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UNIQUENESS OF FILIPPOV SLIDING VECTOR FIELD ON THE INTERSECTION OF TWO SURFACES IN $\mathbb{R}^3$ AND IMPLICATIONS FOR STABILITY OF PERIODIC ORBITS.

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ABSTRACT. In this paper, we consider the class of smooth sliding Filippov vector fields in $\mathbb{R}^3$ on the intersection Σ of two smooth surfaces: $\Sigma = \Sigma_1 \cap \Sigma_2$, where $\Sigma_i = \{x : h_i(x) = 0\}$, and $h_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2$, are smooth functions with linearly independent normals. Although, in general, there is no unique Filippov sliding vector field on Σ, here we prove that –under natural conditions– all Filippov sliding vector fields are orbitally equivalent on Σ. In other words, the aforementioned ambiguity has no meaningful dynamical impact. We also examine the implication of this result in the important case of a periodic orbit a portion of which slides on Σ.

1. Introduction

Consider the following piecewise smooth system in $\mathbb{R}^3$:

$$\dot{x} = f(x)\ ,\ \ f(x) = f_i(x)\ ,\ \ x \in R_i\ ,\ i = 1, 2, 3, 4\ . \tag{1}$$

For $i = 1, 2, 3, 4$, $R_i \subseteq \mathbb{R}^3$ are open, disjoint and connected sets, and (locally) $\mathbb{R}^3 = \overline{\bigcup_i R_i}$. Moreover, each f_i is smooth (at least $\mathcal{C}^1$) in an open neighborhood of each R_i, $i = 1, \dots, 4$, and $\mathbb{R}^3 \setminus \bigcup_i R_i$ has zero (Lebesgue) measure. Further, we will assume that the R_i's are separated (locally) by an implicitely defined smooth surface Σ of co-dimension 2, as follows. We have $\Sigma = \Sigma_1 \cap \Sigma_2$, where $\Sigma_1 = \{x : \ h_1(x) = 0\}$, and $\Sigma_2 = \{x : \ h_2(x) = 0\}$, with $\nabla h_j(x) \neq 0$, $h_j \in \mathcal{C}^k$, $k \geq 2$, $j = 1, 2$, and $\nabla h_1(x), \nabla h_2(x)$, are linearly independent for $x \in \Sigma$. Compactly, we can write

$$\Sigma = \{x \in \mathbb{R}^3\ :\ h(x) = 0\ , \quad h :\ \mathbb{R}^3 \to \mathbb{R}^2\}\ , \quad h(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}\ .$$

Without loss of generality, we will henceforth use the following labeling of the four regions R_i, $i = 1, 2, 3, 4$:

$$\begin{aligned} &R_1: \quad \text{when} \quad h_1 < 0\ , h_2 < 0\ , \qquad R_2: \quad \text{when} \quad h_1 < 0\ , h_2 > 0\ , \\ &R_3: \quad \text{when} \quad h_1 > 0\ , h_2 < 0\ , \qquad R_4: \quad \text{when} \quad h_1 > 0\ , h_2 > 0\ . \end{aligned} \tag{2}$$

For later use, we will also adopt the notation Σ_1^+, respectively Σ_1^- (and similarly, $\Sigma_2^\pm$) to denote the set of points $x \in \Sigma_1$ for which we also have $h_2(x) > 0$, respectively $h_2(x) < 0$ (similarly, $\Sigma_2^\pm$ are the set of points $x \in \Sigma_2$ for which $h_1(x) > 0$ or $h_1(x) < 0$). See Figure 1. Finally, for $i = 1, 2$, and $j = 1, 2, 3, 4$, we will use the notation

$$w_j^i(x) \ := \ \nabla h_i^T(x) f_j(x) \tag{3}$$

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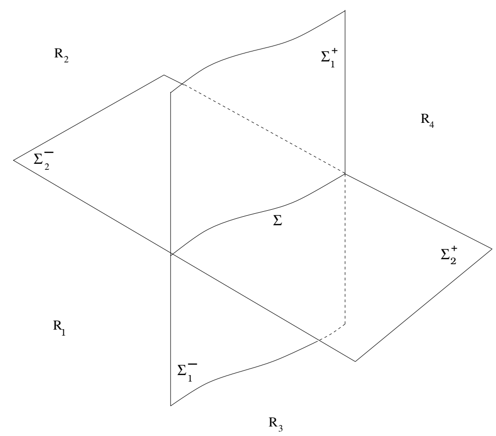

FIGURE 1. Regions $\Sigma^{\pm}_{1,2}$ and the co-dimension 2 discontinuity surface Σ.

for the component of f_j along the normal to Σ_i.

The interesting case is when Σ *attracts* nearby dynamics, and it is reached in finite time by solution trajectories. In this case, upon reaching Σ, trajectories are constrained to remain on Σ, giving rise to so-called *sliding motion*. Presently, we will consider smooth sliding motion of Filippov type, which is defined next.

We call smooth Filippov sliding vector field on Σ (see [10]) any smooth (at least $\mathcal{C}^1$) vector field f_Σ in the convex hull of the f_i's. That is, for each $x \in \Sigma$:

(4)
$$f_\Sigma(x) \in \mathcal{F}(x) \ := \ \{ \sum_{i=1}^{4} \lambda_i(x) f_i(x) \ , \ \lambda_i(x) \geq 0 \ , \ \text{and smooth} \ , \ i = 1,2,3,4, \ \sum_{i=1}^{4} \lambda_i(x) = 1 \, \} \, ,$$

subject to the constraint that f_Σ lies in T_Σ, the tangent plane to Σ at x:

$$(\nabla h_j(x))^T f_\Sigma(x) = 0 \ , \ \text{ for } \ j = 1,2 \, . \tag{5}$$

Obviously, (4-5) can be rewritten as the linear system (to be solved at each $x \in \Sigma$)

$$\begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & w_4^1 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ , \tag{6}$$

and clearly in general one cannot select uniquely the coefficients λ_i, $i = 1,2,3,4$, since we have an undertermined system to solve.

Remark 1. *To avoid the above ambiguity, several authors have considered special Filippov vector fields; e.g., see* [2, 7, 12, 14] *for the so-called* bilinear interpolant *approach, and* [5, 6] *for the so-called* moments method*. But, regardless of the merits of any of these choices, the basic ambiguity intrinsic in* (6) *has to be contended with. In particular, it is obviously important to understand to what extent the choice of a specific Filippov sliding vector field produces different dynamical behavior.*

A main result in this paper is to show that, under appropriate conditions, all smooth Filippov sliding vector fields determine the same orbit on Σ. In other words, the aforementioned ambiguity has no meaningful dynamical impact. We also examine the implication of this result in the important case of a periodic orbit a portion of which slides on Σ.

In what follows, we will also use the following notation. We denote with $f^{\pm}_{\Sigma_1}$ and $f^{\pm}_{\Sigma_2}$ the sliding vector fields on $\Sigma^{\pm}_1$, $\Sigma^{\pm}_2$, respectively. These are smooth Filippov sliding vector fields on the co-dimension 1 manifolds $\Sigma^{\pm}_{1,2}$, and are defined according to the standard Filippov convexification method in this case. Namely (see Figure 1), we have (when they are well defined):

$$
\begin{aligned}
x \in \Sigma_1^- : \quad f^-_{\Sigma_1} &= (1-\alpha^-)f_1 + \alpha^- f_3 \ , \ \ 0 \le \alpha^- \le 1 \ : \ \nabla h_1^T f^-_{\Sigma_1} = 0 \ ; \\
x \in \Sigma_1^+ : \quad f^+_{\Sigma_1} &= (1-\alpha^+)f_2 + \alpha^+ f_4 \ , \ \ 0 \le \alpha^+ \le 1 \ : \ \nabla h_1^T f^+_{\Sigma_1} = 0 \ ; \\
x \in \Sigma_2^- : \quad f^-_{\Sigma_2} &= (1-\beta^-)f_1 + \beta^- f_2 \ , \ \ 0 \le \beta^- \le 1 \ : \ \nabla h_2^T f^-_{\Sigma_2} = 0 \ ; \\
x \in \Sigma_2^+ : \quad f^+_{\Sigma_2} &= (1-\beta^+)f_3 + \beta^+ f_4 \ , \ \ 0 \le \beta^+ \le 1 \ : \ \nabla h_2^T f^+_{\Sigma_2} = 0 \ .
\end{aligned}
\tag{7}
$$

Finally, we let $w^i_{\Sigma^{\pm}_j}$ be the component of $f^{\pm}_{\Sigma_j}$ along the normal to Σ_i, $i, j = 1, 2$. By definition of the $f^{\pm}_{\Sigma_j}$'s, note that we have $\nabla h_i(x)^T f^{\pm}_{\Sigma_i}(x) = 0$, $i = 1, 2$.

2. Uniqueness of sliding on Σ

In light of the "algebraic ambiguity" of (6) in selecting a Filippov sliding vector field on Σ, it is natural to ask what criteria should guide us in the selection of a certain Filippov sliding vector field, and if a certain choice is better than other choices. Specifically, we are concerned with understanding if the choice of a certain sliding vector field can impact the overall dynamics. As it turns out, under natural conditions, it does not.

First of all, we make the simple, but key, observation: in $\mathbb{R}^3$, $\Sigma = \Sigma_1 \cap \Sigma_2$ is a smooth curve (or union of smooth arcs). Therefore, given that all Filippov sliding vector fields on Σ must lie on the tangent plane to Σ, all Filippov vector fields are parallel (they could have different orientation, or vanish, of course).

Secondly, we will assume that Σ, or at least some connected part of it, is *attractive in finite time* for the dynamics, that is, it is reached by solution trajectories in finite time. Moreover, once on Σ, a solution trajectory is forced to slide on it until either an equilibrium or an *exit point* is reached. Insofar as exiting Σ, we are interested in *first order exit points*, i.e., points at which one of the sub-sliding vector fields on $\Sigma^{\pm}_1$ or $\Sigma^{\pm}_2$ is tangent to Σ as well. The formal definition follows (see [7]).

Definition 1. Assume that, while sliding on Σ, the solution trajectory reaches a point $\bar{x} \in \Sigma$, where one –and only one– of the following four conditions is satisfied.

(i) Exiting on Σ^-_2 or Σ^+_2:
  (a) $w^1_{\Sigma^-_2}(\bar{x}) = \nabla h_1(\bar{x})^T f^-_{\Sigma_2}(\bar{x}) = 0$, or
  (b) $w^1_{\Sigma^+_2}(\bar{x}) = \nabla h_1(\bar{x})^T f^+_{\Sigma_2}(\bar{x}) = 0$.

(ii) Exiting on Σ^-_1 or Σ^+_1:
  (a) $w^2_{\Sigma^-_1}(\bar{x}) = \nabla h_2(\bar{x})^T f^-_{\Sigma_1}(\bar{x}) = 0$, or
  (b) $w^2_{\Sigma^+_1}(\bar{x}) = \nabla h_2(\bar{x})^T f^+_{\Sigma_1}(\bar{x}) = 0$.

Then, we say that $\bar{x}$ is a (first order) *generic tangential exit point*. Further, we will call *exit vector field* respectively: $f^-_{\Sigma_2}$ in case (i)-(a), $f^+_{\Sigma_2}$ in case (i)-(b), $f^-_{\Sigma_1}$ in case (ii)-(a), and $f^+_{\Sigma_1}$ in case (ii)-(b). □

We clarify (i)-(a), the other conditions are analogous. Condition (i)-(a) says that at $\bar{x}$, the sliding vector field on Σ^-_2, that is $f^-_{\Sigma_2}$, is also tangent to Σ_1 and hence to Σ. Now, suppose that $x(\cdot)$ is a solution trajectory on Σ that reaches $\bar{x}$, say $x(\bar{t})=\bar{x}$. Note that, since Σ is attractive, for $t<\bar{t}$ and near $\bar{t}$, $f_{\Sigma_2}^-$ could not give a sliding motion on Σ_2 away from Σ. Then, generically, $\frac{d}{dt}w^1_{\Sigma_2^-}(x(t))$ changes sign at $\bar{t}$, and the sliding vector field $f_{\Sigma_2}^-$ on Σ_2^- points away from Σ for t in a right neighborhood of $\bar{t}$. This implies a loss of attractivity for Σ and makes $\bar{x}$ a first order generic tangential exit point from Σ to Σ_2^-.

Remark 2. *The conditions of Definition 1 depend only on $f^{\pm}_{\Sigma_1}$ and $f^{\pm}_{\Sigma_2}$, that are unambiguously defined (see* (7)*).*

Remark 3. *In Section 3, we will assume that if a tangential exit point is reached, then a trajectory which was sliding on Σ* will exit *from Σ, no matter how sliding motion on Σ had been taking place. It is important to observe immediately that, in our case, although all Filippov vector fields on Σ are parallel, in general they will have different norms. For this reason, a trajectory exiting at a generic tangential exit point will do so tangentially, but not necessarily smoothly (the latter property will depend on which particular sliding vector field one is considering). In other words, suppose that the solution has reached a point $\bar{x}$ where case (i)-(a) of Definition 1 holds (this case is considered for illustration only, any other case of Definition 1 would give similar conclusions). At $\bar{x}$, f_Σ (the selected sliding vector field on Σ) will be parallel to $f^-_{\Sigma_2}(\bar{x})$, but not necessarily of the same magnitude; hence, requiring the trajectory to exit at $\bar{x}$ will produce a tangential exit, but the corresponding vector field will not necessarily be continuous. See Section 3 for the impact of this observation, but see also Theorem 11.*

Thirdly, we make the following assumption which legitimizes the last remark.

Assumption 1. *No smooth Filippov sliding vector field $f_\Sigma \in \mathcal{F}$ (see* (4)*) has an equilibrium on Σ.*

As the two examples below show, when Assumption 1 is violated, different dynamics can be observed.

Example 4. *Consider the following example with constant vector fields*

$$f_1=\begin{bmatrix}1\\1\\0.25\end{bmatrix},\quad f_2=\begin{bmatrix}1\\-1\\0.5\end{bmatrix},\quad f_3=\begin{bmatrix}-1\\1\\-1\end{bmatrix},\quad f_4=\begin{bmatrix}-1\\-1\\0.25\end{bmatrix}.$$

Let $\Sigma_1=\{x\in\mathbb{R}^3\ ,\ x_1=0\}$ and $\Sigma_2(x)=\{x\in\mathbb{R}^3\ ,\ x_2=0\}$ and note that $\Sigma=\Sigma_1\cap\Sigma_2$ is just the x_3-axis, which attracts all trajectories. On Σ, the family of Filippov sliding vector fields is the set $\mathcal{F}(x)=\begin{bmatrix}0\\0\\c-0.25\end{bmatrix}$, with $c\in[0,0.5]$ (here we consider only smooth vector fields, hence c can be any smooth function of x_3). Consider the following choices for c:

(i) $c=1/10$,

(ii) $c=-\frac{\arctan(x_3)}{2\pi}+0.25$,

(iii) $c=0.25$, *and*

(iv) $c=\frac{2}{5}$.

With c as in (i) and (iv), f_Σ has no equilibria on Σ but the resulting Filippov vector fields have opposite orientation on Σ. If we choose c as in (ii), the origin is an asymptotically *stable equilibrium for f_Σ. Finally, if we choose c as in (iii), every point on Σ is an equilibrium for f_Σ.* □

Example 5. *Here we have a situation where some sliding vector fields have an equilibrium before the exit point, and others after the exit point. Take*

$$f_1 = \begin{bmatrix} 1-3x_1/4 \\ x_2/2-1 \\ x_2-x_1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1-3x_1/4 \\ -x_2/2 \\ -x_2 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 1-3x_1/4 \\ 1-x_2/2 \\ (x_2+x_1)/2 \end{bmatrix}, \quad f_4 = \begin{bmatrix} -1/4-3x_1/4+x_3/2 \\ 1-x_2/2 \\ x_1/2 \end{bmatrix},$$

and $\Sigma_1 = \{x \in \mathbb{R}^3\ ,\ x_1 = 1\}$ and $\Sigma_2 = \{x \in \mathbb{R}^3\ ,\ x_2 = 1\}$. Here, $\Sigma = \Sigma_1 \cap \Sigma_2$ is the line $\{(1,1,x_3)\}$, it is attractive and it is reached upon sliding on Σ_1^+, for $x_3 < 3/2$. At $x_3 = 3/2$, there is a tangential exit point on Σ_1^+. On Σ, the family of Filippov sliding vector fields is the set

$$\mathcal{F}(x) = \begin{bmatrix} 0 \\ 0 \\ -\lambda_2 + \frac{2-x_3}{5-2x_3} \end{bmatrix},$$

where λ_2 (which may be any smooth function of x_3) must satisfy $0 \le \lambda_2 \le 1/2$. The coefficients of Filippov's convex combination must satisfy: $\lambda_1 + \lambda_2 = \frac{1}{2}$, $\lambda_3 = \frac{3-2x_3}{2(5-2x_3)}$, $\lambda_4 = \frac{1}{5-2x_3}$. Choosing different values of λ_2 we obtain different behaviors. For example:

(i) $\lambda_2 = 1/4$ *gives the equilibrium point at the exit point $x_3 = 3/2$;*
(ii) $\lambda_2 = 1/8$ *gives the equilibrium point at $x_3 = 11/6$, that is after the exit point;*
(iii) $\lambda_2 = 3/8$ *gives the equilibrium point at $x_3 = 1/2$, that is before the exit point;*
(iv) *A choice of λ_2 as a quadratic function of x_3 such as $\lambda_2 = -\frac{8}{9}x_3(x_3 - 3/2)$ gives two equilibria before the exit point.* □

Finally, we give the anticipated result that, under some (natural) conditions, the dynamics on Σ are equivalent for all sliding vector fields.

Theorem 6. *Let Γ be a connected arc of Σ, and consider the differential inclusion on Γ*

$$\dot{x} \in \mathcal{F}(x)\ ,\quad x \in \Gamma\ ,\quad \text{and} \quad \dot{x} \in T_\Gamma\ , \tag{8}$$

where $\mathcal{F}$ is the (Filippov) convex hull of f_1, f_2, f_3, f_4, in (4). Assume that there are no equilibria on $\Sigma \cap \Gamma$ for any smooth function in $\mathcal{F}$. Then, the systems $\dot{x} = f_\Sigma(x)$, with $f_\Sigma(x)$ any smooth selection in $\mathcal{F}(x) \cap T_\Gamma$, are all orbitally equivalent.

Proof. Let $n(x)$ be the cross product of $\nabla h_1(x)$ and $\nabla h_2(x)$, the two normals to Σ_1 and Σ_2, respectively: $n(x) = \nabla h_1(x) \times \nabla h_2(x)$, $x \in \Gamma$. Then, for $x \in \Gamma$, any smooth element of $\mathcal{F}$ can be represented as $f_\Sigma(x) = \gamma(x)n(x)$, for some (smooth) function γ. The hypothesis of no equilibria guarantees that $\gamma(x) \neq 0$, and therefore all vector fields are oriented in the same way. Indeed, if there were two Filippov vector fields that had opposite direction at a point $\hat{x} \in \Gamma$, then there would be a third Filippov vector field in the convex combination having an equilibrium at $\hat{x}$. This contradicts the assumption that no smooth function in $\mathcal{F}$ has an equilibrium on $\Sigma \cap \Gamma$. Let $f_{S_1}(\cdot) = \gamma_1(\cdot)n(\cdot)$ be any such vector field, and $f_{S_2}(\cdot) = \gamma_2(\cdot)n(\cdot)$ be another one. Then:

$$f_{S_2}(x) = \frac{\gamma_2(x)}{\gamma_1(x)} f_{S_1}(x) =: \omega(x) f_{S_1}(x)\ ,$$

where $\omega(x) = \frac{\gamma_2(x)}{\gamma_1(x)}$ and thus $\omega(x) > 0$, for all $x \in \Gamma$. Obviously, ω is a smooth function for all $x \in \Gamma$, and the result follows. □

Remarks 7.

(i) *In the above proof, the hypothesis of no equilibria on Σ is used to guarantee that one cannot have $\omega = 0$. However, the assumption of no equilibria on Σ can be weakened. It is sufficient to restrict to those smooth Filippov vector fields from $\mathcal{F}$ that have no equilibria on the arc Γ where sliding motion is taking place, and that have the same orientation. [The main concern caused by an equilibrium is that directionality of motion on the curve may be different for different vector fields; one could go "right-or-left"].*

(ii) *In some situations one may be able to establish a-priori that there are no equilibria, for example when the components of the vector fields f_i, $i = 1, 2, 3, 4$, in the direction of $\nabla h_1 \times \nabla h_2$, all have same sign. However, in general, the absence of equilibria is not easy to establish ahead of time.*

(iii) *Theorem 6 holds true also for other sliding vector fields of non-Filippov type, as long as one has a smooth sliding vector field (hence, tangent to Σ), with no equilibria on Σ.*

(iv) *With much the same assumptions, Theorem 6 also holds in $\mathbb{R}^n$, $n > 3$, in case of sliding on the curve given by the intersection of $(n-1)$ surfaces.*

2.1. **Time reparametrization, convexity, and smooth exits.** Here we look at some important consequences of Theorem 6.

2.1.1. *Time reparametrization.* The orbital equivalence of Theorem 6 means that a reparametrization of time takes a sliding vector field into another. That is to say, solutions associated to different sliding vector fields are tracing the same orbit, but at different speeds. Indeed, for two different sliding vector fields f_{S_1} and f_{S_2} we must have

$$\frac{dx}{dt} = f_{S_1} \iff \frac{dx}{d\tau} = f_{S_2} = \omega(x) f_{S_1} \text{ and } \omega(x) = \frac{dt}{d\tau} . \tag{9}$$

From (9), we can interpret the two vector fields as follows:

$$\begin{aligned} f_{S_1} &= \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 \ , \\ f_{S_2} &= \lambda_1(\omega f_1) + \lambda_2(\omega f_2) + \lambda_3(\omega f_3) + \lambda_4(\omega f_4) \ , \quad \text{or} \\ f_{S_2} &= \nu_1 f_1 + \nu_2 f_2 + \nu_3 f_3 + \nu_4 f_4 \ , \quad \nu_i = \omega \lambda_i \ , \ \ i = 1, 2, 3, 4 \ . \end{aligned} \tag{10}$$

2.1.2. *Convexity.* Now, observe that –if the λ_i's are used to obtain a convex combination of the f_i's and to form the sliding system $\frac{dx}{dt} = f_{S_1} = \sum_i \lambda_i f_i$– in (10) the coefficients ν_i's, albeit positive, cannot give just a convex combination of the f_i's, unless $\omega(x) \equiv 1$. This is at first puzzling, given that we are obtaining the coefficients λ_i's or ν_i's by imposing convexity requirement (and tangency), and then form a convex combination of the f_i's (see f_{S_1} and the second expression for f_{S_2} in (10)). However, what we are witnessing is in fact natural. What is happening is that
"*We are obtaining a Filippov sliding vector field, and a convex combination, but for (modified) vector fields f_i's. The (modified) vector fields are given by ωf_i, $i = 1, 2, 3, 4$, and they do depend on the parametrization of time.*"
Note that, since the system is autonomous, there is no specific meaning attached to the time variable. Hence, in a more emphatic way, we may say that
"*Convexity (i.e., the coefficients in the convex combination) depends on the parametrization of time.*"

The above point is consistent with the Filippov construction, and can be finally summarized as follows.

Theorem 8. *Under the assumptions of Theorem 6, any smooth Filippov sliding vector field in $\mathcal{F}$ can be interpreted as having always the same convex combination coefficients, but for modified vector fields. Namely, the modified vector fields are given by ωf_i, $i = 1, 2, 3, 4$, where $\omega = \frac{dt}{d\tau}$ accounts for the reparametrization of time having taken place.*

Remark 9. *Note that the function ω remains well defined and positive in a neighborhood of Σ. In other words, the time parametrization expressed by time τ extends to a neighborhood of Σ. Therefore, the modified vector fields are defined in a neighborhood of Σ as well, not just on Σ. This means that, around Σ, one could consider the problem* (1) *rewritten as*

$$\frac{dx}{d\tau} = f(x) \ , \ \ f(x) = f_i(x) \ , \ \ x \in R_i \ , \ i = 1, 2, 3, 4 \ , \tag{11}$$

where everything continues to be defined as before (here, τ is such that $\frac{dt}{d\tau} = \omega$).

2.1.3. *Smooth exits.* A final consequence of the above considerations, in particular of Remark 9, is that *all first order exits are smooth*, in the appropriate time parametrization.

This is the content of Theorem 11 below, where we consider a first order tangential exit point satisfying Condition (i)-(a) of Definition 1; naturally, an exit point satisfying any of the other conditions would give the same outcome. First, we have the following.

Lemma 10. *Under the assumptions of Theorem 6, let $\bar{x}$ be a first order tangential exit point on Γ satisfying condition (i)-(a) of Definition 1. Let $f_S \in \mathcal{F}$ be any given smooth sliding vector field, and $x(\cdot)$ be the solution trajectory of $\frac{dx}{dt} = f_S(x)$, $x(0) = x_0 \in \Gamma$. Then, for each such sliding vector field, there exists a value $\bar{t}$ (which depends on the vector field) such that $x(\bar{t}) = \bar{x}$.*

Moreover, there always exists a smooth Filippov vector field $f_S \in \mathcal{F}$ such that $f_S(x(\bar{t})) = f_{\Sigma_2^-}(\bar{x})$.

Proof. We can assume $x(0) \neq \bar{x}$, otherwise the claim is trivial. Since there are no equilibria on Σ, and all systems are orbitally equivalent, all the trajectories associated to different sliding vector fields, starting at x_0, will need to reach $\bar{x}$, and all of them will need to do so either for $t > 0$ or for $t < 0$.

As for the last statement, notice that the moments method of [6] satisfies it. ☐

Finally, we have the anticipated result.

Theorem 11. *Under the assumptions of Theorem 6, let $x(t)$, $t \geq 0$, be the solution trajectory of $\frac{dx}{dt} = f_S(x)$, on Γ, where f_S is a given smooth Filippov sliding vector field. Suppose that the trajectory reaches a first order tangential exit point $\bar{x}$, satisfying condition (i)-(a) of Definition 1, and that the trajectory exits smoothly at $\bar{x}$, that is $f_S(\bar{x}) = f_{\Sigma_2^-}(\bar{x})$. Let f_U be another Filippov sliding vector field on Γ, leading to a non-smooth exit at $\bar{x}$, that is $f_U(\bar{x}) \neq f_{\Sigma_2^-}(\bar{x})$.*

Let f_U and f_S be related through ω (see (9)*): $f_U(x) = \omega(x) f_S(x)$, and –with $\omega = \frac{dt}{d\tau}$– also $\frac{dx}{d\tau} = f_U(x)$. If, in a neighborhood of Σ, we consider the system in the time variable τ as in* (11)*, and the sliding vector fields on the co-dimension 1 surface Σ_2^- also with respect to τ, we have*

$$f_U(\bar{x}) = \left[\frac{dx}{d\tau}\right]_{\Sigma_2^-(\bar{x})} .$$

Proof. The proof follows from Theorem 8, by modifying all vector fields in a neighborood of Σ as in (8) (with respect to time "t"), and then $f_U(\bar{x})$ will be equal to $\omega(\bar{x})f_{\Sigma_2^-}(\bar{x})$, and the latter is the sliding vector field at $\bar{x} \in \Sigma_2^-$ for the problem written with respect to time "τ". □

In the next section, we look at the practical impact of the orbital equivalence result of Theorem 6 in the case of periodic orbits.

3. Stability of periodic orbits with a sliding portion on Σ

Our aim in this section is to show that, under suitable assumptions, if a given choice of Filippov vector field determines a periodic solution tracing an orbit (a closed curve) γ, with partial sliding on Σ, all other Filippov vector fields determine the same orbit γ and, moreover, the stability properties of γ are unaffected by the choice of sliding vector field on Σ. The main consequence of this result is that the algebraic ambiguity in the construction of a smooth Filippov sliding vector field is not a concern insofar as the dynamics.

Throughout this section we will assume Σ to be *attractive in finite time upon sliding* (see [7]). With this, we mean that Σ (or a finite union of arcs of Σ) is stable and attracts nearby orbits in finite time, and there is attractive sliding motion towards Σ on at least one[1] of Σ_1^+, Σ_1^-, Σ_2^+, or Σ_2^-. It must be clarified that attractivity through sliding is not the only characterization under which Σ attracts nearby trajectories; to witness, Σ may be spirally attractive (see [4]).

We **assume** that **for a given smooth choice** f_P in the Filippov convex combination, system (1) has a periodic solution with partial sliding on Σ, tracing a periodic orbit γ. In what follows we investigate under which conditions a periodic solution exists for any sliding vector field in Σ and has periodic orbit given by γ, and, if this is the case, whether the stability properties of the orbit γ are independent of the chosen vector field on Σ. We will assume that γ has a unique arc Γ on Σ. However, the results in this section extend easily to the case in which γ has in common with Σ a finite number of disjoint arcs. Our argument below is based on the following assumption.

Assumption 2. *Assume that, while sliding on Σ, the solution trajectory $x = x(t)$ meets a (first order) generic tangential exit point $\bar{x}$ (see Definition 1), and that, at $\bar{x}$, $x(t)$ leaves Σ regardless of whether such exit is smooth or not.*

To visualize, we will assume that the periodic orbit γ looks like in Figure 2. That is, let $\bar{x} \in \Sigma$ be a tangential exit point that satisfies Condition (i)-(a) in Definition 1 above. Take the initial condition as $x(0) = \bar{x}$. The corresponding trajectory starts sliding on Σ_2^+ with vector field $f_{\Sigma_2^+}$. At $x = x_1 \in \Sigma_2^+$, the trajectory exits Σ_2^+ smoothly and enters in R_4. At $x = x_2$, the trajectory reaches Σ_1^+ transversally and starts sliding on it. At $x = x_3$ it reaches Σ transversally, and starts sliding on it with vector field f_P, up to the exit point $x_4 = \bar{x}$. We can (and will) assume that $f_P(\bar{x}) = f_{\Sigma_2^+}(\bar{x})$ (Lemma 10 shows that this is always possible), hence $x = x(t)$ exits Σ at $\bar{x}$ with continuous vector field. Denote with t_j the time t such that $x(t_j) = x_j$, $j = 1, 2, 3, 4$. Then $\gamma = \{x \in \mathbb{R}^3,\ \ x = x(t),\ \ t \geq 0\}$, is a periodic orbit with period t_4.

The following assumption is a relaxed version of Assumption 1, and it is the condition we need for the aforementioned equivalence result.

[1]The simplest case is that of *nodal attractivity*, when there is sliding motion toward Σ, on each of $\Sigma_{1,2}^{\pm}$

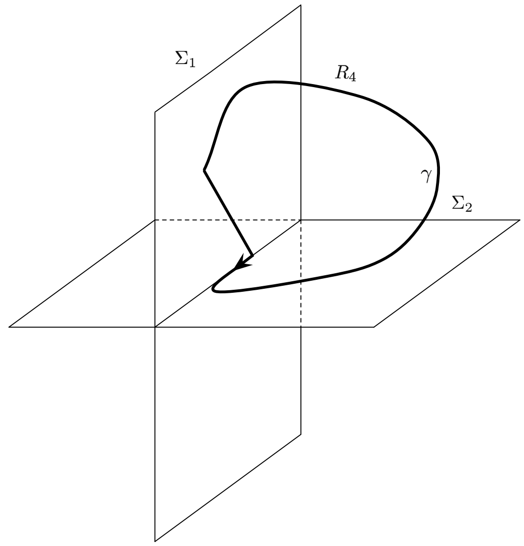

FIGURE 2. Model periodic solution.

Assumption 3. *Assume that no smooth Filippov vector field in $\mathcal{F}$ (see (4)) has an equilibrium on Γ.*

Under Assumption 2 and Assumption 3, the existence of γ is guaranteed for any choice of Filippov vector field. This is because of the following observations:

i) the result on orbital equivalence of the Filippov vector fields in Theorem 6 applies;
ii) we always exit Σ when we reach a tangential exit point.

Hence, every choice of Filippov vector field on Σ will lead to the same closed curve γ as periodic orbit, though the time parametrization of this curve depends on how sliding on Σ takes place.

The question is whether or not the stability properties of γ depend on the specific vector field f_Σ. The answer to this question is not immediate. Indeed, it must be emphasized that, while Theorem 6 implies that any differential equation in (8) can be obtained from $\dot{x} = f_P(x)$ through a reparametrization of time $\frac{d\tau}{dt} = \omega(x)$, the time reparametrization does not carry outside (a neighborhood of) Σ. To study the stability of γ, we will compute its associated Floquet multipliers and hence we need to form the monodromy matrix X. At the points where the trajectory is only continuous (hence, certainly at x_2 and x_3), the discontinuity of the vector field determines a jump in the fundamental matrix solution that must be taken into account, and this is done through a suitable *saltation matrix* S. At $x = x_j$, the saltation matrix S can be thought of as the fundamental matrix solution between t_j^- and t_j^+, and this characterization allows one to derive the explicit expression for S. Below we give the saltation matrices at $x = x_2$

$$S_{\Sigma_1^+} = I + (f_{\Sigma_1^+}(x_2) - f_4(x_2)) \frac{\nabla h_1(x_2)^\top}{\nabla h_1(x_2)^\top f_4(x_2)} \tag{12}$$

and at $x = x_3$

$$S_\Sigma = I + (f_\Sigma(x_3) - f_{\Sigma_1^+}(x_3)) \frac{\nabla h_2(x_3)^\top}{\nabla h_2(x_3)^\top f_{\Sigma_1^+}(x_3)} \tag{13}$$

and we refer the reader to [1, 8, 13, 11, 10] for a derivation of the formulae above.

At $x_4 = \bar{x}$, if the vector field f_Σ in the Filippov convex combination is exactly $f_P(\bar{x}) = f_{\Sigma_2^+}(\bar{x})$, then the corresponding solution exits Σ with continuos vector field and the saltation matrix is the identity matrix. If, instead, $f_\Sigma(\bar{x}) \neq f_{\Sigma_2^+}(\bar{x})$, the corresponding solution trajectory still exits Σ at $\bar{x}$ with vector field $f_{\Sigma_2^+}(\bar{x})$, but the exit will be tangential and not smooth. Hence, in the latter case, when forming the fundamental matrix solution relative to the vector field f_Σ, we need to take into account a saltation matrix at $\bar{x}$. Because of Theorem 6, for x in the sliding region there exists a positive differentiable scalar-valued function ω such that $f_\Sigma(\bar{x}) = \omega(\bar{x}) f_P(\bar{x}) = \omega(\bar{x}) f_{\Sigma_2^+}(\bar{x})$. Relying on this result, we can give the exact form of the saltation matrix at $\bar{x}$.

Proposition 12. *Under Assumption 3, let $x(\cdot)$ be a sliding trajectory on Σ with vector field f_Σ and let $\bar{x}$ be a generic first order tangential exit point; say, $\bar{x}$ satisfies* (i)-(a) *in Definition 1. Let $f_P(x)$ be a sliding vector field on Σ such that $f_P(\bar{x}) = f_{\Sigma_2^+}(\bar{x})$, and let ω be such that $f_\Sigma(\bar{x}) = \omega(\bar{x}) f_P(\bar{x})$ as in Theorem 6. Then, the saltation matrix S, such that $S f_\Sigma(\bar{x}) = f_P(\bar{x})$, is given by*

$$S = \frac{1}{\omega(\bar{x})} I \,, \tag{14}$$

where $I \in \mathbb{R}^{3\times 3}$ is the identity matrix.

Proof. Consider the initial condition $x_0 \in \Sigma$, and a perturbed value $y_0 = x_0 + \Delta_0 \in \Sigma$. Denote with $\varphi^t(\cdot)$ the flow of the system $\dot{x} = f_\Sigma(x)$, and let t_0 be such that $\varphi^{t_0}(x_0) = \bar{x}$. Also, let Δt be such that $\varphi^{t_0+\Delta t}(y_0) = \bar{x}$. Without loss of generality, we can assume $\Delta t > 0$. Our purpose is to find a matrix S such that, in first approximation,

$$(\varphi^{t_0+\Delta t}(x_0) - \varphi^{t_0+\Delta t}(y_0)) = S(\varphi^{t_0}(x_0) - \varphi^{t_0}(y_0)) \,.$$

By Taylor expansion:

$$\varphi^{t_0+\Delta t}(x_0) = \varphi^{t_0}(x_0) + f_{\Sigma_2}(\varphi^{t_0}(x_0))\Delta t + \text{ h.o.t. } = \bar{x} + f_{\Sigma_2^+}(\bar{x})\Delta t + \text{ h.o.t.} \tag{15}$$

$$\varphi^{t_0+\Delta t}(y_0) = \varphi^{t_0}(y_0) + f_\Sigma(\varphi^{t_0}(y_0))\Delta t + \text{ h.o.t .} = \varphi^{t_0}(y_0) + f_\Sigma(\bar{x})\Delta t + \text{ h.o.t. } , \tag{16}$$

where "h.o.t." denote higher order terms (in Δt). The last equality in (16) is obtained considering the Taylor expansion of $f_\Sigma(\varphi^{t_0}(y_0))$ at $\bar{x}$.

From (16), in first approximation, $f_\Sigma(\bar{x})\Delta t = \bar{x} - \varphi^{t_0}(y_0) = \varphi^{t_0}(x_0) - \varphi^{t_0}(y_0)$. Using this, and the difference between (15) and (16), we have $\varphi^{t_0+\Delta t}(x_0) - \varphi^{t_0+\Delta t}(y_0) = (\varphi^{t_0}(x_0) - \varphi^{t_0}(y_0)) + (\frac{1}{\omega(\bar{x})} - 1) f_\Sigma(\bar{x})\Delta t = \frac{1}{\omega(\bar{x})}(\varphi^{t_0}(x_0) - \varphi^{t_0}(y_0))$, so that the theorem is proven. □

We are now ready to give the monodromy matrix of (1) along γ :

$$X(t_4, 0) = X_\Sigma(t_4, t_3) S_\Sigma(x_3) X_{\Sigma_1^+}(t_3, t_2) S_{\Sigma_1^+}(x_2) X_4(t_2, t_1) X_{\Sigma_2^+}(t_1, 0) S(\bar{x}) \,, \tag{17}$$

and below we explain the different factors in (17) .

- $S(\bar{x})$ is the saltation matrix at $\bar{x}$, given in Proposition 12;
- $X_{\Sigma_2^+}(t_1, 0)$ is the solution at $t = t_1$ of the following Cauchy problem on Σ_2^+:

$$\dot{X}_{\Sigma_2^+}(t, 0) = D f_{\Sigma_2^+}(x(t)) X_{\Sigma_2^+}(t, 0), \;\; X_{\Sigma_2^+}(0, 0) = I \,;$$

- $X_4(t_2, t_1)$ is the solution at $t = t_2$ of the following Cauchy problem in R_4:

$$\dot{X}_4(t, t_1) = D f_4(x(t)) X_4(t, t_1), \;\; X_4(t_1, t_1) = I \,;$$

- $S_{\Sigma_1^+}(x_2)$ is the saltation matrix at x_2 given in (12);
- $X_{\Sigma_1^+}(t_3, t_2)$ is the solution at $t = t_3$ of the following Cauchy problem on Σ_1^+:

$$\dot{X}_{\Sigma_1^+}(t_3, t_2) = Df_{\Sigma_1^+}(x(t))X_{\Sigma_1^+}(t_3, t_2), \;\; X_{\Sigma_1^+}(t_2, t_2) = I \;;$$

- $S_{\Sigma}(x_3)$ is the saltation matrix at x_3 given in (13);
- $X_{\Sigma}(t_4, t_3)$ is the solution at $t = t_4$ of the following Cauchy problem on Σ:

$$\dot{X}_{\Sigma}(t, t_3) = Df_{\Sigma}(x(t))X_{\Sigma}(t, t_3), \;\; X_{\Sigma}(t_3, t_3) = I \;.$$

The following Lemma is needed to establish the number of Floquet multipliers equal to 0 for the monodromy matrix of (17).

Lemma 13. *The fundamental matrix solution $X_{\Sigma_1^+}(t, t_2)$ takes $T_{x_2}(\Sigma_1)$ into $T_{x(t)}\Sigma_1$, where, with $T_x\Sigma_1$ we denote the tangent space of Σ_1 at x.*

Proof. The key ingredient of the proof is the definition of the tangent map of $\varphi^t(\cdot)$, see for example [3].

Since the intersection of $x(t)$ with Σ_1 is transversal, there is a neighborhood I_{x_2} of x_2 in Σ_1, such that $I_{x_2} \cap \Sigma_1$ is attractive and there is sliding motion on it. By construction, the sliding motion on $I_{x_2} \cap \Sigma_1$ must be towards Σ. Hence there is a neighborhood of $\gamma \cap \Sigma_1$ that is invariant under $\varphi^t_{\Sigma_1^+}(\cdot)$, where with $\varphi^t_{\Sigma_1^+}(\cdot)$ we denote the flow of $\dot{x} = f_{\Sigma_1^+}(x)$. Let now v be a vector in $T_{x_2}\Sigma_1$ and let $\psi(s)$ be a curve on Σ_1 such that, $\psi(0) = x_2$ and $\frac{d}{ds}\psi(s)|_{s=0} = v$. Then $\varphi^t_{\Sigma_1^+}(\psi(s))$ is in Σ_1 and in particular its derivative with respect to s, computed at $s = 0$, must be in $T_{\varphi^t_{\Sigma_1}\psi(0)}\Sigma_1 = T_{x(t)}\Sigma_1$. But $\frac{d}{ds}\varphi^t_{\Sigma_1^+}(\psi(s))|_{s=0} = \frac{d}{dx}\varphi^t_{\Sigma_1^+}(x_2)\frac{d}{ds}\psi(s)|_{s=0} = X_{\Sigma_1^+}(t, t_2)v$. Hence the lemma is proved. □

Our main result, Theorem 14 below, says that the monodromy matrix associated to a periodic orbit with partial sliding on Σ (and Σ attractive upon sliding) has two Floquet multipliers equal to 0 and one equal to 1, regardless of how we selected the Filippov sliding vector field on Σ and hence regardless of whether the exit at $\bar{x}$ is smooth or not, as long as one exits at $\bar{x}$.

Theorem 14. *Consider system* (1)*. Assume that a subset of $\Sigma = \Sigma_1 \cup \Sigma_2$ is attractive in finite time upon sliding, and that, for a given choice of f_Σ on Σ, the corresponding solution trajectory $x = x(t)$ is periodic. We denote with $\gamma = \{x \in \mathbb{R}^3, \;\; x = x(t), t \geq 0\}$ the corresponding periodic orbit. Assume, moreover, that γ intersects Σ in at most a finite number of (disjoint) arcs and that $x(t)$ always reaches Σ through sliding along Σ_1 or Σ_2 and that it leaves Σ at generic tangential exit points as in Definition 1. Then, the monodromy matrix associated to γ has two Floquet multipliers equal to* 0 *and one equal to* 1.

Proof. Without loss of generality we can assume that the monodromy matrix associated to γ is the one given in (17). Clearly (17) has an eigenvalue at 1, since $X(t_4, 0)f_{\Sigma_2^+}(\bar{x}) = f_{\Sigma_2^+}(\bar{x})$. That the monodromy matrix in (17) has two eigenvalues at 0 will follow from these facts (which are immediate consequences of the previous forms of saltation matrices, and recalling that $\nabla h_2^T f_\Sigma(x_3) = 0$ and $\nabla h_1^T f_{\Sigma_1^+}(x_1) = 0$):

i) $\ker(S_{\Sigma_1^+}(x_2)) = \text{span}\{(f_{\Sigma_1^+} - f_4)(x_2)\}$,

ii) $\ker(S_\Sigma(x_3)) = \text{span}\{(f_\Sigma - f_{\Sigma_1^+})(x_3)\}$,

iii) $\text{range}(S_{\Sigma_1^+}(x_2)) = \nabla h_1^\perp(x_2)$, where $\nabla h_1^\perp(x_2)$ denotes the orthogonal complement of ∇h_1 at x_2, and

iv) $T_{x(t)}\Sigma_1 = \nabla h_1^\perp(x(t))$.

Now, let us look at the various pieces in (17).

Since $X_4(t_2,t_1)X_{\Sigma_2^+}(t_1,0)S(\bar{x})$ is non singular, there is always a (unique up to normalization) vector v such that $X_4(t_2,t_1)X_{\Sigma_2^+}(t_1,0)S(\bar{x})v$ is in $\text{span}\{(f_{\Sigma_1^+} - f_4)(x_2)\}$. This shows that $X(t_4,0)$ has one eigenvalue at 0. To find a second (independent) eigenvector associated to the eigenvalue 0, because of ii), we just need to show that there exists a vector v such that $X_{\Sigma_1^+}(t_3,t_2)S_{\Sigma_1^+}(x_2)X_4(t_2,t_1)X_{\Sigma_2^+}(t_1,0)S(\bar{x})v$ is in $\text{span}\{(f_\Sigma - f_{\Sigma_1^+})(x_3)\}$. To do this, we use iii) above together with Lemma 13, use fact iv) above, and the fact that $(f_\Sigma - f_{\Sigma_1})(x_3) \in \nabla h_1^\perp(x_3)$. □

Remark 15. *In* [8, Lemma 3.1]*, the authors show that,* generically*, the saltation matrix obtained when the trajectory reaches Σ from one of the R_i's (and not upon sliding on one of Σ_1 or Σ_2) has a 2-dimensional kernel. This, together with the proof of Theorem 14, allows us to say that when there is a periodic orbit comprising a sliding motion on a codimension* 2 *surface Σ as in this work, generically there will be two Floquet multipliers at* 0.

4. Numerical Examples

We illustrate computation of a periodic trajectory, and Theorem 14, with two examples.

Example 16. *In the following example there are two co-dimension 1 discontinuity surfaces, namely $\Sigma_1 = \{x \in \mathbb{R}^3 \ : \ x_2 - 0.2 = 0\}$ and $\Sigma_2 = \{x \in \mathbb{R}^3 \ : \ x_3 - 0.4 = 0\}$. Let Σ denote their intersection, that is Σ is just the x_1-axis. Σ_1 and Σ_2 divide the phase space in four subregions denoted as follows $R_1 = \{x \in \mathbb{R}^3 \ : \ x_2 < 0.2, \ x_3 < 0.4\}$, $R_2 = \{x \in \mathbb{R}^3 \ : \ x_2 < 0.2, \ x_3 > 0.4\}$, $R_3 = \{x \in \mathbb{R}^3 \ : \ x_2 > 0.2, \ x_3 < 0.4\}$ and $R_4 = \{x \in \mathbb{R}^3 \ : \ x_2 > 0.2, \ x_3 > 0.4\}$. In each subregion R_j we have the vector fields below*

$$
\begin{aligned}
&f_1(x) = \begin{pmatrix} (x_2+x_3)/2 \\ -x_1 + \frac{1}{1.2-x_2} \\ -x_1 + \frac{1}{1.4+\eta-x_3} \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} (x_2+x_3)/2 \\ -x_1 + \frac{1}{1.2-x_2} \\ -x_1 - \frac{1}{0.6+x_3}, \end{pmatrix}, \\
&f_3(x) = \begin{pmatrix} (x_2+x_3)/2 \\ -x_1 - \frac{1}{0.8+x_2} \\ -x_1 + \frac{1}{1.4-x_3}, \end{pmatrix}, \quad f_4(x) = \begin{pmatrix} (x_2+x_3)/2 + x_1(x_2+0.8)(x_3+0.6) \\ -x_1 - \frac{1}{0.8+x_2} \\ -x_1 - \frac{1}{0.6+x_3} \end{pmatrix},
\end{aligned}
\tag{18}
$$

where we will take $\eta = 0.1$, or $\eta = -0.1$, in the expression for f_1. For $\eta = 0.1$, Σ is attractive for $-1 < x_1 < 4.2/4.6$, and for $\eta = -0.1$, Σ is attractive for $-1 < x_1 < 1$.

For each of these two values of η, the system has a periodic solution that slides on Σ upon sliding on Σ_1 ($\eta = 0.1$) or Σ_2 ($\eta = -0.1$). Theorem 14 applies and the corresponding periodic orbit has Floquet multipliers equal to $(1,0,0)$. Here we want to numerically compute the Floquet multipliers of the periodic orbit for different choices of vector fields in the Filippov convex combination. We first use the bilinear vector field during sliding motion on Σ.

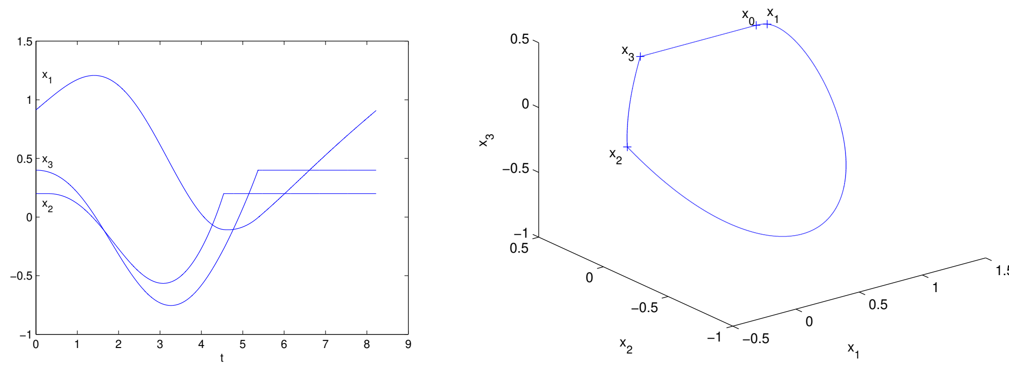

FIGURE 3. Periodic solution when $\eta = 0.1$.

The numerical solution of the system is computed by an event driven method based on the classic 4th order Runge Kutta scheme (RK4). That is, in the smooth regions R_j, $j = 1, 2, 3, 4$, the solution is approximated with RK4, the entry event points to a discontinuity surface are computed by a zero finding routine (we used the secant method), and during the sliding motion a projection technique is used to constraint the numerical solution to the surface, while the exit points from a surface will be found again by a zero finding routine (see for instance [9] *for details). The monodromy matrices have been approximated by solving the corresponding linearized problems by the explicit Euler method. Throughout, the time step is fixed at the value $\Delta t = 10^{-3}$.*

For $\eta = 0.1$, we start with initial condition $x_0 = (4.2/4.6, 0.2, 0.4)$. This is a generic tangential exit point satisfying Definition 1 (ii)-(a) and hence the vector field $f_{\Sigma_1^-}$ is tangent to Σ at x_0. The corresponding solution slides on Σ_1^- until it reaches the first event point $x_1 \approx (1, 0.2, 0.3855)$, then it enters the region R_1. After evolving in R_1, the solution hits Σ_1 transversally at the second event point $x_2 \approx (-0.1067, 0.2, -0.2706)$, and starts sliding on Σ_1^-, until it hits Σ (again, transversally) at the third event point $x_3 \approx (-0.0042, 0.2, 0.4)$. It then starts sliding on Σ upon reaching $x_4 = x_0$. In Table 1 we show (the first five digits of) the computed Floquet multipliers m_1, m_2, m_3: as expected, two Floquet multipliers are 0, and one is 1 (within numerical accuracy). In Figure 3 we show the three components of the solution during one period and the periodic trajectory; the event points x_1, x_2, x_3, x_4, are marked by '+' in the right plot. Notice that, for this example, all Filippov vector fields exit smoothly at x_4.

We repeated the experiment for $\eta = -0.1$. Here we choose the initial condition $x_0 = (1, 0.2, 0.4)$, that is a generic tangential exit point satisfying Definition 1 (i)-(a). Hence the vector field $f_{\Sigma_2^-}$ is tangent to Σ at x_0 and the corresponding solution slides on Σ_2^- until it reaches the first event point $x_1 \approx (1.1111, 0.1761, 0.4)$, where the solution enters the region R_1. After evolving in R_1, the trajectory hits Σ_2 transversally at the second event point $x_2 \approx (-0.1456, -0.6535, 0.4)$, and starts sliding on Σ_2^- until it enters Σ (again, transversally) at the third event point $x_3 \approx (-0.0739, 0.2, 0.4)$. At this point the solution starts slides on Σ upon reaching $x_4 = x_0$. The computed Floquet multipliers for the periodic orbit are shown in Table 1. In Figure 4 we show the three components of the solution during one period and the periodic trajectory.

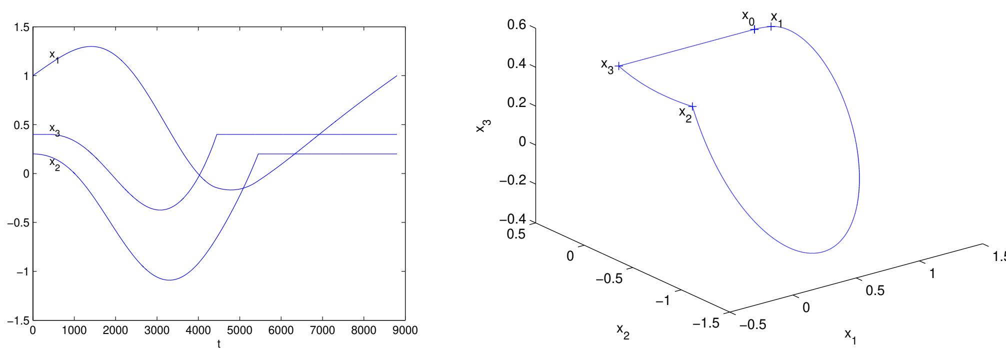

FIGURE 4. Periodic solution when $\eta = -0.1$.

TABLE 1. Computed Floquet multipliers

	m_1	m_2	m_3
$\eta = 0.1$	1.0032	0.0	0.0
$\eta = -0.1$	1.0020	0.0	0.0

We now consider a different Filippov vector field on Σ, still for the case of $\eta = -0.1$. Any Filippov sliding vector on Σ, i.e. an admissible solution of (6), can be written in terms of λ_4 as:

$$\lambda_1 = c(\lambda_4 + x_1)\ , \quad \lambda_2 = 0.5(1 + x_1) - c(\lambda_4 + x_1)\ , \quad \lambda_3 = 0.5(1 - x_1) - \lambda_4\ ,$$

where $c = \frac{2}{\frac{1}{1+\eta}+1}$. Notice that any choice of (admissible) λ_4 guarantees a smooth exit at $x_0 = (1, 0.2, 0.4)$. For our experiment we choose

$$\lambda_4 = -0.0745(x_1 - 1),$$

which gives a smoothly varying Filippov vector field on the portion of Σ of interest, further exiting Σ at $x_1 = 1$. Using this vector field, and repeating the previous computations, we obtain the same periodic orbit and once more find Floquet multipliers $\{0, 0, \approx 1\}$, as predicted by our theory. The only difference, as predicted in Section 2.1.1, is the travel "time" on Σ: for the bilinear method, it takes $t \approx 3.35$, while in the present case it takes $t \approx 3.45$.

In the previous example, all Filippov vector fields exit Σ smoothly, hence, in computing the Floquet multipliers, we do not need to take into account the saltation matrix defined in Proposition 12. In the example below, instead, we modify the vector fields of Example 16, so that not all the vector fields in the Filippov convex combination exit Σ smoothly at a generic tangential exit point.

Example 17. *We use same notation as Example 16, and take f_1, f_3 and f_4 as there, with $\eta = 0.1$, but now take f_2 as*

$$f_2(x) = \begin{pmatrix} \frac{x_2+x_3}{2} \\ -x_1 + \frac{1}{1.2-x_2} \\ -x_1 + \frac{1}{0.65+x_3} \end{pmatrix}.$$

This problem has the same periodic orbit as the one described in Example 16 for $\eta = 0.1$*, and* Σ *is attractive for* $-1 < x_1 < 4.2/4.6$*.*

As initial condition we take $x_0 = (\frac{4.2}{4.6}\ \ 0.2\ \ 0.4)$*. This is a generic first order exit point from* Σ *into* Σ_1^-*. However,* $f_{\Sigma_1^-}(x_0)$ *is not the unique vector field in the Filippov convex combination at* x_0*. For our simulations, we choose two different smooth vector fields on* Σ*: the bilinear vector field,* f_B*, that exhibits a smooth exit at* x_0*, and a second vector field,* f_F*, that does not lead to a smooth exit. To define* f_F*, notice that the coefficients for the Filippov convex combination can be expressed in function of* x_1*, for* $-1 < x_1 < 4.2/4.6$*, as*

$$\lambda_1 = \frac{1+x_1}{2} - \lambda_2\ ,\ \ \lambda_2 = \frac{2\lambda_4 + \frac{2.3}{2.2}x_1 - \frac{2.1}{2.2}}{\frac{2.05}{1.05} - \frac{2.1}{1.1}}\ ,\ \ \lambda_3 = -\lambda_4 + \frac{1-x_1}{2},$$

and we use $\lambda_4 = \frac{-2.15}{4.2}x_1 + \frac{2.05}{4.2}$*. The corresponding vector field is not equal to* $f_{\Sigma_1^-}(x_0)$ *at* x_0*, nonetheless we use Assumption 2 and the solution exits* Σ_1^- *at* x_0 *even if just continuously. This notwithstanding, requiring the trajectory (which is the same as in Figure 3) to exit at* x_0 *gives a saltation matrix at* x_0 *as in Proposition 12. In Table 2 we show (the first five digits of) the computed Floquet multipliers in the case of the bilinear sliding vector with a smooth exit and in the case of the Filippov sliding vector field* f_F *with a nonsmooth exit (*$\lambda_4 = 0.0207$*). As expected, they coincide.*

Table 2. Computed Floquet multipliers

	m_1	m_2	m_3
f_B	1.0043	0.0	0.0
f_F	1.0055	0.0	0.0

5. Conclusions

In this work we examined smooth sliding motion (in the Filippov sense) on a co-dimension 2 surface Σ, intersection of two smooth co-dimension 1 surfaces in $\mathbb{R}^3$. In this case, it is well understood that there is an **algebraic ambiguity** on how to select a Filippov sliding vector field. Our main result has been to show that –under appropriate assumptions– this algebraic ambiguity **bears no dynamical impact**, since we have shown that all sliding motions are orbitally equivalent. We further examined the implications of this fact insofar as the stability properties of a periodic orbit having a portion of its trajectory on Σ. Again, we proved that there is no impact on stability caused by the different sliding vector fields. In conclusion, what at first appeared to be an ill-posed problem, in fact is not. More pragmatically, our results imply that –as long as our assumptions are verified, and if the interest is to understand the dynamics of the discontinuous system– one can select whichever smooth sliding vector field is most convenient. At the same time, our results also clarify the importance of following a trajectory which exits at first order exit points.

We believe that our effort is a first step towards removal of the algebraic ambiguity inherent in the selection of a Filippov vector field when sliding motion takes place on a discontinuity surface of co-dimension 2. And, although it does not appear easy to generalize our results (and techniques) to the case of state space $\mathbb{R}^n$, with $n > 3$, or to higher co-dimenion singularity surfaces, this very problem of "understanding whether and when the algebraic ambiguity in the selection process of a sliding vector field bears a dynamical impact" is one that we believe ought to be addressed.

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