Research Article

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Existence and multiplicity results for some Lane–Emden elliptic systems: Subquadratic case

Abstract: We study the nonlinear elliptic system of Lane–Emden type

$$\begin{cases} -\Delta u = \operatorname{sgn}(v)|v|^{p-1} & \text{in } \Omega, \\ -\Delta v = f(x,u) & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, $p > 1$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying suitable growth assumptions. Existence and multiplicity results are proved by means of a generalized Weierstrass Theorem and a variant of the Symmetric Mountain Pass Theorem.

Keywords: Nonlinear elliptic system of Lane–Emden type, subquadratic growth, fourth order elliptic equation, variational tools

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1 Introduction

In the last years many authors have studied elliptic systems of two coupled semilinear Poisson equations

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 2$, and $f, g : \mathbb{R} \to \mathbb{R}$ are given functions.

In the model case $g(s) = s^{p-1}$ and $f(s) = s^{q-1}$, $p, q > 1$ (here and in the following $s^\alpha = \operatorname{sgn}(s)|s|^\alpha$ denotes the odd extension of the power function), the previous problem is referred to as the Lane–Emden system because it is a natural extension of the classical Lane–Emden equation

$$-\Delta u = u^{p-1} \quad \text{in } \Omega,$$

arising in Astronomy. It has been proved that Lane–Emden type systems have nontrivial solutions for all $p, q > 1$ either in the so-called superquadratic but subcritical case, i.e.

$$1 - \frac{2}{N} < \frac{1}{p} + \frac{1}{q} < 1$$

(see [6, 8, 9, 13]) and in the subquadratic case, i.e.

$$\frac{1}{p} + \frac{1}{q} > 1$$

(see [4, 11]). On the contrary, if p and q belong to the critical hyperbola

$$\frac{1}{p} + \frac{1}{q} = 1 - \frac{2}{N},$$

because of the lack of compactness of the problem, non-existence of solutions has been stated in [15] and [20] by using Pohozaev type arguments.

However, as concerns the results of multiplicity, in literature there is a substantial difference between the subquadratic and the superquadratic case: indeed, different authors have found multiple solutions in the first case (see [2, 17, 18]) while, to our knowledge, no multiplicity results have been proved in the second case.

Therefore, aim of this paper is to state the existence of infinitely many solutions to the nonlinear elliptic system (1.1) in the subquadratic case. It is well known that the standard functional associated to system (1.1) is strongly indefinite in interpolation spaces of infinite dimension. In order to overcome this problem, following [7] we will apply a decoupling technique which works when one of the two nonlinear terms f or g is an increasing nonlinearity. For sake of simplicity, we restrict ourselves to the pure power case $g(s) = s^{p-1}$ and we study the nonlinear elliptic system

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \Omega, \\ -\Delta v = f(x,u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(f_1) f is a Carathéodory function (i.e., $f(\cdot, s)$ is measurable on Ω for all $s \in \mathbb{R}$ and $f(x,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$) such that

$$\sup_{|s|\le r} |f(\cdot, s)| \in L^\infty(\Omega) \quad \text{for all } r > 0,$$

(f_2) there exist two real constants l_- and l_+ small enough (see Remark 2.3) such that

$$\lim_{s\to-\infty} \frac{f(x,s)}{|s|^{\frac{1}{p-1}}} = l_- \quad \text{uniformly with respect to a.e. } x \in \Omega,$$

$$\lim_{s\to+\infty} \frac{f(x,s)}{|s|^{\frac{1}{p-1}}} = l_+ \quad \text{uniformly with respect to a.e. } x \in \Omega,$$

(f_3) $f(x, 0) = 0$ for a.e. $x \in \Omega$,

(f_4) there exist $q \in (1, \frac{p}{p-1})$, $C > 0$ and $\delta > 0$ such that

$$f(x,s)s \ge C|s|^q \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}, |s| \le \delta.$$

We prove the following existence result.

Theorem 1.1. *Let f verify assumptions (f_1) and (f_2). Then, system* (1.2) *admits a weak solution* $(\overline{u}, (-\Delta\overline{u})^{\frac{1}{p-1}})$. *Moreover, if f satisfies also (f_3) and (f_4), it is $\overline{u} \neq 0$.*

Now, in order to give a multiplicity result, let us consider the following additional conditions:

(f_5) $f(x,\cdot)$ is odd for a.e. $x \in \Omega$,

(f_6) there exists a constant $C_1 > 0$ such that

$$f(x,s)s \le C_1|s|^q \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}, |s| \le \delta.$$

In the following theorem, I denotes the energy functional associated to the problem (see Section 2 for its definition).

Theorem 1.2. *Suppose that (f_1), (f_4) and (f_5) hold. Moreover, if $p \ge \frac{N}{N-2}$, assume that also (f_6) holds. Then, system* (1.2) *has a sequence* $(\overline{u}_k, (-\Delta\overline{u}_k)^{\frac{1}{p-1}})$ *of solutions such that* $\overline{u}_k \neq 0$, $I(\overline{u}_k) \le 0$, $I(\overline{u}_k) \to 0$ *and* $\overline{u}_k \to 0$ *uniformly in Ω as $k \to +\infty$.*

Remark 1.3. Clearly, condition $1 < q < \frac{p}{p-1}$ in (f_4) is equivalent to $\frac{1}{p} + \frac{1}{q} > 1$, therefore taking in mind the model function

$$f(x,s) = ls^{\frac{1}{p-1}} + s^{q-1},$$

our results apply either in the subquadratic case if $l = 0$ and in the “asymptotically quadratic” case if $l \neq 0$ with l small enough. Then, as already pointed out, the multiplicity result stated in Theorem 1.2 is new while the existence result in Theorem 1.1 has been just proved, among other properties of the solution, in [11] by P. Felmer and S. Martinez even for a more general nonlinearity $f(x,s)$ and by D. Bonheure, E. M. Dos Santos and M. Ramos in [4, Theorems 1.5 and 1.6] only in the case $f(x,s) = s^{q-1}$.

Remark 1.4. Let us observe that, denoting

$$F(x,s) = \int_0^s f(x,t)\,dt,$$

from (f_4) and (f_6) by integration it follows that

$$F(x,s) \geq \frac{C}{q}|s|^q \quad \text{for a.e. } x \in \Omega \text{ and } s \in \mathbb{R},\ |s| \leq \delta, \tag{1.3}$$

$$F(x,s) \leq \frac{C_1}{q}|s|^q \quad \text{for a.e. } x \in \Omega \text{ and } s \in \mathbb{R},\ |s| \leq \delta.$$

We emphasize that, really, the hypotheses necessary to obtain the multiplicity result are only local since Theorem 1.2 still holds if we consider $f : \Omega \times [-\delta, \delta] \to \mathbb{R}$ an odd bounded Carathéodory function verifying (f_4) and, eventually, (f_6).

Example 1.5. From Theorem 1.1, system (1.2) admits a nontrivial solution if

$$f(x,s) = g(x) + h(s)$$

where g is a nontrivial bounded measurable function on Ω and h is a continuous function on $\mathbb{R}$ such that, for l_- and l_+ small enough, it results

$$\lim_{s \to -\infty} \frac{h(s)}{|s|^{\frac{1}{p-1}}} = l_- \quad \text{and} \quad \lim_{s \to +\infty} \frac{h(s)}{|s|^{\frac{1}{p-1}}} = l_+.$$

Example 1.6. System (1.2) has a nontrivial solution if

$$f(x,s) = g(x)h(s)$$

with g and h as in the previous example and moreover, if $h(0) = 0$, satisfying

$$g(x)h(s)s \geq C|s|^q \quad \text{for a.e. } x \in \Omega \text{ and for } |s| \text{ small enough.}$$

Furthermore, if we assume h odd with respect to $s \in \mathbb{R}$ and, if $p \geq \frac{N}{N-2}$,

$$g(x)h(s)s \leq C_1|s|^q \quad \text{for a.e. } x \in \Omega \text{ and for } |s| \text{ small enough,}$$

Theorem 1.2 guarantees the existence of infinitely many solutions to system (1.2).

The paper is organized as follows: In Section 2 we introduce the variational formulation of the problem and we recall a variant of the Symmetric Mountain Pass Theorem stated in [14]. In Section 3 we prove Theorem 1.1 and Theorem 1.2. In particular, in order to state the multiplicity result, we introduce a new modified problem which admits a sequence of solutions uniformly converging to zero. Finally, we prove that these solutions provide solutions to the original system (1.2).

Notations. We will use the following notations:

- $L^t(\Omega)$, with $1 \leq t \leq +\infty$ denotes the Lebesgue space with the usual norm $|\cdot|_t$,
- $W^{k,\sigma}(\Omega)$, with $k \in \mathbb{N}$, $\sigma \in \mathbb{R}$, $1 \leq k, \sigma \leq \infty$, is the usual Sobolev space equipped with the norm

$$\|u\|_{W^{k,\sigma}} = \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^\sigma \,dx + \int_\Omega |u|^\sigma dx \right)^{\frac{1}{\sigma}},$$

- $W_0^{1,\sigma}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{1,\sigma}}$ endowed with the equivalent norm

$$\|u\|_{W_0^{1,\sigma}} = \left(\int_\Omega |\nabla u|^\sigma \right)^{\frac{1}{\sigma}},$$

- $C_B(\Omega)$ is the space of the continuous bounded functions on Ω equipped with the usual norm $|\cdot|_\infty$,
- c denotes a real positive constant changing line from line.

2 Variational tools

Let $N \geq 2$ and $p > 1$. Arguing as in [7], it is possible to transform system (1.2) in an equivalent quasilinear scalar problem. Indeed, the system (1.2) can be rewritten as

$$\begin{cases} (-\Delta u)^{\frac{1}{p-1}} = v & \text{in } \Omega, \\ -\Delta v = f(x,u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

that is equivalent to the fourth order quasilinear elliptic equation

$$\begin{cases} -\Delta(-\Delta u)^{\frac{1}{p-1}} = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Clearly, if u is a weak solution of (2.1), we define weak solution of system (1.2) the couple $(u, (-\Delta u)^{\frac{1}{p-1}})$. In order to prove that problem (2.1) has a variational structure, let us consider the space

$$E = W^{2,\frac{p}{p-1}}(\Omega) \cap W_0^{1,\frac{p}{p-1}}(\Omega)$$

endowed with the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^{\frac{p}{p-1}}\, dx \right)^{\frac{p-1}{p}}$$

equivalent to the usual intersection norm equal to $\max\{\|u\|_{W^{2,\frac{p}{p-1}}}, \|u\|_{W_0^{1,\frac{p}{p-1}}}\}$. From now on, we will denote by $(E', \|\cdot\|_{E'})$ its dual space. Then, if we set

$$\left(\frac{p}{p-1}\right)^{**} = \begin{cases} \frac{Np}{(N-2)p-N} & \text{if } p > \frac{N}{N-2}, \\ +\infty & \text{if } 1 < p \leq \frac{N}{N-2}, \end{cases} \tag{2.2}$$

the Sobolev Embedding Theorems give the following result (see e.g. [5, Corollary 9.13]).

Proposition 2.1. *The following continuous embeddings hold:*
(A) *If $p > \frac{N}{N-2}$, i.e. $\frac{p}{p-1} < \frac{N}{2}$, then*

$$E \hookrightarrow L^t(\Omega) \quad \text{if } \frac{p}{p-1} \leq t \leq \left(\frac{p}{p-1}\right)^{**}.$$

(B) *If $p = \frac{N}{N-2}$, i.e. $\frac{p}{p-1} = \frac{N}{2}$, then*

$$E \hookrightarrow L^t(\Omega) \quad \text{if } \frac{p}{p-1} \leq t < \left(\frac{p}{p-1}\right)^{**}.$$

(C) *If $1 < p < \frac{N}{N-2}$, i.e. $\frac{p}{p-1} > \frac{N}{2}$, then*

$$E \hookrightarrow L^t(\Omega) \quad \text{if } \frac{p}{p-1} \leq t \leq \left(\frac{p}{p-1}\right)^{**}$$

and, for every $u \in E$,

$$|u(x) - u(x')| \leq C\|u\| \cdot |x - x'|^\alpha \quad \text{for a.e. } x, x' \in \Omega,$$

where α and C are suitable constants depending on p and N. Moreover,

$$E \hookrightarrow C_B(\Omega). \tag{2.3}$$

Proposition 2.2. *The following compact embeddings hold:*

$$E \hookrightarrow\hookrightarrow L^t(\Omega) \quad \text{if } \frac{p}{p-1} \leq t < \left(\frac{p}{p-1}\right)^{**}.$$

From now on, let us denote by k_t the embedding constant of E in $L^t(\Omega)$.

Remark 2.3. As we will see in Section 3, the constants l_- and l_+ in hypothesis (f_2) need to satisfy

$$l_- < \frac{1}{k_{\frac{p-1}{p}}} \quad \text{and} \quad l_+ < \frac{1}{k_{\frac{p-1}{p}}}.$$

Remark 2.4. Since $q < \frac{p}{p-1} < (\frac{p}{p-1})^{**}$, it is $E \hookrightarrow\hookrightarrow L^q(\Omega)$.

Remark 2.5. In the superquadratic but subcritical case $1 - \frac{2}{N} < \frac{1}{p} + \frac{1}{q} < 1$, it results

$$\frac{p}{p-1} < q < \left(\frac{p}{p-1}\right)^{**}$$

and existence and multiplicity results are found, respectively, by means of the Mountain Pass Theorem and the Symmetric Mountain Pass Theorem.

Before stating the variational principle, we prove the following result.

Lemma 2.6. *Assume that f satisfies (f_1) and (f_2). Then, for every $\varepsilon > 0$ there exist two real constant $b_\varepsilon, c_\varepsilon > 0$ such that*

$$|f(x,s)| \le b_\varepsilon |s|^{\frac{1}{p-1}} + c_\varepsilon, \tag{2.4}$$

$$f(x,s) \le (\max\{l_-, l_+\} + \varepsilon)|s|^{\frac{1}{p-1}} + c_\varepsilon, \tag{2.5}$$

$$F(x,s) \le \frac{p-1}{p}(\max\{l_-, l_+\} + \varepsilon)|s|^{\frac{p}{p-1}} + c_\varepsilon |s| \tag{2.6}$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$.

Proof. By the definition of limit, it is

$$\lim_{s\to-\infty} \frac{f(x,s)}{|s|^{\frac{1}{p-1}}} = l_- \iff \forall \varepsilon > 0\ \exists R'_\varepsilon > 0 \text{ such that } l_- - \varepsilon < \frac{f(x,s)}{|s|^{\frac{1}{p-1}}} < l_- + \varepsilon \text{ for a.e. } x \in \Omega \text{ and for all } s < -R'_\varepsilon,$$

$$\lim_{s\to+\infty} \frac{f(x,s)}{|s|^{\frac{1}{p-1}}} = l_+ \iff \forall \varepsilon > 0\ \exists R''_\varepsilon > 0 \text{ such that } l_+ - \varepsilon < \frac{f(x,s)}{|s|^{\frac{1}{p-1}}} < l_+ + \varepsilon \text{ for a.e. } x \in \Omega \text{ and for all } s > R''_\varepsilon.$$

Therefore, for every $\varepsilon > 0$ there exists an $R_\varepsilon = \max\{R'_\varepsilon, R''_\varepsilon\} > 0$ such that for a.e. $x \in \Omega$, for every $|s| > R_\varepsilon$:

$$\min\{l_-, l_+\} - \varepsilon < \frac{f(x,s)}{|s|^{\frac{1}{p-1}}} < \max\{l_-, l_+\} + \varepsilon.$$

Hence, taking $b_\varepsilon = \max\{|\min\{l_-, l_+\} - \varepsilon|, |\max\{l_-, l_+\} + \varepsilon|\} > 0$, for a.e. $x \in \Omega$, for all $|s| > R_\varepsilon$ it is

$$|f(x,s)| \le b_\varepsilon |s|^{\frac{1}{p-1}}.$$

Now, by (f_1), denoted $c_\varepsilon = \sup \operatorname{ess}\{|f(x,s)| : x \in \Omega,\ |s| \le R_\varepsilon\}$, we obtain inequalities (2.4) and (2.5) and, by integration, also (2.6). □

Now, it is possible to state the following variational principle.

Proposition 2.7. *Let f verify (f_1) and (f_2). Then, the weak solutions of the equation in (2.1) are the critical points of the energy functional defined on E by*

$$I(u) = \frac{p-1}{p} \int_\Omega |\Delta u|^{\frac{p}{p-1}}\, dx - \int_\Omega F(x,u)\, dx.$$

More precisely, $I \in C^1(E)$ and its differential $dI : E \to E'$ is defined as

$$dI(u)[\zeta] = \int_\Omega \left[(-\Delta u)^{\frac{1}{p-1}}(-\Delta \zeta) - f(x,u)\zeta\right] dx$$

for all $u, \zeta \in E$. Moreover, the function $u \mapsto f(\cdot, u(\cdot))$ is compact from E to E'.

Proof. Since Ω is bounded, the proof follows by classical arguments. It is standard to prove that the map $\varphi_0(u) = \frac{p-1}{p}\|u\|^{\frac{p}{p-1}}$ is of class $C^1(E)$ with Fréchet differential

$$d\varphi_0(u)[\zeta] = \int_\Omega (-\Delta u)^{\frac{1}{p-1}}(-\Delta\zeta)\,dx \quad \text{for all } u,\zeta \in E.$$

By (2.4) and Proposition 2.1 we have that $\varphi_1 \in C^1(E)$ with

$$d\varphi_1(u)[\zeta] = \int_\Omega f(x,u)\zeta\,dx \quad \text{for all } u,\zeta \in E;$$

moreover by Proposition 2.2 it is $E \hookrightarrow\hookrightarrow L^{\frac{p}{p-1}}(\Omega)$ and $E \hookrightarrow\hookrightarrow L^1(\Omega)$, therefore, by using again (2.4) it follows that $d\varphi_1 : E \to E'$ is compact (see [19, Theorem 1.22]). □

Now, we recall a suitable version stated by R. Kajikiya in [14] of the classical Symmetric Mountain Pass Theorem (see [1]).

Let X be an infinite-dimensional Banach space and let $J : X \to \mathbb{R}$ be a C^1 functional. Let us recall that J satisfies the Palais–Smale, briefly (PS), condition, if any (PS) sequence, i.e. any sequence $\{u_k\}$ in X such that $\{J(u_k)\}$ is bounded and $dJ(u_k) \to 0$ in X' as $k \to +\infty$, has a convergent subsequence.

For any integer k, let

$$\Gamma_k = \{A \subset X - \{0\} : A \text{ is closed and symmetric, } \gamma(A) \geq k\},$$

where, as usual, $\gamma(A)$ denotes the genus of the set A (for the definition and relative properties see e.g. [16]).

The following result has been proved in [14, Theorem 1].

Theorem 2.8. *Let $J \in C^1(X,\mathbb{R})$ satisfy*
(A_1) *J is even, bounded from below, $J(0) = 0$ and J satisfies the* (PS) *condition,*
(A_2) *for every $k \in \mathbb{N}$ there exists an $A_k \in \Gamma_k$ such that $\sup_{A_k} J(u) < 0$.*
Then, one of the following holds:
(i) *there exists a sequence $\{u_k\}$ such that $dJ(u_k) = 0$, $J(u_k) < 0$ and $\{u_k\}$ converges to zero,*
(ii) *there exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $dJ(u_k) = 0$, $J(u_k) = 0$, $u_k \neq 0$, $\lim_k u_k = 0$, $dJ(v_k) = 0$, $J(v_k) < 0$, $\lim_k J(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.*

Remark 2.9. In any case (i) or (ii), Theorem 2.8 gives the existence of a sequence $\{u_k\}$ of critical points such that $J(u_k) \leq 0$, $u_k \neq 0$, $\lim_k u_k = 0$ and, consequently, $\lim_k J(u_k) = 0$.

3 Proof of existence and multiplicity results

First, we prove the existence result.

Proof of Theorem 1.1. From inequality (2.6) and Sobolev embeddings, fixing $\varepsilon > 0$ small enough, there exists a constant $c_\varepsilon > 0$ such that

$$I(u) = \frac{p-1}{p}\int_\Omega |\Delta u|^{\frac{p}{p-1}}\,dx - \int_\Omega F(x,u)\,dx \geq \frac{p-1}{p}\big(1 - (\max\{l_-,l_+\} + \varepsilon)k_{\frac{p}{p-1}}\big)\|u\|^{\frac{p}{p-1}} - c_\varepsilon k_1\|u\|,$$

then, since $p/(p-1) > 1$, choosing l_-, l_+ as in Remark 2.3 it follows that I is bounded from below and coercive on the reflexive Banach space E.

Moreover, the functional $I = \varphi_0 - \varphi_1$ is weakly lower semicontinuous on E since φ_0 is weakly lower semi-continuous by the norm properties while φ_1 is weakly continuous as it is $C^1(E)$ and its derivative $d\varphi_1$ is compact by Proposition 2.7. Then, by a generalized Weierstrass Theorem there exists some $\overline{u} \in E$ such that $I(\overline{u}) = \min_{u\in E} I(u)$. Hence, the first part of the thesis follows by applying again Proposition 2.7.

Clearly, if (f_3) holds, problem (2.1) admits always the trivial solution $u = 0$ with $I(0) = 0$.

Anyway, under the additional assumption (f_4), condition (1.3) holds with $1 < q < \frac{p}{p-1}$, hence the solution $\overline{u}$ is nontrivial since, fixed $u_1 \in E \cap L^\infty(\Omega)$ with $u_1 \neq 0$, we get

$$I(\varepsilon u_1) = \frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\|u_1\|^{\frac{p}{p-1}} - \int_\Omega F(x, \varepsilon u_1)\,dx \le \frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\|u_1\|^{\frac{p}{p-1}} - \frac{C}{q}\varepsilon^q|u_1|_q^q < 0 = I(0)$$

for $\varepsilon > 0$ small enough. □

Before establishing our multiplicity result, we modify the term f by introducing a new function $\overline{f}$ satisfying the same hypotheses of f with (f_4), and eventually (f_6), globally with respect to s.

First, fixed $K \in \mathbb{R}$ with $0 < K < \delta < K + 1$, let us consider a cut-off function φ such that $0 \le \varphi(s) \le 1$, $\varphi(s) = 1$ if $|s| \le K$, $\varphi(s) = 0$ if $|s| \ge K + 1$ and φ is even, continuous and strictly decreasing on $K < |s| < K + 1$. Then, let us define

$$\overline{f}(x, s) = \varphi(s) f(x, s) + (1 - \varphi(s))R|s|^{q-2}s \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}.$$

It is possible to prove the following proposition.

Proposition 3.1. *Assume that f verifies assumptions (f_1), $(f_4)-(f_6)$. Then $\overline{f}$ satisfies (f_1), (f_2) with $l_- = l_+ = 0$ and (f_5). Moreover, for a suitable choice of the constant R, $\overline{f}$ satisfies also*

$(\overline{f}_4)$ *there exists a constant $\overline{C} > 0$ such that*

$$\overline{f}(x, s)s \ge \overline{C}|s|^q \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R},$$

$(\overline{f}_6)$ *there exists a constant $\overline{C}_1 > 0$ such that*

$$\overline{f}(x, s)s \le \overline{C}_1|s|^q \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}.$$

Proof. By (f_1) and the definition of φ it follows that $\overline{f}$ is a Carathéodory function and for all $r > 0$ it is

$$\sup_{|s|\le r} |\overline{f}(x, s)| \le \sup_{|s|\le r} |f(x, s)| + Rr^{q-1} < +\infty$$

for a.e. $x \in \Omega$, then also $\overline{f}$ verifies (f_1). Now, since for $|s| \ge K + 1$ it is $\overline{f}(x, s) = R|s|^{q-2}s$ with $q < \frac{p}{p-1}$ or equivalently $q - 1 < \frac{1}{p-1}$, it follows that

$$\lim_{|s|\to+\infty} \frac{\overline{f}(x, s)}{|s|^{\frac{1}{p-1}}} = \lim_{|s|\to+\infty} \frac{R|s|^{q-2}s}{|s|^{\frac{1}{p-1}}} = 0,$$

uniformly with respect to a.e. $x \in \Omega$, then (f_2) holds with $l_- = l_+ = 0$. Moreover, from (f_5) and the evenness of φ, we have that also $\overline{f}(x, \cdot)$ is odd for a.e. $x \in \Omega$.

In order to prove $(\overline{f}_4)$, let us point out that if $|s| \le K < \delta$, by (f_4) it is $\overline{f}(x, s)s = f(x, s)s \ge C|s|^q$ while if $|s| \ge K + 1$, it is $\overline{f}(x, s)s = R|s|^q \ge C|s|^q$ by choosing $R \ge C$. If $K \le |s| \le K + 1$, two cases can be occur:

- If $K \le |s| \le \delta$, by (f_4) and the monotonicity of φ it follows

$$\overline{f}(x, s)s \ge \varphi(\delta) f(x, s)s \ge \varphi(\delta)C|s|^q.$$

- If $\delta \le |s| \le K + 1$, let $m = \inf\text{ess}\{\varphi(s) f(x, s)s : x \in \Omega,\ \delta \le s \le K + 1\}$. Clearly, $m > -\infty$ by (f_1) and, by using again the monotonicity of φ, it is

$$\overline{f}(x, s)s \ge m + (1 - \varphi(\delta))R\,\delta^q \ge C(K + 1)^q \ge C|s|^q$$

choosing R large enough, more precisely

$$R \ge R_1 = \frac{C(K + 1)^q - m}{\delta^q(1 - \varphi(\delta))}.$$

Hence, $(\overline{f}_4)$ holds with $\overline{C} = \varphi(\delta)C$ if $R \ge \max\{C, \frac{C(K+1)^q - m}{\delta^q(1-\varphi(\delta))}\}$.

Similar arguments prove that, by (f_1) and (f_6), $\overline{f}$ satisfies $(\overline{f}_6)$ choosing R not too much large. More precisely, setting

$$M = \sup\text{ess}\{\varphi(s)f(x,s)s : x \in \Omega,\ \delta \le s \le K+1\}$$

and choosing

$$R < R_2 = \min\left\{C_1, \frac{C_1\delta^q - M}{(K+1)^q(1-\varphi(K+1))}\right\},$$

$(\overline{f}_6)$ holds with $\overline{C}_1 = C_1 + (1-\varphi(\delta))R$. Clearly, for C_1 large enough it results $R_1 < R_2$, hence for $R \in (R_1, R_2)$ we conclude that $\overline{f}$ satisfies $(\overline{f}_4)$ and $(\overline{f}_6)$ if f verifies respectively (f_4) and (f_6). □

At this point we can consider the new problem

$$\begin{cases} -\Delta(-\Delta u)^{\frac{1}{p-1}} = \overline{f}(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and the associated energy functional defined on E by

$$\overline{I}(u) = \frac{p-1}{p}\int_\Omega |\Delta u|^{\frac{p}{p-1}}\,dx - \int_\Omega \overline{F}(x,u)\,dx,$$

with

$$\overline{F}(x,t) = \int_0^t \overline{f}(x,s)\,ds.$$

Since $(\overline{f})$ verifies (f_1) and (f_2), by Proposition 2.7 it follows that $\overline{I} \in C^1(E)$ and its critical points are the weak solutions to problem (3.1). By integration, from $(\overline{f}_4)$ and from $(\overline{f}_6)$ we obtain

$$\overline{F}(x,s) \ge \frac{\overline{C}}{q}|s|^q \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}, \tag{3.2}$$

$$\overline{F}(x,s) \le \frac{\overline{C}_1}{q}|s|^q \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}.$$

The following proposition will be crucial in the statement of our multiplicity result since it allows us to obtain solutions of system (1.2) by studying problem (3.1).

Proposition 3.2. *Assume that f satisfies (f_1) and, if $p \ge \frac{N}{N-2}$ also (f_6). Let $\{u_k\}$ be a sequence in E of solutions of problem (3.1) such that $u_k \to 0$ in E as $k \to +\infty$. Thus, $u_k \to 0$ uniformly in Ω and therefore u_k solves problem (2.1) for all k large enough.*

Proof. If $1 < p < N/(N-2)$, from (2.3) it follows that $u_k \to 0$ uniformly in Ω.

Let $p > N/(N-2)$ (simpler arguments work if $p = N/(N-2)$). Since f satisfies conditions (f_1) and (f_6), Proposition 3.1 implies that $\overline{f}$ satisfies (f_1) and $(\overline{f}_6)$. We recall that $(u_k, (-\Delta u_k)^{\frac{1}{p-1}})$ is a solution of

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \Omega, \\ -\Delta v = \overline{f}(x,u) & \text{in } \Omega, \\ u = v = 0 & \text{in } \partial\Omega. \end{cases}$$

Fix $k \in \mathbb{N}$ and, according to (2.2), denote by $\sigma^{**} = \frac{N\sigma}{N-2\sigma}$ the critical exponent for the embedding of $W^{2,\sigma}(\Omega)$ in the spaces $L^t(\Omega)$. By $(\overline{f}_6)$ and $u_k \in E$, it follows that

$$\overline{f}(x,u_k) \in L^{(\frac{p}{p-1})^{**}\frac{1}{q-1}}(\Omega),$$

then [12, Theorem 9.15] guarantees the existence of a unique solution $w_k \in W^{2,(\frac{p}{p-1})^{**}\frac{1}{q-1}}(\Omega)$ of the problem

$$\begin{cases} -\Delta w_k = \overline{f}(x,u_k) & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by [12, Lemma 9.17] it is

$$\|w_k\|_{W^{2,(\frac{p}{p-1})^{**}\frac{1}{q-1}}} \le c\,|\overline{f}(x,u_k)|_{(\frac{p}{p-1})^{**}\frac{1}{q-1}}.$$

Hence, from $(\overline{f}_6)$, $u_k \to 0$ in $L^{(\frac{p}{p-1})^{**}}(\Omega)$ and Sobolev embeddings, we have

$$w_k \to 0 \quad \text{in } L^{((\frac{p}{p-1})^{**}\frac{1}{q-1})^{**}}(\Omega). \tag{3.3}$$

Since $w_k \in L^{((\frac{p}{p-1})^{**}\frac{1}{q-1})^{**}}(\Omega)$, applying again [12, Theorem 9.15] let $z_k \in W^{2,r}(\Omega)$ be the solution of

$$\begin{cases} -\Delta z_k = w_k^{p-1} & \text{in } \Omega, \\ z_k = 0 & \text{on } \partial\Omega, \end{cases}$$

with $r = ((\frac{p}{p-1})^{**}\frac{1}{q-1})^{**}\frac{1}{p-1}$. By [12, Lemma 9.17] it results

$$\|z_k\|_{W^{2,r}} \le c\,|w_k|_r; \tag{3.4}$$

hence (3.3), (3.4) and Sobolev embeddings imply

$$z_k \to 0 \quad \text{in } L^{r^{**}}(\Omega). \tag{3.5}$$

On the other hand, arguing as in the proof of [3, Proposition 4.1] (see also [10]) it results that

$$z_k = u_k \in L^{r^{**}}(\Omega) \quad \text{and} \quad w_k = (-\Delta u_k)^{\frac{1}{p-1}}.$$

Hence, (3.5) means that $u_k \to 0$ in $L^{r^{**}}(\Omega)$. Let us point out that direct calculations give, since $\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N}$,

$$r^{**} > \Big(\frac{p}{p-1}\Big)^{**}$$

Then, bootstrap arguments and Sobolev embeddings imply that $u_k \to 0$ uniformly in Ω.

Finally, let us point out that, taken $K > 0$ as in the definition of $\overline{f}$, there exists a $\overline{k} \in \mathbb{N}$ such that for every $k \ge \overline{k}$ one has $|u_k|_\infty \le K$, namely $|u_k(x)| \le K$ for every $x \in \Omega$ and for every $k \ge \overline{k}$. It follows that

$$\overline{f}(x,u_k(x)) = f(x,u_k(x)) \quad \text{and} \quad \overline{F}(x,u_k(x)) = F(x,u_k(x)),$$

therefore we have

$$\overline{I}(u_k) = I(u_k) \quad \text{and} \quad d\overline{I}(u_k) = dI(u_k),$$

hence by Proposition 2.7, u_k is a solution to problem (2.1) for every $k \ge \overline{k}$. □

At this point, we are able to prove our multiplicity result.

Proof of Theorem 1.2. Our aim is to apply Theorem 2.8 to the functional $\overline{I}$. Let us remark that, since by Proposition 3.1 the function $\overline{f}$ satisfies (f_1) and (f_2) with $l_- = l_+ = 0$, arguing as in the proof of Theorem 1.1 it follows that $\overline{I}$ is coercive and bounded from below; moreover, $\overline{I}(0) = 0$ and (f_5) implies that $\overline{I}$ is even. Let us point out that $\overline{I}$ satisfies the (PS) condition. Indeed, if $\{u_k\}$ is a (PS) sequence, $\{u_k\}$ is bounded by the coerciveness of $\overline{I}$. Thus, up to subsequence, there exists some $u \in E$ such that $u_k \rightharpoonup u$. Now, by Proposition 2.7 we have that the function $u \to \overline{f}(\cdot, u(\cdot))$ is compact from E to E' and, reasoning as in [10, Section 3] we conclude that $u_k \to u$ in E. Hence, $\overline{I}$ satisfies assumption (A_1) in Theorem 2.8.

Let us denote by $\{e_j\}$ a Schauder basis of the separable Banach space E. For $k \in \mathbb{N}$ fixed, let $E_k = \{e_1, \dots, e_k\}$ be a k-dimensional subspace of E. Since we are in finite dimension, there exists a constant $c_k > 0$ such that $\|u\| \le c_k |u|_q$ for every $u \in E_k$. Clearly,

$$c_k = \sup_{u \in E_k,\, |u|_q = 1} \|u\|,$$

hence the sequence $\{c_k\}$ is increasing. Moreover, $c_k \to +\infty$ if $k \to +\infty$. Indeed, if by contradiction $\{c_k\}$ was bounded, taken $u \in E$ and u_k the component of u along E_k, it is $u = \lim_k u_k$ in E and in $L^q(\Omega)$. Since for every k

it is $\|u_k\| \le c_k|u_k|_q$, passing to the limit we have $\|u\| \le c|u|_q$, for c suitable constant independent of u. Hence, $L^q(\Omega) \hookrightarrow E$ which gives the contradiction. Therefore, taken $u \in E_k$ from (3.2) we get

$$\begin{aligned}\bar{I}(u) &= \frac{p-1}{p}\int_\Omega |\Delta u|^{\frac{p}{p-1}}\,dx - \int_\Omega \bar{F}(x,u)\,dx \\ &\le \frac{p-1}{p}\|u\|^{\frac{p}{p-1}} - \frac{\bar{C}}{q}|u|_q^q \\ &\le \frac{p-1}{p}\|u\|^{\frac{p}{p-1}} - \frac{\bar{C}}{q}c_k^{-q}\|u\|^q \\ &\le -\frac{p-1}{p}\|u\|^{\frac{p}{p-1}}\end{aligned}$$

if we choose

$$2\frac{p-1}{p}\|u\|^{\frac{p}{p-1}} \le \frac{\bar{C}}{q}c_k^{-q}\|u\|^q$$

or equivalently

$$\|u\| \le \left(\frac{p\bar{C}}{2(p-1)qc_k^q}\right)^{\frac{1}{\frac{p}{p-1}-q}}.$$

Chosen

$$0 < d_k \le \left(\frac{p\bar{C}}{2(p-1)qc_k^q}\right)^{\frac{1}{\frac{p}{p-1}-q}} = r_k^{\frac{1}{\frac{p}{p-1}-q}},$$

it results that $r_k \to 0$ as $k \to +\infty$ and

$$\{u \in E_k : \|u\| = d_k\} \subset \left\{u \in E : \bar{I}(u) \le -\frac{1}{p}(p-1)d_k^{\frac{p}{p-1}}\right\}.$$

So, denoted by

$$A_k = \left\{u \in E : \bar{I}(u) \le -\frac{1}{p}(p-1)d_k^{\frac{p}{p-1}}\right\},$$

as $\bar{I}$ is even and continuous, A_k is closed and symmetric, i.e. $A_k \in \Gamma_k$ and, by well-known properties of the genus, $\gamma(A_k) \ge \gamma(E_k \cap S_{d_k}) = k$, where $S_{d_k} = \{u \in E : \|u\| = d_k\}$. Consequently, for every $k \in \mathbb{N}$ there exists an $A_k \in \Gamma_k$ such that

$$\sup_{A_k} \bar{I} \le -\frac{1}{p}(p-1)d_k^{\frac{p}{p-1}} < 0.$$

Hence, (A_2) holds and by Theorem 2.8 (see also Remark 2.9), there exists a sequence $\{\bar{u}_k\}$ in E such that $\bar{u}_k \ne 0$, $d\bar{I}(\bar{u}_k) = 0$, $\lim_k \bar{u}_k = 0$ and $\lim_k \bar{I}(\bar{u}_k) = 0$. Therefore, by Proposition 2.7 applied to the functional $\bar{I}$, $\{\bar{u}_k\}$ is a sequence of nontrivial solutions to (3.1) such that $\bar{I}(\bar{u}_k) \le 0$, $\lim_k \bar{u}_k = 0$ in E and $\lim_k \bar{I}(\bar{u}_k) = 0$.

Finally, by applying Proposition 3.2, $\bar{u}_k \to 0$ uniformly in Ω and for k large enough $\bar{u}_k$ is a solution to problem (2.1), hence $(\bar{u}_k, (-\Delta\bar{u}_k)^{\frac{1}{p-1}})$ is a solution to system (1.2) with $\bar{u}_k \ne 0$, $\lim_k \bar{u}_k = 0$ and $\lim_k I(\bar{u}_k) = 0$. □

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