+\sqrt{(n+\mu-2)^2+8(n+\mu-1)}}{n+\mu-1}, \tag{10}$$

i.e. $p_1(n,\mu)$ is the solution to

$$\frac{n+\mu-1}{2}(p-1)^2 + (n+\mu-2)(p-1) - 4 = 0. \tag{11}$$

Again, $p_1(n,\mu)$ is nonincreasing with respect to n and μ (see later, Section 1.2). As $\mu \to \bar\mu(n)$, it holds $p_1(n,\mu) \to 1+2/(n-1)$. We remark that $1+2/(n-1) < p_S(n)$.

Comparing $p_0(n,\mu)$ and $p_1(n,\mu)$ one finds that $p_0(n,\mu) \geq p_1(n,\mu)$ if, and only if, $\mu \geq \mu^*$ (see later, Section 1.2). Therefore, summarizing, we prove global existence of small data solutions to (1) for $\mu > \bar\mu$ and $p > \max\{p_0(n,\mu), p_1(n,\mu)\}$ (and $p \leq 1+2/(n-1)$ if $n \geq 2$).

We are now ready to state our result.

Theorem 1 *Let $n \geq 1$, $\mu > \bar\mu(n)$ and $\delta < 0$ in* (2). *Assume*

$$p > \max\{p_0(n,\mu), p_1(n,\mu)\},$$

where p_0, p_1 are defined in (6), (10). *Moreover, let $p \leq 1+2/(n-1)$ if $n \geq 2$.*

Then, there exists a constant $\epsilon_0 > 0$ such that for any

$$(u_0,u_1) \in \mathcal{A} := \left(H^1 \cap W^{1,1+1/p}\right) \times \left(L^2 \cap L^{1+1/p}\right) \quad \text{with } \|(u_0,u_1)\|_{\mathcal{A}} \leq \epsilon_0 \tag{12}$$

there is a uniquely determined energy solution $u \in \mathcal{C}([0,\infty),H^1) \cap \mathcal{C}^1([0,\infty),L^2)$ to Cauchy problem (1). *Moreover, the energy of the solution verifies the estimate*

$$\frac{1}{2}\|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\|\nabla u(t,\cdot)\|_{L^2}^2 \leq C\,(1+t)^{-\mu}\|(u_0,u_1)\|_{\mathcal{A}}^2 \tag{13}$$

and the L^{p+1} norm of $u(t,\cdot)$ verifies the estimate

$$\|u(t,\cdot)\|_{L^{p+1}} \leq C\,(1+t)^{-\frac{\mu}{2}} b(t) \|(u_0,u_1)\|_{\mathcal{A}} \tag{14}$$

where

$$b(t) = (1+t)^{-\min\left\{\frac{n-1}{2}\left(1-\frac{2}{p+1}\right), n\left(1-\frac{2}{p+1}\right)-\frac{1}{2}\right\}}. \tag{15}$$

Here $C > 0$ is a fixed constant which does not depend on the initial data.

Remark 2 The expression of $b(t)$ in (15) is determined as follows:

$$b(t) = \begin{cases} (1+t)^{-n\left(1-\frac{2}{p+1}\right)+\frac{1}{2}} & \text{if } p \leq 1+2/n, \\ (1+t)^{-\frac{n-1}{2}\left(1-\frac{2}{p+1}\right)} & \text{if } p \geq 1+2/n. \end{cases} \tag{16}$$

In other words, there appears a competition in estimate (14) between two different decay rates. If $p \geq 1+2/n$, the decay rate given by (14) is $(1+t)^{-\frac{\mu}{2}-\frac{n-1}{2}\left(\frac{1}{q'}-\frac{1}{q}\right)}$, where $q = p+1$ and $q' = q/(q-1)$ is its Hölder conjugate. That is, we have the same type of decay rate of an estimate on the conjugate line $L^{q'} - L^q$ for a wave equation, with an additional decay term $(1+t)^{-\frac{\mu}{2}}$. If $p \leq 1+2/n$, the decay rate given by (14) is $(1+t)^{-\frac{\mu-1}{2}-n\left(\frac{1}{q'}-\frac{1}{q}\right)}$, that is, we have the same decay rate of an estimate on the conjugate line $L^{q'} - L^q$ for the heat equation $v_t - (1+t)\Delta v = 0$, with an additional decay term $(1+t)^{-\frac{\mu-1}{2}}$.

Remark 3 We recall that

$$\max\{p_0(n,\mu), p_1(n,\mu)\} = \begin{cases} p_0(n,\mu) & \text{if } \mu \geq \mu^*(n), \\ p_1(n,\mu) & \text{if } \mu \leq \mu^*(n), \end{cases}$$

in Theorem 1. The role played by the threshold $\mu^*(n)$ is to separate the two different decay rates profiles which regulate the $L^{1+\frac{1}{p}} - L^{p+1}$ estimate on the conjugate line, when p tends to the critical exponent. The fact that the role played by the decay rate for the $L^{1+\frac{1}{p}} - L^{p+1}$ is crucial, is motivated by the convergence of the integral (29) in the proof of Theorem 1.

The two different decay rates profiles are determined by the function $b(t)$ in Lemma 1, i.e. $b(t)$ in (16). Due to

$$p_0(n,\mu) = p_1(n,\mu) = 1 + \frac{2}{n} \iff \mu = \mu^*(n),$$

we may now see why $\mu^*(n)$ is the threshold for the two critical exponents. Indeed, for $\mu > \mu^*(n)$, the $L^{1+\frac{1}{p}} - L^p$ decay rate is determined by the first alternative in (16), for "powers p close enough to the critical exponent" (being this latter smaller than $1+2/n$). On the other hand, for $\mu < \mu^*(n)$, the $L^{1+\frac{1}{p}} - L^p$ decay rate is determined by the second alternative in (16), for "powers p close enough to the critical exponent" (being this latter larger than $1+2/n$).

Remark 4 The threshold value $\bar{\mu}(n)$ in Theorem 1 solves the equality $p_1(n,\mu) = 1+2/(n-1)$ for $n \geq 2$. Therefore, in Theorem 1 it is necessary to require $\mu > \bar{\mu}(n)$ in order to guarantee that the range for the exponent p is nonempty.

1.1 Comparison with the linear problem

The estimates obtained for the L^{p+1} norm of the solution to (1) and for the energy, in Theorem 1 are consistent with the following estimates, obtained for the corresponding linear problem (3) by the second author [16].

Lemma 1 *Let $n \geq 1$, $\mu \geq 0$ and $\delta < 1$ in (2). If we fix initial data in $H^1 \times L^2$, then the solution u to Cauchy problem (3) satisfies the following estimates*

$$\|(u_t, \nabla u)(t,\cdot)\|_{L^2} \leq C(1+t)^{-\frac{\mu}{2}} \Big(\|u_0\|_{H_1} + \|u_1\|_{L^2} \Big). \tag{17}$$

If we fix $q \in [2,\infty)$ and we assume initial data $(u_0,u_1) \in H^r_{q'} \times H^{(r-1)_+}_{q'}$ with $r = n(1-2/q)$ in (3), then

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\frac{\mu}{2}} b(t) \Big(\|u_0\|_{H^r_{q'}} + \|u_1\|_{H^{(r-1)_+}_{q'}} \Big), \tag{18}$$

where

$$b(t) := \begin{cases} (1+t)^{-\min\left\{\frac{n-1}{2}\left(1-\frac{2}{q}\right), n\left(1-\frac{2}{q}\right)-\frac{1}{2}\right\}} & \text{if } \delta < 0, \\ C_\varepsilon (1+t)^{-\min\left\{\frac{n-1}{2}\left(1-\frac{2}{q}\right), n\left(1-\frac{2}{q}\right)-\frac{1}{2}-\varepsilon\right\}} & \text{if } \delta = 0, \\ (1+t)^{-\min\left\{\frac{n-1}{2}\left(1-\frac{2}{q}\right), n\left(1-\frac{2}{q}\right)-\frac{1}{2}-\frac{\sqrt{\delta}}{2}\right\}} & \text{if } \delta \in (0,1). \end{cases} \tag{19}$$

Here $q' = q/(q-1)$ is the Hölder conjugate of q, $\varepsilon > 0$ is an arbitrarily small positive constant and C, C_ε do not depend on the initial data.

In Lemma 1, $H^r_q = (1-\Delta)^{-\frac{r}{2}} L^q$ is the Bessel potential space of order r of L^q. In particular, for $q \in (1,\infty)$ and integer r, it holds $H^r_q = W^{q,r}$, the classical Sobolev space.

To study the semilinear problem (1), it is more convenient to fix $r \leq 1$ in (18), so that no derivative appears in the norm of the power nonlinearity, once Duhamel's principle is applied. However, since we plan to apply the previous estimates with $q = p+1$, this restriction leads to the upper bound $p \leq 1+2/(n-1)$ in space dimension $n \geq 2$.

Remark 5 At $\delta = 0$, an arbitrarily small polynomial loss of decay $(1+t)^\varepsilon$ appears in estimate (18) for linear problem (3). As a consequence, the critical exponent for problem (1) remains the same given in Theorem 1, whereas the same small polynomial loss of decay may appear in the decay estimates for the solution to (1).

On the other hand, for $\delta \in (0,1)$, a possibly substantial loss of decay $(1+t)^{\frac{\sqrt{\delta}}{2}}$ appears in the decay estimates for (1), due to the fact that $m \in (0,1/4)$ in (5). This loss of decay corresponds to a possibly worse, i.e., larger, critical exponent. Namely, the critical exponent becomes $\max\{p_0(n,\mu-\sqrt{\delta}), p_1(n,\mu)\}$. We omit the details of this case, since for $\delta \in (0,1)$, the nature of the equation is hybrid between a noneffective Klein-Gordon equation and a wave equation with time-dependent coefficients (see [17,18,20,21]).

1.2 Some properties of the critical exponents

In this section, we collect some properties for the critical exponents $p_0(n,\mu)$ and $p_1(n,\mu)$ given in (6) and (10).

To show that $p_0(n,\mu)$ and $p_1(n,\mu)$ are nonincreasing with respect to μ and n, it is sufficient to apply the following result with $x = p-1$, setting $\alpha = \mu$ and $\alpha = n$ respectively.

Lemma 2 *Let $a_0, c, \alpha, \gamma > 0$, $b_0 \in \mathbb{R}$, and consider the equation*

$$ax^2 + 2bx - c = 0, \quad a = a_0 + \gamma\alpha, \quad b = b_0 + \alpha \tag{20}$$

Then the positive solution $x(\alpha) = -b + \sqrt{b^2 + ac}$ to (20) *is nondecreasing with respect to α.*

Proof By taking the derivative of the equation in (20) with respect to α, we get:

$$2(ax + b)x'(\alpha) = -(\gamma x^2 + 2x) < 0.$$

However, $ax + b > 0$ if, and only if, $2ax^2 + 2bx > 0$; replacing $2bx = c - ax^2$, we see that this latter reduces to the true inequality $ax^2 + c > 0$. Therefore, $x'(\alpha) < 0$.

To find $\max\{p_0(n, \mu), p_1(n, \mu)\}$, it is sufficient to find μ such that $p_0(n, \mu) = p_1(n, \mu)$, due to the fact that the two exponents continuously depend on μ. Taking the difference of the equation in (7) and in (11), and dividing by $p - 1$, we find that $p_0(n, \mu) = p_1(n, \mu)$ solves

$$p - 1 = \frac{(n + \mu - 2) - (n + \mu - 3)}{(n + (\mu - 1)/2) - (n + \mu - 1)/2} = \frac{2}{n}.$$

Due to the fact that $p_0(n, \mu), p_1(n, \mu)$ are injective, this implies that they are equal if, and only if, $\mu = \mu^*$. In particular, $p_0(n, \mu) > p_1(n, \mu)$ if, and only if, $\mu \geq \mu^*$, due to

$$p_0(n, \mu) \sim 1 + \frac{4}{n + \mu - 3}, \qquad p_1(n, \mu) \sim 1 + \frac{4}{n + \mu - 2},$$

as $\mu \to \infty$.

2 Proof of Theorem 1

In order to prove Theorem 1, we will employ a standard contraction argument.

Let us denote by $\varphi(t, x)$ the solution to linear Cauchy problem (3). By Duhamel's principle, a function $u(t, x)$ solves (1) in a space X if it is a fixed point for the application

$$u \mapsto Nu = \varphi + Ju, \quad Ju = \int_0^t E(t, s, x) *_{(x)} |u(s, x)|^p \, ds,$$

where by $E(t, s, x) * g(s, x)$ we denote the solution to Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = 0, & t > s, \ x \in \mathbb{R}^n, \\ u(s, x) = 0, & x \in \mathbb{R}^n, \\ u_t(s, x) = g(s, x), & x \in \mathbb{R}^n. \end{cases} \tag{21}$$

Our problem is reduced to find a suitable space X in which we may prove, under suitable assumptions on p, the inequalities

$$\|Nu\|_X \leq C_1 \|(u_0, u_1)\|_{\mathcal{A}} + C_2 \|u\|_X^p, \tag{22}$$

$$\|Ju - Jv\|_X \leq C_3 \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1} \right), \tag{23}$$

where in general the space for initial data $\mathcal{A}$ depends on the definition of X.

For any $T>0$, we fix

$$X := \mathcal{C}([0,T],H^1)\cap\mathcal{C}^1([0,T],L^2)$$

i.e., we look for the solution in the energy space, equipped with the norm

$$\|u\|_X := \sup_{t\in[0,T]}(1+t)^{\frac{\mu}{2}}\Big[\|(\nabla u,u_t)(t,\cdot)\|_{L^2}+b(t)^{-1}\|u(t,\cdot)\|_{L^{p+1}} \\ +(1+t)^{-1}\|u(t,\cdot)\|_{L^2}\Big],$$

where $b(t)$ is as in (15).

By using the contraction mapping principle, it follows that (22) and (23) imply local in time existence of small data solutions. Indeed, by setting $\|(u_0,u_1)\|_{\mathcal{A}}=\epsilon$, then, for some $R_0\geq 2C_1$ and for any $0<\epsilon\leq\epsilon_0:=\min\{R_0^{-1}(2C_2)^{1-p},R_0^{-1}(4C_3)^{1-p}\}$, it follows by (22) and (23) that $N:\mathfrak{B}(R_0\epsilon)\to\mathfrak{B}(R_0\epsilon)$ is a contraction, where $\mathfrak{B}(R_0\epsilon):=\{u\in X:\|u\|_X\leq R_0\epsilon\}$.

Moreover, if C_1,C_2,C_3 are independent of T in (22), (23), then the solution is global. Moreover, it satisfies estimates (13) and (14).

By Lemma 1, the estimate in (22) immediately follows for $\varphi(t,x)$, where $\mathcal{A}$ is as in (12). Therefore, it remains to prove (22) for Ju and (23).

Due to the fact that problem (3) is not invariant by time translations, it does not hold $E(t,s,x)=E(t-s,0,x)$ for the fundamental solution to (21). Therefore, we use the following extension of Lemma 1 (see [16]).

Lemma 3 *Let $n\geq 1$, $\mu\geq 0$ and $\delta<1$ in* (2)*. If we fix $g(s,\cdot)\in L^2$, then the solution u to Cauchy problem* (21) *satisfies the following estimates*

$$\|(u_t,\nabla u)(t,\cdot)\|_{L^2}\leq C(1+t)^{-\frac{\mu}{2}}(1+s)^{\frac{\mu}{2}}\|g(s,\cdot)\|_{L^2}, \tag{24}$$

for $t\geq s$. If we fix $q\in[2,\infty)$ if $n=1$, or $q\in[2,2n/(n-1)]$ if $n\geq 2$, and $g(s,\cdot)\in L^{q'}$, with $q'=q/(q-1)$, then the solution u to Cauchy problem (21) *satisfies the following estimates*

$$\|u(t,\cdot)\|_{L^q}\leq C\,(1+t)^{-\frac{\mu}{2}}\,b(t)\,(1+s)^{\frac{\mu}{2}}\,b(s)^{-1}(1+s)^{1-r}\|g(s,\cdot)\|_{L^{q'}}, \tag{25}$$

for $t\geq s$, where the functions $b(t)$ and $r=r(q)$ are defined as in Lemma 1 and C does not depend on $s,g(s,x)$.

In the following, we use the notation $f_1\lesssim f_2$, for two functions $f_1,f_2:I\to\mathbb{R}$, if there exists $C>0$ such that $0\leq f_1(y)\leq Cf_2(y)$, for any $y\in I$.

2.1 Proof of Theorem 1 in space dimension $n=1$

For the ease of reading, we first consider the easier case of space dimension $n=1$.

We define $\beta=\mu/2$ if $p\geq 3$ and $\beta=(\mu-1)/2-2/(p+1)$ if $p<3$, so that $(1+t)^{-\beta}=(1+t)^{-\frac{\mu}{2}}\,b(t)$.

By Gagliardo-Nirenberg inequality, it holds

$$\|u(t,\cdot)\|_{L^{2p}}\lesssim\|u_x(t,\cdot)\|_{L^2}^{\theta}\|u(t,\cdot)\|_{L^{p+1}}^{1-\theta}\leq(1+t)^{-\theta\frac{\mu}{2}-(1-\theta)\beta}\|u\|_X$$

for any $u \in X$, for some $\theta \in [0,1]$. However, since $\beta \le \mu/2$ for any $p > 1$, we may estimate

$$\|u(t,\cdot)\|_{L^{2p}} \lesssim (1+t)^{-\beta}\|u\|_X. \tag{26}$$

By Lemma 3, using (26), it follows:

$$\begin{aligned}(1+t)^{\frac{\mu}{2}}\,\|(\partial_t,\partial_x)Ju(t,\cdot)\|_{L^2} &\lesssim \int_0^t (1+s)^{\frac{\mu}{2}}\,\|u(s,\cdot)\|_{L^{2p}}^p\,ds\\ &\lesssim \|u\|_X^p\int_0^t (1+s)^{\frac{\mu}{2}-p\beta}\,ds\,,\\ (1+t)^{\beta}\,\|Ju(t,\cdot)\|_{L^{p+1}} &\lesssim \int_0^t (1+s)^{\beta+\frac{2}{p+1}}\,\|u(s,\cdot)\|_{L^{p+1}}^p\,ds\\ &\lesssim \|u\|_X^p\int_0^t (1+s)^{\frac{2}{p+1}-(p-1)\beta}\,ds\,.\end{aligned}$$

Similarly, we may prove

$$\begin{aligned}(1+t)^{\frac{\mu}{2}-1}\|Ju(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{1}{2}}\int_0^t (1+s)^{\frac{\mu+1}{2}}\,\|u(s,\cdot)\|_{L^{2p}}^p\,ds\\ &\lesssim \|u\|_X^p\int_0^t (1+s)^{\frac{\mu}{2}-p\beta}\,ds\,,\end{aligned}$$

where we estimated $(1+s)^{\frac{1}{2}} \le (1+t)^{\frac{1}{2}}$ in the last term, in order to reduce to the same integral obtained for $(\partial_t,\partial_x)Ju$.

The convergence of the two integrals implies the desired estimate for $\|Ju\|_X$. If $p \ge 3$, then $\beta = \mu/2$ and

$$\frac{\mu}{2} - p\beta = -(p-1)\beta < \frac{2}{p+1} - (p-1)\beta = \frac{2}{p+1} - \frac{\mu}{2}\,(p-1) < -1,$$

where the last inequality is equivalent to $p > p_1$. If $p \in (1,3]$, then

$$\beta = \frac{\mu+1}{2} - \frac{2}{p+1},$$

and consequently we find

$$\frac{\mu}{2} - p\beta < \frac{\mu+1}{2} - p\beta = \frac{2}{p+1} - \beta(p-1) < -1,$$

where the last inequality is equivalent to $p > p_0$.

Summarizing, we proved (22). Finally, since the estimate

$$|u|^p - |v|^p \le p|u-v|(|u|^{p-1} + |v|^{p-1}) \tag{27}$$

and Hölder's inequality for $h = 2, 1+1/p$ imply

$$\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^h} \lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{hp}}\Big(\|u(s,\cdot)|\|_{L^{hp}}^{p-1} + \|u(s,\cdot)|\|_{L^{hp}}^{p-1}\Big),$$

the proof of (23) similarly follows. This concludes the proof.

2.2 Proof of Theorem 1 in space dimension $n \geq 2$

Now we consider the general case $n \geq 2$. By Gagliardo-Nirenberg inequality, it holds

$$\|u(t,\cdot)\|_{L^{2p}} \lesssim \|\nabla u(t,\cdot)\|_{L^2}^{\theta} \|u(t,\cdot)\|_{L^{p+1}}^{1-\theta} \leq (1+t)^{-\frac{\mu}{2}} b(t)^{1-\theta} \|u\|_X, \tag{28}$$

for any $u \in X$, where $\theta \in [0,1]$ solves

$$\frac{1-\theta}{p+1} + \frac{\theta}{2^*} = \frac{1}{2p},$$

where $2^* = 2n/(n-2)$ if $n \geq 3$, and $2^* = \infty$ if $n = 2$.

By Lemma 3, using (28), it follows:

$$\begin{aligned}
(1+t)^{\frac{\mu}{2}} \|(\partial_t, \nabla) Ju(t,\cdot)\|_{L^2} &\lesssim \int_0^t (1+s)^{\frac{\mu}{2}} \|u(s,\cdot)\|_{L^{2p}}^p \, ds \\
&\lesssim \|u\|_X^p \int_0^t (1+s)^{-(p-1)\frac{\mu}{2}} b(s)^{p(1-\theta)} \, ds \,, \\
(1+t)^{\frac{\mu}{2}} b(t)^{-1} \|Ju(t,\cdot)\|_{L^{p+1}} &\lesssim \int_0^t (1+s)^{1-r+\frac{\mu}{2}} b(s)^{-1} \|u(s,\cdot)\|_{L^{p+1}}^p \, ds \\
&\lesssim \|u\|_X^p \int_0^t (1+s)^{-(p-1)\frac{\mu}{2}+1-r} b(s)^{(p-1)} \, ds \,.
\end{aligned}$$

Similarly, we may prove

$$\begin{aligned}
(1+t)^{1-\frac{\mu}{2}} \|Ju(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{1}{2}} \int_0^t (1+s)^{\frac{\mu+1}{2}} \|u(s,\cdot)\|_{L^{2p}}^p \, ds \\
&\lesssim \|u\|_X^p \int_0^t (1+s)^{-(p-1)\frac{\mu}{2}} b(s)^{p(1-\theta)} \, ds \,,
\end{aligned}$$

where we estimated $(1+s)^{\frac{1}{2}} \leq (1+t)^{\frac{1}{2}}$ in the last term, in order to reduce to the same integral obtained for $(\partial_t, \nabla) Ju$.

We now claim that

$$b(s)^{p(1-\theta)} \leq (1+s)^{1-r} b(s)^{p-1}, \quad \text{i.e.,} \qquad b(s)^{-p\theta} \leq (1+s)^{1-r} b(s)^{-1},$$

so that the first and the third integral can be estimated by the second one, as it happened in space dimension $n = 1$.

In order to show that, we distinguish two cases. If b is increasing, that is, $r \leq 1/2$, this is trivial. Indeed, $b(s)^{-p\theta}$ is bounded by 1, whereas

$$(1+s)^{1-r} b(s)^{-1} = (1+s)^{1-r+(r-1/2)} = (1+s)^{\frac{1}{2}}.$$

Now let b be decreasing. It is sufficient to show that $p\theta$ is an increasing function with respect to p, which tends to 1 as $p \to 1 + 2/(n-1)$. As a consequence, we immediately derive

$$b(s)^{-p\theta} \leq b(s)^{-1} \leq (1+s)^{1-r} b(s)^{-1}.$$

Let us prove our claim. Since $p\theta$ solves

$$p\theta\left(\frac{1}{p+1}-\frac{1}{2^*}\right)=\frac{p}{p+1}-\frac{1}{2}=\frac{1}{2}-\frac{1}{p+1},$$

we may write

$$p\theta=\frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{p+1}-\frac{1}{2^*}}=-1+\frac{1}{n\left(\frac{1}{p+1}-\frac{1}{2^*}\right)}.$$

It immediately follows that $p\theta$ is increasing with respect to p. Moreover,

$$(p\theta)|_{p=1+2/(n-1)}=-1+\frac{1}{n\left(\frac{n-1}{2n}-\frac{n-2}{2n}\right)}=1.$$

This proves our claim. Now it is sufficient to prove the convergence of the integral

$$\int_0^t (1+s)^{-(p-1)\frac{\mu}{2}+1-r}\, b(s)^{(p-1)}\, ds\,. \tag{29}$$

As in the case of space dimension $n=1$, we shall distinguish two cases. If $p\leq 1+2/n$, then

$$b(s)=(1+s)^{-(r-1/2)}=(1+s)^{\frac{1}{2}-n\left(1-\frac{2}{p+1}\right)}.$$

In this case, the integral converges if, and only if,

$$-(p-1)\frac{\mu}{2}+\frac{p+1}{2}-pn\left(1-\frac{2}{p+1}\right)<-1.$$

This latter is equivalent to $p>p_0(n,\mu)$. Now let $p\in(2,\infty)$ if $n=2$ or $p\in(1+2/n,1+2/(n-1)]$ if $n\geq 3$. In this case,

$$b(s)=(1+s)^{-\frac{n-1}{2}\left(1-\frac{2}{p+1}\right)};$$

hence, the integral converges if, and only if,

$$-(p-1)\frac{\mu}{2}+1-n\left(1-\frac{2}{p+1}\right)-(p-1)\frac{n-1}{2}\left(1-\frac{2}{p+1}\right)<-1.$$

This latter is equivalent to $p>p_1(n,\mu)$.

Summarizing, we proved (22). Once again we use (27) to prove (23) similarly. This concludes the proof.

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