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CANONICAL MODELS ON STRONGLY CONVEX DOMAINS VIA THE SQUEEZING FUNCTION

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Abstract. We prove that if a holomorphic self-map $f\colon \Omega \to \Omega$ of a bounded strongly convex domain $\Omega \subset \mathbb{C}^q$ with smooth boundary is hyperbolic then it admits a natural semi-conjugacy with a hyperbolic automorphism of a possibly lower dimensional ball $\mathbb{B}^k$. We also obtain the dual result for a holomorphic self-map $f\colon \Omega \to \Omega$ with a boundary repelling fixed point. Both results are obtained by rescaling the dynamics of f via the squeezing function.

Contents

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1. Introduction

When studying the dynamics of a holomorphic self-map f of the unit ball $\mathbb{B}^q \subset \mathbb{C}^q$, an important role is played by fixed points at the boundary, where the map is not necessarily continuous. A point $\zeta \in \partial\mathbb{B}^q$ is a *boundary regular fixed point* if

(1) for every sequence (z_n) converging to ζ inside a Koranyi region
$$K(\zeta, M) := \left\{ z \in \mathbb{B}^q \,\middle|\, \frac{|1 - \langle z, \zeta\rangle|}{1 - \|z\|} < M \right\},$$
where $M > 1$, we have that $f(z_n)$ converges to ζ, and

(2) the *dilation* λ_ζ defined as
$$\lambda_\zeta := \liminf_{z \to \zeta} \frac{1 - \|f(z)\|}{1 - \|z\|}, \tag{1.1}$$
is finite. If $\lambda_\zeta > 1$ the point ζ is called a *boundary repelling fixed point*.

The classical Denjoy–Wolff Theorem illustrates the relevance of this notion.

Theorem 1.1. *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic self-map without interior fixed points. Then there exists a boundary regular fixed point $\xi \in \partial\mathbb{B}^q$ with dilation $0 < \lambda_\xi \le 1$, called the Denjoy–Wolff point, such that the sequence of iterates (f^n) converges to ξ.*

As a consequence, the family of holomorphic self-maps of $\mathbb{B}^q$ is partitioned in three classes: f is *elliptic* if it admits a fixed point $z \in \mathbb{B}^q$, and if f is not elliptic, it is *parabolic* if the dilation λ_ξ at its Denjoy–Wolff point is 1 and it is *hyperbolic* if $\lambda_\xi < 1$.

The automorphisms of $\mathbb{B}^q$ have explicit normal forms, which show that they have a simple dynamical behaviour. For example, a hyperbolic automorphism has only two boundary regular fixed points, one is the Denjoy–Wolff ξ, and the other is a boundary repelling fixed point ζ with dilation $\lambda_\zeta = 1/\lambda_\xi$. The normal form of hyperbolic automorphisms is easily described. Recall that the *Siegel half-space* $\mathbb{H}^q := \{(z_1, z') \in \mathbb{C} \times \mathbb{C}^{q-1} \colon \operatorname{Im} z_1 > \|z'\|^2\}$ is biholomorphic to the ball $\mathbb{B}^q$. Given any hyperbolic automorphism τ of the ball there exists a biholomorphism $\Psi\colon \mathbb{B}^q \to \mathbb{H}^q$ sending the Denjoy–Wolff point ξ to ∞, such that
$$\Psi \circ \tau \circ \Psi^{-1}(z_1, z') = \left(\frac{1}{\lambda_\xi} z_1, \frac{e^{it_1}}{\sqrt{\lambda_\xi}} z'_1, \ldots, \frac{e^{it_{q-1}}}{\sqrt{\lambda_\xi}} z'_{q-1} \right),$$
with $t_j \in \mathbb{R}$ (a similar normal form can be obtained sending the repelling point to ∞).

In order to understand the forward or backward dynamics of a holomorphic self-map f it is natural to search for a semi-conjugacy between f and an automorphism of the ball. In this direction, the following results were recently proved in [6, 7, 8, 9] using the theory of canonical models (the cases $q = 1$ are the classical results of Valiron [23] and Poggi-Corradini [21]).

Theorem 1.2 (Forward iteration)**.** *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a hyperbolic holomorphic self-map with Denjoy–Wolff point ξ. Then there exists an integer $1 \le k \le q$, a holomorphic map $h\colon \mathbb{B}^q \to \mathbb{H}^k$ and a hyperbolic automorphism τ of $\mathbb{H}^k$ of the form*
$$\tau(z_1, z') = \left(\frac{1}{\lambda_\xi} z_1, \frac{e^{it_1}}{\sqrt{\lambda_\xi}} z'_1, \ldots, \frac{e^{it_{k-1}}}{\sqrt{\lambda_\xi}} z'_{k-1} \right)$$
such that
$$h \circ f = \tau \circ h.$$

Theorem 1.3 (Backward iteration). *Let $f\colon \mathbb{B}^q \to \mathbb{B}^q$ be a holomorphic self-map and let ζ be a boundary repelling fixed point. Then there exists an integer $1 \le k \le q$, a holomorphic map $h\colon \mathbb{H}^k \to \mathbb{B}^q$ and a hyperbolic automorphism τ of $\mathbb{H}^k$ of the form*

$$\tau(z_1, z') = \left(\frac{1}{\lambda_\zeta} z_1, \frac{e^{it_1}}{\sqrt{\lambda_\zeta}} z_1', \ldots, \frac{e^{it_{k-1}}}{\sqrt{\lambda_\zeta}} z_{k-1}'\right)$$

such that

$$f \circ h = h \circ \tau.$$

In both theorems the function h intertwines the map f with a hyperbolic holomorphic automorphism of $\mathbb{H}^k$. In Theorem 1.2 one obtains an automorphism with Denjoy–Wolff point at ∞ and dilation $\lambda_\infty = \lambda_\xi$. In Theorem 1.3 one obtains an automorphism with a repelling boundary fixed point at ∞ with dilation $\lambda_\infty = \lambda_\zeta$. The semi-conjugacy provided by h satisfies a universal property and thus is unique up to biholomorphisms. Such a semi-conjugacy is called a *canonical model* for f, see Sections 3, 5 for definitions.

In this paper we are interested in extending these results to the case where $f\colon \Omega \to \Omega$ is a holomorphic self-map of a strongly convex domain $\Omega \subset\subset \mathbb{C}^q$ whose boundary is C^3. For such map f, the concepts of boundary regular fixed point and dilation can be defined intrinsically in terms of the Kobayashi distance k_Ω, see Section 2 for definitions. In [2] Abate proved that the Denjoy–Wolff theorem still holds in this setting. Hence we can partition the family of holomorphic self-maps of Ω in elliptic, parabolic and hyperbolic maps exactly as in the case of the ball.

Let thus $f\colon \Omega \to \Omega$ be a holomorphic self-map of a strongly convex domain, which is either hyperbolic or which admits a boundary repelling fixed point ζ. When trying to generalize Theorems 1.2 and 1.3, the first obstacle that one encounters is that the proofs of these theorems rely heavily on the fact that the automorphism group of the ball is transitive, whereas by Wong–Rosay's theorem any strongly convex domain which is not biholomorphic to the ball cannot have a transitive group of automorphisms. Moreover, it is natural to search for a semi-conjugacy of f with a hyperbolic automorphism of a (possibly lower-dimensional) strongly convex domain $\Lambda \subset\subset \mathbb{C}^k$, but it follows again by Wong–Rosay's theorem that if a strongly convex domain Λ admits a non-elliptic automorphism, then Λ is biholomorphic to the ball $\mathbb{B}^k$.

Indeed, we prove that f admits a natural semi-conjugacy with a hyperbolic automorphism of a ball $\mathbb{B}^k$, where $1 \le k \le q$. To cope with the lack of transitivity, we use the fact that the squeezing function S_Ω of Ω converges to 1 at the boundary $\partial\Omega$, which roughly speaking means that the geometry of Ω resembles more and more the geometry of the ball as we approach the boundary. Hence we can rescale the dynamics of f obtaining in the limit the desired intertwining mappings with automorphisms of a (possibly lower-dimensional) ball.

Our main results are the following.

Theorem 1.4 (Forward iteration). *Let $\Omega \subset\subset \mathbb{C}^q$ be a strongly convex domain with C^3 boundary. Let $f\colon \Omega \to \Omega$ be a hyperbolic holomorphic self-map with Denjoy–Wolff point ξ. Then there exists an integer $1 \le k \le q$, a holomorphic map $h\colon \Omega \to \mathbb{H}^k$ and a hyperbolic automorphism τ of $\mathbb{H}^k$ of the form*

$$\tau(z_1, z') = \left(\frac{1}{\lambda_\xi} z_1, \frac{e^{it_1}}{\sqrt{\lambda_\xi}} z_1', \ldots, \frac{e^{it_{k-1}}}{\sqrt{\lambda_\xi}} z_{k-1}'\right)$$

such that

$$h \circ f = \tau \circ h.$$

Theorem 1.5 (Backward iteration)**.** *Let $\Omega \subset\subset \mathbb{C}^q$ be a strongly convex domain with C^4 boundary. Let $f\colon \Omega \to \Omega$ be a holomorphic self-map and let ζ be a boundary repelling fixed point. Then there exists an integer $1 \leq k \leq q$, a holomorphic map $h\colon \mathbb{H}^k \to \Omega$ and a hyperbolic automorphism τ of $\mathbb{H}^k$ of the form*

$$\tau(z_1, z') = \left(\frac{1}{\lambda_\zeta} z_1, \frac{e^{it_1}}{\sqrt{\lambda_\zeta}} z'_1, \ldots, \frac{e^{it_{k-1}}}{\sqrt{\lambda_\zeta}} z'_{k-1} \right)$$

such that

$$f \circ h = h \circ \tau.$$

In both cases the semi-conjugacy provided by h is again a canonical model for f and as such it is unique up to biholomorphisms. Notice also that in the forward iteration case we obtain a similar result also for parabolic nonzero-step maps, see Theorem 4.6.

The strong convexity of the domain Ω, which implies the fact that the squeezing function converges to 1 at the boundary $\partial\Omega$, is essential in our proofs. Indeed, a key ingredient of the proof of Theorem 1.4 is that every holomorphic self-map of Ω with an orbit converging to the boundary admits a canonical model biholomorphic to a (possibly lower-dimensional) ball. This is not true on a weakly convex domain. As an example, consider the egg domain

$$\Omega := \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^4 < 1\},$$

which is strongly convex at every point of $\partial\Omega$ except for those with $\{z_2 = 0\}$ (where the squeezing function S_Ω does not converge to 1). Consider the automorphism $f\colon \Omega \to \Omega$ defined changing holomorphic coordinates to the unbounded realization $\{(z_1, z_2)\colon \mathsf{Im}\, z_1 > |z_2|^4\}$ of Ω and considering $(z_1, z_2) \mapsto (\frac{1}{\lambda} z_1, \frac{1}{\sqrt[4]{\lambda}} z_2)$, with $0 < \lambda < 1$. Every forward orbit of the automorphim $f\colon \Omega \to \Omega$ converges to the point $(1, 0)$. A canonical model for the automorphism f is simply given by the identity map $\mathsf{id}\colon \Omega \to \Omega$ intertwining f with itself. But the canonical model is unique up to biholomorphisms, and Ω is not biholomorphic to the ball. Similar considerations hold in the backward iteration case.

When dealing with the backward iteration case, an additional difficulty arises in the proof of Theorem 1.5. Indeed, in order to apply Theorem 5.5 one needs to construct a backward orbit (z_n) converging to ζ and satisfying

$$\lim_{n\to\infty} k_\Omega(z_n, z_{n+1}) = \log \lambda_\zeta.$$

In the case of the ball $\mathbb{B}^q$, such orbit is obtained in [9] using a fixed horosphere centered in ζ to define the stopping time of an iterative process. Transitivity of the automorphism group of $\mathbb{B}^q$ guarantees the convergence of this process. In the case of a strongly convex domain Ω we need first to find a change of coordinates in $\mathrm{Aut}(\mathbb{C}^q)$ so that in the new coordinates the domain Ω contains a $\mathbb{B}^q$-horosphere centered in e_1, and is locally contained in $\mathbb{B}^q$ near e_1. This is done assuming that $\partial\Omega$ is C^4-smooth, composing Fefferman's change of coordinates with a parabolic "push", and using Andersén-Lempert jet interpolation to obtain an automorphism of $\mathbb{C}^q$. As a consequence we obtain that $k_{\mathbb{B}^q}$ and k_Ω are very close on small $\mathbb{B}^q$-horospheres centered in e_1. We then define an iterative process using smaller and smaller $\mathbb{B}^q$-horospheres as stopping times, and we prove the convergence of the process by rescaling it with automorphisms of the ball.

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2. Background

2.1. **Real geodesics.**

Definition 2.1. Let (X,d) be a metric space. A *real geodesic* is a map γ from an interval $I \subset \mathbb{R}$ to X which is an isometry with respect to the euclidean distance on I and the distance on X, that is for all $s,t \in I$,

$$d(\gamma(s),\gamma(t)) = |t-s|.$$

If the interval is closed and bounded (resp. $[0,+\infty)$, $(-\infty,+\infty)$) we call γ a *geodesic segment* (resp. *geodesic ray*, *geodesic line*).

2.2. **The ball.** Horospheres and Koranyi regions play a central role in the study of strongly convex domains. They are generalizations of corresponding notions appearing in the setting of the unit ball $\mathbb{B}^q$, where they are defined in euclidean terms. Here we recall their definitions.

The *horosphere* of *center* $\zeta \in \partial\mathbb{B}^q$ and *radius* $R > 0$ is defined as

$$E(\zeta,R) := \left\{ z \in \mathbb{B}^q \colon \frac{|1-\langle z,\zeta\rangle|^2}{1-\|z\|^2} < R \right\}.$$

The *Koranyi region* of *center* $\zeta \in \partial\mathbb{B}^q$ and *amplitude* $M > 1$ is defined as

$$K(\zeta,M) := \left\{ z \in \mathbb{B}^q \colon \frac{|1-\langle z,\zeta\rangle|}{1-\|z\|} < M \right\}.$$

When working with horospheres, it is sometime convenient to consider their expression in the *Siegel half-space* $\mathbb{H}^q := \{(z_1,z') \in \mathbb{C}\times\mathbb{C}^{q-1} \colon \operatorname{Im} z_1 > \|z'\|^2\}$, which is biholomorphic to $\mathbb{B}^q$ under the *Cayley transform* $\mathcal{C}\colon \mathbb{B}^q \to \mathbb{H}^q$

$$\mathcal{C}(z_1,z') := \left(i\frac{1+z_1}{1-z_1}, \frac{z'}{1-z_1} \right). \tag{2.1}$$

Notice that also the bihomolomorphism $(z_1,z') \mapsto \left(i\frac{1+z_1}{1-z_1}, i\frac{z'}{1-z_1} \right)$ from $\mathbb{B}^q$ to $\mathbb{H}^q$ is commonly referred to as the Cayley transform.

With this change of holomorphic coordinates, the point $e_1 = (1,0,\ldots,0)$ is sent to ∞ and the horosphere $E(e_1,R)$ becomes

$$E(\infty,R) := \left\{ (z_1,z') \in \mathbb{H}^q \colon \operatorname{Im} z_1 > \|z'\|^2 + \frac{1}{R} \right\}.$$

The automorphism group $\operatorname{Aut}(\mathbb{B}^q)$ is transitive, and this property characterizes the ball among strongly pseudoconvex domains.

Theorem 2.2 (Wong–Rosay [24]). *Let $\Omega \subset \mathbb{C}^q$ be a domain. Suppose that there exist $x_0 \in \Omega$ and a sequence (φ_n) in $\operatorname{Aut}(\Omega)$ such that $\varphi_n(x_0) \to \zeta \in \partial\Omega$, and that $\partial\Omega$ is C^2 and strongly pseudoconvex near ζ. Then Ω is biholomorphic to $\mathbb{B}^q$.*

2.3. **Strongly convex domains.** We start by recalling the definition of *strong convexity.*

Definition 2.3. A bounded convex domain $\Omega \subset \mathbb{C}^q$ with C^2 boundary is *strongly convex* at $\zeta \in \partial\Omega$ if for some (and hence for any) defining function ρ for Ω at ζ, the Hessian $H_\zeta\rho$ is positive definite on the tangent space $T_\zeta\partial\Omega$. The domain Ω is *strongly convex* if it is strongly convex at every point $\zeta \in \partial\Omega$.

Remark 2.4. It is well known that a strongly convex domain is also strongly pseudoconvex.

The analysis of strongly convex domains relies extensively on Lempert's theory of complex geodesics [20].

Definition 2.5. A *complex geodesic* in a Kobayashi hyperbolic manifold X is a holomorphic map $\varphi\colon \mathbb{D} \to X$ which is an isometry with respect to the Kobayashi distance of the disc $\mathbb{D} \subset \mathbb{C}$ and the Kobayashi distance of X.

Theorem 2.6 (See e.g. [1])**.** *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary.*

(1) For every pair of distinct points $z, w \in \Omega$, let $r := k_\Omega(z, w)$. Then there exists a unique complex geodesic φ such that $\varphi(0) = z$ and $\varphi(r) = w$.
(2) Every complex geodesic $\varphi\colon \mathbb{D} \to \Omega$ extends to a continuous map on $\overline{\mathbb{D}}$, and the extension is injective.
(3) For every $z \in \Omega$ and $\zeta \in \partial\Omega$, there exists a unique complex geodesic with $\varphi(0) = z$ and $\varphi(1) = \zeta$.

Complex geodesics are isometries between $(\mathbb{D}, k_\mathbb{D})$ and (Ω, k_Ω), and therefore map real geodesic of $\mathbb{D}$ to real geodesic in Ω. On the other hand every real geodesic of Ω is contained in some complex geodesic [17, Lemma 3.3], and its pullback is a real geodesic in $\mathbb{D}$. Hence every real geodesic in Ω is C^∞, and for all $x \neq y \in \Omega$ the geodesic segment joining x to y is unique up to isometries of the interval I. Moreover for every $p \in \Omega$ and $\zeta \in \partial\Omega$ there exists a unique geodesic ray connecting the two points.

Let $\zeta \in \partial\Omega$ and choose a *pole* $p \in \Omega$. It is proved in [1, Theorem 2.6.47] that the limit

$$\lim_{w\to\zeta} [k_\Omega(z, w) - k_\Omega(p, w)] \tag{2.2}$$

exists. We denote by $h_{\zeta,p}\colon \Omega \to \mathbb{R}_{>0}$ (sometimes by $h^\Omega_{\zeta,p}$) the continuous function defined as

$$h_{\zeta,p}(z) := \exp\left(\lim_{w\to\zeta} [k_\Omega(z, w) - k_\Omega(p, w)]\right).$$

If instead of p we choose a different pole $p' \in \Omega$ the function changes by a multiplicative constant:

$$h_{\zeta,p'}(z) = h_{\zeta,p}(z)h_{\zeta,p'}(p). \tag{2.3}$$

The concepts of horosphere, Koranyi region, boundary regular fixed points and dilation can be carried over to the case of strongly convex domains with C^3 boundary, giving intrinsic definitions in terms of the Kobayashi distance.

Definition 2.7. The *horosphere* of *center* $\zeta \in \partial\Omega$, *pole* $p \in \Omega$ and *radius* $R > 0$ is the set

$$E_\Omega(p, \zeta, R) := \{z \in \Omega \,|\, h_{\zeta,p}(z) < R\}.$$

The *Koranyi region* of *center* $\zeta \in \partial\Omega$, *pole* $p \in \Omega$ and *amplitude* $M > 1$ is the set

$$K_\Omega(p, \zeta, M) := \{z \in \Omega \,|\, \log h_{\zeta,p}(z) + k_\Omega(p, z) < 2 \log M\}.$$

Remark 2.8. When $\Omega = \mathbb{B}^q$, horospheres and Koranyi regions with pole $p = 0$ and center $\zeta \in \partial\mathbb{B}^q$ coincide with the regions $E(\zeta, R)$ and $K(\zeta, M)$ defined previously (see [1, Propositions 2.2.20 and 2.7.3]).

Definition 2.9. Let $\Omega \subset \mathbb{C}^q$ be a strongly convex domain with C^3 boundary. A holomorphic map $f\colon \Omega \to \mathbb{C}^m$ has K-limit σ at $\zeta \in \partial\Omega$ if for every sequence (z_n) converging to ζ inside a

Koranyi region we have that $f(z_n)$ converges to σ. If $f\colon \Omega \to \Omega$ is a holomorphic self-map, a point $\zeta \in \partial\Omega$ is a *boundary fixed point* if

$$K\text{-}\lim_{z\to\zeta} f(z) = \zeta.$$

Given $\zeta \in \partial\Omega$, the *dilation* of f at ζ with *pole* $p \in \Omega$ is the number $\lambda_{\zeta,p} \in \mathbb{R}_{>0}$ defined as

$$\log \lambda_{\zeta,p} = \liminf_{z\to\zeta} [k_\Omega(p,z) - k_\Omega(p, f(z))].$$

Remark 2.10. By [4, Lemma 1.3] the dilation coefficient $\lambda_{\zeta,p}$ at a boundary fixed point does not depend on $p \in \Omega$, thus we can write $\lambda_\zeta = \lambda_{\zeta,p}$. The proof of this fact is based on the existence of complex geodesics and of the limit (2.2).

Definition 2.11. A boundary fixed point $\zeta \in \partial\Omega$ for $f\colon \Omega \to \Omega$ is *regular* if its dilation λ_ζ is finite.

Remark 2.12. If $\Omega = \mathbb{B}^q$, then a straightforward calculation shows that

$$\liminf_{z\to\zeta} [k_\Omega(0,z) - k_\Omega(0, f(z))] = \liminf_{z\to\zeta} \log \frac{1 - \|f(z)\|}{1 - \|z\|},$$

in agreement with (1.1).

We will need the following version of Julia's Lemma (see e.g. [1, Theorem 2.4.16, Proposition 2.7.15])

Proposition 2.13. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary, and let $f\colon \Omega \to \Omega$ be a holomorphic self-map. Let $\zeta \in \partial\Omega$ be a boundary regular fixed point, and let $p \in \Omega$. Then*

$$f(E_\Omega(p,\zeta,R)) \subset E_\Omega(p,\zeta,\lambda_\zeta R), \quad \forall R > 0.$$

Complex geodesics are also useful in order to compute dilation coefficients. Recall for example the following result [4, Lemma 3.1].

Lemma 2.14. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary, and let $f\colon \Omega \to \Omega$ be a holomorphic self-map. Let $\zeta \in \partial\Omega$ be a boundary regular fixed point of f, and let $\varphi\colon \mathbb{D} \to \Omega$ be a complex geodesic with $\varphi(1) = \zeta$. Then*

$$\lim_{t\to 1, t\in\mathbb{R}\cap\mathbb{D}} k_\Omega(\varphi(t), f(\varphi(t))) = |\log \lambda_\zeta|.$$

The Denjoy–Wolff theorem also carries over to this setting (see e.g. [2, Theorem 0.6]).

Theorem 2.15. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex C^3 domain. Let $f\colon \Omega \to \Omega$ be a holomorphic self-map without interior fixed points. Then there exists a boundary regular fixed point $\xi \in \partial\Omega$ with dilation $0 < \lambda_\xi \le 1$, called the* Denjoy–Wolff point, *such that the sequence of iterates (f^n) converges to ξ.*

This allows to partition the family of holomorphic self-maps of Ω as in the ball.

Definition 2.16. Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex C^3 domain. A holomorphic self-map $f\colon \Omega \to \Omega$ is called *elliptic* if it admits an interior fixed point. Otherwise it is called *hyperbolic* if the dilation λ_ξ at its Denjoy–Wolff point satisfies $\lambda_\xi < 1$, and it is called *parabolic* if $\lambda_\xi = 1$.

2.4. **Squeezing function.** The squeezing function $S_\Omega \colon \Omega \to (0,1]$ of a bounded domain $\Omega \subset \mathbb{C}^q$ measures how much Ω resembles the ball $\mathbb{B}^q$.

Definition 2.17. Let $\Omega \subset \mathbb{C}^q$ be a bounded domain and $z \in \Omega$. If $\varphi : \Omega \to \mathbb{B}^q$ is an injective holomorphic function with $\varphi(z) = 0$ we set

$$S_{\Omega,\varphi}(z) := \sup\{r > 0\colon B(0,r) \subset \varphi(\Omega)\},$$

and

$$S_\Omega(z) := \sup_\varphi \{S_{\Omega,\varphi}(z)\}.$$

The function $S_\Omega : \Omega \to (0,1]$ is called the *squeezing function* of the domain Ω.

By a normality argument it follows that the sup is actually attained.

Proposition 2.18. *Let $\Omega \subset \mathbb{C}^q$ be a bounded domain and $z \in \Omega$. Then there exists an injective holomorphic map $\varphi\colon \Omega \to \mathbb{B}^q$, with $\varphi(z) = 0$ such that*

$$S_{\Omega,\varphi}(z) = S_\Omega(z).$$

We will need the following result proved in [14].

Theorem 2.19. *If $\Omega \subset \mathbb{C}^q$ is a bounded strongly pseudoconvex domain with C^2 boundary, then*

$$\lim_{z \to \partial\Omega} S_\Omega(z) = 1.$$

Part 1. Forward iteration

3. Canonical Kobayashi hyperbolic semi-models

In this section we construct a Canonical Kobayashi hyperbolic semi-model for a holomorphic self-map f of a bounded domain Ω, assuming that the squeezing function converges to 1 along an orbit.

Definition 3.1. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. Let $x \in X$, and let $m \geq 1$. The *forward m-step* $s_m(x)$ of f at x is the limit

$$s_m(x) := \lim_{n\to\infty} k_X(f^n(x), f^{n+m}(x)).$$

Such a limit exists since the sequence $(k_X(f^n(x), f^{n+m}(x)))_{n\geq 0}$ is non-increasing. The *divergence rate* $c(f)$ of f is the limit

$$c(f) := \lim_{m\to\infty} \frac{k_X(f^m(x), x)}{m}. \tag{3.1}$$

It is shown in [8] that the limit above exists, does not depend on the point $x \in X$ and equals $\inf_{m\in\mathbb{N}} \frac{k_X(f^m(x),x)}{m}$.

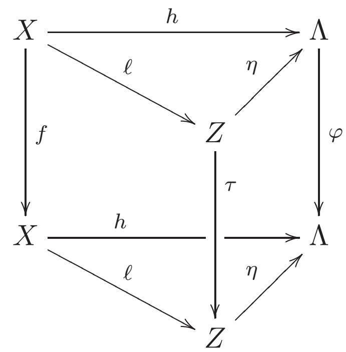
Definition 3.2. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. A *semi-model* for f is a triple (Λ, h, φ) where Λ is a complex manifold called the *base space*, $h\colon X \to \Lambda$ is a holomorphic mapping, and $\varphi\colon \Lambda \to \Lambda$ is an automorphism such that

$$h \circ f = \varphi \circ h, \tag{3.2}$$

and

$$\bigcup_{n\geq 0} \varphi^{-n}(h(X)) = \Lambda. \tag{3.3}$$

Let (Z, ℓ, τ) and (Λ, h, φ) be two semi-models for the map f. A *morphism of semi-models* $\hat{\eta}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$ is given by a holomorphic map $\eta : Z \to \Lambda$ such that the following diagram commutes:



If the mapping $\eta\colon Z \to \Lambda$ is a biholomorphism, then we say that $\hat{\eta}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$ is an *isomorphism of semi-models*. Notice that then $\eta^{-1}\colon \Lambda \to Z$ induces a morphism $\hat{\eta}^{-1}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$.

Definition 3.3. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. Let (Z, ℓ, τ) be a semi-model for f whose base space Z is Kobayashi hyperbolic. We say that (Z, ℓ, τ) is a *canonical Kobayashi hyperbolic semi-model* for f if for any semi-model (Λ, h, φ) for f such that the base space Λ is Kobayashi hyperbolic, there exists a unique morphism of semi-models $\hat{\eta}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$.

Remark 3.4. If (Z, ℓ, τ) and (Λ, h, φ) are two canonical Kobayashi hyperbolic semi-models for f, then they are isomorphic.

In this section we prove the following result.

Theorem 3.5. *Let $\Omega \subset \mathbb{C}^q$ be a bounded domain, $f\colon \Omega \to \Omega$ be a holomorphic self-map and assume that there exists an orbit (z_m) with $S_\Omega(z_m) \to 1$. Then there exists a canonical Kobayashi hyperbolic semi-model $(\mathbb{B}^k, \ell, \tau)$ for f with $0 \le k \le q$. Moreover, the following holds:*

(1) for all $n \ge 0$,

$$\lim_{m\to\infty} (f^m)^* k_\Omega = (\tau^{-n} \circ \ell)^* k_{\mathbb{B}^k},$$

(2) the divergence rate of τ satisfies

$$c(\tau) = c(f) = \lim_{m\to\infty} \frac{s_m(x)}{m} = \inf_{m\in\mathbb{N}} \frac{s_m(x)}{m}.$$

Remark 3.6. By Theorem 2.19 the assumptions of the theorem are satisfied when Ω is bounded strongly pseudoconvex with C^2 boundary and (z_m) converges to $\partial\Omega$.

The proof of Theorem 3.5 is based on the following result.

Proposition 3.7. *Let $\Omega \subset \mathbb{C}^q$ be a bounded domain, $f\colon \Omega \to \Omega$ be a holomorphic self-map and assume that there exists an orbit (z_m) such that $S_\Omega(z_m) \to 1$. Then there exists a family of holomorphic maps $(\alpha_n\colon \Omega \to Z)$, where Z is an holomorphic retract of $\mathbb{B}^q$, such that the following hold:*

(a) *for all $m \ge n \ge 0$,*

$$\alpha_m \circ f^{m-n} = \alpha_n,$$

(b) *for every* $n \geq 0$ *we have* $\alpha_n(\Omega) \subset \alpha_{n+1}(\Omega)$ *and*

$$\bigcup_{n\in\mathbb{N}} \alpha_n(\Omega) = Z, \tag{3.4}$$

(c) *for all* $n \geq 0$,

$$\lim_{m\to\infty} (f^m)^* k_\Omega = \alpha_n^* k_Z, \tag{3.5}$$

(d) **Universal property***: let* Q *be a Kobayashi hyperbolic complex manifold and* $(\gamma_n\colon \Omega \to Q)$ *a family of holomorphic mappings satisfying* $\gamma_m \circ f^{m-n} = \gamma_n$ *for all* $m \geq n \geq 0$*. Then there exists a unique holomorphic map* $\Gamma\colon Z \to Q$ *such that* $\gamma_n = \Gamma \circ \alpha_n$ *for all* $n \geq 0$*.*

Remark 3.8. Such family (α_n) is a canonical Kobayashi hyperbolic direct limit for the sequence of iterates $(f^{m-n}\colon \Omega \to \Omega)$, see [6, Definition 2.7].

Once Proposition 3.7 is proved, the proof of Theorem 3.5 is the same as that of [6, Theorem 4.6]. We present a sketch of the construction of the semi-model for the convenience of the reader.

Proof of Theorem 3.5. Define $\ell := \alpha_0$ and $\gamma_n := \alpha_n \circ f$. It is not hard to show that $(\gamma_n\colon \Omega \to Z)$ is a family of holomorphic mappings satisfying $\gamma_m \circ f^{m-n} = \gamma_n$ for all $m \geq n \geq 0$. Therefore by the universal property of the family (α_n) there exists a unique holomorphic map $\tau\colon Z \to Z$ such that for all $n \geq 0$,

$$\tau \circ \alpha_n = \gamma_n = \alpha_n \circ f,$$

in particular $\tau \circ \ell = \ell \circ f$.

Similarly if we define $\tilde{\gamma}_n := \alpha_{n+1}$ we obtain a holomorphic map $\delta : Z \to Z$ such that $\tilde{\gamma}_n = \delta \circ \alpha_n$ for all $n \geq 0$. It is easy to see that

$$\tau \circ \delta \circ \alpha_n = \delta \circ \tau \circ \alpha_n = \alpha_n.$$

By the universal property of the family (α_n) described in Proposition 3.7, we conclude that $\delta = \tau^{-1}$, proving that τ is an automorphism of Z. Since for all $n \geq 0$,

$$\tau^n \circ \alpha_n = \alpha_n \circ f^n = \ell,$$

it follows that $\alpha_n = \tau^{-n} \circ \ell$. The triple (Z, ℓ, τ) is a semi-model thanks to (3.4) Notice that Z being a holomorphic retract of $\mathbb{B}^q$, it is biholomorphic to a ball of dimension $0 \leq k \leq q$. The universal property of the canonical Kobayashi hyperbolic semi-model (Z, ℓ, τ) is a direct consequence of the universal property of the family (α_n).

Point (1) follows immediately from (3.5), hence we are left with proving (2). From [8, Proposition 2.7] it follows that $c(f) = \lim_{m\to\infty} \frac{s_m(x)}{m} = \inf_{m\in\mathbb{N}} \frac{s_m(x)}{m}$. Equation (3.5) immediately gives the following formula for the forward m-step, which implies (2),

$$s_m(x) = k_Z(\alpha_0(x), \alpha_0 \circ f^m(x)) = k_Z(\ell(x), \tau^m \circ \ell(x)).$$

□

The proof of Proposition 3.7 is articulated in several intermediate lemmas. Let (z_m) be an f-orbit in Ω with $S_\Omega(z_m) \to 1$. By Proposition 2.18 there exists a sequence $(\psi_m\colon \Omega \to \mathbb{B}^q)$ of holomorphic injective maps with $\psi_m(z_m) = 0$ and

$$\psi_m(\Omega) \supset B(0, S_\Omega(z_m)).$$

Notice that for every compact subset $K \subset \mathbb{B}^q$ the inverse map ψ_m^{-1} is defined on K for m sufficiently large.

Since $\psi_m \circ f^m(z_0) = 0$ for all $m \geq 0$ and since $\mathbb{B}^q$ is taut, there exists a subsequence $(m_0(h))$ such that the sequence $(\psi_{m_0(h)} \circ f^{m_0(h)})$ converges uniformly on compact subsets to a holomorphic map $\alpha_0\colon \Omega \to \mathbb{B}^q$ and $\alpha_0(z_0) = 0$. Similarly, there exists a subsequence $(m_1(h))$ of $(m_0(h))$ such that the sequence $(\psi_{m_1(h)} \circ f^{m_1(h)-1})$ converges uniformly on compact subsets to a holomorphic map $\alpha_1 : \Omega \to \mathbb{B}^q$ and $\alpha_1(z_1) = 0$. Iterating this procedure we obtain a family of subsequences $\{(m_n(h))_{h\geq 0}\}_{n\geq 0}$ and a family of holomorphic maps

$$(\alpha_n : \Omega \to \mathbb{B}^q)_{n\geq 0}$$

such that

$$\psi_{m_n(h)} \circ f^{m_n(h)-n} \overset{h\to\infty}{\longrightarrow} \alpha_n$$

uniformly on compact subsets and $\alpha_n(z_n) = 0$. Notice that for all $m \geq n \geq 0$,

$$\alpha_m \circ f^{m-n} = \alpha_n. \tag{3.6}$$

Let $\nu(h) := m_h(h)$ be the diagonal subsequence, which for all $j \geq 0$ is eventually a subsequence of $(m_j(h))_{h\geq 0}$.

Consider the sequence $\beta_{\nu(h)} := \alpha_{\nu(h)} \circ \psi_{\nu(h)}^{-1}$. Given a compact subset $K \subset \mathbb{B}^q$ and h large enough, the map $\beta_{\nu(h)}$ is well defined on K and $\beta_{\nu(h)}(K) \subset \mathbb{B}^q$. Notice that $\beta_{\nu(h)}(0) = 0$ for all $h \geq 0$. By the tautness of $\mathbb{B}^q$ up to extracting a further subsequence of $\nu(h)$ we have that the sequence $(\beta_{\nu(h)})$ converges uniformly on compact subsets to a holomorphic map $\alpha\colon \mathbb{B}^q \to \mathbb{B}^q$.

Lemma 3.9. *For all $j \geq 0$,*

$$\alpha \circ \alpha_j = \alpha_j. \tag{3.7}$$

Proof. Let $z \in \mathbb{B}^q$. For all positive integers h such that $\nu(h) \geq j$, we have, using (3.6),

$$\alpha_j(z) = (\alpha_{\nu(h)} \circ f^{\nu(h)-j})(z) = (\alpha_{\nu(h)} \circ \psi_{\nu(h)}^{-1} \circ \psi_{\nu(h)} \circ f^{\nu(h)-j})(z) \overset{h\to\infty}{\longrightarrow} (\alpha \circ \alpha_j)(z).$$

□

Lemma 3.10. *The map $\alpha\colon \mathbb{B}^q \to \mathbb{B}^q$ is a holomorphic retraction, that is*

$$\alpha \circ \alpha = \alpha.$$

Proof. Let $z \in \mathbb{B}^q$. From (3.7) we get, for all $h \geq 0$ big enough,

$$(\alpha \circ \beta_{\nu(h)})(z) = (\alpha \circ \alpha_{\nu(h)} \circ \psi_{\nu(h)}^{-1})(z) = (\alpha_{\nu(h)} \circ \psi_{\nu(h)}^{-1})(z) = \beta_{\nu(h)}(z),$$

and the result follows since $\beta_{\nu(h)} \to \alpha$. □

Define $Z := \alpha(\mathbb{B}^q)$. Being a holomorphic retract, it is a closed complex submanifold of $\mathbb{B}^q$, biholomorphic to a k-dimensional ball $\mathbb{B}^k$, with $0 \leq k \leq q$. By (3.7) it follows that, for all $j \geq 0$,

$$\alpha_j(\Omega) \subset Z.$$

Let (A, Λ_n) be the direct limit of the dynamical system $(f^n : \Omega \to \Omega)$. Recall that $A := (\Omega \times \mathbb{N})/_\sim$, where $(x, n) \sim (y, u)$ if and only if $f^{m-n}(x) = f^{m-u}(y)$ for m large enough, and the equivalence class of (x, n) is denoted by $[x, n]$. The map $\Lambda_n : \Omega \to A$ is defined by $\Lambda_n(x) = [x, n]$.

By the universal property of direct limits, there exists a unique map $\Psi : A \to Z$ such that, for all $n \geq 0$,

$$\alpha_n = \Psi \circ \Lambda_n.$$

The mapping Ψ sends the point $[x,n] \in A$ to $\alpha_n(x)$. Define on A the following equivalence relation:

$$[x,n] \simeq [y,u] \quad \iff \quad k_\Omega(f^{m-n}(x), f^{m-u}(y)) \stackrel{m\to\infty}{\longrightarrow} 0.$$

Lemma 3.11. *The map $\Psi\colon A \to Z$ is surjective and $\Psi([x,n]) = \Psi([y,u])$ if and only if $[x,n] \simeq [y,u]$.*

Proof. We first prove surjectivity. We have to prove that for all $z \in Z$ there exists $x \in \Omega$ and $n \geq 0$ such that $\alpha_n(x) = z$. Let $U \subset Z$ be a relatively compact neighborhood in Z of z. The sequence $(\beta_{\nu(h)}|_U) = (\alpha_{\nu(h)} \circ \psi^{-1}_{\nu(h)}|_U)$ is well defined for h big enough and converges uniformly to $\alpha|_U = \mathsf{id}_U$, and therefore is eventually injective and its image eventually contains z.

If $[x,n] \simeq [y,u]$, then since the Kobayashi distance is non-expansive with respect to holomorphic maps, we have

$$k_Z(\Psi[x,n], \Psi[y,u]) = k_Z(\alpha_m \circ f^{m-n}(x), \alpha_m \circ f^{m-u}(y)) \leq k_\Omega(f^{m-n}(x), f^{m-u}(y)) \stackrel{m\to\infty}{\longrightarrow} 0.$$

As Z is Kobayashi hyperbolic, it follows that $\Psi[x,n] = \Psi[y,u]$.

Conversely, assume that $\Psi([x,n]) = \Psi([y,u])$, and fix $j \geq \max\{n,u\}$. We have

$$\alpha_j \circ f^{j-n}(x) = \alpha_j \circ f^{j-u}(y).$$

By definition of the map α_j it follows that

$$\lim_{h\to\infty} \psi_{m_j(h)} \circ f^{m_j(h)-n}(x) = \lim_{h\to\infty} \psi_{m_j(h)} \circ f^{m_j(h)-u}(y).$$

We claim that this implies that $[x,n] \simeq [y,u]$. Notice that, since the sequence

$$\left(k_\Omega(f^{m-n}(x), f^{m-u}(y))\right)_{m\geq \max\{n,u\}}$$

is decreasing, it suffices to show that

$$k_\Omega(f^{m_j(h)-n}(x), f^{m_j(h)-u}(y)) \stackrel{h\to\infty}{\longrightarrow} 0.$$

Denote $z_h := \psi_{m_j(h)} \circ f^{m_j(h)-n}(x)$ and $w_h := \psi_{m_j(h)} \circ f^{m_j(h)-u}(y)$. Then the sequences (z_h) and (w_h) converge to the same point $a \in \mathbb{B}^q$.

Let $B \subset\subset \mathbb{B}^q$ be a ball centered in a. When h is sufficiently large we have that $\psi^{-1}_{m_j(h)}\colon B \to \Omega$ is well defined and since the Kobayashi distance is not expanding, we conclude that

$$k_\Omega(f^{m_j(h)-n}(x), f^{m_j(h)-u}(y)) \leq k_B(z_h, w_h) \to 0.$$

□

Lemma 3.12. *For all $n \geq 0$,*

$$\lim_{m\to\infty} (f^m)^* k_\Omega = \alpha_n^* k_Z.$$

Proof. For all $m \geq n \geq 0$ we have

$$k_Z(\alpha_n(x), \alpha_n(y)) = k_Z(\alpha_m \circ f^{m-n}(x), \alpha_m \circ f^{m-n}(y)) \leq k_\Omega(f^{m-n}(x), f^{m-n}(y)).$$

Hence $k_Z(\alpha_n(x), \alpha_n(y)) \leq \lim_{m\to\infty} k_\Omega(f^m(x), f^m(y))$.

To obtain the inverse inequality, denote $z_h := \psi_{m_n(h)} \circ f^{m_n(h)-n}(x)$ and $w_h := \psi_{m_n(h)} \circ f^{m_n(h)-n}(y)$. Then (z_h) converges to $\alpha_n(x)$ and (w_h) converges to $\alpha_n(y)$. Fix $\epsilon > 0$, then there exist a ball $B = B(0,r) \subset \mathbb{B}^q$, with radius close enough to 1 such that it contains both $\alpha_n(x), \alpha_n(y)$, and such that for some $h_0 \geq 0$ we have

$$k_B(z_h, w_h) \leq k_{\mathbb{B}^q}(\alpha_n(x), \alpha_n(y)) + \epsilon, \qquad \forall h \geq h_0.$$

Let $h_1 \geq 0$ be such that for all $h \geq h_1$ we have that $\psi_{m_n(h)}(\Omega) \supset B$. Then for all $h \geq \max\{h_0, h_1\}$ we have

$$k_\Omega(f^{m_n(h)-n}(x), f^{m_n(h)-n}(y)) \leq k_B(z_h, w_h) \leq k_{\mathbb{B}^q}(\alpha_n(x), \alpha_n(y)) + \epsilon.$$

proving that for every $\varepsilon > 0$ we have

$$\lim_{m\to\infty} k_\Omega(f^m(x), f^m(y)) \leq k_Z(\alpha_n(x), \alpha_n(y)) + \epsilon,$$

where we used the fact that $\alpha_n(x), \alpha_n(y) \in Z$ and that $k_{\mathbb{B}^q}|_Z = k_Z$. □

We are now ready to prove Proposition 3.7. Points (a) and (c) correspond precisely to (3.6) and Lemma 3.12. By (3.6) it is clear that $\alpha_n(\Omega) = \alpha_{n+1}(f(\Omega)) \subset \alpha_{n+1}(\Omega)$, and by Lemma 3.11 we obtain that the union of the sets $\alpha_n(\Omega)$ coincides with Z, which proves point (b).

It remains to prove the universal property (d). Let Q be a Kobayashi hyperbolic complex manifold and let $(\gamma_n \colon \Omega \to Q)$ be a family of holomorphic maps satisfying $\gamma_m \circ f^{m-n} = \gamma_n$ for all $m \geq n \geq 0$. By the universal property of the direct limit, there exists a unique map $\Phi \colon A \to Q$ such that $\gamma_n = \Phi \circ \Lambda_n$ for all $n \geq 0$. The map Φ passes to the quotient to a map $\hat{\Phi} \colon A/_{\simeq} \to Q$. Indeed, if $[(x,n)] \simeq [(y,u)]$, for all $m \geq n, u$ we have that $\Phi([(x,n)]) = \gamma_m \circ f^{m-n}(x)$ and $\Phi([(y,u)]) = \gamma_m \circ f^{m-u}(y)$. Hence

$$k_Q(\Phi([(x,n)]), \Phi([(y,u)])) \leq k_\Omega(f^{m-n}(x), f^{m-u}(y)) \overset{m\to\infty}{\longrightarrow} 0,$$

and thus $\Phi([(x,n)]) = \Phi([(y,u)])$. Set

$$\Gamma := \hat{\Phi} \circ \hat{\Psi}^{-1} \colon Z \to Q.$$

The mapping Γ acts in the following way: if $z \in Z$, then there exists $x \in \Omega$ and $n \geq 0$ such that $\alpha_n(x) = z$, and then $\Gamma(z) = \gamma_n(x)$. It is thus clear that it is the unique map satisfying $\Gamma \circ \alpha_n = \gamma_n$ for all $n \geq 0$. The map Γ is holomorphic. Indeed, if $z \in Z$, by the proof of Lemma 3.11, there exist a neighborhood U of z in Z, a point $w \in U$ and $m' \geq 0$ such that $(\alpha_{m'} \circ \psi_{m'}^{-1}|_U \colon U \to Z)$ is defined, holomorphic and injective and $\alpha_{m'} \circ \psi_{m'}^{-1}(w) = z$. Thus there exists an open neighborhood $V \subset Z$ of z and a holomorphic map $\sigma \colon V \to U$ such that

$$\alpha_{m'} \circ \psi_{m'}^{-1} \circ \sigma = \mathsf{id}_V.$$

Then, for all $y \in V$,

$$\Gamma(y) = \Gamma \circ \alpha_{m'} \circ \psi_{m'}^{-1} \circ \sigma(y) = \gamma_{m'} \circ \psi_{m'}^{-1} \circ \sigma(y),$$

that is, Γ is holomorphic in V, which concludes the proof of (d) and of Proposition 3.7.

4. Main result on strongly convex domains

In this section we apply the results of the previous section to the case of strongly convex domains, and we prove Theorem 1.4. We start with the following proposition in which we compare the dilation λ_ξ with the divergence rate $c(f)$.

Proposition 4.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly convex domain with C^3 boundary. Let $f \colon \Omega \to \Omega$ be a holomorphic self-map without interior fixed points, and let ξ be its Denjoy–Wolff point. Then*

$$\log \lambda_\xi = -c(f).$$

Proof. Let $p, z \in \Omega$. We have

$$\begin{aligned} -c(f) &= \lim_{n\to+\infty} \frac{-k_\Omega(p, f^n(z))}{n} \\ &\geq \liminf_{n\to+\infty} [-k_\Omega(p, f^{n+1}(z)) + k_\Omega(p, f^n(z))] \\ &\geq \liminf_{z\to\xi} [k_\Omega(p, z) - k_\Omega(p, f(z))], \end{aligned}$$

where we used that for all real sequences (a_n)

$$\liminf_{n\to+\infty} \frac{a_n}{n} \geq \liminf_{n\to+\infty} [a_{n+1} - a_n].$$

Hence $\log \lambda_\xi \leq -c(f)$.

If $\lambda_\xi = 1$, the result follows. If $0 < \lambda_\xi < 1$, we obtain the converse inequality in the following way. We claim that if $z \in E_\Omega(p, \xi, R)$, then $k_\Omega(p, z) \geq -\log R$. Indeed we have that $k_\Omega(p, z) \geq k_\Omega(p, w) - k_\Omega(z, w)$ for all $w \in \Omega$ and thus

$$k_\Omega(p, z) \geq \lim_{w\to\xi} [k_\Omega(p, w) - k_\Omega(z, w)] > -\log R.$$

Let $z \in E_\Omega(p, \xi, 1)$. It follows from Proposition 2.13 that $f^n(z) \in E_\Omega(p, \xi, \lambda_\xi^n)$. Hence

$$\frac{k_\Omega(p, f^n(z))}{n} \geq -\frac{\log \lambda_\xi^n}{n} = -\log \lambda_\xi.$$

□

Remark 4.2. Proposition 4.1 shows why the concept of divergence rate is relevant in this context. Indeed, let $f : \Omega \to \Omega$ be a holomorphic self-map without interior fixed points, and let $(\mathbb{B}^k, \ell, \tau)$ be the canonical Kobayashi hyperbolic semi-model given by Theorem 3.5. Assume that τ has no interior fixed points. Since $c(f) = c(\tau)$ it follows that the dilation of f and τ at their respective Denjoy-Wolff points is the same.

We are now ready to state the main result of this section.

Theorem 4.3. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary. Let $f : \Omega \to \Omega$ be a hyperbolic holomorphic self-map, with Denjoy–Wolff point ξ. Then there exist*

(1) an integer k such that $1 \leq k \leq q$,
(2) a hyperbolic automorphism $\tau : \mathbb{H}^k \to \mathbb{H}^k$ of the form

$$\tau(z_1, z') = \left(\frac{1}{\lambda_\xi} z_1, \frac{e^{it_1}}{\sqrt{\lambda_\xi}} z'_1, \ldots, \frac{e^{it_{k-1}}}{\sqrt{\lambda_\xi}} z'_{k-1} \right), \tag{4.1}$$

where $t_j \in \mathbb{R}$ for $1 \leq j \leq k-1$,
(3) a holomorphic mapping $\ell : \Omega \to \mathbb{H}^k$,

such that the triple $(\mathbb{H}^k, \ell, \tau)$ is a canonical Kobayashi hyperbolic semi-model for f.

Proof. Thanks to Theorem 3.5 we obtain the existence of a canonical Kobayashi hyperbolic semi-model $(\mathbb{H}^k, \ell, \tau)$ for f with $c(\tau) = c(f) > 0$. It immediately follows that $k > 0$. Moreover, by Proposition 4.1 it follows that τ is a hyperbolic automorphism of $\mathbb{H}^k$ with dilation λ_ξ at its Denjoy–Wolff point. Now we can change variables in $\mathbb{H}^k$ to put τ the form (4.1), concluding the proof. □

Remark 4.4. It is natural to ask whether

$$K\text{-}\lim_{z\to\xi} h(z) = \infty, \tag{4.2}$$

which is the case when $\Omega = \mathbb{B}^q$. If there exists an orbit (z_n) which enters eventually a Koranyi region with vertex at the Denjoy-Wolff point ξ, then (4.2) follows as in [8, Theorem 5.6]. Notice that the proof of [8, Theorem 5.6] is given for the ball $\mathbb{B}^q$, but a similar proof works for bounded strongly convex domains with smooth boundary. If $\Omega = \mathbb{B}^q$ then all orbits eventually enter a Koranyi region. It is an open question whether such an orbit (z_n) exists when Ω is a bounded strongly convex domain with C^3 boundary.

We end this section giving a similar result for parabolic nonzero-step maps.

Definition 4.5. Let $\Omega \subset\subset \mathbb{C}^q$ be a strongly convex domain with C^3 boundary. If $f\colon \Omega \to \Omega$ is a parabolic holomorphic self-map, we say that it is *nonzero-step* if for all $z \in \Omega$ we have that $s_1(z) > 0$.

Theorem 4.6. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary. Let $f\colon \Omega \to \Omega$ be a parabolic nonzero-step holomorphic self-map with Denjoy–Wolff point ξ. Then there exist*

(1) an integer k such that $1 \le k \le q$,

(2) a parabolic automorphism $\tau\colon \mathbb{H}^k \to \mathbb{H}^k$ of the form

$$\tau(z_1, z') = (z_1 \pm 1, e^{it_1} z'_1, \dots e^{it_{k-1}} z'_{k-1}), \tag{4.3}$$

where $t_j \in \mathbb{R}$ for $1 \le j \le k-1$, or of the form

$$\tau(z_1, z') = (z_1 - 2z'_1 + i, z'_1 - i, e^{it_2} z'_2, \dots e^{it_{k-1}} z'_{k-1}), \tag{4.4}$$

where $t_j \in \mathbb{R}$ for $2 \le j \le k-1$,

(3) a holomorphic mapping $\ell\colon \mathbb{B}^q \to \mathbb{H}^k$,

such that the triple $(\mathbb{H}^k, \ell, \tau)$ is a canonical Kobayashi hyperbolic semi-model for f.

Proof. Let $(\mathbb{H}^k, \ell, \tau)$ be the canonical Kobayashi hyperbolic model for f given by Theorem 3.5. Then for all $z \in \mathbb{H}^k$, it follows from (3.5) that $k_{\mathbb{H}^k}(z, \tau(z)) = s_1(z) > 0$. Hence $k \ge 1$, and τ is not elliptic. Moreover, since $c(\tau) = c(f)$, it follows from Proposition 4.1 that τ is parabolic. Finally we can change holomorphic coordinates and put τ in the form (4.3) or (4.4). □

Part 2. Backward iteration

5. Canonical pre-models

In this section we construct a canonical pre-model for a holomorphic self-map f of a bounded taut domain Ω assuming the existence of a backward orbit with bounded step along which the squeezing function converges to 1.

Definition 5.1. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. Let $\beta = (x_m)_{m\ge 0}$ be a backward orbit for f, meaning that $f(x_{m+1}) = x_m$ for all $m \ge 0$. The *backward m-step* $s_m(\beta)$ of f at β is the limit

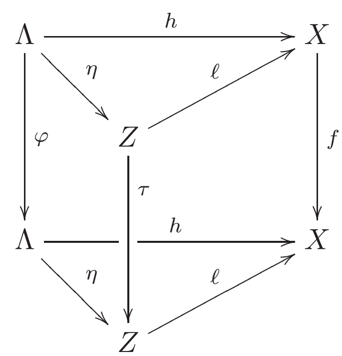
$$s_m(\beta) := \lim_{n\to\infty} k_X(x_n, x_{n+m}).$$

Such a limit exists since the sequence $(k_X(x_n, x_{n+m}))_{n\geq 0}$ is non-decreasing. We will say that the backward orbit has bounded step if $s_1(\beta) < \infty$. If (y_m) is a backward orbit for f, we denote by $[y_m]$ the family of all backward orbits (z_m) of f such that the sequence $(k_X(z_m, y_m))$ is bounded.

Definition 5.2. Let X be a complex manifold and let $f\colon X \to X$ be a holomorphic self-map. A *pre-model* for f is a triple (Λ, h, φ) where Λ is a complex manifold called the *base space*, $h\colon \Lambda \to X$ is a holomorphic mapping, and $\varphi\colon \Lambda \to \Lambda$ is an automorphism such that

$$f \circ h = h \circ \varphi. \tag{5.1}$$

Let (Λ, h, φ) and (Z, ℓ, τ) be two pre-models for f. A *morphism of pre-models* $\hat{\eta}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$ is given by a holomorphic mapping $\eta\colon \Lambda \to Z$ such that the following diagram commutes:



If the mapping $\eta\colon \Lambda \to Z$ is a biholomorphism, then we say that $\hat{\eta}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$ is an *isomorphism of pre-models*. Notice then that $\eta^{-1}\colon Z \to \Lambda$ induces a morphism $\hat{\eta}^{-1}\colon (Z, \ell, \tau) \to (\Lambda, h, \varphi)$.

Definition 5.3. Let X be a complex manifold, let $f\colon X \to X$ be a holomorphic self-map and let (Λ, h, φ) be a pre-model for f. If $[y_m]$ is a class of backward orbits, we say that (Λ, h, φ) is *associated* with $[y_m]$ if for some (and hence for any) $x \in X$ we have that $(h \circ \varphi^{-m}(x)) \in [y_m]$.

We say that (Z, ℓ, τ) is a *canonical pre-model* for f associated with $[y_m]$ if

(1) (Z, ℓ, τ) is a pre-model for f associated with $[y_m]$, and
(2) for any other pre-model (Λ, h, φ) for f associated with $[y_m]$ there exists a unique morphism of pre-models $\hat{\eta}\colon (\Lambda, h, \varphi) \to (Z, \ell, \tau)$.

Remark 5.4. If (Z, ℓ, τ) and (Λ, h, φ) are two canonical pre-models for f associated with the same class $[y_m]$, then they are isomorphic. Moreover it is is easy to see (see e.g. [6, Lemma 7.4]) that if a pre-model is associated with a class $[y_m]$ then every backward orbit in $[y_m]$ has bounded step.

Let $\Omega \subset\subset \mathbb{C}^q$ be a taut domain and $f\colon \Omega \to \Omega$ be a holomorphic self-map. Let (Θ, V_n) be the inverse limit of the sequence of iterates $(f^{m-n}\colon \Omega \to \Omega)$. Recall that Θ is defined as

$$\Theta := \{(z_m)_{m\geq 0} \in \Omega^{\mathbb{N}}\colon (z_m) \text{ is a backward orbit for } f\},$$

and the map $V_n\colon \Theta \to \Omega$ is defined as $V_n((z_m)) = z_n$.

The following theorem is the analogous of Theorem 3.5 for the backward dynamics. Notice that here we assume that the domain Ω is taut. This condition will be used in the proof of Proposition 5.7 in order to construct the sequence of maps $\alpha_n : Z \to \Omega$.

Theorem 5.5. *Let $\Omega \subset \mathbb{C}^q$ be a bounded taut domain and $f\colon \Omega \to \Omega$ be a holomorphic self-map. Assume that there exists a backward orbit (z_m) with bounded step and $S_\Omega(z_m) \to 1$.*

Then there exists a canonical pre-model $(\mathbb{B}^k, \ell, \tau)$ for f associated with $[z_m]$, where $0 \leq k \leq q$. Moreover, the following holds:

(1) the image of the map ℓ is

$$\ell(\mathbb{B}^k) = V_0([z_m]),$$

and for all $(w_m) \in [z_m]$, there exists a unique $z \in \mathbb{B}^k$ such that $(\ell \circ \tau^{-m}(z)) = (w_m)$,

(2) we have

$$\lim_{m\to\infty} (\ell \circ \tau^{-m})^* k_\Omega = k_{\mathbb{B}^k},$$

(3) if β is a backward orbit in the class $[z_m]$, then the divergence rate of τ satisfies

$$c(\tau) = \lim_{m\to\infty} \frac{s_m(\beta)}{m} = \inf_{m\in\mathbb{N}} \frac{s_m(\beta)}{m}.$$

Remark 5.6. The assumptions of the theorem are satisfied when Ω is bounded strongly pseudoconvex with C^2 boundary and (z_m) converges to $\partial\Omega$, since in this case the domain Ω is taut (see [1, Corollary 2.1.14]) and $S_\Omega(z_m) \to 1$ by Theorem 2.19.

The proof of Theorem 5.5 is based on the following result.

Proposition 5.7. *Let $\Omega \subset \mathbb{C}^q$ be a bounded taut domain and $f\colon \Omega \to \Omega$ be a holomorphic self-map. Assume that there exists a backward orbit (z_m) with bounded step such that $S_\Omega(z_m) \to 1$.*

Then there exists a family of holomorphic maps $(\alpha_n\colon Z \to \Omega)$, where Z is an holomorphic retract of $\mathbb{B}^q$, such that the following hold:

(a) *for all $m \geq n \geq 0$,*

$$\alpha_n = f^{m-n} \circ \alpha_m,$$

(b) *let $\Psi : Z \to \Theta$ be the map defined as $\Psi(z) = (\alpha_m(z))$. Then Ψ is injective, $\Psi(0) = (z_m)$ and*

$$\Psi(Z) = [z_m], \tag{5.2}$$

(c)

$$\lim_{m\to\infty} \alpha_m^* k_\Omega = k_Z, \tag{5.3}$$

(d) **Universal property***: let Q be a complex manifold and $(\gamma_n\colon Q \to \Omega)$ a family of holomorphic mappings satisfying $\gamma_n = f^{m-n} \circ \gamma_m$ for all $m \geq n \geq 0$, and such that $(\gamma_m(x)) \in [z_m]$ for some (and hence for any) $x \in Q$. Then there exists a unique holomorphic map $\Gamma\colon Q \to Z$ such that $\alpha_m \circ \Gamma = \gamma_m$ for all $m \geq 0$.*

Remark 5.8. Such family (α_n) is a canonical inverse limit for the sequence of iterates $(f^{m-n}\colon \Omega \to \Omega)$ associated with $[z_m]$, see [6, Definition 6.9].

Once Proposition 5.7 is proved, the proof of Theorem 5.5 is the same as the proof of [6, Theorem 8.7]. We present a sketch of the construction of the pre-model for the convenience of the reader.

Proof of Theorem 5.5. Following the proof of [6, Theorem 8.7] we define $\ell := \alpha_0$ and $\gamma_n := f \circ \alpha_n$. It is not hard to show that $(\gamma_n\colon Z \to \Omega)$ is a family of holomorphic mappings satisfying $\gamma_n = f^{m-n} \circ \gamma_m$ for all $m \geq n \geq 0$. Furthermore since (z_m) has bounded step, we have

$$\sup_m k_\Omega(\gamma_m(0), z_m) = \sup_m k_\Omega(z_{m-1}, z_m) < \infty.$$

It follows that $(\gamma_m(x)) \in [z_m]$ for every $x \in Q$.

By the universal property of the family (α_n) there exists a unique holomorphic map $\tau : Z \to Z$ such that for all $n \geq 0$,

$$\alpha_n \circ \tau = \gamma_n = f \circ \alpha_n, \tag{5.4}$$

in particular $\ell \circ \tau = f \circ \ell$.

Similarly if we define $\tilde{\gamma}_n := \alpha_{n+1}$ we obtain a holomorphic map $\delta : Z \to Z$ such that $\tilde{\gamma}_n = \alpha_n \circ \delta$ for all $n \geq 0$. It is easy to see that, for all n,

$$\alpha_n \circ \tau \circ \delta = \alpha_n \circ \delta \circ \tau = \alpha_n.$$

By the universal property of the family (α_n) described in Proposition 5.7, we conclude that $\delta = \tau^{-1}$, proving that τ is an automorphism of Z. For all $n \geq 0$ we have that

$$\alpha_n \circ \tau^n = f^n \circ \alpha_n = \ell,$$

and thus $\alpha_n = \ell \circ \tau^{-n}$ for all $n \geq 0$. The triple (Z, ℓ, τ) is a pre-model associated with $[z_m]$ thanks to (5.2). Since Z is a holomorphic retract of $\mathbb{B}^q$, it is biholomorphic to a ball of dimension $0 \leq k \leq q$.

The universal property of the canonical pre-model (Z, ℓ, τ) is a direct consequence of the universal property of the family (α_n). Since $\ell = V_0 \circ \Psi$, point (1) is an immediate consequence of (5.2). Point (2) easily follows from (5.3).

Finally, let $\beta := (w_m)$ be a backward orbit in the class $[z_m]$. By (5.2) there exists $z \in Z$ so that $(w_m) = (\alpha_m(z))$. Using again (5.3), we obtain the following formula for the backward m-step of β, which directly implies (3)

$$s_m(\beta) = \lim_{n\to\infty} k_\Omega(\alpha_{n-m}(z), \alpha_n(z)) = \lim_{n\to\infty} k_\Omega(\alpha_n \circ \tau^m(z), \alpha_n(z)) = k_Z(z, \tau^m(z)).$$

□

The proof of Proposition 5.7 is articulated in several intermediate lemmas. Let (z_m) be a backward orbit with bounded step satisfying $S_\Omega(z_m) \to 1$. Let $(\psi_m : \Omega \to \mathbb{B}^q)$ be a sequence of holomorphic injective maps with $\psi_m(z_m) = 0$ and

$$\psi_m(\Omega) \supset B(0, S_\Omega(z_m)).$$

Given a compact subset $K \subset \mathbb{B}^q$, the map ψ_m^{-1} is well defined on K when m is large enough. Since $f^m \circ \psi_m^{-1}(0) = z_0$ for all $m \geq 0$ and since Ω is taut, there exists a subsequence $(m_0(h))$ such that the sequence $(f^{m_0(h)} \circ \psi^{-1}_{m_0(h)})$ converges uniformly on compact subsets to a holomorphic map $\alpha_0 : \mathbb{B}^q \to \Omega$ and $\alpha_0(0) = z_0$. Similarly, there exists a subsequence $(m_1(h))$ of $(m_0(h))$ such that the sequence $(f^{m_1(h)-1} \circ \psi^{-1}_{m_1(h)})$ converges uniformly on compact subsets to a holomorphic map $\alpha_1 : \mathbb{B}^q \to \Omega$ and $\alpha_1(0) = z_1$. Iterating this procedure we obtain a family of subsequences $\{(m_n(h))_{h\geq 0}\}_{n\geq 0}$ and a family of holomorphic maps

$$(\alpha_n : \mathbb{B}^q \to \Omega)_{n\geq 0}$$

such that

$$f^{m_n(h)-n} \circ \psi^{-1}_{m_n(h)} \overset{h\to\infty}{\longrightarrow} \alpha_n$$

uniformly on compact subsets and $\alpha_n(0) = z_n$. Notice that for all $m \geq n \geq 0$,

$$\alpha_n = f^{m-n} \circ \alpha_m. \tag{5.5}$$

Consider the diagonal sequence $\nu(h) := m_h(h)$ which for all $j \geq 0$ is eventually a subsequence of $(m_j(h))_{h\geq 0}$.

Consider the sequence $\beta_{\nu(h)} := \psi_{\nu(h)} \circ \alpha_{\nu(h)}$. Notice that $\beta_{\nu(h)}(0) = 0$ for all $h \geq 0$. By the tautness of Ω, up to extracting a further subsequence of $\nu(h)$ if necessary, we may assume that the sequence $(\beta_{\nu(h)})$ converges uniformly on compact subsets to a holomorphic map $\alpha\colon \mathbb{B}^q \to \mathbb{B}^q$.

Lemma 5.9. *For all $j \geq 0$, we have*

$$\alpha_j \circ \alpha = \alpha_j. \tag{5.6}$$

Proof. Let $z \in \mathbb{B}^q$. For all positive integers h such that $\nu(h) \geq j$,

$$\alpha_j(z) = f^{\nu(h)-j} \circ \alpha_{\nu(h)}(z) = f^{\nu(h)-j} \circ \psi^{-1}_{\nu(h)} \circ \psi_{\nu(h)} \circ \alpha_{\nu(h)}(z) \stackrel{h\to\infty}{\longrightarrow} \alpha_j \circ \alpha(z).$$

□

Lemma 5.10. *The map $\alpha\colon \mathbb{B}^q \to \mathbb{B}^q$ is a holomorphic retraction, that is*

$$\alpha \circ \alpha = \alpha.$$

Proof. Let $z \in \mathbb{B}^q$. From (5.6) we get, for all $h \geq 0$ big enough,

$$\beta_{\nu(h)} \circ \alpha(z) = \psi_{\nu(h)} \circ \alpha_{\nu(h)} \circ \alpha(z) = \psi_{\nu(h)} \circ \alpha_{\nu(h)}(z) = \beta_{\nu(h)}(z),$$

and the result follows since $\beta_{\nu(h)} \to \alpha$. □

Define $Z := \alpha(\mathbb{B}^q)$. Being a holomorphic retract, it is a closed complex submanifold of $\mathbb{B}^q$, biholomorphic to a k-dimensional ball $\mathbb{B}^k$, with $0 \leq k \leq q$.

By the universal property of the inverse limit (Θ, V_n), there exists a unique map $\Psi\colon Z \to \Theta$ such that

$$\alpha_n = V_n \circ \Psi, \quad \forall\, n \geq 0.$$

The mapping Ψ sends the point $z \in Z$ to the backward orbit $(\alpha_m(z))_{m\geq 0}$.

Lemma 5.11. *The map $\Psi\colon Z \to \Theta$ is injective and $\Psi(Z) = [z_m]$.*

Proof. We first prove injectivity. Let $z, w \in Z$ such that $\alpha_m(z) = \alpha_m(w)$ for all $m \geq 0$. It follows that

$$\alpha(z) = \lim_{h\to\infty} \psi_{\nu(h)} \circ \alpha_{\nu(h)}(z) = \lim_{h\to\infty} \psi_{\nu(h)} \circ \alpha_{\nu(h)}(w) = \alpha(w).$$

Since $\alpha|_Z$ is the identity, we conclude that $z = w$ and that Ψ is injective.

Given $z \in Z$ it follows that

$$\sup_m k_\Omega(\alpha_m(z), z_m) = \sup_m k_\Omega(\alpha_m(z), \alpha_m(0)) \leq k_Z(z, 0) < \infty,$$

proving that $\Psi(Z) \subset [z_m]$. On the other hand, given $(w_m) \in [z_m]$ we have

$$\sup_m k_{\mathbb{B}^q}(\psi_m(w_m), 0) = \sup_m k_{\mathbb{B}^q}(\psi_m(w_m), \psi_m(z_m)) < \infty,$$

By taking a subsequence of $\nu(h)$ if necessary, we may therefore assume that the sequence $\psi_{\nu(h)}(w_{\nu(h)})$ converges to a point $w \in \mathbb{B}^q$. Now we notice that for all $m \geq 0$,

$$\alpha_m(w) = \lim_{h\to\infty} f^{\nu(h)-m} \circ \psi^{-1}_{\nu(h)} \circ \psi_{\nu(h)}(w_{\nu(h)}) = \lim_{h\to\infty} f^{\nu(h)-m}(w_{\nu(h)}) = w_m.$$

Now notice that

$$\alpha(w) = \lim_{h\to\infty} \psi_{\nu(h)} \circ \alpha_{\nu(h)}(w) = \lim_{h\to\infty} \psi_{\nu(h)}(w_{\nu(h)}) = w,$$

proving that $w \in Z$, and therefore that $\Psi(Z) = [z_m]$. □

Lemma 5.12. *For all $n \geq 0$,*

$$\lim_{m\to\infty} \alpha_m^* \, (f^n)^* \, k_\Omega = k_Z.$$

Proof. First of all we notice that for every $m \geq 0$ and $x, y \in Z$

$$k_\Omega(\alpha_m(x), \alpha_m(y)) = k_\Omega((f \circ \alpha_{m+1})(x), (f \circ \alpha_{m+1})(y)) \leq k_\Omega(\alpha_{m+1}(x), \alpha_{m+1}(y)),$$

therefore the limit for $m \to \infty$ is well defined,

For all $n \geq 0$ we have that

$$\lim_{m\to\infty} k_\Omega(f^n \circ \alpha_m(x), f^n \circ \alpha_m(y)) = \lim_{m\to\infty} k_\Omega(\alpha_{m-n}(x), \alpha_{m-n}(y)),$$

proving that

$$\lim_{m\to\infty} \alpha_m^* \, (f^n)^* \, k_\Omega = \lim_{m\to\infty} \alpha_m^* \, k_\Omega.$$

By non-expansiveness of the Kobayashi distance we have

$$\lim_{m\to\infty} k_\Omega(\alpha_m(x), \alpha_m(y)) \leq k_Z(x, y).$$

To obtain the inverse inequality denote $z_h = \psi_{\nu(h)} \circ \alpha_{\nu(h)}(x)$ and $w_h = \psi_{\nu(h)} \circ \alpha_{\nu(h)}(y)$. Then z_h converges to x and w_h converges to y. Fix $\varepsilon > 0$, then there exists a ball $B = B(0, r) \subset\subset \mathbb{B}^q$, with radius close enough to 1 that contains both x, y, and such that for some $h_0 \geq 0$, we have

$$k_B(z_h, w_h) \leq k_{\mathbb{B}^q}(x, y) + \varepsilon, \qquad \forall h \geq h_0.$$

Let $h_1 \geq 0$ such that for all $h \geq h_1$ we have $\psi_{\nu(h)}(\Omega) \supset B$. Then for all $h \geq \max\{h_0, h_1\}$ we have

$$k_\Omega(\alpha_{\nu(h)}(x), \alpha_{\nu(h)}(y)) \leq k_B(z_h, w_h) \leq k_{\mathbb{B}^q}(x, y) + \varepsilon,$$

proving that for every $\varepsilon > 0$

$$\lim_{m\to\infty} k_\Omega(\alpha_m(x), \alpha_m(y)) \leq k_Z(x, y) + \varepsilon,$$

where we used that fact that $x, y \in Z$ and $k_{\mathbb{B}^q}|_Z = k_Z$. $\square$

We are now ready to prove Proposition 5.7. Points (a), (b), and (c) correspond precisely to (5.5), Lemma 5.11 and Lemma 5.12.

It remains to prove the Universal property (d). Let $(\beta_n \colon Q \to \Omega)$ be a family of holomorphic maps satisfying $\beta_n = f^{m-n} \circ \beta_m$ for all $m \geq n \geq 0$ and $(\beta_m(x)) \in [z_m]$ for every $x \in Q$. By the universal property of the inverse limit, there exists a unique map $\Phi \colon Q \to \Theta$ such that $\beta_n = V_n \circ \Phi$ for all $n \geq 0$. It is not hard to show that $\Phi(x) = (\beta_m(x)) \in [z_m]$.

By Lemma 5.11 the map $\Psi \colon Z \to [z_m]$ is a bijection. Therefore we may set

$$\Gamma := \Psi^{-1} \circ \Phi \colon Q \to Z.$$

Given $x \in Q$ it follows that

$$(\beta_m(x)) = (\Psi \circ \Gamma)(x) = \Phi(x) = ((\alpha_m \circ \Gamma)(x)),$$

proving that $\beta_m = \alpha_m \circ \Gamma$ for all $m \geq 0$. Given another Γ' with the same properties it is immediate to show that $\Gamma' = \Psi^{-1} \circ \Phi$, proving uniqueness of the map Γ.

Finally the map Γ is holomorphic. Indeed given $x \in Q$, since $(\beta_m(x)) \in [z_m]$ it follows that

$$\sup_m k_{\mathbb{B}^q}(\psi_m \circ \beta_m(x), 0) = \sup_m k_{\mathbb{B}^q}(\psi_m \circ \beta_m(x), \psi_m(z_m)) \leq k_\Omega(\beta_m(x), z_m) < \infty.$$

The domain Ω is taut and the sequence $\psi_m \circ \beta_m$ is not compactly divergent. By taking a subsequence of $\nu(h)$ if necessary, we may therefore assume that $\psi_{\nu(h)} \circ \beta_{\nu(h)} \to \beta$ where $\beta\colon Q \to \mathbb{B}^q$ is an holomorphic function. Finally for every $x \in Q$ we have that

$$\Gamma(x) = \alpha \circ \Gamma(x) = \lim_{h\to\infty} \psi_{\nu(h)} \circ \alpha_{\nu(h)} \circ \Gamma(x) = \beta(x),$$

proving that Γ is holomorphic, which concludes the proof of point (d) and of Proposition 5.7.

6. Main result on strongly convex domains

In this section we apply the results of the previous section to the case of strongly convex domains, and we prove Theorem 1.5. First of all, on a strongly convex domain it is easy to characterize when a canonical pre-model is 0-dimensional (and thus its base space is a point $\{\star\}$). Recall that a self-map f is called *strongly elliptic* if it is elliptic and its limit manifold is a fixed point $\{p\}$.

Lemma 6.1. *Let $\Omega \subset\subset \mathbb{C}^q$ be a strongly convex domain with C^3 boundary, and let (Z, ℓ, τ) be a canonical pre-model associated with a class $\mathscr{C}$ of backward orbits. Then Z is 0-dimensional if and only if f is strongly elliptic and the class $\mathscr{C}$ contains only the constant orbit p, where $\{p\}$ is the limit manifold of f.*

Proof. Assume $Z = \{\star\}$, and set $p := \ell(\star)$. Clearly p is fixed. By assumption the backward orbit $(\ell \circ \tau^{-m}(\star)))$, which is contantly equal to p, is in the class $\mathscr{C}$.

Let $\mathcal{M}$ be the limit manifold where the forward dynamics of f converges. The restriction $f|_{\mathcal{M}}$ is an automorphism of $\mathcal{M}$ and $(\mathcal{M}, \mathsf{id}, f|_{\mathcal{M}})$ is a pre-model for f. Since $p \in \mathcal{M}$ is a fixed point for f, the pre-model $(\mathcal{M}, \mathsf{id}, f|_{\mathcal{M}})$ is associated with the class $\mathscr{C}$. By the universal property of the canonical pre-model (Z, ℓ, τ), there exists a morphism $\hat\eta\colon (\mathcal{M}, \mathsf{id}, f|_{\mathcal{M}}) \to (Z, \ell, \tau)$, which means that the identity map $\mathsf{id}\colon \mathcal{M} \to \mathcal{M}$ is equal to the constant map $\mathcal{M} \to \{p\}$. Hence $\mathcal{M} = \{p\}$ and the map f is strongly elliptic.

By [4, Lemma 2.9] any other backward orbit with bounded step (w_m) converges to $\partial\Omega$. Since Ω is complete hyperbolic, it follows that $k_\Omega(w_m, p) \to +\infty$, and thus $(w_m) \notin \mathscr{C}$. The converse is immediate. □

Let $\Omega \subset\subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary. Let $f\colon \Omega \to \Omega$ be a holomorphic self-map, and let $\zeta \in \partial\Omega$ be a repelling boundary point with dilation $\lambda_\zeta > 1$.

Definition 6.2. The *stable subset* $\mathcal{S}(\zeta)$ of ζ is the set of starting points of backward orbits with bounded step converging to ζ. We say that a pre-model (Λ, h, φ) is *associated* with the boundary repelling point ζ if for some (and hence for any) $x \in \Lambda$ we have

$$\lim_{n\to\infty} h \circ \varphi^{-n}(x) = \zeta.$$

We will later prove the two following results.

Theorem 6.3 (Uniqueness of backward orbits)**.** *Let (x_m) and (y_m) be two backward orbits with bounded step, both converging to the boundary repelling fixed point $\zeta \in \partial\Omega$. Then*

$$\lim_{m\to\infty} k_\Omega(x_m, y_m) < \infty.$$

Theorem 6.4 (Existence of backward orbits)**.** *Assume further that $\partial\Omega$ is C^4. Then there exists a backward orbit (z_m) with step $\log\lambda_\zeta$ converging to ζ.*

As a consequence, the family of backward orbits with bounded step converging to ζ is non-empty and consists of a unique equivalence class $[z_m]$. This, together with Theorem 5.5 gives the following.

Theorem 6.5. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^4 boundary. Let $f\colon \Omega \to \Omega$ be a holomorphic self-map, and let $\zeta \in \partial\Omega$ be a boundary repelling fixed point. Then there exist*

(1) an integer k such that $1 \leq k \leq q$,
(2) a hyperbolic automorphism $\tau\colon \mathbb{H}^k \to \mathbb{H}^k$ of the form

$$\tau(z_1, z') = \left(\frac{1}{\lambda_\zeta} z_1, \frac{e^{it_1}}{\sqrt{\lambda_\zeta}} z'_1, \dots, \frac{e^{it_{k-1}}}{\sqrt{\lambda_\zeta}} z'_{k-1} \right), \tag{6.1}$$

where $t_j \in \mathbb{R}$ for $1 \leq j \leq k-1$,
(3) a holomorphic mapping $\ell\colon \mathbb{H}^k \to \Omega$,

such that the triple $(\mathbb{H}^k, \ell, \tau)$ is a pre-model for f associated with ζ satisfying the following universal property: if (Λ, h, φ) is a pre-model associated with ζ, then there exists a unique morphism $\hat{\eta}\colon (\Lambda, h, \varphi) \to (\mathbb{H}^k, \ell, \tau)$.

Moreover, the image $\ell(\mathbb{H}^k)$ is equal to the stable subset $\mathcal{S}(\zeta)$, and

$$K\text{-}\lim_{z \to \zeta} \ell(z) = \zeta.$$

Proof. By Theorems 6.4 and 6.3, the family of backward orbits with bounded step converging to ζ is non-empty and consists of a unique equivalence class $[z_m]$. Let $(\mathbb{H}^k, \ell, \tau)$ be the canonical pre-model for f associated with $[z_m]$. Notice that

$$c(\tau) \leq s_1(z_m) = \log \lambda.$$

To obtain the opposite inequality, notice that, if $p \in \Omega$,

$$\log \lambda \leq \liminf_{m \to \infty} k_\Omega(p, z_{m+1}) - k_\Omega(p, z_m).$$

Hence, if $n \geq 0$ is fixed,

$$n \log \lambda \leq \liminf_{m \to \infty} k_\Omega(p, z_{m+n}) - k_\Omega(p, z_m) \leq \liminf_{m \to \infty} k_\Omega(z_{m+n}, z_m) \leq s_n(z_m).$$

Thus

$$c(\tau) = \inf_{n \in \mathbb{N}} \frac{s_n(z_m)}{n} \geq \log \lambda.$$

Now we can change variables in $\mathbb{H}^k$ to put τ in the form (6.1).

We finally study the regularity at ∞ of the intertwining mapping ℓ. Consider the backward orbit with bounded step $((\lambda^n i, 0))$ in $\mathbb{H}^k$ for τ. Clearly $((\lambda^n i, 0))$ converges to ∞ and $(\ell(\lambda^n i, 0))$ is a backward orbit for f which converges to $\zeta \in \partial\Omega$. Then [8, Theorem 5.6] yields $K\text{-}\lim_{z \to \infty} \ell(z) = \zeta$.

□

7. Uniqueness of backward orbits

In this section we prove Theorem 6.3. We remark that the C^4-smoothness of $\partial\Omega$ is only required in the proof of Theorem 6.4), which is done in the next section. Here it will be sufficient to assume that the domain Ω has C^3 boundary.

Given a backward orbit (x_m) one can always assume that it is indexed by integers $m \in \mathbb{Z}$, by defining $x_{-m} := f^m(x_0)$ for all $m \geq 0$.

Lemma 7.1. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary. Let $f\colon \Omega \to \Omega$ be a holomorphic self-map, and let $\zeta \in \partial\Omega$ be a boundary repelling fixed point.*

Let (x_m) and (y_m) be two backward orbits with bounded step, both converging to ζ. Then $\lim_{m\to+\infty} k_\Omega(x_m, y_m) < \infty$ if and only if

$$\lim_{n\to+\infty} \inf_{m\in\mathbb{Z}} k_\Omega(x_n, y_m) < \infty.$$

Proof. The proof is the same as in [9, Lemma 2]. □

We recall some definitions and classical results (see e.g. [13]).

Definition 7.2. A metric space (X, d) is *geodesic* if any two points are joined by a geodesic segment.

Theorem 7.3 (Generalized Hopf-Rinow)**.** *If an inner metric space (X, d) is complete and locally compact, it is geodesic.*

Hence, a complete Kobayashi hyperbolic manifold is geodesic.

Definition 7.4. Let $\delta > 0$. A geodesic metric space (X, d) is δ-*Gromov hyperbolic* if every geodesic triangle is δ-*slim*, meaning that every side is contained in a δ-neighborhood of the union of the other two sides. A geodesic metric space (X, d) is *Gromov hyperbolic* if it is δ-Gromov hyperbolic for some $\delta > 0$.

Theorem 7.5 ([10, Theorem 1.4])**.** *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly pseudoconvex domain with C^2 boundary and k_Ω be its Kobayashi distance. Then (Ω, k_Ω) is Gromov hyperbolic.*

Proposition 7.6. *Let (X, d) be a geodesic δ-Gromov hyperbolic metric space. If γ is a geodesic line, $x_0 \in \gamma$, $z \in X$, and if z_γ denotes a point in γ such that $d(z, z_\gamma) = d(z, \gamma)$, then*

$$d(x_0, z) \geq d(x_0, z_\gamma) + d(z_\gamma, z) - 6\delta.$$

Proof. If $d(z_\gamma, z) \leq 3\delta$, the result follows from the triangular inequality. Assume thus that $d(z_\gamma, z) > 3\delta$. Let (x_0, z_γ) denote the portion of γ between x_0 and z_γ. Let (z_γ, z) be a geodesic segment connecting z_γ to z, and let (x_0, z) be a geodesic segment connecting x_0 to z. Let x be a point in (z_γ, z) such that $d(x, z_\gamma) = 2\delta$. Since every geodesic triangle is δ-slim, there exists a point y in $(x_0, z_\gamma) \cup (x_0, z)$ such that $d(y, x) < \delta$. From $d(x, \gamma) = 2\delta$ it follows that $y \in (x_0, z)$, and from the triangle inequality we have $d(y, z_\gamma) < 3\delta$. Using twice more the triangular inequality we obtain

$$d(x_0, y) \geq d(x_0, z_\gamma) - d(y, z_\gamma),$$
$$d(y, z) \geq d(z, z_\gamma) - d(y, z_\gamma).$$

Summing the two inequalities yields the result. □

Definition 7.7. Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary. Let $p \in \Omega$, $\zeta \in \partial\Omega$ and let $\gamma \subset \Omega$ be the geodesic ray connecting p to ζ. Given $M > 1$ we denote

$$A(\gamma, M) := \{z \in \Omega \colon k_\Omega(z, \gamma) < \log M\}.$$

We now show that the Koranyi regions are comparable to the regions $A(\gamma, M)$. Let $\delta > 0$ be such that (Ω, k_Ω) is δ-Gromov hyperbolic.

Lemma 7.8. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary. Let $\zeta \in \partial\Omega$, $p \in \Omega$ and let $\gamma \subset \Omega$ be the geodesic ray connecting p to ζ. Then for every $M > 1$,*

$$A(\gamma, M) \subset K_\Omega(p, \zeta, M) \subset A(\gamma, Me^{6\delta}).$$

Proof. Let $z \in A(\gamma, M)$, and let $y \in \gamma$ be a point such that $k_\Omega(z, y) < \log M$. Let $w \in \gamma$ close enough to ζ. Then

$$k_\Omega(p,z)+k_\Omega(z,w)-k_\Omega(p,w) \leq k_\Omega(p,y)+k_\Omega(y,z)+k_\Omega(z,y)+k_\Omega(y,w)-k_\Omega(p,w) = 2\,k(y,z).$$

Letting w go to ζ on the geodesic ray γ, we obtain that

$$\log h_{\zeta,p}(z) + k_\Omega(z,p) \leq 2\,k(y,z) < 2\log M,$$

and therefore that $z \in K_\Omega(p, \zeta, M)$.

Conversely, let $z \in K_\Omega(p, \zeta, M)$. Denote by $\tilde\gamma$ the geodesic line containing γ and let $y \in \tilde\gamma$ be the closest point to z. Let $w \in \gamma$. Applying Proposition 7.6 twice we obtain

$$k_\Omega(p,z) + k_\Omega(z,w) - k_\Omega(p,w) \geq k_\Omega(p,y) + k_\Omega(y,w) + 2k_\Omega(y,z) - 12\delta - k_\Omega(p,w). \quad (7.1)$$

We now have two cases. If $y \in \gamma$, from (7.1) we get

$$k_\Omega(p,z) + k_\Omega(z,w) - k_\Omega(p,w) \geq 2k_\Omega(y,z) - 12\delta.$$

If $y \notin \gamma$, then $k_\Omega(y,w) - k_\Omega(p,w) = k(y,p)$, and thus from (7.1) we get

$$k_\Omega(p,z) + k_\Omega(z,w) - k_\Omega(p,w) \geq 2k_\Omega(p,y) + 2k_\Omega(y,z) - 12\delta \geq 2k_\Omega(z,p) - 12\delta.$$

In both cases, letting w go to ζ on the geodesic ray γ, we obtain that

$$k_\Omega(z,\gamma) < \log M + 6\delta,$$

and thus that $z \in A(\gamma, Me^{6\delta})$. □

Lemma 7.9. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary. Let $f\colon \Omega \to \Omega$ be a holomorphic self-map, and let ζ be a boundary repelling fixed point. Let (z_m) be a backward orbit with bounded step converging to ζ. Then for every $p \in \Omega$ there exists $M > 1$ so that*

$$z_m \in K_\Omega(p, \zeta, M), \qquad \forall m \geq 0.$$

Proof. Let $p \in \Omega$. By the definition of the dilation λ_ζ we have

$$\liminf_{n\to\infty} (k_\Omega(p, z_{n+1}) - k_\Omega(p, z_n)) \geq \log \lambda_\zeta.$$

For all $n \geq 0$ define s_n by $-\log s_n := k_\Omega(p, z_n)$. It follows that there exist $\lambda_\zeta^{-1} < c < 1$ and $n_0 \geq 0$ such that $s_{n+1} \leq c s_n$ for all $n \geq n_0$. Up to shifting the sequence (z_n) we may thus assume that

$$s_{n+k} \leq c^k s_n, \quad \forall n \geq 0, k \geq 0.$$

The proof now follows as in [4, Errata Corrige–Lemma 2.16].

□

We are now ready to prove Theorem 6.3.

Proof of Theorem 6.3. Let $(x_m)_{m\in\mathbb{Z}}$ and $(y_m)_{m\in\mathbb{Z}}$ be two backward orbits with bounded step, both converging to the boundary repelling fixed point $\zeta \in \partial\Omega$. Let $p \in \Omega$ and γ be the geodesic ray starting at p and ending in ζ. By Lemma 7.9 the sequence $(x_m)_{m\in\mathbb{N}}$ is contained in a Koranyi region $K_\Omega(p, \zeta, M)$ for some $M > 1$, and thus by Lemma 7.8 it is contained in the region $A(\gamma, Me^{6\delta})$. We claim that there exists $R > 0$ such that

$$A(\gamma, Me^{6\delta}) \subset \left\{ z \in \Omega\colon \inf_{m\in\mathbb{Z}} k_\Omega(z, y_m) < R \right\}.$$

Once the claim is proved, the result follows by Lemma 7.1.

It is enough to show that there exists a constant $R' > 0$ such that for all $w \in \gamma$, we have $\inf_{m\in\mathbb{Z}} k_\Omega(w, y_m) < R'$. Since $(y_m)_{m\in\mathbb{N}}$ is also contained in a Koranyi region $K_\Omega(p, \zeta, M')$, if we write $C = 6\delta + \log M'$ it follows that $k_\Omega(y_m, \gamma) < C$ for all $m \in \mathbb{N}$. Let a_m be a point in γ such that $k_\Omega(y_m, a_m) < C$. Clearly $a_m \to \zeta$. Let w be a point in the portion of γ which connects a_0 to ζ. Then there exists $m(w)$ such that w belongs to the portion of γ which connects $a_{m(w)}$ to $a_{m(w)+1}$. Hence

$$\inf_m k_\Omega(w, y_m) \le C + k_\Omega(a_{m(w)}, a_{m(w)+1}) \le C + 2C + k_\Omega(y_{m(w)}, y_{m(w)+1}) \le 3C + \sigma(y_m).$$

Letting $R' = 3C + \sigma(y_m)$ we obtain that $\inf_{m\in\mathbb{Z}} k_\Omega(w, y_m) < R'$ holds for every $w \in \gamma$ sufficiently close to ζ. By taking a bigger R' if necessary, we may therefore assume that the inequality holds for all $w \in \gamma$, concluding the proof of the theorem. □

8. Existence of backward orbits

In this section we prove Theorem 6.4. In the case of the ball $\mathbb{B}^q$ this result was first proved in [22] assuming that the boundary repelling fixed point is isolated, and then for a general boundary repelling fixed point in [9]. For strongly convex domains with C^3 boundary, it was proved in [4] in the case of an isolated boundary repelling fixed point.

8.1. **Preparatory results in the ball.** In the following result we reformulate the crucial part of the proof of [9, Theorem 2] as a purely geometric statement.

Proposition 8.1. *Let $(x_n), (y_n) \in \mathbb{B}^q$ be two sequences satisfying*

$$\lim_{n\to\infty} [k_{\mathbb{B}^q}(0, x_n) - k_{\mathbb{B}^q}(0, y_n)] = \lim_{n\to\infty} k_{\mathbb{B}^q}(x_n, y_n) = L > 0. \tag{8.1}$$

Suppose further that there exists $R > 0$ and $\zeta \in \partial\mathbb{B}^q$ so that $x_n \in E_{\mathbb{B}^q}(0, \zeta, R)$ and $y_n \notin E_{\mathbb{B}^q}(0, \zeta, R)$ for every $n \in \mathbb{N}$. Then (x_n) and (y_n) are relatively compact in $\mathbb{B}^q$.

Proof. Assume that this were not the case. Since $x_n \in E_{\mathbb{B}^q}(0, \zeta, R)$ and since the Kobayashi distance between x_n and y_n is bounded, by taking a subsequence of x_n if necessary, we may then assume that $x_n, y_n \to \zeta$.

The automorphism group of the unit ball is transitive. Therefore for every positive integer n we may find $\sigma_n \in \mathrm{Aut}(\mathbb{B}^q)$ so that $\sigma_n(x_n) = 0$. By composing such map with a rotation, we may further suppose that $\sigma_n(\zeta) = e_1$.

By (2.3) and by invariance of the Kobayashi distance under automorphisms, we obtain that

$$h_{e_1,\sigma_n(0)}(z) = h_{e_1,0}(z) h_{e_1,\sigma_n(0)}(0) = h_{e_1,0}(z) h_{\zeta,0}(x_n), \tag{8.2}$$

and therefore that

$$E_{\mathbb{B}^q}(0, e_1, 1) \subset E_{\mathbb{B}^q}(\sigma_n(0), e_1, R) = \sigma_n(E_{\mathbb{B}^q}(0, \zeta, R)). \tag{8.3}$$

We claim that $\sigma_n(0) \to e_1$. Notice that for every $n \in \mathbb{N}$ large enough we have $h_{\zeta,0}(x_n) > Re^{-2L}$. Indeed if for some n sufficiently large this were not the case, then we would have

$$h_{\zeta,0}(y_n) \leq e^{k_{\mathbb{B}^q}(x_n,y_n)} h_{\zeta,0}(x_n) < R,$$

contradicting the fact that $y_n \notin E_{\mathbb{B}^q}(0,\zeta,R)$. Letting $z = \sigma_n(0)$ in (8.2) we conclude that for n large enough

$$h_{e_1,0}(\sigma_n(0)) = h_{\zeta,0}(x_n)^{-1} < e^{2L}/R,$$

showing that the sequence $\sigma_n(0)$ is eventually contained in $E_{\mathbb{B}^q}(0,e_1,e^{2L}/R)$. The point x_n converges to ζ, and therefore $k_{\mathbb{B}^q}(0,x_n) \to \infty$. By invariance of the Kobayashi distance it follows that $k_{\mathbb{B}^q}(0,\sigma_n(0)) \to \infty$, and thus that $\sigma_n(0) \to e_1$, proving the claim.

Let $0 < \alpha < 1$ and define $z_n \in \mathbb{B}^q$ as

$$z_n := -\alpha \frac{\sigma_n(0)}{\|\sigma_n(0)\|}.$$

If we write $\beta := \log \frac{1+\alpha}{1-\alpha}$, then for every positive integer n we have $k_{\mathbb{B}}(0,z_n) = \beta$. Since $\sigma_n(0) \to e_1$ it follows that $z_n \to z_\infty = (-\alpha, 0, \dots, 0)$.

By (8.1) the sequence $\sigma_n(y_n)$ is relatively compact in $\mathbb{B}^q$. Therefore by taking a subsequence if necessary, we may assume that $\sigma_n(y_n) \to y_\infty \in \mathbb{B}^q$. Notice that since $y_n \notin E_{\mathbb{B}^q}(0,\zeta,R)$, then by (8.3) we must have $\sigma_n(y_n) \notin E_{\mathbb{B}^q}(0,e_1,1)$, thus that $y_\infty \notin E_{\mathbb{B}^q}(0,e_1,1)$.

We claim that $k_{\mathbb{B}^q}(z_\infty,y_\infty) < L + \beta$. Indeed, since $k_{\mathbb{B}^q}(0,y_\infty) = L$ and $k_{\mathbb{B}}(0,z_\infty) = \beta$, we get by triangular inequality that $k_{\mathbb{B}^q}(z_\infty,y_\infty) \leq L + \beta$. Equality holds if and only if y_∞ is contained in the geodesic ray connecting the origin to e_1. But this is not possible since such geodesic is contained in $E_{\mathbb{B}^q}(0,e_1,1)$. Let thus $\delta > 0$ be such that $k_{\mathbb{B}^q}(z_\infty,y_\infty) < L + \beta - 2\delta$.

By the last inequality and by (8.1) we may choose n big enough such that $k_{\mathbb{B}^q}(z_n,\sigma_n(y_n)) < L + \beta - \delta$ and

$$k_{\mathbb{B}^q}(\sigma_n(0),0) - k_{\mathbb{B}^q}(\sigma_n(0),\sigma_n(y_n)) = k_{\mathbb{B}^q}(0,x_n) - k_{\mathbb{B}^q}(0,y_n) \geq L - \delta.$$

We conclude that

$$\begin{aligned} k_{\mathbb{B}^q}(\sigma_n(0),z_n) - k_{\mathbb{B}^q}(\sigma_n(0),\sigma_n(y_n)) &= k_{\mathbb{B}^q}(\sigma_n(0),0) + k_{\mathbb{B}^q}(0,z_n) - k_{\mathbb{B}^q}(\sigma_n(0),\sigma_n(y_n)) \\ &\geq \beta + L - \delta \\ &> k_{\mathbb{B}^q}(z_n,\sigma_n(y_n)), \end{aligned}$$

contradicting the triangular inequality. □

Before starting the proof of Theorem 6.4, we need to estimate the Kobayashi distance of horospheres near the center ζ.

Lemma 8.2. *Let $\zeta \in \partial\mathbb{B}^q$ and $R > 0$. Write k_E for the Kobayashi distance of the horosphere $E_{\mathbb{B}^q}(0,\zeta,R)$. Then for all $\varepsilon > 0$ there exists $0 < R_\varepsilon < R$ such that*

$$k_E(x,y) \leq k_{\mathbb{B}^q}(x,y) + \varepsilon, \qquad \forall x,y \in E_{\mathbb{B}^q}(0,\zeta,R_\epsilon).$$

Proof. Consider the change of coordinates from the unit ball to the Siegel half-space given by (2.1). The horosphere $E_{\mathbb{B}^q}(0,\zeta,R)$ is mapped by such biholomorphism to the horosphere

$$E_{\mathbb{H}^q}(I,\infty,R) = \left\{ (z_1,z') \in \mathbb{H}^q \colon \operatorname{Im} z_1 > \|z'\|^2 + \frac{1}{R} \right\}.$$

Consider the biholomorphism $T\colon E_{\mathbb{H}^q}(I,\infty,R) \to \mathbb{H}^q$ given by $T(z) := (z_1 - i/R, z')$. For all $x,y \in E_{\mathbb{H}^q}(I,\infty,R)$ we have that $k_E(x,y) = k_{\mathbb{H}^q}(T(x),T(y))$.

Let $S > 1/R > 0$ be such that $k_{\mathbb{H}}(\zeta, \zeta - i/R) \leq \varepsilon/2$ when $\operatorname{Im} \zeta > S$. Set $R_\varepsilon = 1/S$. Given $x, y \in E_{\mathbb{H}^q}(I, \infty, R_\varepsilon)$ we obtain that

$$k_E(x, y) = k_{\mathbb{H}^q}(T(x), T(y)) \leq k_{\mathbb{H}^q}(T(x), x) + k_{\mathbb{H}^q}(x, y) + k_{\mathbb{H}^q}(y, T(y)),$$

hence the result follows if we show that

$$k_{\mathbb{H}^q}(T(z), z) \leq \varepsilon/2, \qquad \forall z \in E_{\mathbb{H}^q}(I, \infty, R_\varepsilon).$$

Let thus $z = (z_1, z') \in E_{\mathbb{H}^q}(I, \infty, R_\varepsilon)$. Consider the complex geodesic $i_{z'} : \mathbb{H} \to \mathbb{H}^q$ given by $i_{z'}(\xi) = (\xi + i\|z'\|^2, z')$. Set $\zeta := z_1 - i\|z'\|^2$. Then ζ satisfies $\operatorname{Im} \zeta > S$ and $i_{z'}(\zeta) = z$, and $i_{z'}(\zeta - i/R) = T(z)$. Therefore we have

$$k_{\mathbb{H}^q}(z, T(z)) = k_{\mathbb{H}}(\zeta, \zeta - i/R) \leq \varepsilon/2.$$

□

8.2. **Localization of the Kobayashi distance near the boundary.** In this subsection we show that, up to changing coordinates, we can compare $k_{\mathbb{B}^q}$ and k_Ω in little $\mathbb{B}^q$-horospheres centered at ζ, as the following result shows.

Proposition 8.3. *Let Ω be a bounded strongly convex domain with C^4 boundary and let $\zeta \in \partial\Omega$. Then there exists a change of coordinates in* $\operatorname{Aut}(\mathbb{C}^q)$ *so that in the new coordinates $\zeta = e_1$ and the following holds: for every $\varepsilon > 0$ we may find $R_\varepsilon > 0$ so that $E_{\mathbb{B}^q}(0, e_1, R_\varepsilon) \subset \Omega$ and*

$$k_{\mathbb{B}^q}(z, w) - \varepsilon \leq k_\Omega(z, w) \leq k_{\mathbb{B}^q}(z, w) + \varepsilon, \qquad \forall z, w \in E_{\mathbb{B}^q}(0, e_1, R_\varepsilon).$$

We first need to prove some preparatory results. The following is proved in [15, Lemma 2] (see also [19, Proposition 9.7.7]).

Proposition 8.4. *Let $\Omega \subset \mathbb{C}^q$ be a domain and let $\zeta \in \partial\Omega$ be a C^4-smooth strongly pseudoconvex point. There exists a biholomorphic mapping w defined on a neighborhood V of ζ sending ζ to the origin, and sending $\Omega \cap V$ to a region with local defining function at the origin of the form*

$$\psi(w) = -\mathrm{Im}\, w_1 + \|w'\|^2 - P_4(\mathrm{Re}\, w_1, w', \overline{w'}) + (\text{5-th and h.o.t. in } \mathrm{Re}\, w_1, w', \overline{w'}), \qquad (8.4)$$

with $(w_1, w') \in w(V)$. Here P_4 is a real-valued 4th-degree homogeneous polynomial in $\mathrm{Re}\, w_1, w'$ and $\overline{w'}$ satisfying $P_4(\mathrm{Re}\, w_1, w', \overline{w'}) \geq C(|\mathrm{Re}\, w_1|^4 + \|w'\|^4)$, for some $C > 0$.

After applying the (local) change of variables $w = w(z)$, the boundaries of Ω and of the Siegel half-space $\mathbb{H}^q$ have a 3-th order contact at the origin.

Consider the biholomorphism $\tilde{\mathcal{C}} : \mathbb{B}^q \to \mathbb{H}^q$ given by

$$\tilde{\mathcal{C}}(z_1, z') = \left(i\frac{1 - z_1}{1 + z_1}, \frac{z'}{1 + z_1} \right), \qquad (8.5)$$

which is equal to the Cayley transform $\mathcal{C}$ defined by (2.1) precomposed with the automorphism $(z_1, z') \mapsto (-z_1, z')$ of $\mathbb{B}^q$, and as such it maps e_1 to 0 and $-e_1$ to infinity.

The horospheres of $\mathbb{H}^q$ with pole $I = (i, 0, \dots, 0)$ and center the origin are the images under the map $\tilde{\mathcal{C}}$ of the horospheres of the ball with pole 0 and center e_1. Therefore for every $R > 0$ we can write them as

$$E_{\mathbb{H}^q}(I, 0, R) = \left\{ w \in \mathbb{C}^q \,|\, \operatorname{Im} w_1 > \|w'\|^2 + \frac{|w_1|^2}{R} \right\}.$$

Choose a constant $D > 0$ so that, whenever w is sufficiently close to the origin, we have

$$C(|\operatorname{Re} w_1|^4 + \|w'\|^4) \leq P_4(\operatorname{Re} w_1, w', \overline{w'}) \leq \frac{D}{2}(|\operatorname{Re} w_1|^4 + \|w'\|^4). \tag{8.6}$$

Fix $R > 0$ and let $t := \frac{1}{R}$. The holomorphic map defined by $\Phi(w_1, z') := (w_1 + it, w')$ is an automorphism of $\mathbb{CP}^q$ which fixes the line at infinity and satisfies $\Phi(\mathbb{H}^q) = E_{\mathbb{H}^q}(I, \infty, R)$. Conjugating Φ with the involution $(w_1, w') \mapsto (-\frac{1}{w_1}, -\frac{iw'}{w_1})$ we obtain the automorphism of $\mathbb{CP}^q$

$$T(w_1, w') := \left(\frac{Rw_1}{R - iw_1}, \frac{Rw'}{R - iw_1}\right),$$

which fixes the line $w_1 = 0$ and satisfies $T(\mathbb{H}^q) = E_{\mathbb{H}^q}(I, 0, R)$.

Fix $0 < R < D^{-1}$. Let V and $w(z)$ as in the previous proposition and, up to taking a smaller V if necessary, define the local biholomorphism $\eta\colon V \to \mathbb{C}^q$ as $\eta = T \circ w$. It is not hard to show that in the coordinates $\eta = \eta(z)$ a local defining function of $\Omega \cap V$ at the origin can be written as

$$\psi(\eta) = -\operatorname{Im} \eta_1 + \|\eta'\|^2 + \frac{|\eta_1|^2}{R} - P_4(\operatorname{Re} \eta_1, \eta', \overline{\eta'}) + (\text{5-th and h.o.t. in } \operatorname{Re} \eta_1, \eta', \overline{\eta'}), \tag{8.7}$$

where P_4 is the same polynomial as in Proposition 8.4. Notice that the higher order terms still do not depend on $\operatorname{Im} \eta_1$.

Lemma 8.5. *There exists $\rho > 0$ so that,*

$$E_{\mathbb{H}^q}(I, 0, R) \cap B(0, \rho) \subset \eta(\Omega \cap V) \cap B(0, \rho) \subset \mathbb{H}^q \cap B(0, \rho)$$

Proof. The local defining function of $\Omega' := \eta(\Omega \cap V)$ at the origin is of the form

$$\psi(\eta) = -\operatorname{Im} \eta_1 + \|\eta'\|^2 + \frac{|\eta_1|^2}{R} - r(\operatorname{Re} \eta_1, \eta', \overline{\eta'}),$$

where r is the sum of P_4 and the higher order terms. By (8.6) it follows that whenever η is sufficiently close to the origin, we have that

$$0 \leq r(\operatorname{Re} \eta_1, \eta', \overline{\eta'}) \leq D(|\operatorname{Re} \eta_1|^4 + \|\eta'\|^4).$$

Suppose first that $\eta \in E_{\mathbb{H}^q}(I, 0, R)$ is close to the origin. Since r is non negative it follows immediately that $\psi(\eta) < 0$, proving that $\eta \in \Omega'$.

If on the other hand we have that if $\eta \in \Omega'$ is close to the origin, then

$$\begin{aligned} 0 &< \operatorname{Im} \eta_1 - \frac{|\eta_1|^2}{R} - \|\eta'\|^2 + r(\operatorname{Re} \eta_1, \eta', \overline{\eta'}) \\ &< \operatorname{Im} \eta_1 - \frac{|\eta_1|^2}{R} - \|\eta'\|^2 + D(|\operatorname{Re} \eta_1|^4 + \|\eta'\|^4), \end{aligned}$$

and therefore

$$D\|\eta'\|^4 - \|\eta'\|^2 + \operatorname{Im} \eta_1 - R^{-1}|\eta_1|^2 + D|\operatorname{Re} \eta_1|^4 > 0.$$

As η converges to the origin the corresponding second degree equation in $\|\eta'\|^2$ has two solutions $0 < t_1 < t_2$. The solution t_2 converges to $1/D$, while

$$t_1 = \frac{1 - \sqrt{1 - 4D\operatorname{Im} \eta_1 + 4DR^{-1}|\eta_1|^2 - 4D^2|\operatorname{Re} \eta_1|^4}}{2D} \to 0.$$

Hence if η is small enough, $\|\eta'\|^2 < t_1$, which immediately implies $\eta_1 \neq 0$. Moreover,

$$\|\eta'\|^2 < \operatorname{Im} \eta_1 + (D - R^{-1})|\eta_1|^2 + O(|\eta_1|^3),$$

proving that whenever $\eta \in \Omega'$ is sufficiently close to the origin, we have $\eta \in \mathbb{H}^q$. ☐

Consider now the biholomorphism $\tilde{\eta} := \tilde{\mathcal{C}}^{-1} \circ \eta$ sending ζ to e_1. If φ is a biholomorphism defined in a neighborhood of ζ such that

$$\varphi(z) - \tilde{\eta}(z) = O(\|z - \zeta\|^{d+1}),$$

with d sufficiently large, then the expression (8.7) remains unchanged when we replace η with $\tilde{\mathcal{C}} \circ \varphi$.

The proof of the previous lemma relies uniquely on the form of the boundary defining function (8.7). Therefore, up to taking a smaller V if necessary, the lemma remains valid when we consider $\tilde{\mathcal{C}} \circ \varphi$ instead of η. We conclude that, for every given map φ as above, there exists $\rho > 0$ sufficiently small such that

$$E_{\mathbb{B}^q}(0, e_1, R) \cap B(0, \rho) \subset \varphi(\Omega \cap V) \cap B(0, \rho) \subset \mathbb{B}^q \cap B(0, \rho).$$

By jet interpolation in Andersén-Lempert theory [5, Proposition 6.3] we may choose φ to be an automorphism of $\mathbb{C}^q$. Since we can always find $0 < R' < R$ so that $E_{\mathbb{B}^q}(0, e_1, R') \subset E_{\mathbb{B}^q}(0, e_1, R) \cap B(0, \rho)$, we conclude the following

Lemma 8.6. *Let $\Omega \subset \mathbb{C}^q$ be a domain and let $\zeta \in \partial\Omega$ be a C^4-smooth strongly pseudoconvex point. Then there exists a change of coordinates in* $\mathrm{Aut}(\mathbb{C}^q)$*, and constants $\rho, R > 0$ so that in the new coordinates we have $\zeta = e_1$ and the two following inclusions hold*

$$E_{\mathbb{B}^q}(0, e_1, R) \subset \Omega \qquad \text{and} \qquad \Omega \cap B(e_1, \rho) \subset \mathbb{B}^q.$$

Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary. Let $\mathscr{F}$ be the family of complex geodesics $\varphi : \mathbb{D} \to \Omega$ that satisfy (see [12, 18])

$$d(\varphi(0), \partial D) = \max_{\tau \in \mathbb{D}} d(\varphi(\tau), \partial\Omega).$$

Then by [18, Proposition 1] it follows that there exists $C > 0$ so that for every $\varphi \in \mathscr{F}$ and $\tau_1, \tau_2 \in \mathbb{D}$ we have

$$\|\varphi^{(j)}(\tau_1) - \varphi^{(j)}(\tau_2)\| \leq C|\tau_1 - \tau_2|^{1/4}, \quad j = 0, 1. \tag{8.8}$$

Lemma 8.7. *Let $\Omega \subset \mathbb{C}^q$ be a bounded strongly convex domain with C^3 boundary and let $\zeta \in \partial\Omega$. Then given $\varepsilon > 0$ we may find $\delta > 0$ so that for every $z, w \in \Omega \cap B(\zeta, \delta)$ the geodesic segment for the Kobayashi distance γ connecting z and w is contained in $\Omega \cap B(\zeta, \varepsilon)$ and has euclidean length $\ell(\gamma) < \varepsilon$.*

Proof. The fact that γ is contained in $\Omega \cap B(\zeta, \varepsilon)$ for δ sufficiently small is an immediate consequence of [1, Lemma 2.3.64].

To prove the statement concerning $\ell(\gamma)$ it is enough to show that, given a sequence $(z_n, w_n) \to (\zeta, \zeta)$ there exist a subsequence (still denoted (z_n, w_n)) such that the geodesic segment $\gamma_n \colon [0, a_n] \to \Omega$ joining z_n to w_n has euclidean length converging to 0 as $n \to \infty$.

Let φ_n be a complex geodesic passing through z_n and w_n. Up to composing φ_n with an automorphism of $\mathbb{D}$, we may assume that $\varphi_n \in \mathscr{F}$. Up to passing to a subsequence we have that $\varphi_n \to \varphi_\infty$ uniformly on compact subsets of $\mathbb{D}$. It follows from (8.8) that actually $\varphi_n \to \varphi_\infty$ and $\varphi_n' \to \varphi_\infty'$ uniformly on $\overline{\mathbb{D}}$. Since strongly convex domains have simple boundary [1, Corollary 2.1.14], either $\varphi_\infty \colon \mathbb{D} \to \Omega$ or $\varphi_\infty \equiv \zeta$. Let $\tau_n, \sigma_n \in \mathbb{D}$ be defined by $\tau_n := \varphi_n^{-1}(z_n)$ and that $\sigma_n := \varphi_n^{-1}(w_n)$.

Assume that $\varphi_\infty \colon \mathbb{D} \to \Omega$. After taking a subsequence of φ_n if necessary, we may assume that $\tau_n \to \tau_\infty \in \overline{\mathbb{D}}$, that $\sigma_n \to \sigma_\infty \in \overline{\mathbb{D}}$. Clearly τ_∞ and σ_∞ belong to $\partial\mathbb{D}$, and $\varphi_\infty(\tau_\infty) = \varphi_\infty(\sigma_\infty) = \zeta$. By the continuity of the Kobayashi distance, it follows that φ_∞ is a complex geodesic, and since the extension of a complex geodesic to $\overline{\mathbb{D}}$ is injective, we obtain that $\tau_\infty = \sigma_\infty$. Let $\eta_n : [0, a_n] \to \mathbb{D}$ be the geodesic segment connecting τ_n and σ_n. Notice that since τ_n and σ_n converge to the same point we must have $\ell(\eta_n) \to 0$. By the uniqueness of real geodesics we have $\gamma_n = \varphi_n \circ \eta_n$ and therefore

$$\ell(\gamma_n) = \int_0^{a_n} \|\gamma_n'(t)\| \, dt \le \sup_{t\in[0,a_n]} \|\varphi_n'(\eta_n(t))\| \ell(\eta_n).$$

Since the value of $\sup_{\tau\in\mathbb{D}} \|\varphi_n'(\tau)\|$ is uniformly bounded it follows that $\ell(\gamma_n) \to 0$.

Assume now that $\varphi_\infty \equiv \zeta$. Let $\eta_n : [0, a_n] \to \mathbb{D}$ be the geodesic segment connecting τ_n and σ_n. As before

$$\ell(\gamma_n) \le \sup_{t\in[0,a_n]} \|\varphi_n'(\eta_n(t))\| \ell(\eta_n).$$

Since the euclidan length of geodesics lines in the disk is bounded from above, and φ_n' converges uniformly to 0 on $\overline{\mathbb{D}}$, we have the result.

□

We are now ready to prove Proposition 8.3. We denote by κ the Kobayashi-Royden metric.

Proof of Proposition 8.3. Consider the change of coordinates and the constants $\rho, R > 0$ given by Lemma 8.6. Given $0 < \delta < \rho$ we define the bounded sets $D := \Omega \cup \mathbb{B}^q$, $D_1 := \mathbb{B}^q$, $D_0 := \mathbb{B}^q \cap B(e_1, \delta)$. Then by [16, Theorem 2.1] we conclude, up to taking a smaller δ so that $d(\,\cdot\,, \partial D) = d(\,\cdot\,, \partial\mathbb{B}^q)$ on D_0, that there exists a constant $c > 0$ so that for all $z \in \mathbb{B}^q \cap B(e_1, \delta)$ and $v \in T_z\mathbb{C}^q$

$$\kappa_\Omega(z; v) \ge \kappa_D(z; v) \ge (1 - c\, d(z, \partial\mathbb{B}^q))\kappa_{\mathbb{B}^q}(z; v) \ge \kappa_{\mathbb{B}^q}(z; v) - c\|v\|,$$

where the estimate $d(z, \partial\mathbb{B}^q)\kappa_{\mathbb{B}^q}(z; v) \le \|v\|$ follows from the definition of Kobayashi-Royden metric.

For every $\varepsilon > 0$, by the previous lemma we can choose $0 < \delta_1 < \delta$ so that for every $z, w \in \Omega \cap B(e_1, \delta_1)$ the geodesic segment γ for the Kobayashi distance connecting z and w is contained in $\Omega \cap B(e_1, \delta)$ and has euclidean length $\ell(\gamma) \le \varepsilon/c$ (notice that Ω is not necessarily strongly convex, but the lemma still holds after a change of coordinate in $\mathrm{Aut}(\mathbb{C}^q)$). It follows that

$$k_\Omega(z, w) \ge k_{\mathbb{B}^q}(z, w) - c\,\ell(\gamma) \ge k_{\mathbb{B}^q}(z, w) - \varepsilon.$$

Given $R, \varepsilon > 0$ as above we define choose $R_\varepsilon > 0$ as in Lemma 8.2. If k_E denotes the Kobayashi distance of the horosphere $E_{\mathbb{B}^q}(0, e_1, R)$ we conclude that

$$k_\Omega(z, w) \le k_E(z, w) \le k_{\mathbb{B}^q}(z, w) + \varepsilon, \qquad \forall z, w \in E_{\mathbb{B}^q}(0, e_1, R_\varepsilon).$$

By taking R_ε smaller if necessary we may further assume that $E_{\mathbb{B}^q}(0, e_1, R_\varepsilon) \subset \Omega \cap B(e_1, \delta_1)$, concluding the proof of the Proposition. □

8.3. **Proof of Theorem 6.4.** Consider the change of coordinates given by Proposition 8.3. We remark that in the new coordinates the domain Ω is strongly pseudoconvex but not necessarily strongly convex. On the other hand all the properties of strongly convex domains we will use in this last section are invariant under automorphisms of $\mathbb{C}^q$.

Given a decreasing sequence $0 < \varepsilon_n < 1/2$ converging to 0, by Lemma 8.3 we may find another decreasing sequence $R_n > 0$ so that, for all $n \ge 0$, we have $E_{\mathbb{B}^q}(0, e_1, R_n) \subset \Omega$ and

$$k_{\mathbb{B}^q}(z, w) - \varepsilon_n \le k_\Omega(z, w) \le k_{\mathbb{B}^q}(z, w) + \varepsilon_n, \qquad \forall z, w \in E_{\mathbb{B}^q}(0, e_1, R_n). \tag{8.9}$$

The point e_1 is a boundary repelling fixed point for (the conjugate of) the map f with dilation $\lambda := \lambda_\zeta$. If the map f has no interior fixed point, then its Denjoy-Wolff point ξ does not coincide with e_1, and therefore, by taking a smaller R_0 if necessary, we may assume that $\xi \notin \overline{E_{\mathbb{B}^q}(0, e_1, R_0)}$. On the other hand, if f admits interior fixed points, there exists a limit manifold $\mathcal{M}$ which is a holomorphic retract of Ω. Thanks to [3, Proposition 3.4], we have that $e_1 \notin \overline{\mathcal{M}}$. Hence, by taking a smaller R_0 if necessary, we may assume that $\overline{E_{\mathbb{B}^q}(0, e_1, R_0)} \cap \mathcal{M} = \emptyset$. We conclude that

Lemma 8.8. *Every orbit starting in $E_{\mathbb{B}^q}(0, e_1, R_0)$ eventually leaves the same set.*

Set $\widetilde{R} := \lambda e$, $r_n := R_n/\widetilde{R}$ and choose $z_n \in E_{\mathbb{B}^q}(0, e_1, r_n)$. Since $r_n < R_n < R_0$, we have $z_n \in \Omega$, and it is easy to see using (2.3) that we have the two following chains of strict inclusion:

$$E_{\mathbb{B}^q}(0, e_1, R_n) \supset E_{\mathbb{B}^q}(0, e_1, r_n) \supset E_{\mathbb{B}^q}(z_n, e_1, 1). \tag{8.10}$$

$$E_{\mathbb{B}^q}(0, e_1, R_n) \supset E_{\mathbb{B}^q}(z_n, e_1, \tilde{R}) \supset E_{\mathbb{B}^q}(z_n, e_1, 1). \tag{8.11}$$

Since $z_n \in \Omega$ there exists a unique complex geodesic φ_n of the domain Ω so that $\varphi_n(0) = z_n$ and $\varphi_n(1) = e_1$. As a consequence of [17, Lemma 3.5], the restriction $\alpha_n \colon [0, 1) \to \mathbb{C}^q$ of φ_n extends C^1-smoothly to the closed interval $[0, 1]$ and $\alpha'(1) \notin T_{e_1}\partial\Omega$, hence as the real number t increases to 1, the point $\varphi_n(t)$ converges to e_1 non-tangentially.

It follows that every real t sufficiently close to 1 we have $\varphi_n(t) \in E_{\mathbb{B}^q}(0, e_1, r_n)$. After rescaling the complex geodesic φ_n via an appropriate automorphism of the unit disk and eventually replacing z_n with $\varphi_n(0)$, we may therefore assume that $\varphi_n([0, 1)) \subset E_{\mathbb{B}^q}(0, e_1, r_n)$.

We now show that $\varphi_n([t_0, 1)) \subset E_{\mathbb{B}^q}(z_n, e_1, 1)$, where $t_0 := \frac{e-1}{e+1}$. Indeed, if $0 < t < t' < 1$ by (8.9) we obtain that

$$\begin{aligned}
\log h^{\mathbb{B}^q}_{e_1, z_n}(\varphi_n(t)) &= \lim_{\mathbb{B}^q \ni w \to e_1} [k_{\mathbb{B}^q}(\varphi_n(t), w) - k_{\mathbb{B}^q}(z_n, w)] \\
&= \lim_{t' \to 1} [k_{\mathbb{B}^q}(\varphi_n(t), \varphi_n(t')) - k_{\mathbb{B}^q}(z_n, \varphi_n(t'))] \\
&\le \lim_{t' \to 1} [k_\Omega(\varphi_n(t), \varphi_n(t')) - k_\Omega(z_n, \varphi_n(t'))] + 2\varepsilon_n \\
&\le -k_\Omega(z_n, \varphi_n(t)) + 2\varepsilon_n \\
&< -\log \frac{1+t}{1-t} + 1,
\end{aligned}$$

where we used the fact that the three points $\varphi_n(t), \varphi_n(t')$ and z_n lie on the same real geodesic of Ω. Notice that we could use (8.9) since $\varphi_n(t), \varphi_n(t')$ and z_n all belong to $E_{\mathbb{B}^q}(0, e_1, R_n)$.

Choose an increasing sequence $t_0 < t_k < 1$, converging to 1. Since every orbit starting from a point in $E_{\mathbb{B}^q}(0, e_1, R_0)$ eventually leaves the same set, it follows that for every $n \ge 0$ we may define $m_{n,k}$ as the first positive integer so that $f^{m_{n,k}} \circ \varphi_n(t_k) \notin E_{\mathbb{B}^q}(z_n, e_1, 1)$.

Lemma 8.9. *There exists $n \in \mathbb{N}$ so that $f^{m_{n,k}} \circ \varphi_n(t_k)$ has a convergent subsequence in Ω.*

Proof. Since $\varphi_n(t_k) \in E_{\mathbb{B}^q}(z_n, e_1, 1)$, we must have $m_{n,k} \ge 1$, and thus we may write

$$\begin{aligned}
x_{n,k} &:= f^{m_{n,k}-1} \circ \varphi_n(t_k) \in E_{\mathbb{B}^q}(z_n, e_1, 1) \\
y_{n,k} &:= f^{m_{n,k}} \circ \varphi_n(t_k) \notin E_{\mathbb{B}^q}(z_n, e_1, 1),
\end{aligned}$$

By Lemma 2.14, for every $n \in \mathbb{N}$ we have

$$\limsup_{k \to \infty} k_\Omega(x_{n,k}, y_{n,k}) \le \lim_{k \to \infty} k_\Omega(\varphi_n(t_k), f \circ \varphi_n(t_k)) = \log \lambda. \tag{8.12}$$

Suppose now that the statement of Lemma 8.9 is false. Then for every fixed n we would have that the sequences $(x_{n,k})$ and $(y_{n,k})$ both converge to e_1. Indeed, for the sequence $(x_{n,k})$, which is contained in $E_{\mathbb{B}^q}(z_n, e_1, 1)$, this follows from

$$\overline{E_{\mathbb{B}^q}(z_n, e_1, 1)} \setminus \{e_1\} \subset E_{\mathbb{B}^q}(0, e_1, R_0) \subset \Omega. \tag{8.13}$$

Since $k_\Omega(x_{n,k}, y_{n,k})$ is bounded, it follows that $(y_{n,k})$ converges to e_1 too. This is a direct consequence of [1, Corollary 2.3.55] and the fact that Ω is biholomorphic to a bounded strongly convex domain via an automorphism of $\mathbb{C}^q$.

By Definition 2.9 and by (8.12) we have that

$$\begin{aligned}\log\lambda &\leq \liminf_{k\to\infty} k_\Omega(z_n, x_{n,k}) - k_\Omega(z_n, y_{n,k})\\ &\leq \liminf_{k\to\infty} k_\Omega(x_{n,k}, y_{n,k})\\ &\leq \limsup_{k\to\infty} k_\Omega(x_{n,k}, y_{n,k})\\ &\leq \log\lambda,\end{aligned}$$

proving that $\lim_{k\to\infty} k_\Omega(x_{n,k}, y_{n,k}) = \log\lambda$. It is now easy to show that

$$\lim_{k\to\infty} k_\Omega(z_n, x_{n,k}) - k_\Omega(z_n, y_{n,k}) = \lim_{k\to\infty} k_\Omega(x_{n,k}, y_{n,k}) = \log\lambda. \tag{8.14}$$

Since $\log\tilde{R} = \log\lambda + 1$, for all $n \geq 0$ we may choose a positive integer k_n so that for all $k \geq k_n$,

$$k_\Omega(x_{n,k}, y_{n,k}) < \log\widetilde{R} - 2\varepsilon_n. \tag{8.15}$$

We now show that

$$z_n, x_{n,k}, y_{n,k} \in E_{\mathbb{B}^q}(0, e_1, R_n), \qquad \forall k \geq k_n. \tag{8.16}$$

This is clear for z_n and $x_{n,k}$ by (8.10). Let $\gamma_{n,k} : [0, a_{n,k}] \to \Omega$ be the geodesic segment connecting $x_{n,k}$ and $y_{n,k}$. Clearly for all $t \in [0, a_{n_k})$ we have $k_\Omega(x_{n,k}, \gamma_{n,k}(t)) < k_\Omega(x_{n,k}, y_{n,k})$. By (8.11), as long as $\gamma_{n,k}(t) \in E_{\mathbb{B}^q}(z_n, e_1, \widetilde{R})$ we have that

$$\begin{aligned}\log h^{\mathbb{B}^q}_{e_1,z_n}(\gamma_{n,k}(t)) &= \lim_{w\to e_1} [k_{\mathbb{B}^q}(\gamma_{n,k}(t), w) - k_{\mathbb{B}^q}(z_n, w)]\\ &\leq \lim_{w\to e_1} [k_{\mathbb{B}^q}(x_{n,k}, w) - k_{\mathbb{B}^q}(z_n, w)] + k_{\mathbb{B}^q}(x_{n,k}, \gamma_{n,k}(t))\\ &< k_\Omega(x_{n,k}, \gamma_{n,k}(t)) + \varepsilon_n\\ &< \log\widetilde{R} - \varepsilon_n,\end{aligned}$$

where we used (8.9),(8.15) and the fact that $x_{n,k} \in E_{\mathbb{B}^q}(z_n, e_1, 1)$. We conclude that $\gamma_{n,k}(t) \in E_{\mathbb{B}^q}(z_n, e_1, \widetilde{R})$ for every $t \in [0, a_{n,k}]$, and thus (8.16) follows.

Thus, using (8.9) and (8.14), we obtain for all $k \geq k_n$,

$$\begin{aligned}\log\lambda - 2\varepsilon_n &\leq \liminf_{k\to\infty} k_{\mathbb{B}^q}(z_n, x_{n,k}) - k_{\mathbb{B}^q}(z_n, y_{n,k})\\ &\leq \limsup_{k\to\infty} k_{\mathbb{B}^q}(x_{n,k}, y_{n,k})\\ &\leq \log\lambda + \varepsilon_n\end{aligned}$$

Let $\sigma_n \in \mathrm{Aut}(\mathbb{B}^q)$ be such that $\sigma_n(z_n) = 0$ and $\sigma_n(e_1) = e_1$. Notice that for every n the sequences $\sigma_n(x_{n,k}), \sigma_n(y_{n,k}) \to e_1$ as $k \to \infty$. We may therefore choose a sequence $K_n \geq k_n$ so that $x'_n := \sigma_n(x_{n,K_n}) \to e_1$, $y'_n := \sigma_n(y_{n,K_n}) \to e_1$ and

$$\log\lambda - 3\varepsilon_n \leq k_{\mathbb{B}^q}(0, x'_n) - k_{\mathbb{B}^q}(0, y'_n) \leq k_{\mathbb{B}^q}(x'_n, y'_n) \leq \log\lambda + 2\varepsilon_n.$$

Finally notice that $x'_n \in E_{\mathbb{B}^q}(0, e_1, 1)$ and that $y'_n \notin E_{\mathbb{B}^q}(0, e_1, 1)$, which contradicts Proposition 8.1 since $\varepsilon_n \to 0$. □

By the previous lemma there exists $n \in \mathbb{N}$, which from now on will be fixed, such that, up to passing to a subsequence of t_k if necessary, the sequence $f^{m_{n,k}} \circ \varphi_n(t_k) \to z_0 \in \Omega$. By Lemma 2.14 we have that for all $j \in \mathbb{N}$ there exists $C_j > 0$ such that

$$k_\Omega(\varphi_n(t_k), f^j \circ \varphi_n(t_k)) \leq C_j, \quad \forall k \geq 0.$$

Therefore, since $\varphi_n(t_k) \to e_1$, the sequence $m_{n,k}$ is divergent.

The remaining of the proof is similar to [9] and [22], but we add it for the convenience of the reader. Consider the sequence $(f^{m_{n,k}-1} \circ \varphi_n(t_k))$. Since

$$k_\Omega(f^{m_{n,k}} \circ \varphi_n(t_k), f^{m_{n,k}-1} \circ \varphi_n(t_k)) \leq k_\Omega(f \circ \varphi_n(t_k), \varphi_n(t_k)) \to \log \lambda,$$

we can extract a subsequence $k_1(h)$ such that $f^{m_{n,k_1(h)}-1} \circ \varphi_n(t_{k_1(h)}) \to z_1 \in \Omega$. Iterating this procedure, we obtain for every $\nu \geq 1$ a subsequence $k_{\nu+1}(h)$ of $k_\nu(h)$ such that

$$f^{m_{n,k_{\nu+1}(h)}-\nu-1} \circ \varphi_n(t_{k_{\nu+1}(h)}) \to z_{\nu+1} \in \Omega,$$

and $f(z_{\nu+1}) = z_\nu$. Hence (z_ν) is a backward orbit.

We now show that $z_\nu \to e_1$. Recall that $f^{m_{n,k}-\nu} \circ \varphi_n(t_k) \in E_{\mathbb{B}^q}(z_n, e_1, 1)$, which implies that $z_\nu \in \overline{E_{\mathbb{B}^q}(z_n, e_1, 1)} \setminus \{e_1\}$. Therefore either $z_\nu \to e_1$ or there exists a subsequence $z_{\nu_m} \to z' \in \overline{E_{\mathbb{B}^q}(z_n, e_1, 1)} \setminus \{e_1\} \subset \Omega$. In the second case for every $i \in \mathbb{N}$ we have that

$$f^i(z') = \lim_{m\to\infty} f^i(z_{\nu_m}) = \lim_{m\to\infty} z_{\nu_m - i} \in \overline{E_{\mathbb{B}^q}(z_n, e_1, 1)} \setminus \{e_1\},$$

and thus by (8.13) it follows that the orbit of the point z' is contained in $E_{\mathbb{B}^q}(0, e_1, R_0)$, contradicting Lemma 8.8.

We are left with showing that the step of (z_ν) is $\log \lambda$. Let $p \in \Omega$. We have that

$$k_\Omega(z_\nu, z_{\nu-1}) = k_\Omega(z_\nu, f(z_\nu)) \geq k_\Omega(p, z_\nu) - k_\Omega(p, f(z_\nu)),$$

and since $z_\nu \to e_1$, it follows that $s_1(z_\nu) \geq \log \lambda$. Moreover,

$$\begin{aligned} k_\Omega(z_\nu, z_{\nu-1}) &= \lim_{h\to\infty} k_\Omega(f^{m_{n,k_\nu(h)}-\nu} \circ \varphi_n(t_{k_\nu(h)}), f^{m_{n,k_\nu(h)}-\nu+1} \circ \varphi_n(t_{k_\nu(h)})) \\ &\leq \lim_{h\to\infty} k_\Omega(\varphi_n(t_{k_\nu(h)}), f \circ \varphi_n(t_{k_\nu(h)})) \\ &= \log \lambda. \end{aligned}$$

This ends the proof of Theorem 6.4.

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