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Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday

SOME MULTIPLICITY RESULTS OF HOMOCLINIC SOLUTIONS FOR SECOND ORDER HAMILTONIAN SYSTEMS

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Abstract. We look for homoclinic solutions $q : \mathbb{R} \to \mathbb{R}^N$ to the class of second order Hamiltonian systems

$$-\ddot{q} + L(t)q = a(t)\nabla G_1(q) - b(t)\nabla G_2(q) + f(t) \quad t \in \mathbb{R},$$

where $L : \mathbb{R} \to \mathbb{R}^{N\times N}$ and $a, b : \mathbb{R} \to \mathbb{R}$ are positive bounded functions, $G_1, G_2 : \mathbb{R}^N \to \mathbb{R}$ are positive homogeneous functions and $f : \mathbb{R} \to \mathbb{R}^N$. Using variational techniques and the Pohozaev fibering method, we prove the existence of infinitely many solutions if $f \equiv 0$ and the existence of at least three solutions if f is not trivial but small enough.

Keywords: second order Hamiltonian systems, homoclinic solutions, variational methods, compact embeddings.

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1. INTRODUCTION

In this paper, we consider the following second order Hamiltonian system

$$-\ddot{q}(t) = \nabla W(t, q(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $q : \mathbb{R} \to \mathbb{R}^N$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $\nabla W(t, q)$ denotes the gradient of W with respect to $q \in \mathbb{R}^N$ for every $t \in \mathbb{R}$. We look for homoclinic solutions of (1.1), i.e., solutions q to (1.1) such that $q(t) \to 0$ as $|t| \to +\infty$.

The existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been recognized from Poincaré [12]. From their existence, one may, under certain conditions, infer the existence of chaos nearby on the bifurcation behavior of periodic orbits.

In the last four decades, the existence and multiplicity of homoclinic solutions for (1.1) have been widely studied via the critical points theory. If $W(t,q)$ is periodic in t, many authors obtained homoclinic solutions by passing to the limit of periodic solutions of the approximating problems (see e.g., [2, 3, 13, 14, 17]). If $W(t,q)$ is not periodic in t, the problem is quite different because of the lack of compactness of the Sobolev embeddings (see e.g., [4, 9, 15, 16]).

Aim of this paper is to consider a non-periodic superquadratic function $W(t,q)$ verifying suitable conditions in order to generalize or to give complementary results to the ones known in literature. Following an idea of Omana and Willem [9], we will overcome the lack of compactness by taking W of a particular form in such a way that the space in which we work is a subspace of $H^1(\mathbb{R},\mathbb{R}^N)$ compactly embedded in $L^2(\mathbb{R},\mathbb{R}^N)$ (see also [15]).

More precisely, we study the problem

$$-\ddot{q}(t) + L(t)q(t) = a(t)\nabla G_1(q) - b(t)\nabla G_2(q) + f(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $L : \mathbb{R} \to \mathbb{R}^{N\times N}$, $a, b : \mathbb{R} \to \mathbb{R}$, $G_1, G_2 : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^N$. In the following, $(\cdot,\cdot)$ denotes the usual inner product in $\mathbb{R}^N$ and $|\cdot|$ the induced norm. Moreover, $|\cdot|_\mu$ denotes the usual norm in the Lebesgue space $L^\mu(\mathbb{R},\mathbb{R}^N)$ and μ' the conjugate exponent to μ.

Throughout the paper we assume

(L_1) $L \in C(\mathbb{R},\mathbb{R}^{N\times N})$, $L(t)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ satisfying
$$l(t) = \inf_{|q|=1} (L(t)q,q) \to +\infty \quad \text{as } |t| \to +\infty;$$

(H_1) $G_i \in C^1(\mathbb{R}^N,\mathbb{R})$, $i = 1,2$, and there exist two constants μ,ν with $1 < \nu \le \max\{2,\nu\} < \mu$ such that
$$G_1(sq) = |s|^\mu G_1(q) \quad \text{and} \quad G_2(sq) = |s|^\nu G_2(q) \quad \text{for every } (s,q) \in \mathbb{R}\times\mathbb{R}^N,$$
that is, G_1 is homogeneous of degree μ and G_2 is homogeneous of degree ν;

(H_2) $G_i(q) > 0$ for every $i = 1,2$ and for every $q \in \mathbb{R}^N$ with $|q| = 1$;

(H_3) $a \in C(\mathbb{R},\mathbb{R})$ is a nontrivial positive bounded function;

(H_4) $b \in C(\mathbb{R},\mathbb{R})$ is a positive bounded function; furthermore, $b \in L^{\frac{2}{2-\nu}}(\mathbb{R})$ if $1 < \nu < 2$.

If $f = 0$, the problem (1.2) is symmetric and we can state the following multiplicity result.

Theorem 1.1 (Symmetric case)**.** *Let $f = 0$. Suppose that (L) and (H_1)–(H_4) hold. Then, problem* (1.2) *admits infinitely many homoclinic solutions.*

On the other hand, if $f \neq 0$, the problem loses its symmetry. Anyway, denoting by c_μ the embedding constant of $H^1(\mathbb{R},\mathbb{R}^N)$ in $L^\mu(\mathbb{R},\mathbb{R}^N)$, we are able to state again a multiplicity result as follows.

Theorem 1.2 (Nonsymmetric case)**.** *Let $f \in L^{\mu'}(\mathbb{R}, \mathbb{R}^N)$. Assume that (L) and (H_1)–(H_4) hold. Then, there exists a strictly positive constant $\delta = \delta(\mu, c_\mu, a, G_1)$ such that, if $|f|_{\mu'} < \delta$, problem* (1.2) *has at least three homoclinic solutions.*

The following result gives an additional property for the homoclinic solutions found in the previous theorems.

Corollary 1.3. *Assume that all the hypotheses of Theorem 1.1 or Theorem 1.2 are verified. Moreover, suppose that $L(t)$ satisfies the additional condition*

(L_2) *there exist $\overline{a} > 0$ and $\overline{r} > 0$ such that one of the following is true:*
(i) *$L \in C^1(\mathbb{R}, \mathbb{R}^{N\times N})$ and $|L'(t)| \le \overline{a}|L(t)|$ for any $|t| \ge \overline{r}$*
or
(ii) *$L \in C^2(\mathbb{R}, \mathbb{R}^{N\times N})$ and $|L''(t)| \le \overline{a}|L(t)|$ for any $|t| \ge \overline{r}$,*
where $L'(t) = \frac{d}{dt}L(t)$ and $L''(t) = \frac{d^2}{dt^2}L(t)$.

Then, the homoclinic solutions to system (1.2) *are such that $\dot{q}(t) \to 0$ as $|t| \to +\infty$.*

First of all, we list some properties of the homogeneous functions which are useful in the following.

Proposition 1.4. *Let $G \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\alpha > 0$.*

(i) *G is homogeneous of degree α if and only if G is even and positively homogeneous of degree α;*
(ii) *if G is positively homogeneous of degree α, then there exist $m_G, M_G \in \mathbb{R}$ such that*
$$m_G|q|^\alpha \le G(q) \le M_G|q|^\alpha \quad \text{for all } q \in \mathbb{R}^N \tag{1.3}$$
with $m_G = \min_{|q|=1} G(q)$ and $M_G = \max_{|q|=1} G(q)$. Clearly, $m_G > 0$ if $G(q) > 0$ for all $q \in \mathbb{R}^N$, $|q| = 1$.
(iii) *if G is homogeneous of degree $\alpha > 1$ then $\frac{\partial G}{\partial q_i}$ is positively homogeneous of degree $\alpha - 1$ for all $i = 1, \dots, N$; moreover, $(\nabla G(q), q) = \alpha G(q)$ for all $q \in \mathbb{R}^N$;*
(iv) *if G is homogeneous of degree $\alpha > 1$ and $G(q) > 0$ for all $q \in \mathbb{R}^N$, $|q| = 1$, then there exist $m_{\nabla G}, M_{\nabla G} > 0$ such that*
$$m_{\nabla G}\,|q|^{\alpha-1} \le |\nabla G(q)| \le M_{\nabla G}\,|q|^{\alpha-1} \quad \text{for all } q \in \mathbb{R}^N \tag{1.4}$$
with $m_{\nabla G} = \min_{|q|=1} |\nabla G(q)|$ and $M_{\nabla G} = \max_{|q|=1} |\nabla G(q)|$;

Proof.

(i) It is enough to observe that
$$G(-q) = G((-1)q) = |-1|^\alpha G(q) = G(q) \quad \text{for every } q \in \mathbb{R}^N.$$

(ii) The proof of (1.3) follows by the fact that G is continuous and strictly positive on the compact set $\{q \in \mathbb{R}^N : |q| = 1\}$ and
$$G(q) = G\left(|q|\frac{q}{|q|}\right) = |q|^\alpha G\left(\frac{q}{|q|}\right) \quad \text{for all } q \in \mathbb{R}^N,\ q \neq 0.$$

(iii) For simplicity, we consider the case $N = 1$. If $N \geq 2$, we can apply similar arguments involving partial derivatives. It results

$$\begin{aligned} G'(sq) &= \lim_{h\to 0} \frac{G(sq+h) - G(sq)}{h} = \lim_{h\to 0} \frac{|s|^\alpha}{s} \frac{G(q+h/s) - G(q)}{h/s} \\ &= |s|^{\alpha-2} s\, G'(q) = s^{\alpha-1} G'(q) \end{aligned}$$

for every $q \in \mathbb{R}^N$ and $s > 0$. For the equality, known as Euler's Theorem, see e.g. [1, p. 699].

(iv) From (iii), it follows that $|\nabla G|$ is positively homogeneous of degree $\alpha - 1$. Hence, (ii) implies (1.4). We remark that $m_{\nabla G} > 0$ since, otherwise, a $\overline{q} \in \mathbb{R}^N$, $|\overline{q}| = 1$ exists such that $|\nabla G(\overline{q})| = 0$, i.e., $\nabla G(\overline{q}) = 0$. Therefore, Euler's Theorem implies that $G(\overline{q}) = \frac{1}{\alpha}(\nabla G(\overline{q}), \overline{q}) = 0$, thus contradicting the hypothesis $G(q) > 0$ for every $q \in \mathbb{R}^N$, $|q| = 1$.

□

From now on, according to (ii) and (iv) of Proposition 1.4, we denote by m_{G_i}, M_{G_i}, $m_{\nabla G_i}$ and $M_{\nabla G_i}$ the constants associated to the functions G_i and ∇G_i for $i = 1, 2$. Moreover, thanks to (H_3) and (H_4), we can consider the real constants

$$m_a = \inf_{\mathbb{R}} a(t), \quad m_b = \inf_{\mathbb{R}} b(t), \quad M_a = \sup_{\mathbb{R}} a(t), \quad M_b = \sup_{\mathbb{R}} b(t).$$

Clearly, it results

$$M_a > 0, \quad 0 \leq m_a \leq M_a \quad \text{and} \quad 0 \leq m_b \leq M_b.$$

Remark 1.5. If $b(t) \neq 0$ and $\nu < 2 < \mu$, the potential $V(t,q) = a(t)G_1(q) - b(t)G_2(q)$ is a combination of a subquadratic term and a superquadratic one. Therefore, $V(t,q)$ does not satisfy the global Ambrosetti–Rabinowitz condition ((AR) condition):

(AR) there exists $p > 2$ such that $0 < pV(t,q) \leq (\nabla V(t,q), q)$ for every $(t,q) \in \mathbb{R} \times \mathbb{R}^N, q \neq 0$.

Indeed, fixing $\bar{t} \in \mathbb{R}$ such that $b(\bar{t}) > 0$ and $q \in \mathbb{R}^N$ with $|q|$ small enough, it is

$$V(\bar{t}, q) \leq M_a\, M_{G_1}\, |q|^\mu - b(\bar{t})\, m_{G_2}\, |q|^\nu = |q|^\nu \left(M_a\, M_{G_1}\, |q|^{\mu-\nu} - b(\bar{t})\, m_{G_2} \right) < 0$$

while, choosing $p = \mu$, from Euler's Theorem for all $q \in \mathbb{R}^N$ it is

$$\begin{aligned} &(\nabla V(t,q), q) - \mu V(t,q) \\ &= a(t)(\nabla G_1(q), q) - b(t)(\nabla G_2(q), q) - \mu a(t) G_1(q) + \mu b(t) G_2(q) \\ &= (\mu - \nu) b(t) G_2(q) \\ &\geq m_b(\mu - \nu) G_2(q) \geq 0. \end{aligned}$$

Remark 1.6. We point out that (AR) condition, used in all the works mentioned above, implies that $V(t,q)$ has a superquadratic growth as $|q| \to \infty$. Recently, if $f = 0$, there are some papers dealing with superquadratic potentials $V(t,q)$ no verifying (AR) condition, but satisfying a set of hypotheses different from ours (see e.g., [19] and [20]). On the other hand, if $f \neq 0$, the existence of a homoclinic solution has been proved in [7, 18] if the potential V does not verify (AR) condition. Anyway, to the best of our knowledge, the only multiplicity result in the inhomogeneous case is contained in [16], where the potential $V(t,q)$ verifies (AR) condition.

Finally, we recall that a different kind of multiplicity results for periodic Hamiltonian systems is contained in [5, 6].

The paper is organized as follows. In Section 2 we introduce the variational formulation of the problem and we recall the Pohozaev fibering method. In Section 3 we prove Theorem 1.1 in the symmetric case and in Section 4 we show Theorem 1.2 in the non-symmetric case.

2. VARIATIONAL FRAMEWORK AND POHOZAEV FIBERING METHOD

In order to introduce the variational structure of problem (1.2), let us consider the Hilbert space

$$E = \left\{ q \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} \left(|\dot{q}(t)|^2 + (L(t)q(t), q(t)) \right) dt < +\infty \right\}$$

endowed with the scalar product

$$(q_1, q_2)_E = \int_{\mathbb{R}} \Big((\dot{q_1}(t), \dot{q_2}(t)) + (L(t)q_1(t), q_2(t)) \Big) dt \quad \text{for any } q_1, q_2 \in E$$

and the associated norm

$$\|q\|_E = \left(\int_{\mathbb{R}} \left(|\dot{q}(t)|^2 + (L(t)q(t), q(t)) \right) dt \right)^{\frac{1}{2}} \quad \text{for any } q \in E.$$

Observe that, by (L_1) the space E is continuously embedded in $H^1(\mathbb{R}, \mathbb{R}^N)$, hence

$$E \hookrightarrow L^p(\mathbb{R}, \mathbb{R}^N) \quad \text{for any } p \in [2, +\infty).$$

In the following we denote by $|\cdot|_p$ the usual norm on $L^p(\mathbb{R}, \mathbb{R}^N)$ for every $p \in [2, +\infty]$ and by $(E', \|\cdot\|_{E'})$ the normed dual space of E.

As ensured by Omana and Willem [9, Lemma 1] (see also Salvatore [16, Proposition 3.1]), we have

$$E \hookrightarrow\hookrightarrow L^p(\mathbb{R}, \mathbb{R}^N) \quad \text{for any } p \in [2, +\infty).$$

Now, let us consider the functional $J_f : E \to \mathbb{R}$ defined by

$$J_f(q) = \frac{1}{2}\|q\|_E^2 - \int_{\mathbb{R}} a(t)G_1(q(t))\,dt + \int_{\mathbb{R}} b(t)G_2(q(t))\,dt - \int_{\mathbb{R}} (f(t), q(t))\,dt$$

for any $q \in E$. Under our assumptions, we get J_f is of class C^1 on E with differential $J_f' : E \to E'$ defined as

$$\begin{aligned}\left\langle J_f'(q), h\right\rangle_E &= \int_{\mathbb{R}} \Big(\big(\dot{q}(t), \dot{h}(t)\big) + (L(t)q(t), h(t)) \Big)\,dt - \int_{\mathbb{R}} a(t)(\nabla G_1(q(t)), h(t))\,dt \\ &\quad + \int_{\mathbb{R}} b(t)(\nabla G_2(q(t)), h(t))\,dt - \int_{\mathbb{R}} (f(t), h(t))\,dt \\ &= (q,h)_E - \int_{\mathbb{R}} a(t)(\nabla G_1(q(t)), h(t))\,dt + \int_{\mathbb{R}} b(t)(\nabla G_2(q(t)), h(t))\,dt \\ &\quad - \int_{\mathbb{R}} (f(t), h(t))\,dt\end{aligned}$$

for any $q, h \in E$. Therefore, homoclinic solutions to (1.2) can be found as critical points of the functional J_f on E.

Remark 2.1. We note that, if $L(t)$ satisfies only the assumption (L_1), we are not able to prove that critical points of J_f verify the property $\dot{q} \to 0$ as $|t| \to \infty$. Anyway, this condition holds under additional assumptions on L (see Corollary 1.3).

In order to prove our results, we will use the spherical fibering method introduced by Pohozaev ([10,11]), so for completeness we recall it in the following.

Let Y be a real Banach space and J a C^1 functional on $Y \setminus \{0\}$. We associate with J a functional $\widetilde{J}$ defined on $\mathbb{R} \times Y$ by

$$\widetilde{J}(t, v) = J(tv) \quad \text{for any } (t, v) \in \mathbb{R} \times Y.$$

Denoted by S the unit sphere in Y, the following result holds (see [11, Theorem 1.2.1]).

Theorem 2.2. *Let Y be a real Banach space with norm differentiable on $Y \setminus \{0\}$ and let $(t, v) \in (\mathbb{R} \setminus \{0\}) \times S$ be a conditionally stationary point of the functional $\widetilde{J}$ on $\mathbb{R} \times S$. Then, the vector $u = tv$ is a non zero stationary point of the functional J, i.e. $J'(u) = 0$.*

In other words, any critical point (t, v) of $\widetilde{J}$ restricted on $(\mathbb{R} \setminus \{0\}) \times S$ generates the free non trivial critical point $u = tv$ of J and vice-versa, that is, the equation

$$J'(u) = 0, \quad u \neq 0$$

is equivalent to the system

$$\begin{cases} \widetilde{J}_t'(t, v) = 0, \\ \widetilde{J}_v'(t, v) = 0 \end{cases}$$

for $\|v\| = 1$. In the following we will call the first scalar equation of the previous system the "bifurcation equation".

3. PROOF OF THEOREM 1.1: SYMMETRIC CASE

In the symmetric case $f = 0$, by Proposition 1.4 (i) we have that the energy functional J_0 is even, where $J_0 : E \to \mathbb{R}$ is defined by

$$J_0(q) = \frac{1}{2}\int_{\mathbb{R}} \left(|\dot{q}(t)|^2 + (L(t)q(t), q(t))\right)\, dt - \int_{\mathbb{R}} a(t)G_1(q(t))\, dt + \int_{\mathbb{R}} b(t)G_2(q(t))\, dt$$

for any $q \in E$. According to the spherical fibering method, we look for critical points $q \in E$ of the functional J_0 of the type

$$q = sw, \quad \text{where } s \in \mathbb{R},\ w \in E,\ \|w\|_E = 1.$$

Thus, the functional J_0 can be extended to the space $\mathbb{R} \times E$ by setting

$$\begin{aligned}\widetilde{J}_0(s, w) = J_0(sw) &= \frac{s^2}{2}\|w\|_E^2 - \int_{\mathbb{R}} a(t)G_1(sw(t))\, dt + \int_{\mathbb{R}} b(t)G_2(sw(t))\, dt \\ &= \frac{s^2}{2}\|w\|_E^2 - |s|^\mu \int_{\mathbb{R}} a(t)G_1(w(t))\, dt + |s|^\nu \int_{\mathbb{R}} b(t)G_2(w(t))\, dt.\end{aligned}$$

Plainly, the restriction of $\widetilde{J}_0$ on $\mathbb{R} \times S$, $S = \{v \in E : \|v\|_E = 1\}$, becomes

$$\widetilde{J}_0(s, w) = \frac{s^2}{2} - |s|^\mu \int_{\mathbb{R}} a(t)G_1(w(t))\, dt + |s|^\nu \int_{\mathbb{R}} b(t)G_2(w(t))\, dt$$

for any $w \in S$, therefore if $s \neq 0$, the bifurcation equation $(\widetilde{J}_0)'_s(s, w) = 0$ takes the form

$$s - \mu\,|s|^{\mu-2}s \int_{\mathbb{R}} a(t)G_1(w(t))\, dt + \nu\,|s|^{\nu-2}s \int_{\mathbb{R}} b(t)G_2(w(t))\, dt = 0$$

or, equivalently,

$$1 - \mu\,|s|^{\mu-2} \int_{\mathbb{R}} a(t)G_1(w(t))\, dt + \nu\,|s|^{\nu-2} \int_{\mathbb{R}} b(t)G_2(w(t))\, dt = 0. \tag{3.1}$$

It is not difficult to observe that for any $w \in S$ equation (3.1) has exactly two nontrivial solutions $\pm s(w)$. Indeed, setting

$$\varphi_w(s) = 1 - \mu\,|s|^{\mu-2} \int_{\mathbb{R}} a(t)G_1(w(t))\, dt + \nu\,|s|^{\nu-2} \int_{\mathbb{R}} b(t)G_2(w(t))\, dt$$

for $s \neq 0$, it results

$$\lim_{|s|\to+\infty} \varphi_w(s) = -\infty$$

since $\nu < \mu$.

Moreover,

$$\lim_{s\to 0}\varphi_w(s)=\begin{cases}+\infty & \text{if } 1<\nu<2,\\ 1+2\displaystyle\int_{\mathbb{R}} b(t)G_2(w(t))\,dt & \text{if } \nu=2,\\ 1 & \text{if } \nu>2.\end{cases}$$

Then, φ_w has at least two zeros. More precisely, φ_w admits exactly two zeros since for $s\neq 0$ it follows

$$\varphi_w'(s)=-\mu(\mu-2)|s|^{\mu-4}s\int_{\mathbb{R}} a(t)G_1(w(t))\,dt+\nu(\nu-2)|s|^{\nu-4}s\int_{\mathbb{R}} b(t)G_2(w(t))\,dt.$$

Clearly, the equation $\varphi_w'(s)=0$ does not admit solutions if $1<\nu\leq 2$ while if $\nu>2$ it has two solutions

$$\overline{s}(w)=\pm\left(\frac{\nu(\nu-2)\int_{\mathbb{R}} b(t)G_2(w(t))\,dt}{\mu(\mu-2)\int_{\mathbb{R}} a(t)G_1(w(t))\,dt}\right)^{1/(\mu-\nu)}.$$

The functional $\hat{J}_0(w)=\widetilde{J}_0(s(w),w)$ on the unit sphere S reduces to

$$\hat{J}_0(w)=\frac{s^2(w)}{2}-|s(w)|^{\mu}\int_{\mathbb{R}} a(t)G_1(w(t))\,dt+|s(w)|^{\nu}\int_{\mathbb{R}} b(t)G_2(w(t))\,dt.$$

From equation (3.1) we have

$$\hat{J}_0(w)=\left(\frac{1}{2}-\frac{1}{\mu}\right)s^2(w)+\nu\left(\frac{1}{\nu}-\frac{1}{\mu}\right)|s(w)|^{\nu}\int_{\mathbb{R}} b(t)G_2(w(t))\,dt.$$

Since $b(t)\geq 0$ and $G_2(w)\geq 0$, we get $\int_{\mathbb{R}} b(t)G_2(w(t))\,dt\geq 0$ and then $\hat{J}_0$ is bounded from below since it is the sum of two non-negative terms.

At this point, we prove that $\hat{J}_0$ is weakly continuous on S. Let $\{w_n\}\subset S$ be a weakly convergent sequence to $w\in E$. Since $E\hookrightarrow\hookrightarrow L^p(\mathbb{R},\mathbb{R}^N)$ for any $p\geq 2$, it follows that $w_n\to w$ in $L^p(\mathbb{R},\mathbb{R}^N)$ for any $p\geq 2$. Now, we prove that, as $n\to+\infty$,

$$\int_{\mathbb{R}} a(t)G_1(w_n(t))\,dt\to\int_{\mathbb{R}} a(t)G_1(w(t))\,dt, \tag{3.2}$$

$$\int_{\mathbb{R}} b(t)G_2(w_n(t))\,dt\to\int_{\mathbb{R}} b(t)G_2(w(t))\,dt. \tag{3.3}$$

Indeed, from (H_1), (H_3) and the Lagrange Theorem for any $n\in\mathbb{N}$ and for any $t\in\mathbb{R}$ there exists $\theta_n(t)\in[0,1]$ such that

$$\begin{aligned}
&\left|\int_{\mathbb{R}} a(t)G_1(w_n(t))\,dt - \int_{\mathbb{R}} a(t)G_1(w(t))\,dt\right| \\
&\leq M_a \int_{\mathbb{R}} |G_1(w_n(t)) - G_1(w(t))|)\,dt \qquad (3.4)\\
&= M_a \int_{\mathbb{R}} \left|\Big(\nabla G_1\Big(w_n(t) + \theta_n(t)(w(t) - w_n(t))\Big)\Big)\cdot(w_n(t) - w(t))\right| dt.
\end{aligned}$$

Now, by Proposition 1.4 (iv) and applying the Hölder inequality to the last integral in (3.4) we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}} a(t)G_1(w_n(t))\,dt - \int_{\mathbb{R}} a(t)G_1(w(t))\,dt\right| \\
&\leq M_a \int_{\mathbb{R}} \left|\nabla G_1\Big(w_n + \theta_n(w - w_n)\Big)\right| |w_n - w|\,dt \\
&\leq M_a\, M_{\nabla G_1} \int_{\mathbb{R}} |w_n + \theta_n(w - w_n)|^{\mu-1}\,|w_n - w|\,dt \\
&\leq M_a\; M_{\nabla G_1}\,|w_n + \theta_n(w - w_n)|_\mu^{\mu-1}\,|w - w_n|_\mu \\
&\leq M_a\, M_{\nabla G_1}\, 2^{\mu-2}\,\left(|w_n|_\mu^{\mu-1} + |w - w_n|_\mu^{\mu-1}\right)\,|w - w_n|_\mu \\
&\leq c_1\,|w_n - w|_\mu,
\end{aligned}$$

where the term in the last line goes to zero since $w_n \to w$ in $L^\mu(\mathbb{R}, \mathbb{R}^N)$. Therefore, we conclude that (3.2) is satisfied. In a similar way, we can prove (3.3) if $\nu \geq 2$. On the other hand, if $1 < \nu < 2$, from (H_1), (H_4) and using the Lagrange Theorem and the Hölder inequality, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}} b(t)G_2(w_n)\,dt - \int_{\mathbb{R}} b(t)G_2(w)\,dt\right| \\
&\leq \int_{\mathbb{R}} b(t)\left|\nabla G_2\Big(w_n + \theta_n(t)(w - w_n)\Big)\right| |w_n - w|\,dt \\
&\leq c_2 \int_{\mathbb{R}} b(t)\left(|w_n|^{\nu-1} + |w - w_n|^{\nu-1}\right)\,|w_n - w|\,dt \\
&= c_2 \left(\int_{\mathbb{R}} b(t)\,|w_n|^{\nu-1}\,|w_n - w|\,dt + \int_{\mathbb{R}} b(t)\,|w - w_n|^{\nu}\,dt\right) \\
&\leq c_2\,|b|_{\frac{2}{2-\nu}}\,\left(|w_n|_{\frac{2}{\nu-1}}\,|w_n - w|_2 + |w_n - w|_2^{\nu}\right),
\end{aligned}$$

where the term in the last line goes to zero since $w_n \to w$ in $L^2(\mathbb{R},\mathbb{R}^N)$ and $\{|w_n|_{\frac{2}{\nu-1}}\}$ is bounded as $\frac{2}{2-\nu} > 2$. Thus, (3.3) is satisfied.

From the implicit functions theorem the sequence $s(w_n)$ of solutions of equation (3.1) converges to the corresponding solution $s(w)$, therefore

$$\hat{J}_0(w_n) \to \hat{J}_0(w).$$

By the Weierstrass theorem, $\hat{J}_0$ attains its minimum at a point $\overline{w} \in B$. Now, it remains to prove that $\overline{w} \in S$. Indeed, by using Euler's Theorem and the bifurcation equation (3.1), we get

$$\begin{aligned}
&\frac{d}{d\theta}\hat{J}_0(\theta\, w)\\
&= \frac{d}{d\theta}\Big(\frac{s^2(\theta w)}{2} - |s(\theta w)|^\mu \int_{\mathbb{R}} a(t)G_1(\theta w(t))\,dt + |s(\theta w)|^\nu \int_{\mathbb{R}} b(t)G_2(\theta w(t))\,dt\Big)\\
&= s(\theta w)\,\frac{d}{d\theta}\,(s(\theta w)) - \mu\,|s(\theta w)|^{\mu-2}\,s(\theta w)\frac{d}{d\theta}\,(s(\theta w)) \int_{\mathbb{R}} a(t)G_1(\theta w(t))\,dt\\
&\quad - \frac{|s(\theta w)|^\mu}{\theta} \int_{\mathbb{R}} a(t)\,(\nabla G_1(\theta w(t)), \theta w(t))\;dt\\
&\quad + \nu\,|s(\theta w)|^{\nu-2}\,s(\theta w)\frac{d}{d\theta}\,(s(\theta w)) \int_{\mathbb{R}} b(t)G_2(\theta w(t))\,dt\\
&\quad + \frac{|s(\theta w)|^\nu}{\theta} \int_{\mathbb{R}} b(t)\,(\nabla G_2(\theta w(t)), \theta w(t))\;dt\\
&= -\mu\,\frac{|s(\theta w)|^\mu}{\theta} \int_{\mathbb{R}} a(t)G_1(\theta w(t))\,dt + \nu\,\frac{|s(\theta w)|^\nu}{\theta} \int_{\mathbb{R}} b(t)G_2(\theta w(t))\,dt\\
&\quad + \frac{d}{d\theta}\,(s(\theta w))\,\Big(s(\theta w) - \mu\,|s(\theta w)|^{\mu-2}s(\theta w) \int_{\mathbb{R}} a(t)G_1(\theta w(t))\\
&\quad + \nu\,|s(\theta w)|^{\nu-2}s(\theta w) \int_{\mathbb{R}} b(t)G_2(\theta w(t))\Big)\\
&= -\mu\,\frac{|s(\theta w)|^\mu}{\theta} \int_{\mathbb{R}} a(t)G_1(\theta w(t))\,dt + \nu\,\frac{|s(\theta w)|^\nu}{\theta} \int_{\mathbb{R}} b(t)G_2(\theta w(t))\,dt\\
&= -\frac{|s(\theta w)|^2}{\theta} < 0
\end{aligned}$$

for every $\theta \in]0,1]$. Thus, $\hat{J}_0(\theta\, w)$ decreases with respect to $\theta \in [0,1]$ and the minimum is attained for $\theta = 1$, that is, $\min \hat{J}_0(\theta\, w)$ is achieved on S so $\overline{w} \in S$. According to the fibering method, we get $\pm s(\overline{w})\overline{w}$ are two solutions of problem (1.2) with $f = 0$.

Now, since $\hat{J}_0$ is even, bounded from below, weakly continuous and of class C^1 on S, by the theory of Lusternik–Schnirelmann introduced in [8], it follows that $\hat{J}_0$

admits a sequence of conditionally critical points $w_1, w_2, \ldots, w_n, \ldots \in S$ such that $\hat{J}_0(w_n) \to +\infty$ as $n \to +\infty$. From Theorem 2.2 we conclude that problem (1.2) in the case $f = 0$ has a sequence of distinct solutions $\pm q_1, \pm q_2, \ldots \pm q_n, \ldots$ with $q_n = s(w_n)w_n$ and $J_0(q_n) \to +\infty$ as $n \to +\infty$.

4. PROOF OF THEOREM 1.2: NON-SYMMETRIC CASE

In this section we consider the case $f \neq 0$. The energy functional J_f extended to $\mathbb{R} \times E$ becomes

$$\widetilde{J}_f(s, w) = \widetilde{J}_0(s, w) - s \int_{\mathbb{R}} (f(t), w(t))\, dt$$

and its restriction to $\mathbb{R} \times S$ is

$$\widetilde{J}_f(s, w) = \frac{s^2}{2} - |s|^\mu \int_{\mathbb{R}} a(t) G_1(w(t))\, dt + |s|^\nu \int_{\mathbb{R}} b(t) G_2(w(t))\, dt - s \int_{\mathbb{R}} (f(t), w(t))\, dt.$$

The bifurcation equation involves

$$\begin{aligned} &\frac{\partial \widetilde{J}_f}{\partial s}(s, w) = 0 \iff \\ &s - \mu\, |s|^{\mu-2} s \int_{\mathbb{R}} a(t) G_1(w(t))\, dt + \nu\, |s|^{\nu-2} s \int_{\mathbb{R}} b(t) G_2(w(t))\, dt = \int_{\mathbb{R}} (f(t), w(t))\, dt. \end{aligned} \tag{4.1}$$

At this point, we prove that equation (4.1) admits at least three different roots $s_i(w)$ for $i = 1, 2, 3$ if f is sufficiently small. Let $\psi_w : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$\psi_w(s) = s - \mu\, |s|^{\mu-2} s \int_{\mathbb{R}} a(t) G_1(w(t))\, dt + \nu\, |s|^{\nu-2} s \int_{\mathbb{R}} b(t) G_2(w(t))\, dt$$

for any $s \in \mathbb{R}$. Clearly, ψ_w is odd, $\lim_{s \to +\infty} \psi_w(s) = -\infty$ and $\psi'_w(s) = 0$ has exactly two distinct solutions $\pm s(w)$. Indeed, for $s > 0$ by derivation we obtain

$$1 - \mu(\mu - 1)\, s^{\mu-2} \int_{\mathbb{R}} a(t)\, G_1(w(t))\, dt + \nu(\nu - 1)\, s^{\nu-2} \int_{\mathbb{R}} b(t) G_2(w(t))\, dt = 0. \tag{4.2}$$

This equation is similar to (3.1) in Section 3. It results

$$\lim_{s \to +\infty} \psi'_w(s) = -\infty$$

being $\nu < \mu$. Moreover,

$$\lim_{s \to 0} \psi'_w(s) = \begin{cases} +\infty & \text{if } 1 < \nu < 2, \\ 1 + 2 \displaystyle\int_{\mathbb{R}} b(t) G_2(w(t))\, dt & \text{if } \nu = 2, \\ 1 & \text{if } \nu > 2. \end{cases}$$

Now, we note that

$$(\psi'_w)'(s)$$
$$= -\mu(\mu-1)(\mu-2)s^{\mu-3}\int_{\mathbb{R}} a(t)G_1(w(t))\,dt + \nu(\nu-1)(\nu-2)s^{\nu-3}\int_{\mathbb{R}} b(t)G_2(w(t))\,dt.$$

Therefore, if $1<\nu\leq 2$ it results $(\psi'_w)'(s)<0$ for all $s>0$ while if $\nu>2$ it follows

$$(\psi'_w)'(s)=0 \Longleftrightarrow \overline{s}(w)=\pm\left(\frac{\nu(\nu-1)(\nu-2)\displaystyle\int_{\mathbb{R}} b(t)G_2(w(t))\,dt}{\mu(\mu-1)(\mu-2)\displaystyle\int_{\mathbb{R}} a(t)G_1(w(t))\,dt}\right)^{1/(\mu-\nu)}.$$

In particular, it follows that ψ'_w has exactly two zeros then ψ_w has a local minimum and a local maximum such that $M_w=-m_w$. Thus, the bifurcation equation (4.1)

$$\psi_w(s)=\int_{\mathbb{R}}(f(t),w(t))\,dt$$

has three distinct solutions if

$$\left|\int_{\mathbb{R}}(f(t),w(t))\,dt\right|<M_w.$$

We are not able to evaluate explicitly M_w but we can observe that

$$\psi_w(s)\geq\overline{\psi}_w(s)=s-\mu\,s^{\mu-1}\int_{\mathbb{R}} a(t)G_1(w(t))\,dt$$

for every $w\in S$ and $s\geq 0$ since $b(t)\geq 0$ and $G_2(w(t))\geq 0$. Direct calculations show that, denoting by $\overline{M}_w$ the local maximum of $\overline{\psi}_w$, it is

$$\begin{aligned}\overline{M}_w&=\overline{\psi}_w\left(\mu(\mu-1)\int_{\mathbb{R}} a(t)G_1(w(t))\,dt\right)^{-\frac{1}{\mu-2}}\\&=(\mu-2)(\mu-1)^{-\frac{\mu-1}{\mu-2}}\left(\mu\int_{\mathbb{R}} a(t)G_1(w(t))\,dt\right)^{-\frac{1}{\mu-2}}\end{aligned}$$

Since $M_w\geq\overline{M}_w$, if we suppose

$$\sup_{w\in S}\left(\left|\int_{\mathbb{R}}(f(t),w(t))\,dt\right|\left(\int_{\mathbb{R}} a(t)G_1(w(t))\,dt\right)^{\frac{1}{\mu-2}}\right)<(\mu)^{-\frac{1}{\mu-2}}(\mu-2)(\mu-1)^{-\frac{\mu-1}{\mu-2}},\tag{4.3}$$

the bifurcation equation admits three isolated smooth branches of solutions $s_i = s_i(w)$ for $i = 1, 2, 3$. We note that inequality (4.3) holds if $f \in L^{\mu'}(\mathbb{R}, \mathbb{R}^N)$ with $|f|_{\mu'}$ small. Indeed,

$$\left| \int_{\mathbb{R}} (f(t), w(t))\, dt \right| \left(\int_{\mathbb{R}} a(t) G_1(w(t))\, dt \right)^{\frac{1}{\mu-2}} \leq |f|_{\mu'}\, |w|_{\mu}\, \left(M_a\, M_{G_1}\, |w|_{\mu}^{\mu} \right)^{\frac{1}{\mu-2}}$$
$$\leq |f|_{\mu'}\, \left(M_a\, M_{G_1}\, c_{\mu}^{2\mu-2} \right)^{\frac{1}{\mu-2}},$$

where we have exploited the fact that $|w|_{\mu} \leq c_{\mu} \|w\|_E = c_{\mu}$ for all $w \in S$. Hence, it is enough to choose $|f|_{\mu'}$ such that

$$|f|_{\mu'}\, \left(M_a\, M_{G_1}\, c_{\mu}^{2\mu-2} \right)^{\frac{1}{\mu-2}} \leq (\mu)^{-\frac{1}{\mu-2}} (\mu-2)(\mu-1)^{-\frac{(\mu-1)}{\mu-2}},$$

that is,

$$|f|_{\mu'} \leq \delta(\mu, c_{\mu}, M_a, M_{G_1}) := (\mu)^{-\frac{1}{\mu-2}} (\mu-2)(\mu-1)^{-\frac{\mu-1}{\mu-2}}\, c_{\mu}^{-\frac{2(\mu-1)}{\mu-2}}\, \left(M_a\, M_{G_1} \right)^{-\frac{1}{\mu-2}}.$$

Now, let us consider the three induced functionals

$$\hat{J}_{f,i}(w) = \widetilde{J}_{f,i}(s_i(w), w) = \frac{1}{2}\, s_i^2(w) - |s_i(w)|^{\mu} \int_{\mathbb{R}} a(t) G_1(s_i(w(t))\, dt$$
$$+ |s_i(w)|^{\nu} \int_{\mathbb{R}} b(t) G_2(s_i(w(t))\, dt - s_i(w) \int_{\mathbb{R}} (f(t), w(t))\, dt$$

which are defined and distinct on $B \setminus \{0\}$. Reasoning in a similar way as in the case $f = 0$, it is possible to prove that for each $i = 1, 2, 3$ the functional

$$\widetilde{J}_{f,i}(w) = \hat{J}_0(w) - s_i(w) \int_{\mathbb{R}} (f(t), w(t))\, dt$$

is bounded from below and weakly continuous, then it achieves its minimum at a point $\overline{w}_i \in B$ with $s_i(\overline{w}_i) \neq 0$. Moreover, we can see that $\overline{w}_i \in S$ since, using again

Euler's Theorem and the bifurcation equation (4.1), it results

$$\begin{aligned}
\frac{d}{d\theta}\hat{J}_{f,i}(\theta\, w) &= \frac{d}{d\theta}\Big(\frac{1}{2}\, s_i^2(\theta w) - |s_i(\theta w)|^\mu \int_{\mathbb{R}} a(t)G_1(\theta(w(t))\, dt \\
&\quad + |s_i(\theta w)|^\nu \int_{\mathbb{R}} b(t)G_2(\theta(w(t))\, dt - s_i(\theta w) \int_{\mathbb{R}} (f(t), \theta\, w(t))\, dt\Big) \\
&= -\mu\frac{|s_i(\theta w)|^\mu}{\theta} \int_{\mathbb{R}} a(t)G_1(\theta\, w(t))\, dt + \nu\frac{|s_i(\theta w)|^\nu}{\theta} \int_{\mathbb{R}} b(t)G_2(\theta\, w(t))\, dt \\
&\quad - s_i(\theta w) \int_{\mathbb{R}} (f(t), w(t))\, dt + \frac{d}{d\theta}s(\theta w)\Big(s_i(\theta w) \\
&\quad - \mu|s_i(\theta w)|^{\mu-2}s_i(\theta w) \int_{\mathbb{R}} a(t)G_1(\theta\, w(t))\, dt \\
&\quad + \nu|s_i(\theta w)|^{\nu-2}s_i(\theta w) \int_{\mathbb{R}} b(t)G_2(\theta\, w(t))\, dt - \int_{\mathbb{R}} (f(t), \theta\, w(t))\, dt\Big) \\
&= -\frac{s_i^2(\theta w)}{\theta} < 0
\end{aligned}$$

for every $\theta \in]0, 1]$. Thus, $\min \hat{J}_{f,i}(\theta\, w)$ is achieved on S. Consequently, $w_i = s_i(\overline{w}_i)\overline{w}_i$ are three critical points of the functional J_f and then three solutions of the assigned equation (1.2). Since the sign of $s_i(\overline{w}_i)$ depends on the sign of $\int_{\mathbb{R}}(f(t), \overline{w}_i(t))\, dt$ we get

$$\int_{\mathbb{R}} (f(t), w_1(t))\, dt \le 0, \quad \int_{\mathbb{R}} (f(t), w_2(t))\, dt \ge 0 \quad \text{and} \quad \int_{\mathbb{R}} (f(t), w_3(t))\, dt \ge 0.$$

Proof of Corollary 1.3. It is enough to apply [4, Lemma 2.3]. □

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