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NECESSARY AND SUFFICIENT CONDITIONS FOR HYPERBOLICITY AND WEAK HYPERBOLICITY OF SYSTEMS WITH CONSTANT MULTIPLICITY, PART I

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Abstract. We consider a linear system of partial differential equations, whose principal symbol is hyperbolic with characteristics of constant multiplicities. We define necessary and sufficient invariant condition in order the Cauchy problem to be well-posed in $\mathcal{C}^\infty$. These conditions generalize the Levi conditions for scalar operators. The proof is based on the construction of a new non commutative determinant adapted to this case (and to the holomorphic case).

Contents

1. Introduction

Let $x = (x_0, x') = (x_0, x_1, \dots, x_n) \in \Omega$, and Ω be a neighborhood of 0 in $\mathbb{R}^{n+1}$. We consider an $N \times N$ linear first order system of differential operators

$$h(x, D) = a(x, D) + b(x) ,$$

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where $D = (D_0, D') = (D_0, D_1, \dots, D_n)$, $D_0 = \frac{\partial}{\partial x_0}$, $D_j = \frac{\partial}{\partial x_j}$, $a(x, \xi)$ is the principal symbol of h, $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \dots, \xi_n)$, a and b are $N \times N$ matrices with analytic coefficients.

In the proof of the necessity part, we suppose the coefficients are $\mathcal{C}^\infty$ [25]. We consider the Cauchy problem, for h,

$$\begin{aligned} h(x, D)u(x) &= f(x)\,, \\ u\big|_{x_0=\underline{x_0}} &= u_0(x')\,. \end{aligned} \tag{1.1}$$

Definition 1.1. The operator h is called hyperbolic if the Cauchy problem (1.1) is uniformly well-posed in $\mathcal{C}^\infty(\Omega)$, see [17].

In the analytic case, let $\mathcal{O}[\xi]$ be the ring of homogeneous polynomials in ξ, with coefficients from the ring of analytic germs in x at $x = 0$, and let $\mathcal{M}_N\big(\mathcal{O}[\xi]\big)$ be the set of the $N \times N$ matrices, whose entries belong to $\mathcal{O}[\xi]$. In $\mathcal{O}[\xi]$ we have the decomposition

$$\det a(x; \xi) = H_1^{m_1}(x; \xi) \cdots H_{\tau_0}^{m_{\tau_0}}(x; \xi)\,. \tag{1.2}$$

where H_τ, $\tau = 1, \dots, \tau_0$, are irreducible polynomials, homogeneous of degree s_τ in ξ, with analytic coefficients in x, and $m_1, \dots, m_{\tau_0} \in \mathbb{N}$ do not depend on $(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\}$.

We assume that $\det a(x; \xi)$ is a hyperbolic polynomial of *constant multiplicity*: the polynomial $H_1 \cdots H_{\tau_0}$ is strictly hyperbolic with respect to $(1, 0, \dots, 0)$ for any $x \in \Omega$, i.e. the solutions in ξ_0 of the equation

$$H_1(x; \xi_0, \xi') \cdots H_{\tau_0}(x; \xi_0, \xi') = 0$$

are real and distinct for any $(x, \xi') \in \Omega \times \mathbb{R}^n \setminus \{0\}$. This assumption is equivalent to the decomposition

$$\det a(x; \xi) = \prod_{j=1}^{r} \big(\xi_0 - \lambda_{(j)}(x; \xi')\big)^{m_{(j)}}\,,$$

where the $\lambda_{(j)}$'s are real analytic functions with

$$\inf_{x\in\Omega,\, |\xi'|=1,\, j\neq k} \big|\lambda_{(j)}(x; \xi') - \lambda_{(k)}(x; \xi')\big| \neq 0\,,$$

and the $m_{(j)}$'s are constant integers (see [15]).

To simplify the presentation, we assume that in (1.2) there is only one multiple factor H, of degree s and multiplicity m, and a simple factor K, of degree χ. The general case can be treated in a similar or equivalently way.

$$\begin{aligned} \det a(x; \xi) = {}& \big(\xi_0 - \lambda_{(1)}(x; \xi')\big)^m \cdots \big(\xi_0 - \lambda_{(s)}(x; \xi')\big)^m \\ & \times \big(\xi_0 - \lambda_{(s+1)}(x; \xi')\big) \cdots \big(\xi_0 - \lambda_{(s+\chi)}(x; \xi')\big)\,. \end{aligned} \tag{1.3}$$

We consider the problem: What are the conditions on a and b in order for h to be hyperbolic?

To answer this question, we use a set of conditions (L) previously defined in [26]. Before to state them, we recall some notation.

Let (H) be the prime ideal of $\mathcal{O}[\xi]$ defined by H, we consider $\mathcal{O}_{(H)}[\xi] := \mathcal{O}[\xi]/(H)$, the localized ring of $\mathcal{O}[\xi]$ with respect to (H). $\mathcal{O}_{(H)}[\xi]$ is a principal ring, whose elements are the fractions whose denominators are not divisible by

H; its ideals are generated by the powers of H. In $\mathcal{O}_{(H)}[\xi]$ the matrix $a(x,\xi)$ is equivalent to the diagonal matrix:

$$\operatorname{diag}\big[H^p, H^{q_1}, \dots, H^{q_\ell}, 1, \dots, 1\big],$$

where $p, q_1, \dots, q_\ell$ are such that

$$p \ge q_1 \ge \dots \ge q_\ell > 0\,, \quad p + q = m\,, \quad q := q_1 + \dots + q_\ell\,.$$

The sequence $(p, q_1, \dots, q_\ell)$ will be called the *type* of the operator. In other words, the minors of a of dimension $m-k$ are divisible by $H^{q_k+\dots+q_\ell}$ [4, §4, no. 6, Prop. 6]. Let A be the cofactor matrix of a, so that

$$aA = Aa = \det a I = H^m K I_N\,. \tag{1.4}$$

Since A is divisible by H^q, we set

$$\mathscr{A} := \frac{1}{H^q} A\,;$$

The entries of $\mathscr{A}(x;\xi)$ are polynomials in ξ, with analytic coefficients in x, homogeneous of degree $\mu_0 := ps + \chi - 1$. From (1.4), we have

$$a\mathscr{A} = \mathscr{A} a = H^p K I_N\,. \tag{1.5}$$

We denote

$$\mu_j := \begin{cases} \mu_0 + j\gamma + \Big(\sum_{1\le k\le j} q_k - j\Big)s\,, & \text{for } 0 \le j \le \ell\,,\\ \mu_0 + j\gamma + (q-\ell)s\,, & \text{for } j \ge \ell+1\,,\end{cases}$$

where $\gamma := s + \chi - 1$.

For a scalar or matrix-valued differential or classical pseudo-differential operator $\Lambda'(x;D)$, of order $\le \nu$, we denote by $\Lambda = \sigma_\nu(\Lambda')$ the homogeneous symbol of order ν, which is equal to the principal part of Λ', if Λ' is of order ν, and 0 if not. It is clear that σ_ν is an additive function. Conversely, to a matrix symbol $\Lambda(x;\xi)$ of polynomials or homogeneous symbols of order ν we associate matrix valued operators denoted by $\Lambda'(x;D)$ so that $\sigma_\nu(\Lambda') = \Lambda$. If $\Lambda(x;\xi)$ is scalar, we associate to Λ a matrix operator $\Lambda'(x;D)$, such that $\sigma_\nu(\Lambda') = \Lambda I_N$, I_N being the $N\times N$ identity matrix.

Thus, for example, H' is any $N\times N$ matrix operator of order s such that $\sigma_s(H') = HI_N$, K' is any $N\times N$ matrix operator of order χ such that $\sigma_\chi(K') = KI_N$, $\mathscr{A}'$ is any $N\times N$ matrix operator of order μ_0 such that $\sigma_{\mu_0}(\mathscr{A}') = \mathscr{A}$.

We are ready to state conditions (L), which are defined by induction.

(L_1) There exist differential operators $\mathscr{A}'$, H', and K', and a symbol $\Lambda_1(x,\xi) \in \mathcal{M}_N\big(\mathcal{O}[\xi]\big)$ whose entries are homogeneous of degree μ_1 in ξ such that

$$S_0 \equiv \mathscr{A}\sigma_{\mu_0}(h\mathscr{A}' - H'^p K') = H^{p-q_1}\,\Lambda_1\,. \tag{1.6}$$

In other words, using the division algorithm by the hyperbolic polynomial H^{p-q_1} we have

$$S_0 = H^{p-q_1}\,\Lambda_1 + T_1\,,$$

and condition (L_1) is equivalent to $T_1 \equiv 0$. Thanks to (1.6) we have

$$a\Lambda_1 = H^{q_1} K \sigma_{\mu_0}(h\mathscr{A}' - H'^p K')\,.$$

(L_2) Assuming (L_1) is satisfied and $\mathscr{A}'$, H', K', Λ'_1 chosen, there exists $\Lambda_2(x,\xi) \in \mathcal{M}_N\big(\mathcal{O}[\xi]\big)$ whose entries are homogeneous of degree μ_1 in ξ such that

$$S_1 \equiv \mathscr{A}\sigma_{\mu_1}\big(h\Lambda'_1 - h\mathscr{A}'H'^{q_1}K' + H'^{p}K'H'^{q_1}K'\big) = H^{p-q_2}\,\Lambda_2\,.$$

Here also, using the division algorithm by the hyperbolic polynomial H^{p-q_1} we have

$$S_1 = H^{p-q_2}\,\Lambda_2 + T_2\,,$$

and condition (L_2) is equivalent to $T_2 \equiv 0$.

(L_ℓ) Assuming conditions (L_1), …, ($L_{\ell-1}$) are satisfied, there exists a symbol $\Lambda_\ell(x,\xi)$ whose entries are homogeneous polynomial of degree μ_ℓ with analytic coefficients such that

$$\begin{aligned} S_{\ell-1} \equiv \mathscr{A}\sigma_{\mu_\ell-1}(&h\Lambda'_{\ell-1} - h\Lambda'_{\ell-2}H'^{q_\ell-1}K' + \dots \\ &+ (-1)^{\ell-1}h\mathscr{A}'H'^{q_1}K'H'^{q_2}K'\dots H'^{q_\ell-1}K' \\ &+ (-1)^{\ell}H'^{p}K'H'^{q_1}K'\dots H'^{q_\ell-1}K') = H^{p-q_\ell}\Lambda_\ell. \end{aligned}$$

As before, we have

$$S_{\ell-1} = H^{p-q_\ell}\,\Lambda_\ell + T_\ell\,,$$

and condition (L_ℓ) is equivalent to $T_\ell \equiv 0$.

($L_{\ell+1}$) Assuming (L_1), …, (L_ℓ) are satisfied, there exists a symbol $\Lambda_{\ell+1}(x,\xi)$ whose entries are homogeneous polynomial of degree $\mu_{\ell+1}$ with analytic coefficients such that

$$\begin{aligned} S_\ell &\equiv \mathscr{A}\sigma_{\mu_\ell}\Big(h\Lambda'_\ell - h\Lambda'_{\ell-1}H'^{q_\ell}K' + \dots \\ &\quad + (-1)^{\ell}h\mathscr{A}'H'^{q_1}K'\dots H'^{q_\ell}K' \\ &\quad + (-1)^{\ell+1}H'^{p}K'.H'^{q_1}K'\dots H'^{q_\ell}K'\Big) \\ &= H^{p-1}\Lambda_{\ell+1}\,. \end{aligned}$$

As before, we have

$$S_\ell = H^{p-q_{\ell+1}}\,\Lambda_{\ell+1} + T_{\ell+1}\,,$$

and condition ($L_{\ell+1}$) is equivalent to $T_{\ell+1} \equiv 0$.

($L_{m'}$) Assuming conditions (L_1), …, ($L_{m'-1}$) are satisfied, there exists a symbol $\Lambda_{m'}(x,\xi)$ whose entries are homogeneous polynomial of degree $\mu_{m'}$ with analytic coefficients such that

$$\begin{aligned} S_{m'_1-1} &\equiv \mathscr{A}\sigma_{\mu_{m'_1}-1}\Big(h\Lambda'_{m'_1-1} - h\Lambda'_{m'_1-2}H'K' + \dots \\ &\quad + (-1)^{m'_1-1}h\mathscr{A}'H'^{q_1}K'\dots H'^{q_\ell}K'(H'K')^{m_1-\ell-1} \\ &\quad + (-1)^{m'_1}H'^{p}K'H'^{q_1}K'\dots H'^{q_\ell}K'(H'K)^{m_1-\ell-1}\Big) \\ &= H^{p-1}\Lambda'_{m'_1}. \end{aligned}$$

As before, we have

$$S_{m'_1-1} = H^{p-q_{m'_1}}\,\Lambda_{m'_1} + T_{m'_1}\,,$$

and condition ($L_{m'_1}$) is equivalent to $T_{m'_1} \equiv 0$.

Definition 1.2 ([27])**.** m_1' is the smallest integer such that all conditions ($\mathrm{L}_{\mathrm{m}_1''}$), with $m_1'' > m_1'$, are consequences of conditions (L_1), ..., ($\mathrm{L}_{\mathrm{m}'}$).

The number m_1' is made precise by the study of Newton's diagram and the Gevrey index (cf. §4.1). Note that Conditions (L) are invariant, since they are defined on principal symbols.

Proposition 1.3. *Conditions* (L) *do not depend on the choice of the operators H', K', $\mathscr{A}'$, Λ_1', ..., $\Lambda_{m'-1}'$.*

This proposition and Proposition 2.4 below, have been proved in the case $p = m$ (or $\ell = 0$) in [25, Chap. II]. The proof in the general case require only some additional technicalities.

Remark 1.4. As a consequence of Proposition 1.3, choosing $H' = H(x, D)\, I$, $K' = K(x, D)I$, $\mathscr{A}' = \mathscr{A}(x, D)$, ..., we can express conditions (L) explicitly as differential relations between the coefficients of h, by using calculus of the symbols and elementary calculus of division.

Remark 1.5. If $m = 2$ conditions (L) reduces to the single (L_1), and this condition is equivalent to the cancelation of the subcharacteristic symbol on the characteristic manifold [7, 22, 24]:

$$ASA + \frac{1}{2}\, A \cdot \{H, A\} \quad \text{is divisible by } H\,.$$

The case $q = 0$ ($\ell = 0$) is well-known, see [25, 9] and the references therein.

Remark 1.6. The sufficiency of conditions (L) were studied in [28], and a more precise proof will be obtained by the present method.

The necessity of conditions (L) has been stated up to multiplicity 5 in [27]. For the necessity we assume only that the coefficients are in $\mathcal{C}^\infty$.

In the complex domain these conditions characterize the solutions with polar singularities on the characteristic manifold when the data have polar singularities.

Theorem 1.7. *We assume $\ell = 1$, that is a is equivalent in $\mathcal{O}_{(H)}[\xi]$ to the diagonal matrix*

$$\mathrm{diag}\big[H^p, H^q, 1, \dots, 1\big],$$

where p, q are such that

$$p \geq q > 0, \quad p + q = m\,.$$

1. *Assuming that the coefficients are $\mathcal{C}^\infty$, Conditions* (L) *are necessary to the hyperbolicity of the operator h.*
2. *Assuming that the coefficients are analytic, Conditions* (L) *are sufficient to the hyperbolicity of the operator h.*
3. *Assuming that the coefficients are analytic, Conditions* (L) *are partially satisfied and h is weakly hyperbolic, we obtain existence and unicity of the solution in a Gevrey class which will be made precise later.*

Definition 1.8. An operator A' with principal symbol A is defined as

$$A' = \mathscr{A}'\, H'^q + \mathscr{A}_1' + \dots + \mathscr{A}_j' + \dots + \mathscr{A}_{N-1}',$$

where $\mathscr{A}_j'$ is a differential operator of order $N - j - 1$ that we can assume homogeneous.

Also we define

$$k = h \circ A' = (a + b) \circ A' .$$

Then we obtain

$$k = K' \, {H'}^m + K'_1 + \dots + K'_w + \dots + K'_N ,$$

where K'_w is a differential operator of order $N - w$ that we can assume homogeneous.

2. Microlocalization and strategy of the proof

To prove Theorem 1.7 we first reduce h to a simple form thanks to the following Proposition, which allows to block diagonalize h (cf. Kajitani [10]).

Proposition 2.1 (Separation of the characteristic roots)**.** *For any $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ there exists a conical neighborhood $\Gamma = \Gamma(x_0, \xi_0)$ and a pseudo-differential operator $\Delta(x, \xi')$ such that*

$$h \circ \Delta \equiv \Delta \circ \widetilde{h} \,, \qquad \bmod S^{-\infty}(\Gamma) \,,$$

where $\widetilde{h}$ is block diagonal, each block corresponding to a characteristic root (cf. (1.3)*):*

$$\widetilde{h} = \widetilde{h}_1 \otimes \dots \otimes \widetilde{h}_s \otimes \widetilde{h}_{s+1} \otimes \dots \otimes \widetilde{h}_{s+\chi} \,,$$

where

- *for $1 \le j \le s$, $\widetilde{h}_j$ is the $m \times m$ operator*

$$\widetilde{h}_j = I_m \, D_0 - \widetilde{\mathrm{a}}_j(x, D') + \widetilde{b}_j(x, D'),$$

  $\widetilde{\mathrm{a}}_j$ has the unique eigenvalue $\lambda_j(x, \xi')$, and $b_j(x, D')$ is a $m \times m$ operator in D' of order ≤ 0,
- *for $j > s$, $\widetilde{h}_j$ is the scalar operator*

$$\widetilde{h}_j = D_0 - \lambda_j(x, D') + \widetilde{b}_j(x, D')$$

  and $b_j(x, D')$ is a scalar operator in D' of order ≤ 0.

To prove Theorem 1.7 a more refined reduction is needed. The normal form by Arnold-Petkov [1, 19, 12, 25, 13] is stated as follows.

Proposition 2.2. *There exists an analytic set $\Sigma \subset \Omega \times \mathbb{R}^n \setminus \{0\}$ of codimension less than of equal 1 such that for any $(x_0, \xi_0) \in (\Omega \times \mathbb{R}^n \setminus \{0\}) \setminus \Sigma$ there exists a conical neighborhood $\Gamma = \Gamma(x_0, \xi_0) \subset (\Omega \times \mathbb{R}^n \setminus \{0\}) \setminus \Sigma$, and a pseudo-differential operator $\Delta(x, \xi')$ such that*

$$h \circ \Delta \equiv \Delta \circ \widetilde{h} \,, \qquad \bmod S^{-\infty}(\Gamma \setminus \Sigma) \,,$$

where $\widetilde{h}$ is block diagonal, each block corresponding to a characteristic root (cf. (1.3)*):*

$$\widetilde{h} = \widetilde{h}_1 \otimes \dots \otimes \widetilde{h}_s \otimes \widetilde{h}_{s+1} \otimes \dots \otimes \widetilde{h}_{s+\chi} \,,$$

where $\widetilde{h}_j$ are pseudo-differential operators such that

- *for $1 \le j \le s$, $\widetilde{h}_j$ is the $m \times m$ matrix*

$$\widetilde{h}_j = \big(D_0 - \lambda_j(x, D') \big) \, I_m + J \, |D'| + \widetilde{b}_0(x, D') + \dots + \widetilde{b}_j(x, D') + \dots ,$$

J is a Jordan nilpotent matrix, with block of size $p = q_0$ and $q_1 = q$,

$$J = \begin{pmatrix} J_0 & 0 \\ 0 & J_1 \end{pmatrix} \quad J_i = \begin{pmatrix} 0 & 1 & \dots & & 0 \\ \vdots & 0 & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad q_i \times q_i$$

and $\widetilde{b}_j$ are pseudo-differential operators of order less than or equal to 0 in D' having the Arnold's normal form

$$\widetilde{b}_j = \begin{array}{cc} \left(\begin{array}{c|c} \multicolumn{2}{c}{0} \\ \hline & \\ & 0 \\ & \\ \hline \end{array} \right) & \begin{array}{l} \\ p^{\text{th}}\ \textit{line} \\ \\ \\ (p+q)^{\text{th}}\ \textit{line} \end{array} \\ 1^{\text{st}}\ \textit{col.}\ (p+1)^{\text{th}}\ \textit{col.} & \end{array} \tag{2.1}$$

the non-zero elements are only on the indicated lines;

- *for $j > s$, $\widetilde{h}_j$ is the scalar operator*

$$\widetilde{h}_j = D_0 - \lambda_j(x, D') + b_j(x, D')$$

and $b_j(x, D')$ is a scalar operator in D' of order ≤ 0.

Thanks to Proposition 2.1 we can assume that h has only one characteristic root λ, and, by using Egorov's theorem, we can also assume that $\lambda \equiv 0$ and the diagonal elements of b are zero. This means that we can reduce the general case to the case in which $H(x,\xi) \equiv \xi_0$ and $K(x,\xi) \equiv 1$:

$$\det a(\xi) = \xi_0^p\,.$$

Example 2.3. If $p = 4$ and $q = 3$, then

$$a = \begin{pmatrix} \xi_0 & \xi_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_0 & \xi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_0 & \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_0 & \xi_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_0 & \xi_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi_0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & 0 & b_5^4 & b_6^4 & b_7^4 \\ b_1^5 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1^7 & 0 & 0 & 0 & b_5^7 & b_6^7 & 0 \end{pmatrix}$$

The reduction to this simple case is made possible by the following statement.

Proposition 2.4 ([25])**.** *Let $\Delta(x; D')$ be an elliptic classical pseudo-differential operator of order* 0*. Then h satisfies conditions* (L)*, if and only if every localized transformed operator $\widetilde{h} := \Delta^{-1} h \Delta$ satisfies the same conditions* ($\widetilde{L}$)*.*

Because of the reduction, the operator $\widetilde{h}$ is pseudo-differential in D' and differential in D_0. To simplify the writing of the proofs, in the following, we omit the tilde, and we assume that $H(x,\xi) = \xi_0$, $K(x,\xi) = 1$ and $M = m$. For $\alpha \in \mathbb{N}^n$ we denote

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_n}^{\alpha_n}}.$$

Proposition 2.5 ([21, p. 34])**.** (1) *If $(b_0)_{p+1}^p \neq 0$, we obtain $(b_0)_{p+1}^p = 1$.*
(2) *If $(b_0)_{p+1}^p = \dots = (b_0)_{p+h-1}^p = 0$ and $(b_0)_{p+h}^p \neq 0$, we obtain $(b_0)_{p+h}^p = 1$.*

Definition 2.6 ([25, 22])**.** For $b = b_0 + b_1 + \dots$:

(1) Let

$$\widehat{b}_0(x,\xi,D) = \sum_{|\alpha|=1} \partial^\alpha a(\xi) D^\alpha + b_0(x,\xi'),$$

$$\widehat{b}_1(x,\xi,D') = \frac{1}{2!}\sum_{|\alpha'|=2} \partial^{\alpha'} a(\xi) D^{\alpha'} + \sum_{|\alpha'|=1} \partial^{\alpha'} b_0(x,\xi') D^{\alpha'} + b_1(x,\xi'),$$

$$\dots$$

$$\begin{aligned}\widehat{b}_k(x,\xi,D') = \frac{1}{(k+1)!}&\sum_{|\alpha'|=k+1} \partial^{\alpha'} a(\xi) D^{\alpha'} + \dots \\ &+ \frac{1}{(k-u)!}\sum_{|\alpha'|=k-u} \partial^{\alpha'} b_u(x,\xi') D^{\alpha'} + \dots \\ &+ \sum_{|\alpha'|=1} \partial^{\alpha'} b_{k-1}(x,\xi') D^{\alpha'} + b_k(x,\xi'),\end{aligned}$$

$$\dots,$$

where the last operator is $\widehat{b}_{m-1}(x,\xi,D')$.

(2) Let

$$\begin{aligned}\mathcal{L}_0(x,\xi,D_x) &= \mathscr{A}(\xi)\,\widehat{b}_0(x,\xi,D), \\ \mathcal{L}_1(x,\xi,D_x) &= \mathscr{A}(\xi)\widehat{b}_1(x,\xi,D'), \\ &\dots, \\ \mathcal{L}_k(x,\xi,D_x) &= \mathscr{A}(\xi)\widehat{b}_k(x,\xi,D'), \\ &\dots,\end{aligned}$$

where the last operator is $\mathcal{L}_{m-1}$.

The coefficients of the operators $\mathcal{L}_k$ use ξ_0 only in $\mathscr{A}$. The b_k with $k \geq m-1$ are not used. In the following we denote $\mathscr{A} = \Lambda_0$.

Proposition 2.7. *With $H(x,\xi) = \xi_0$ and $K(x,\xi) = 1$, Conditions* (L) *are equivalent to the following conditions*

(L_1) *There exists an homogeneous symbol $\Lambda_1(x,\xi)$ polynomial in ξ_0, with analytic coefficients in (x,ξ') in the conical neighborhood considered, of degree $m-2$, such that*

$$S_0 = \mathcal{L}_0(\mathscr{A}) = \mathscr{A} b_0 \mathscr{A} = \xi_0^{p-q}\Lambda_1$$

In other words, if we write

$$S_0 = \mathcal{L}_0(\mathscr{A}) = \xi_0^{p-q}\Lambda_1 + T_1\,,$$

and the degree in ξ_0 of T_1 is less than $p-q$, then condition (L_1) *is equivalent to $T_1 \equiv 0$.*

Then by induction, we have

(L_{k+1}) *For $k \geq 0$, there exists an homogeneous symbol $\Lambda_{k+1}(x,\xi)$ polynomial in ξ_0, with analytic coefficients in (x,ξ') in the conical neighborhood considered, of degree $m-2$ such that*

$$S_k = \mathcal{L}_0(\Lambda_k) - \xi_0\mathcal{L}_1(\Lambda_{k-1}) + \dots + (-1)^u\xi_0^u\mathcal{L}_u(\Lambda_{k-u}) + \dots$$

$$\begin{aligned}&+ (-1)^{k-1}\xi_0^{k-1}\mathcal{L}_{k-1}(\Lambda_1) + (-1)^k\xi_0^{k+q-1}\mathcal{L}_k(\mathscr{A})\\ &= \xi_0^{p-1}\Lambda_{k+1}\,.\end{aligned}$$

As before, if we write

$$S_k = \xi_0^{p-1}\Lambda_{k+1} + T_{k+1}\,,$$

and the degree in ξ_0 *of* T_{k+1} *is less than* $p-1$*, then condition* (L_{k+1}) *is equivalent to* $T_{k+1} \equiv 0$*.*

Proposition 2.8. *When conditions* (L) *are microlocalized they become* $T_1 = \cdots = T_{m'} = 0$*.*

Definition 2.9. Let

$$A' = \mathscr{A}'D_0^q + \mathscr{A}_1' + \cdots + \mathscr{A}_j' + \cdots + \mathscr{A}_{m-1}'\,,$$
$$\sigma(\mathscr{A}') = \mathscr{A} = \xi_0^{p-1}I\,.$$

Operator $\mathscr{A}_j'$ is of total order $m-j-1$, differential in D_0, and pseudo-differential in ξ'.

$$k = h\circ A' = (a+b)\circ\mathscr{A}' = D_0^m I + \cdots + K_j' + \cdots + K_m'$$

and K_j' is of order $m-j$.

In the following we denote $z = p-q$, and in the first part we assume that $z > 0$.

Definition 2.10. (1) For $|\alpha| = m-u-j$ we denote $\xi^\alpha = [\xi']^{m-u-j}$, and

$$K_j = K_{j,0}[\xi']^{m-j} + \cdots + K_{j,u}\xi_0^u[\xi']^{m-u-j} + \cdots + K_{j,m-j}\xi_0^{m-j}$$

(2) When we consider homogeneous symbols in ξ, to make easier the redaction we set $|\xi| = 1$,

$$T_1 = \sum_{0\le u\le z-1} T_{1,u} = T_{1,0} + \cdots + T_{1,u}\xi_0^u + \cdots + T_{1,z-1}\xi_0^{z-1},$$
$$\Lambda_1 = \sum_{0\le u\le m-2}\Lambda_{1,u} = \Lambda_{1,0} + \cdots + \Lambda_{1,u}\xi_0^u + \cdots + \Lambda_{1,m-2}\xi_0^{m-2}\,.$$

Proposition 2.11. *By a suitable choice of* A' *the symbols* K_j *can be obtained diagonally,*

$$K_j = \widehat{K}_j I\,,\quad 1\le j\le m-j\,.$$

If $u < q$, then $K_{1,u} = 0$. If $q\le u\le p-1$, then

$$\widehat{K}_{1,u} = (-1)^{p-1}(T_{1,u-q})_p^1$$

So

$$\widehat{K}_{1,q} = (-1)^p(T_{1,0})_p^1,\dots,\widehat{K}_{1,p-1} = (T_{1,z-1})_p^1,$$
$$(\mathrm{L}_1)\qquad (T_{1,0})_p^1 = \cdots = (T_{1,z-1})_p^1 = 0\,.$$

If $p\le u\le m-2$, then

$$\widehat{K}_{1,p} = (-1)^{p-1}\big[(\Lambda_{1,0})_p^1 + (-1)^j(\Lambda_{1,z})_m^{p+1}\big],$$
$$\widehat{K}_{1,u} = (-1)^{p-1}\big[(\Lambda_{1,u-p})_p^1 + (-1)^z(\Lambda_{1,u-q})_m^{p+1}\big],$$
$$\widehat{K}_{1,m-2} = (-1)^{p-1}\big[(\Lambda_{1,q-2})_p^1 + (-1)^z(\Lambda_{1,p-2})_m^{p+1}\big],$$
$$\widehat{K}_{1,m-1} = 1\,.$$

If $(b_0)^p_{p+1} \neq 0$, and (L_1) is satisfied, then

$$\widehat{K}_{1,p} = (-1)^z \frac{(T_{2,z})^1_m}{(b_0)^p_{p+1}},$$
$$\widehat{K}_{1,u} = (-1)^z \frac{(T_{2,u-q})^1_m}{(b_0)^p_{p+1}},$$
$$\widehat{K}_{1,m-2} = (-1)^z \frac{(T_{2,p-2})^1_m}{(b_0)^p_{p+1}}\,.$$

If $(b_0)^p_{p+1} = 0$, we consider $(b_0)^p_{p+2} \neq 0$, etc., we will show explicitly the calculus in detail in the proof.

Assume $j > 1$. If $u < z - j + 2$, then

$$\widehat{K}_{j,u} = (-1)^{p+1}(T_{j,u+j-2})^1_p\,.$$

Here and in the following, $P \sim Q \mod \mathrm{L}_1$ means that $P - Q$ is equal to a linear combination of the terms $T_{1,0}, \dots, T_{1,z-1}$.

If $z - j + 2 \leq u < p - j + 1$, and $(b_0)^p_{p+1} \neq 0$, then

$$\widehat{K}_{j,u} \sim (-1)^{p+j}\Big[(T_{j,u+j-2})^1_p - \frac{(b_0)^p_{z+1}}{(b_0)^p_{p+1}}(T_{j,u+j-2})^1_m\Big] \sim (T_{j,v})^1_m \quad v < u + j - 2\,.$$

If $(b_0)^p_{p+1} = 0$ and $(b_0)^p_{p+2} \neq 0$, we will get in an analogous manner the terms.

If $p - j + 1 \leq u \leq m - j - 1$ and $(b_0)^p_{p+1} \neq 0$, then

$$\widehat{K}_{j,u} = (-1)^{z+1+s}\frac{(T_{j+1,u+j-q-1})^1_m}{(b_0)^p_{p+1}}\,.$$

Example 2.12. For $(b_0)^p_{p+1} \neq 0$, we have

$$\widehat{K}_{j,m-j-1} = (-1)^{z+1+s}\frac{(T_{j+1,p-2})^1_m}{(b_0)^p_{p+1}}\,.$$

If $(b_0)^p_{p+1} = 0$ and $(b_0)^p_{p+2} \neq 0$, we obtain analogous results in the same manner.

Proposition 2.13. *Assume the coefficients of h are in $\mathcal{C}^\infty$. For the Cauchy problem to be well-posed in $\mathcal{C}^\infty$ it is necessary that conditions* (L) *be satisfied.*

Proposition 2.14. *Assuming that the coefficients of h are analytic, for the Cauchy problem to be well-posed in $\mathcal{C}^\infty$, it is sufficient that conditions* (L) *be satisfied.*

Proposition 2.15. *Assume the coefficients of h are analytic. If the coefficients are partially satisfied, we obtain existence and unicity of the solution in a suitable Gevrey class.*

Remark 2.16. (1) Proposition 2.11 concerns, at first, with the calculus of symbols and it is essentially algebraic. The proof of Proposition 2.13 is completely independent of the Proposition 2.11 and is based on the closed graph's theorem.

Because of its length, the study of Proposition 2.11 is divided in two parts. In this article (part I), we prove the necessity of (L_1) and (L_2). In a forthcoming article (part II), we consider (L_3) and the general case.

(2) The algebraic part defines a determinant over a non-commutative graded ring, adapted to the problem and also to the holomorphic cases.

(3) The construction of the determinant is interesting by itself.

Proposition 2.17. *With $H(x,\xi) = \xi_0$, $K(x,\xi) = 1$ and $q = q_1$, operator h is diagonalizable with a good decomposition with respect to ξ_0 if there exist $\mathscr{A}'_0 = \mathscr{A}'$, $\mathscr{A}'_1$, $\mathscr{A}'_2$, $\dots$, $\mathscr{A}'_{m-1}$, K'_1, K'_2, $\dots$, K'_m such that*

$$\begin{aligned} &h \circ \left[\mathscr{A}'\, D_0^q + \mathscr{A}'_1 + \dots + \mathscr{A}'_j + \dots + \mathscr{A}'_{m-1}\right] \\ &= I D_0^m + K'_1\, D_0^{m-1} + \dots + K'_{m-1}\, D_0 + K'_m\,, \end{aligned} \tag{2.2}$$

where $\operatorname{ord}(\mathscr{A}'_j) = m-j-1$ *for* $j = 1,\dots,m-1$, *and* $\operatorname{ord}(K'_j) \le 0$, *for* $j = 1,\dots,m$.

Our bibliography is reduced to the essentials; a more extensive one can be found in [22, 27].

3. Study of K_1

We have $p \ge 2$, $p > q$, $z = p - q \ge 1$.

3.1. Conditions (L_1), (L_2).

Lemma 3.1. *Let $J^p = 0$. Assume $0 \le u \le p-1$, and $1 \le v \le p-u$ or $p+1 \le v \le m-u$. Then*

$$[J^u]^v = [I]^{u+v}\,.$$

As $a = \xi_0 + J\,|\xi'|$ we have

$\Lambda_0 = \mathcal{A} = \xi_0^{p-1} I + \dots + (-1)^{p-u-1}\xi_0^u|\xi'|^{p-u-1}J^{p-u-1} + \dots + (-1)^{p-1}|\xi'|^{p-1}J^{p-1}$, $0 \le u \le p-1$, $\Lambda_{0,u} = (-1)^{p-u-1}|\xi'|^{p-u-1}J^{p-u-1}$, $a\Lambda_0 = \xi_0^p I$, and $|\xi'|J\Lambda_{0,u} = \Lambda_{0,u}J|\xi'| = -\Lambda_{0,u-1}$.

For easier writing, we consider homogeneous symbols in ξ, and let $|\xi'| = 1$.

3.1.1. *Study of* (L_1).

Note taht

$$\begin{gathered} (\Lambda_{0,0})_p^1 = (-1)^{p-1}, \quad (\Lambda_{0,u})_m^1 = 0, \quad (\Lambda_{0,u})_p^{p+1} = 0, \quad (\Lambda_{0,z})_p^{p+1} = (-1)^{q+1}, \\ S_0 = \mathscr{L}_0(\Lambda_0) = \mathcal{A} b_0 \Lambda_0 = T_1 + \xi_0^z \Lambda_1 = \sum_{0\le u\le z-1} T_{1,u} + \xi_0^z \sum_{0\le u\le m-2} \Lambda_{1,u}\xi_0^u\,. \end{gathered}$$

Then (L_1) is the set of conditions

$$T_{1,u} = 0, \quad 0 \le u \le z-1, \quad T_{1,0} = J^{p-1} b_0 J^{p-1}\,,$$

so that $(T_{1,0})_p^1 = (b_0)_1^p$,

$$\begin{aligned} (-1)^{p-1}T_{1,u} &= J^{p-1}b_0\Lambda_{0,u} - J^{p-2}b_0\Lambda_{0,u-1} + \dots \\ &\quad + (-1)^{k-1}J^{p-k}b_0\Lambda_{0,u-k+1} + \dots + (-1)^{p-1}J^{p-u-1}b_0\Lambda_{0,0}, \\ JT_{1,u} &= T_{1,u}J = -T_{1,u-1}, \end{aligned} \tag{3.1}$$

$(T_{1,u})^{p+1} = 0$, $(T_{1,u})_m^1 = 0$,

$$\begin{gathered} (-1)^{p-1}(T_{1,u})_p^1 = (b_0)^p(\Lambda_{0,u})_p = (b_0)_{u+1}^p(-1)^{p+u+1}, \\ (T_{1,z-1})_1^p = (-1)^{z+1}(b_0)_z^p, \\ (\mathrm{L}_1): \quad (b_0)_1^p = \dots = (b_0)_z^p = 0\,. \end{gathered} \tag{3.2}$$

For $0 \le u \le m-2$,

$$\begin{aligned} (-1)^{p-1}\Lambda_{1,u} &= J^{p-1}b_0\Lambda_{0,u+z} - J^{p-2}b_0\Lambda_{0,u+z-1} + \dots \\ &\quad + (-1)^{k-1}J^{p-k}b_0\Lambda_{0,u+z-k+1} + \dots + (-1)^{p-1}b_0\Lambda_{0,u-q+1}\,. \end{aligned}$$

We have

$$J\Lambda_{1,u} = -\Lambda_{1,u-1} + b_0\Lambda_{0,u-q}, \quad \Lambda_{1,u}J = -\Lambda_{1,u-1}, \quad J\Lambda_{1,0} = \Lambda_{1,0}J = 0\,.$$

If $0 < u < q-1$, then

$$(\Lambda_{1,u})^1_p = (-1)^{z+u}(b_0)^p_{u+z+1}. \tag{3.3}$$

If $u \geq q-1$, then $(\Lambda_{1,u})^1_p = 0$. If $0 < u < q$, then

$$(\Lambda_{1,u})^1_m = (-1)^{z+u}(b_0)^p_{p+u+1}\,. \tag{3.4}$$

If $u \geq q$, then $(\Lambda_{1,u})^1_m = 0$.

Moreover if $1 \leq k-1 \leq p-1$, then

$$(\Lambda_{1,u})^k = -\Lambda_{1,u-1})^{k-1} \tag{3.5}$$

and

$$\begin{aligned}
&(-1)^{p-1}(\Lambda_{1,u})^{p+1}\\
&= (-1)^z\Big[(b_0)^m\Lambda_{0,u} - (b_0)^{m-1}_1(\Lambda_{0,u-1})^1 + \dots + (-1)^{q-1}(b_0)^{p+1}_1(\Lambda_{0,u-q+1})^1\Big]\,.
\end{aligned}$$

If $p+1 \leq k-1 \leq m-1$, then

$$(\Lambda_{1,u})^k = -(\Lambda_{1,u-1})^{k-1} + (b_0)^{k-1}_1(\Lambda_{0,u-q})^1 \tag{3.6}$$

and

$$(-1)^{p-1}(\Lambda_{1,u})^{p+1}_m = (-1)^{q+u+1}(b_0)^m(J^{p-u-1})_m = (-1)^{q+u+1}(b_0)^m_{u+q+1}\,.$$

If $u < z$, then $(\Lambda_{1,u})^{p+1}_m = 0$. If $u \geq z$, then

$$(\Lambda_{1,u})^{p+1}_m = (-1)^{z+u}(b_0)^m_{u+q+1}, \tag{3.7}$$

$$(\Lambda_{1,u})^{p+1}_p = (-1)^{z+u}(b_0)^{m-u}_1 \tag{3.8}$$

Example 3.2. Let $(\Lambda_{1,0})^{p+1}_p = (-1)^z(b_0)^m_1$ and $(\Lambda_{1,q-1})^{p+1}_p = (-1)^{p+1}(b_0)^{p+1}_1$. If $u \geq q$, then $(\Lambda_{1,u})^{p+1}_p = 0$.

3.1.2. *Study of* (L_2). Assuming (L_1) is satisfied, we have

$$S_1 = \mathscr{L}_0(\Lambda_1) - \xi_0^q\mathscr{L}_1(\Lambda_0) = T_2 + \xi_0^{p-1}\Lambda_2\,.$$

If $0 \leq u \leq p-2$, then

$$\begin{aligned}
(-1)^{p-1}T_{2,u} = J^{p-1}&(b_0 + D_0 + JD_1)\Lambda_{1,u} - J^{p-2}(b_0 + D_0 + JD_1)\Lambda_{1,u-1}\\
&+ (-1)^{k-1}J^{p-k}(b_0 + D_0 + JD_1)\Lambda_{1,u-k+1}\\
&+ (-1)^u J^{p-u-1}(b_0 + D_0 + JD_1)\Lambda_{1,0}\\
&- J^{p-1}b_1\Lambda_{0,u-q} + \dots + (-1)^{u+q+1}J^{m-u-1}b_1\Lambda_{0,0}\,,
\end{aligned} \tag{3.9}$$

$$JT_{2,u} = -T_{2,u-1} = -T_{2,u}J, \tag{3.10}$$

$$(-1)^{p-1}(T_{2,u})^1 = (b_0)^p\Lambda_{1,u} - (b_1)^p\Lambda_{0,u-q}\,. \tag{3.11}$$

We have also

$$(-1)^{p-1}(T_{2,u})^1_p = (b_0)^p(\Lambda_{1,u})_p - (b_1)^p(\Lambda_{0,u-q})_p\,. \tag{3.12}$$

Example 3.3. Let

$$\begin{aligned}
(-1)^{p-1}(T_{2,0})^1_p = (b_0)^p(\Lambda_{1,0})_p &= (b_0)^p_1(\Lambda_{1,0})^1_p + (b_0)^p_{p+1}(\Lambda_{1,0})^{p+1}_p\\
&= (-1)^z(b_0)^p_{p+1}(b_0)^m_1,
\end{aligned}$$

$$(-1)^{p-1}(T_{2,u})^1_m = (b_0)^p(\Lambda_{1,u})_m - (b_1)^p(\Lambda_{0,u-q})_m\,.$$

If $u < z$, then

$$(-1)^{p-1}(T_{2,u})^1_m = (b_0)^p_1(\Lambda_{1,u})^1_m - (b_0)^p_2(\Lambda_{1,u-1})^1_m - \dots - (-1)^u(b_0)^p_{u+1}(\Lambda_{1,0})^1_m\,.$$

So if $(b_0)^p_1 = \dots = (b_0)^p_z = 0$, then

$$(T_{2,u})^1_m = 0\,, \quad \forall u < z\,. \tag{3.13}$$

If $z \le u \le p-2$, then using (3.4) and (3.7)

$$\begin{aligned}(-1)^{p-1}(T_{2,z})^1_m &= (b_0)^p_{z+1}(b_0)^p_{p+1} + (b_0)^p_{p+1}(b_0)^m_{p+1}\\ &= (b_0)^p_{p+1}[(b_0)^p_{z+1} + (b_0)^m_{p+1}]\\ &= (-1)^z\,(b_0)^p_{p+1}\big[(\Lambda_{1,0})^1_p + (-1)^z\,(\Lambda_{1,z})^{p+1}_m\big]\,,\end{aligned}$$

$$(-1)^{p-1}(T_{2,z+1})^1_m = -(b_0)^p_{p+1}[(b_0)^p_{z+2} + (b_0)^m_{p+2}] - (b_0)^p_{p+2}[(b_0)^p_{z+1} + (b_0)^m_{p+1}]\,.$$

Afterwards,

$$\begin{aligned}(-1)^{p-1}(T_{2,u})^1_m &= (-1)^{u+z}\Big[(b_0)^p_{p+1}\big((b_0)^p_{u+1} + (b_0)^m_{u+q+1}\big)\\ &\quad + (b_0)^p_{p+2}\big((b_0)^p_u + (b_0)^m_{u+q}\big) + \dots\\ &\quad + (b_0)^p_{u+q+1}\big((b_0)^p_{z+1} + (b_0)^m_{p+1}\big)\Big]\,,\\ (-1)^{p-1}(T_{2,p-2})^1_m &= (-1)^q\Big[(b_0)^p_{p+1}\big((b_0)^p_{p-1} + (b_0)^m_{m-1}\big) + \dots\\ &\quad + (b_0)^p_{m-1}\big((b_0)^p_{z+1} + (b_0)^m_{p+1}\big)\Big]\,.\end{aligned} \tag{3.14}$$

If $(b_0)^p_{p+1} = 0$, then $(T_{2,z})^1_m = 0$ and

$$(-1)^{p-1}(T_{2,z+1})^1_m = -(b_0)^p_{p+2}\big((b_0)^p_{z+1} + (b_0)^m_{p+1}\big)\,.$$

If $(b_0)^p_{p+1} = \dots = (b_0)^p_{u+q} = 0$, then

$$(-1)^{p-1}(T_{2,u})^1_m = (-1)^{u+z}(b_0)^p_{u+q+1}\big((b_0)^p_{z+1} + (b_0)^m_{p+1}\big)\,.$$

If $(b_0)^p_{p+1} = \dots = (b_0)^p_{m-2} = 0$, then

$$(-1)^{p-1}(T_{2,p-2})^1_m = (-1)^q(b_0)^p_{m-1}\big((b_0)^p_{z+1} + (b_0)^m_{p+1}\big)$$

and

$$\begin{aligned}(-1)^{p-1}(T_{2,u})^{p+1} &= (-1)^z\Big[(b_0)^m\Lambda_{1,u-z} - (b_0)^{m-1}_1(\Lambda_{1,u-z-1})^1 + \dots\\ &\quad + (-1)^{z+u}(b_0)^{2p-u}_1(\Lambda_{1,0})^1\Big]\end{aligned} \tag{3.15}$$

If $u < z$, then

$$(T_{2,u})^{p+1} = 0\,. \tag{3.16}$$

We have

$$\begin{aligned}(-1)^{p-1}(T_{2,z})^{p+1}_p &= (b_0)^m_1\big[(b_0)^p_{z+1} + (b_0)^m_{p+1}\big]\\ &= (-1)^z\,(b_0)^m_1\big[(\Lambda_{1,0})^1_p + (-1)^z(\Lambda_{1,z})^{p+1}_m\big]\,.\end{aligned} \tag{3.17}$$

If $u \ge z$, we obtain

$$\begin{aligned}(-1)^{p-1}(T_{2,u})^{p+1}_p &= (-1)^{u+z}\Big[(b_0)^m_1\big((b_0)^p_{u+1} + (b_0)^m_{u+q+1}\big) + \dots\\ &\quad + (b_0)^{2p-u}_1\big((b_0)^p_{z+1} + (b_0)^m_{p+1}\big)\Big]\,.\end{aligned} \tag{3.18}$$

Example 3.4. Let

$$\begin{aligned}(-1)^{p-1}(T_{2,p-2})_p^{p+1} = (-1)^q\Big[&(b_0)_1^m\big((b_0)_{p-1}^p + (b_0)_{m-1}^m\big) + \dots \\ &+ (b_0)_1^{p+2}\big((b_0)_{z+1}^p + (b_0)_{p+1}^m\big)\Big].\end{aligned}$$

If $(b_0)_1^m = 0$, then $(T_{2,z})_p^{p+1} = 0$ and

$$(-1)^{p-1}(T_{2,z+1})_p^{p+1} = -(b_0)_1^{m-1}\big((b_0)_{z+1}^p + (b_0)_{p+1}^m\big).$$

If $(b_0)_1^m = \dots = (b_0)_1^{2p-u+1} = 0$, then

$$(T_{2,u})_p^{p+1} = (-1)^{u+z}(b_0)_1^{2p-u}\big((b_0)_{z+1}^p + (b_0)_{p+1}^m\big).$$

If $(b_0)_1^m = \dots = (b_0)_1^{p+z} = 0$, then

$$(T_{2,p-2})_p^{p+1} = (-1)^q(b_0)_1^{p+2}\big((b_0)_{z+1}^p + (b_0)_{p+1}^m\big).$$

3.2. **Linking the coefficients of $K_{1,u}$ to T_1 and T_2.** Let

$$k = h \circ A' = (a+b)(\mathscr{A} D_0^q + \mathscr{A}_1' + \dots + \mathscr{A}_m').$$

Then we identify the homogeneous parts of the symbols

$$a\mathscr{A}\xi_0^q = \xi_0^m I = K_0,$$

$$\mathscr{A}_1 + b_0\mathscr{A}\xi_0^q = K_1 = K_{1,0} + \dots + K_{1,u}\xi_0^u + \dots + K_{1,m-1}\xi_0^{m-1},$$

and for $u < q$ we let

$$K_{1,u} = 0. \tag{3.19}$$

We identify the terms of degree u in ξ_0.

$$\begin{gathered}J\mathscr{A}_{1,u} = 0 \quad \text{if } u < q, \\ J\mathscr{A}_{1,u} + \mathscr{A}_{1,u-1} + b_0\Lambda_{0,u-q} = K_{1,u} \quad \text{if } q \le u \le m-2,\end{gathered} \tag{3.20}$$

$$\begin{gathered}(\mathscr{A}_{1,u})^k + (\mathscr{A}_{1,u-1})^{k-1} = (K_{1,u})^{k-1} \quad \text{if } 1 \le k-1 < p, \\ (\mathscr{A}_{1,u-1})^p + (b_0)^p\Lambda_{0,u-q} = (K_{1,u})^p,\end{gathered} \tag{3.21}$$

$$\begin{gathered}(\mathscr{A}_{1,u})^k + (\mathscr{A}_{1,u-1})^{k-1} + (b_0)^{k-1}\Lambda_{0,u-q} = (K_{1,u})^{k-1} \quad p+1 \le k-1 < m, \\ (\mathscr{A}_{1,u-1})^m + (b_0)^m\Lambda_{0,u-q} = (K_{1,u})^m\end{gathered} \tag{3.22}$$

3.2.1. *Case (1):* $0 \le u \le p-1$. (α) $u < q$, let

$$(\mathscr{A}_{1,0})^k = 0, \quad k \ne 1, p+1,$$

$$J\mathscr{A}_{1,1} + \mathscr{A}_{1,0} + b_0\Lambda_{0,1-q} = 0, (\mathscr{A}_{11})^2 = -(\mathscr{A}_{10})^1, \quad (\mathscr{A}_{11})^3 = 0, \quad \dots, \quad (\mathscr{A}_{11})^p = 0,$$

$$(\mathscr{A}_{11})^{p+2} = -(\mathscr{A}_{10})^{p+1}(\mathscr{A}_{11})^{p+3} = \dots = (\mathscr{A}_{11})^m = 0.$$

Generally $u \le q-1$, $(\mathscr{A}_{1,u})^1$, $(\mathscr{A}_{1,u})^{p+1}$ are free at this step.

(β) $q \le u \le p-1$, $(z \ge 1)$. At first $u = q$,

$$J\mathscr{A}_{1,q} + \mathscr{A}_{1,q-1} + b_0(-1)^{p-1}J^{p-1} = K_{1,q}$$

line k: $1 \le k-1 < p$,

$$(\mathscr{A}_{1,q})^k + (\mathscr{A}_{1,q-1})^{k-1} = (K_{1,q})^{k-1};$$

line p:

$$(\mathscr{A}_{1,q-1})^p + (-1)^{p-1}(b_0)^pJ^{p-1} = (K_{1,q})^p.$$

Then using (3.19), (3.20) and (3.21) we have

$$(-1)^{p-1}(b_0)_1^p = (K_{1,q})_p^p = (-1)^{p+1}(T_{1,0})_p^1,$$
$$k \neq p, \quad (K_{1,q})_k^p = (\mathscr{A}_{1,q-1})_k^p = 0,$$
$$(\mathscr{A}_{1,q})^2 + (\mathscr{A}_{1,q-1})^1 = (K_{1,q})^1\,.$$

So that

$$(\mathscr{A}_{1,q})_1^2 + (\mathscr{A}_{1,q-1})_1^1 = (K_{1,q})_1^1\,.$$

Definition 3.5. We let

$$(\mathscr{A}_{1,q})_1^2 = (-1)^{p-1}(b_0)_1^p - (\mathscr{A}_{1,q-1})_1^1$$

so that $(K_{1,q})_1^1 = (K_{1,q})_p^p$.

We let $k \neq 1$ and

$$(\mathscr{A}_{1,q})_k^2 = -(\mathscr{A}_{1,q-1})_k^1$$

So that $(K_{1,q})_k^1 = 0$. In the same manner, we obtain

$$(K_{1,q})_j^j = (K_{1,q})_p^p \quad \text{for } j \leq p\,.$$

In this step $(\mathscr{A}_{1,q-1})^1$ is free.

Now consider line m, using (3.22), we have

$$(\mathscr{A}_{1,q-1})^m + (b_0)^m J^{p-1}(-1)^{p-1} = (K_{1,q})^m,$$
$$(-1)^{q-1}(\mathscr{A}_{1,0})^{p+1} + (-1)^{p-1}(b_0)^m J^{p-1} = (K_{1,q})^m\,.$$

So that

$$(-1)^{q-1}(\mathscr{A}_{1,0})_m^{p+1} = (K_{1,q})_m^m\,.$$

Definition 3.6. We let

$$(\mathscr{A}_{1,0})_m^{p+1} = (-1)^z(T_{1,0})_p^1 = (-1)^z(b_0)_1^p\,. \tag{3.23}$$

Then

$$(K_{1,q})_m^m = (K_{1,q})_p^p,$$

and

$$(-1)^{q-1}(\mathscr{A}_{1,0})_p^{p+1} + (-1)^{p-1}(b_0)_1^m = (K_{1,q})_p^m\,.$$

Definition 3.7. We let

$$(\mathscr{A}_{1,0})_p^{p+1} = -(\Lambda_{1,0})_p^{p+1} = (-1)^{z+1}(b_0)_1^m \tag{3.24}$$

So $(K_{1,q})_p^m = 0$. We let also

$$(\mathscr{A}_{1,0})_j^{p+1} = 0 \quad j \neq p, m$$

then $(K_{1,q})_j^m = 0$.

If $p+1 \leq k-1 \leq m-1$, then

$$(\mathscr{A}_{1,q})^k + (\mathscr{A}_{1,q-1})^{k-1} + (b_0)^{k-1}\Lambda_{0,0} = (K_{1,q})^{k-1}\,,$$
$$(\mathscr{A}_{1,q})^{p+2} + (\mathscr{A}_{1,q-1})^{p+1} + (b_0)_1^{p+1}(\Lambda_{0,0})^1 = (K_{1,q})^{p+1}$$

We set

$$(\mathscr{A}_{1,q})_{p+1}^{p+2} + (\mathscr{A}_{1,q-1})_{p+1}^{p+1} = (K_{1,q})_p^p = (-1)^{p-1}(b_0)_1^p,$$

so that

$$(K_{1,q})_{p+1}^{p+1} = (K_{1,q})_p^p\,.$$

We set

$$(\mathscr{A}_{1,q})_p^{p+2} + (\mathscr{A}_{1,q-1})_p^{p+1} + (-1)^{p-1}(b_0)_1^{p+1}, = 0$$

so that

$$(K_{1,q})_p^{p+1} = 0\,.$$

We let

$$j \neq p+1, p, \quad (\mathscr{A}_{1,q})_j^{p+2} + (\mathscr{A}_{1,q-1})_j^{p+1} = 0,$$

so that

$$j \neq p+1, \quad (K_{1,q})_j^{p+1} = 0$$

In the same manner,

$$\begin{gathered} p+1 \leq j \leq m-1, \quad (K_{1,q})_j^j = (K_{1,q})_p^p, \\ j' \neq j, \quad (K_{1,q})_{j'}^j = 0\,. \end{gathered}$$

Finally $(\mathscr{A}_{1,q-1})^{p+1}$ is free, and

$$K_{1,q} = (-1)^{p-1}(T_{1,0})_p^1 I\,.$$

To simplify the redaction we replace D' by D_1 and $|\xi'|$ by ξ_1.

Now we have

$$\begin{aligned} k = {} & I D_0^m + (-1)^{p-1}(T_{1,0})_p^1 D_0^q D_1^{p-1} + K'_{1,q+1} D_0^{q+1} D_1^{p-2} + \dots \\ & + K'_{1,m-2} D_0^{m-2} D_1 + K'_{1,m-1} D_0^{m-1} \\ & + K'_{2,0} D_1^{m-2} + K'_{2,1} D_0 D_1^{m-3} + \dots + K'_3 + \dots + K'_m\,, \end{aligned} \tag{3.25}$$

and we know that

$$\begin{gathered} K_1 = \sum_{q \leq u \leq m-1} K_{1,u} \xi_0^u \xi_1^{m-u-1}\,, \\ K_j = \sum_{0 \leq u \leq m-j} K_{j,u} \xi_0^u \xi_1^{m-j-u} \quad \text{for } j > 1\,. \end{gathered}$$

As in the scalar case, we denote the indexes linked to the Puiseux series, Newton's diagram and Gevrey's indexes:

$$g_{j,u} := \frac{m-j-u}{m-u} = 1 - \frac{j}{m-u}\,.$$

These indices permit to characterize Gevrey's classes of indices $1/g_{ju}$. In particular

$$g_{2,0} = \frac{m-2}{m} = 1 - \frac{2}{m}\,, \quad g_{1,q} = \frac{p-1}{p} = 1 - \frac{1}{p}\,,$$

and it implies $g_{2,0} < g_{1,q}$, since we have

$$-\frac{2}{m} < -\frac{1}{p}\,.$$

Proposition 3.8. *Assume the coefficients of h are $\mathcal{C}^\infty$. Then the condition*

$$(T_{1,0})_p^1 = (b_0)_1^p(x) = 0$$

is necessary for the Cauchy problem in $\mathcal{C}^\infty$ begin well posed.

Proof. This propostion is equivalent to state that if $(b_0)_1^p \neq 0$ in an everywhere dense set, then the Cauchy problem is not well posed.

On the other hand, if the Cauchy problem is well posed near $\underline{x}$, we deduce from the closed graph theorem that there exists a neighborhood $\mathcal{U}$ of $\underline{x}$, an integer τ, a compact $\mathcal{K} \subset \mathcal{U}$, a constant $C > 0$ such that for any $u \in \mathcal{C}^\infty$ we have

$$|u|_{k_0,\mathcal{K}} \le C\big[|hu|_{\tau,\mathcal{K}} + \big|u(\underline{x}_0, x')|_{\tau,\mathcal{K}\cap\{x_0=\underline{x}_0\}}\big],$$

where $|u|_{\tau,\mathcal{K}}$ denotes, as usual, a semi-norm of $\mathcal{C}^\infty$ (the upper bound of the derivatives up to the order τ in the compact $\mathcal{K}$) [23, 27].

We know also that

$$A' = \mathscr{A}' D_0^q + \mathscr{A}_1' + \dots + \mathscr{A}_j' + \dots$$

(where the order of A_j' is $m - j - 1$), and $a\mathscr{A} = \xi_0^p$. We follow Proposition 2.2 and we consider, as before, the notation $\tilde{}$,

$$\widetilde{h} = D_0\, I_m + J\,|D'| + \widetilde{b}_0(x, D') + \widetilde{b}_1(x, D') + \dots .$$

For the sake of simplicity, we denote

$$\widetilde{b}_0(x, \xi') = \widetilde{b}_0(x), \quad \widetilde{b}_1(x, \xi') = \widetilde{b}_0(x)\,|\xi'|^{-1}$$

and we let $|\xi'| = 1$. Then

$$\widetilde{A}(x, \xi_1) = \widetilde{\mathscr{A}}\xi_0^q + \widetilde{\mathscr{A}_1} + \dots .$$

(the order of $\mathscr{A}_j'$ is $m - j - 1$)

$$\widetilde{A}_0 = \xi_0^{p-1} I - J\xi_0^{p-2}|\xi'| + \dots + (-1)^{p-1} J^{p-1}|\xi'|^{p-1}\,.$$

We follow the proof of the necessity of $(b_0)_1^p = 0$ obtained in [25, pp. 306-309], and Kajitani [10, p. 536], (cf. [8, 3, 27]).

We will construct an asymptotic expansion which contradicts the inequality of the closed graph's Theorem. Let

$$u_{k_0} = \exp\Big(i\varphi(x, \xi) + \psi(x)\xi_1^{\frac{p-1}{p}}\Big) \sum_{0\le k\le k_0} Y_k(x)\xi_1^{-k/p}$$

$$D_0\varphi = \lambda(x, D'\varphi)\,, \quad \varphi(\underline{x}_0, x', y') = x'y'\,, \quad \psi(\underline{x}_0, x') = 0\,.$$

After microlocalization, in a canonical neighborhood

$$\widetilde{u}_{k_0} = \exp\Big(ix_1\xi_1 + \psi(x)\xi_1^{\frac{p-1}{p}}\Big) \sum_{0\le k\le k_0} \widehat{Y}_k(x)\xi_1^{-k/p}\,.$$

We recall that $k = h \circ A'$ and $\widetilde{k} = \widetilde{h} \circ \widetilde{A}'$. First we calculate $\widetilde{k}(\widetilde{u}_{k_0})$. The first term of the expansion is (cf. (3.25))

$$\begin{aligned}&\Big[(D_0\psi)^m \xi_1^{\frac{m(p-1)}{p}} + i^{p-1}(\widetilde{b}_0)_1^p (D_0\psi)^q \xi_1^{p-1+q\frac{(p-1)}{p}}\Big]\widetilde{Y}_0\\ &= (D_0\psi)^q \xi_1^{\frac{m(p-1)}{p}}\big[(D_0\psi)^p + i^{p-1}(b_0)_1^p\big]\widetilde{Y}_0\end{aligned}$$

We choose ψ so that

$$D_0\psi = -i^{\frac{p-1}{p}}[(b_0)_1^p]^{1/p},$$
$$\psi_1(\underline{x_0}, x') = 0,$$

where $D_0\psi$ is a complex root such that $\operatorname{Re} D_0\psi > 0$.

So the coefficients of $\widetilde{Y}_0$ is zero, $\widetilde{Y}_0$ is free and can be chosen different from 0. Then we cancel the following terms in the expansion of $\widetilde{k}(\widetilde{u}_{k_0})$; as usual [27, 22] we obtain that $\widetilde{Y}_0$ satisfies an ordinary differential equation of order $p-1$ and we cancel in the same manner the successive terms of the expansion.

For a large enough $k_0 > 0$ we obtain an expansion such that, for some k_0',

$$\big|\widetilde{k}(\widetilde{u}_{k_0})\big|_{\tau,\mathcal{K}} = O(\xi_1^{-k_0'}) \exp\big[\sup_{\mathcal{K}} \operatorname{Re}\psi(x)\xi_1^{\frac{p-1}{p}}\big] \quad \text{when } \xi_1 \to +\infty\,.$$

We have also

$$|\widetilde{u}_{k_0}|_{0,\mathcal{K}} = \exp\big[\sup_{\mathcal{K}} \operatorname{Re}\psi(x)\xi_1^{\frac{p-1}{p}}\big]\big(U + o(\xi_1^{-1})\big) \quad \text{as } \xi_1 \to +\infty\,,$$

for some $U \neq 0$, by the choice of $\widetilde{Y}_0(x) \neq 0$ in $\mathcal{U}$, and

$$\big|\widetilde{u}_{k_0}(\underline{x}_0, x')|_{\tau,\mathcal{K}\cap\{x_0=\underline{x}_0\}} = O(\xi_1^{\tau}) \exp\big[\sup_{\mathcal{K}} \operatorname{Re}\psi(\underline{x}_0, x')\xi_1^{\frac{p-1}{p}}\big]\big(U + o(\xi_1^{-1})\big)\,,$$

as $\xi_1 \to +\infty$.

We apply $\widetilde{A}'$ to the expansion and we obtain an analogous expansion

$$\widetilde{A}'(\widetilde{u}_{k_0}) = \exp\Big(ix_1\xi_1 + \psi(x)\xi_1^{\frac{p-1}{p}}\Big) \sum_{0\le k\le k_0} \widetilde{Z}_k(x)\xi_1^{-k/p}$$

with

$$\begin{aligned}\widetilde{Z}_0 = \xi_1^{\frac{(m-1)(p-1)}{p}} &\big[I(D_0\psi)^{m-1} + \dots + J(D_0\psi)^{m-2}D_1\psi \\ &+ \dots + (-1)^{p-1}J^{p-1}(D_0\psi)^q(D_1\psi)^{p-1}\big]\widetilde{Y}_0\end{aligned}$$

as before we obtain the terms $\widetilde{Z}_k$. $\widetilde{Z}_0$ is free and can be chosen $\neq 0$, as $\widetilde{Y}_0$.

For a large enough k, we have

$$\big|\widetilde{h} \circ \widetilde{A}'(u_{k_0})|_{\tau,\mathcal{K}} = O(\xi_1^{-N_0}) \exp\big[\sup_{\mathcal{K}} \operatorname{Re}\psi(x)\xi_1^{\frac{p-1}{p}}\big] \quad \text{as } \xi_1 \to +\infty\,,$$

and we have $|\widetilde{u}_{k_0}|_{p,k}$ and $|\widetilde{u}_{k_0}(\underline{x}_0, x')|_{\tau,K\cap\{\underline{x}=\underline{x}_0\}\}}$ as before.

We come back to h and we obtain an asymptotic expansion which contradict the inequality of the closed graph theorem. $\square$

The same calculus, correctly interpreted, permits to obtain the following result.

Proposition 3.9. *We assume that the coefficients of h are analytic. Then the condition $(T_{1,0})_p^1 \neq 0$ is sufficient for the Cauchy problem begin well posed in the Gevrey class $\gamma^{\frac{p}{p-1}}$.*

Sketch of the proof. We construct a parametrix from the previous expansion as in [27].

If $z = 1$ then case (1-β) is complete and we have obtained the proof of the necessity of (L_1).

We assume now $(T_{1,0})_p^1 = 0$. If $z > 1$ and $q < u \le p-1$, we have

$$J\mathscr{A}_{1,u} + \mathscr{A}_{1,u-1} + b_0\Lambda_{0,u-q} = K_{1,u}\,. \tag{3.26}$$

We have

$$(\mathscr{A}_{1,u-1})^p + (-1)^{z+u+1}(b_0)^p J^{m-u-1} = (K_{1,u})^p, \tag{3.27}$$

$$(K_{1,u})_p^p = (-1)^{z+u+1}(b_0)_{u-q+1}^p = (-1)^{p+1}(T_{1,u-q})_p^1\,. \tag{3.28}$$

As before we choose $(\mathscr{A}_{1,u})$ so that

$$(K_{1,u})_1^1 = \dots = (K_{1,u})_{p-1}^{p-1} = (K_{1,u})_p^p\,,$$
$$(K_{1,u})_j^p \quad \text{for } j \neq p\,. = 0\,.$$

In the same manner $(K_{1,u})_{j'}^j = 0$, for $j' \neq j$. By (3.22),

$$\begin{gathered}(\mathscr{A}_{1,u})^k + (\mathscr{A}_{1,u-1})^{k-1} + (b_0)_1^{k-1}(\Lambda_{0,u-q})^1 = (K_{1,u})^{k-1},\\ (\mathscr{A}_{1,u})^{p+2} + (\mathscr{A}_{1,u-1})^{p+1} + (b_0)_1^{p+1}(\Lambda_{0,u-q})^1 = (K_{1,u})^{p+1}\,.\end{gathered} \tag{3.29}$$

We let

$$(\mathscr{A}_{1,u})_{p+1}^{p+2} + (\mathscr{A}_{1,u-1})_{p+1}^{p+1} + \underbrace{(b_0)_1^{p+1}(\Lambda_{0,u-q})_{p+1}^1}_{=0} = (K_{1,u})_p^p,$$

so that

$$(K_{1,u})_{p+1}^{p+1} = (K_{1,u})_p^p\,.$$

letting $j \neq p+1$, we have

$$(\mathscr{A}_{1,u})_j^{p+2} + (\mathscr{A}_{1,u-1})_j^{p+1} + (b_0)_1^{p+1}(\Lambda_{0,u-q})_j^1 = 0\,.$$

So that $(K_{1,u})_j^{p+1} = 0$. In the same manner,

$$(K_{1,u})_{p+2}^{p+2} = (K_{1,u})_{m-1}^{m-1} = (K_{1,u})_p^p,$$
$$\text{line } m: \quad (\mathscr{A}_{1,u-1})^m + (b_0)^m\Lambda_{0,u-q} = (K_{1,u})^m,$$
$$(-1)^{q-1}[(\mathscr{A}_{1,u-q})_m^{p+1} + (\Lambda_{1,u-q})_m^{p+1}] = (K_{1,u})_m^m\,.$$

We let

$$(\mathscr{A}_{1,u-q})_m^{p+1} = (-1)^z(T_{1,u-q})_p^1 - (\Lambda_{1,u-q})_m^{p+1}\,,$$

and for $0 \leq v \leq z-1$ we define

$$(K_{1,v})_m^{p+1} = (-1)^z(T_{1,v})_p^1 - (\Lambda_{1,v})_m^{p+1} \tag{3.30}$$

Since

$$(K_{1,u})_m^m = (K_{1,u})_p^p,$$

it follows that

$$\begin{aligned}&(-1)^{q-1}(\mathscr{A}_{1,u-q})_j^{p+1} - (b_0)_1^{m-1}(\Lambda_{0,u-q-1})_j^1 + \dots + (-1)^{u+q}(b_0)_1^{m-u+q}(\Lambda_{0,0})_j^1\\ &= (K_{1,u})_j^m\,.\end{aligned}$$

Then for $j \neq p, m$, we choose $(\mathscr{A}_{1,u-q})_j^{p+1}$ such that $(K_{1,u})_j^m = 0$. We let

$$(\mathscr{A}_{1,u-q})_p^{p+1} = -(\Lambda_{1,u-q})_p^{p+1}\,,$$

and for $0 \leq v \leq z-1$ we define

$$(\mathscr{A}_{1,v})_p^{p+1} = -(\Lambda_{1,v})_p^{p+1} \tag{3.31}$$

So $(K_{1,u})_p^m = 0$, and

$$K_{1,u} = (-1)^{p+1}(T_{1,u-q})_p^1 I \quad \text{for } q \leq u \leq p-1\,. \tag{3.32}$$

□

Proposition 3.10. *If $0 \leq u \leq p-1$, then*

$$K_{1,u} = 0 \iff (\mathrm{L}_1) \text{ is satisfied}.$$

3.2.2. *Case (2): $p \le u \le m-2$.* By (3.20) with $u = p$, we have

$$J\mathscr{A}_{1,p} + \mathscr{A}_{1,p-1} + b_0\Lambda_{0,z} = K_{1,p}\,,$$
$$\text{line } k \text{ with } 1 \le k-1 < p: \quad (\mathscr{A}_{1,p})^k + (\mathscr{A}_{1,p-1})^{k-1} = (K_{1,p})^{k-1},$$
$$\text{line } p: \quad (\mathscr{A}_{1,p-1})^p + (b_0)^p(\Lambda_{0,z})_p = (K_{1,p})^p\,.$$

Using (3.21),

$$\begin{gathered}(-1)^{p-1}(\mathscr{A}_{1,0})_p^1 + (K_{1,p-1})_p^{p-1} + (K_{1,p-2})_p^{p-2} + \dots \\ + (-1)^{z+1}(K_{1,q})_p^q + (b_0)_{z+1}^p(\Lambda_{0,z})_p^{z+1} = (K_{1,p})_p^p,\end{gathered}$$

and

$$(-1)^{p-1}(\mathscr{A}_{1,0})_p^1 + (-1)^z(b_0)_{z+1}^p(\Lambda_{0,0})_p^1 = (K_{1,p})_p^p$$

or

$$(-1)^{p-1}\big[(\mathscr{A}_{1,0})_p^1 + (\Lambda_{1,0})_p^1\big] = (K_{1,p})_p^p\,.$$

Let us define

$$(\mathscr{A}_{1,0})_p^1 = (-1)^z(\Lambda_{1,z})_m^{p+1} = (-1)^z(b_0)_{p+1}^m\,. \tag{3.33}$$

Then

$$\begin{aligned}(K_{1,p})_p^p &= (-1)^{p-1}\big[(-1)^z(\Lambda_{1,z})_m^{p+1} + (\Lambda_{1,0})_p^1\big] \\ &= (-1)^{q-1}\big[(b_0)_{p+1}^m + (b_0)_{z+1}^p\big]\,.\end{aligned}$$

Therefore,

$$\begin{aligned}(K_{1,p})_m^p &= (-1)^{p-1}(\mathscr{A}_{1,0})_m^1 + (b_0)^p(\Lambda_{0,z})_m \\ &= (-1)^{p-1}\big[(\mathscr{A}_{1,0})_m^1 + (\Lambda_{1,0})_m^1\big]\,.\end{aligned}$$

Also we define

$$(\mathscr{A}_{1,0})_m^1 = -(\Lambda_{1,0})_m^1, \quad (K_{1,p})_m^p = 0\,. \tag{3.34}$$

Using (3.22),

$$(K_{1,p})^m = (\mathscr{A}_{1,p-1})^m + (b_0)^m\Lambda_{0,z}\,.$$

Using (3.20),

$$\begin{aligned}(K_{1,p})^m &= (-1)^{q-1}(\mathscr{A}_{1,z})^{p+1} + (b_0)^m\Lambda_{0,z} - (b_0)_1^{m-1}(\Lambda_{0,z-1})^1 \\ &\quad + (K_{1,p-1})^{m-1} + (K_{1,p-1})^{m-2} + \dots\,, \\ (K_{1,p})_m^m &= (-1)^{q-1}\big[(\mathscr{A}_{1,z})_m^{p+1} + (\Lambda_{1,z})_m^{p+1}\big]\end{aligned}$$

Let us define

$$(\mathscr{A}_{1,z})_m^{p+1} = (-1)^z(\Lambda_{1,0})_p^1\,. \tag{3.35}$$

Then

$$\begin{gathered}(K_{1,p})_m^m = (K_{1,p})_p^p\,, \\ (K_{1,p})_p^m = (-1)^{q-1}\big[(\mathscr{A}_{1,z})_p^{p+1} + (\Lambda_{1,z})_p^{p+1}\big]\,.\end{gathered}$$

We let

$$(\mathscr{A}_{1,z})_p^{p+1} = -(\Lambda_{1,z})_p^{p+1}\,; \tag{3.36}$$

so that $(K_{1,p})_p^m = 0$; we have also $(K_{1,p})_j^m = 0$ for $j \ne p, q$. Then we obtain

$$(K_{1,p})_1^1 = (K_{1,p})_{p-1}^{p-1} = (K_{1,p})_{p+1}^{p+1} = (K_{1,p})_{m-1}^{m-1} = (K_{1,p})_p^p\,.$$

Finally

$$K_{1,p} = (-1)^{q-1}\big[(b_0)_{p+1}^m + (b_0)_{z+1}^p\big]I$$

or

$$K_{1,p} = (-1)^p \big[(\Lambda_{1,0})_p^1 + (-1)^z (\Lambda_{1,z})_m^{p+1}\big] \tag{3.37}$$

Now for $p < u \le m-2$, we have

$$J\mathscr{A}_{1,u} + \mathscr{A}_{1,u-1} + b_0 \Lambda_{0,u-q} = K_{1,u},$$

$$\begin{aligned} \text{line } p: \quad & (-1)^{p+1}(\mathscr{A}_{1,u-p})^1 + (b_0)^p \Lambda_{0,u-q} - (K_{1,u-1})^{p-1} + (K_{1,u-2})^{p-2} + \dots \\ & + (-1)^p (K_{1,u-p+11})^1 = (K_{1,u})^p \,. \end{aligned}$$

So

$$(-1)^{p-1}\big[(\mathscr{A}_{1,u-p})_p^1 + (\Lambda_{1,u-p})_p^1\big] = (K_{1,u})_p^p \,.$$

We let

$$(\mathscr{A}_{1,u-p})_p^1 = (-1)^z (\Lambda_{1,u-q})_m^{p+1} \,,$$

and for $0 \le v \le q-1$ we define

$$(\mathscr{A}_{1,v})_p^1 = (-1)^z (\Lambda_{1,v+z})_m^{p+1} \,. \tag{3.38}$$

Then

$$(K_{1,u})_p^p = (-1)^{p-1}\big[(-1)^z (\Lambda_{1,u-q})_m^{p+1} + (\Lambda_{1,u-p})_p^1\big] \tag{3.39}$$

Therefore,

$$\begin{aligned} (K_{1,u})_m^p &= (-1)^{p-1}(\mathscr{A}_{1,u-q})_m^1 + (b_0)^p (\Lambda_{0,u-q})_m \\ &= (-1)^{p-1}\big[(\mathscr{A}_{1,u-q})_m^1 + (\Lambda_{1,u-q})_m^1\big] \end{aligned}$$

Then we let $(\mathscr{A}_{1,u-p})_m^1 = -(\Lambda_{1,u-p})_m^1$.

For $0 \le v \le q-1$, we define

$$(\mathscr{A}_{1,v})_m^1 = -(\Lambda_{1,v})_m^1 \,. \tag{3.40}$$

Then $(K_{1,u})_m^p = 0$, and

$$(K_{1,u})_j^p = 0 \quad \text{for } j \ne p, m \,,$$

$$\begin{aligned} \text{Line } m: \quad & (-1)^{q+1}(\mathscr{A}_{1,u-q})^{p+1} + (b_0)^m \Lambda_{0,u-q} - (b_0)^{m-1}(\Lambda_{0,k-q-1})^1 \\ & + \dots - (K_{1,u-1})^{m-1} + (K_{1,u-2})^{m-2} + \dots \\ & = (K_{1,u})^m, \end{aligned}$$

$$\begin{aligned} (K_{1,u})_m^m &= (-1)^{q-1}(\mathscr{A}_{1,u-q})_m^{p+1} + (-1)^{z+u+1}(b_0)_{u+1}^m \\ &= (-1)^{q-1}\big[(\mathscr{A}_{1,u-q})_m^{p+1} + (\Lambda_{1,u-q})_m^{p+1}\big] \,. \end{aligned}$$

We let

$$(\mathscr{A}_{1,u-q})_m^{p+1} = (-1)^z (\Lambda_{1,u-p})_p^1 \,.$$

So that $(K_{1,u})_m^m = (K_{1,u})_p^p$.

For $z \le v \le p-2$, let us define

$$(K_{1,v})_m^{p+1} = (-1)^z (\Lambda_{1,v-z})_p^1, \tag{3.41}$$

Then

$$(K_{1,u})_p^m = (-1)^{q-1}\big[(\mathscr{A}_{1,u-q})_p^{p+1} + (\Lambda_{1,u-q})_p^{p+1}\big] \,.$$

We let

$$(\mathscr{A}_{1,u-q})_p^{p+1} = -(\Lambda_{1,u-q})_p^{p+1} \,,$$

and for $z \le v \le p-2$, we define

$$(\mathscr{A}_{1,v})_p^{p+1} = -(\Lambda_{1,v})_p^{p+1} \tag{3.42}$$

Also we obtain

$$(K_{1,u})^m_j = 0 \quad \text{for } j \neq p, m \,,$$

and

$$(K_{1,u})^1_1 = \dots = (K_{1,u})^{p-1}_{p-1} = (K_{1,u})^{p+1}_{p+1} = \dots = (K_{1,u})^{m-1}_{m-1} = (K_{1,u})^m_m$$

Finally for $p \le u \le m-2$, we have

$$\begin{aligned} K_{1,u} &= (-1)^{q-1}\big[(\Lambda_{1,u-p})^1_p + (-1)^z(\Lambda_{1,u-q})^{p+1}_m\big] I \\ &= (-1)^{z+u+1}\big[(b_0)^p_{u-q+1} + (b_0)^m_{u+1}\big] \end{aligned} \tag{3.43}$$

By (3.14),

$$(-1)^{p-1}(T_{2,z})^1_m = (b_0)^p_{p+1}\big[(b_0)^p_{z+1} + (b_0)^m_{p+1}\big] \,.$$

So that, if $(b_0)^p_{p+1} \neq 0$, then

$$(K_{1,p})^p_p = (-1)^z \frac{(T_{2,z})^1_m}{(b_0)^p_{p+1}} \,.$$

We have also (cf. (3.17))

$$(-1)^{p-1}(T_{2,z})^{p+1}_p = (-1)^z(b_0)^m_1\big[(\Lambda_{1,0})^1_p + (-1)^z(\Lambda_{1,z})^{p+1}_m\big] \,.$$

So that, if $(b_0)^m_1 \neq 0$, then

$$(K_{1,p})^p_p = (-1)^z \frac{(T_{2,z})^{p+1}_p}{(b_0)^m_1} \,.$$

If $(b_0)^p_{p+1} = 0$ or $(b_0)^m_1 = 0$, then

$$(T_{2,z})^1_m = (T_{2,z})^{p+1}_p = 0$$

Then we have

$$(-1)^{p-1}(T_{2,z+1})^1_m = (-1)^z(b_0)^p_{p+2}\big[(\Lambda_{1,0})^1_p + (-1)^z(\Lambda_z)^{p+1}_m\big] \,.$$

So that, if $(b_0)^p_{p+2} \neq 0$, then

$$(K_{1,p})^p_p = (-1)^z \frac{(T_{2,z+1})^1_m}{(b_0)^p_{p+2}} \,.$$

In the same manner, if $(b_0)^{m-1}_1 \neq 0$, then

$$(K_{1,p})^p_p = (-1)^z \frac{(T_{2,z+1})^{p+1}_p}{(b_0)^{m-1}_1} \,.$$

If $(b_0)^p_{m-1} \neq 0$, then

$$(K_{1,p})^p_p = (-1)^z \frac{(T_{2,p-2})^1_m}{(b_0)^p_{m-1}} \,.$$

If $(b_0)^{p+2}_1 \neq 0$, then

$$(K_{1,p})^p_p = (-1)^z \frac{(T_{2,p-2})^{p+1}_p}{(b_0)^{p+2}_1} \,.$$

More generally for $p \le u \le m-2$, we have

$$(K_{1,u})^p_p = (-1)^{p-1}\big[(-1)^z(\Lambda_{1,u-q})^{p+1}_m + (\Lambda_{1,u-p})^1_p\big] \,. \tag{3.44}$$

If $(T_{2,z})^1_m = \dots = (T_{2,u-q-1})^1_m = 0$ and $(b_0)^p_{p+1} \neq 0$, then

$$(K_{1,u})^p_p = (-1)^z \frac{(T_{2,u-q})^1_m}{(b_0)^p_{p+1}} \,. \tag{3.45}$$

If $(T_{2,z})_p^{p+1} = \dots = (T_{2,u-q-1})_p^{p+1} = 0$ and $(b_0)_1^m \neq 0$ (cf. (3.18)), then

$$(K_{1,u})_p^p = (-1)^z \frac{(T_{2,u-q})_p^{p+1}}{(b_0)_1^m}\,.$$

If $(b_0)_{m-1}^p \neq 0$, then

$$(K_{1,u})_p^p = (-1)^z \frac{(T_{2,u-2})_m^1}{(b_0)_{m-1}^p}\,.$$

If $(b_0)_1^{p+2} \neq 0$, then

$$(K_{1,u})_p^p = (-1)^z \frac{(T_{2,u-2})_p^{p+1}}{(b_0)_1^{p+2}}\,.$$

4. Study of K_2 ($K_{2,u}$ for $u \le p-2$)

In this section we assume that (L$_1$) is satisfied.

4.1. Study of (L$_2$).

We already know (3.9), (3.11), (3.15), and (3.14).

If $1 \le k-1 \le p-1$ or $p+1 \le k-1 \le m-1$, then

$$(T_{2,u})^k = -(T_{2,u-1})^{k-1}\,, \quad (T_{2,u})_k = -(T_{2,u-1})_{k+1}\,.$$

Now we obtain $(T_{2,u})_p^1$ explicitly for $0 < u \le p-2$.

4.1.1. *Case (I): $q \ge z$.* Subcase (I-1) $u < z$: We have

$$\begin{aligned}(-1)^{p-1}(T_{2,u})_p^1 &= (b_0)_{p+1}^p(\Lambda_{1,u})_p^{p+1} - (b_0)_{p+2}^p(\Lambda_{1,u-1})_p^{p+1} + \dots \\ &\quad + (-1)^u(b_0)_{p+u+1}^p(\Lambda_{1,0})_p^{p+1}\,.\end{aligned} \tag{4.1}$$

As an example we have:

$$\begin{gathered}(-1)^{p-1}(T_{2,0})_p^1 = (b_0)_{p+1}^p(\Lambda_{1,0})_p^{p+1}\,,\\ (-1)^{p-1}(T_{2,1})_p^1 = (b_0)_{p+1}^p(\Lambda_{1,1})_p^{p+1} - (b_0)_{p+2}^p(\Lambda_{1,0})_p^{p+1}\,,\\ (-1)^{p-1}(T_{2,z-1})_p^1 = (b_0)_{p+1}^p(\Lambda_{1,z-1})_p^{p+1} + \dots + (-1)^{z+1}(b_0)_{p+z}^p(\Lambda_{1,0})_p^{p+1}\,.\end{gathered}$$

Remark 4.1. (1) Assume $(b_0)_{p+1}^p \neq 0$, and $(T_{2,0})_p^1 = 0 \iff (b_0)_1^m = 0$. Then

$$(-1)^{p-1}(T_{2,1})_p^1 = (b_0)_{p+1}^p(-1)^{z+1}(b_0)_1^{m-1}\,.$$

So that

$$(T_{2,1})_p^1 = 0 \iff (b_0)_1^{m-1} = 0\,.$$

Assume $(b_0)_{p+1}^p \neq 0$ and $(T_{2,0})_p^1 = \dots = (T_{2,u-1})_p^1 = 0$. Then, by induction,

$$(T_{2,u})_p^1 = (-1)^{z+u}(b_0)_{p+1}^p(b_0)_1^{m-u} \quad \text{for } u < z\,.$$

So that

$$(T_{2,u})_p^1 = 0 \iff (b_0)_1^{m-u} = 0\,.$$

If $(b_0)_{p+1}^p \neq 0$ and $(T_{2,0})_p^1 = \dots = (T_{2,z-2})_p^1 = 0$, then

$$(b_0)_1^m = \dots = (b_0)_1^{m-z+1} = 0\,.$$

(2) Assume $(b_0)_{p+1}^p = 0$ and $(b_0)_{p+2}^p \neq 0$. Then

$$\begin{gathered}(-1)^{p+1}(T_{2,1})_p^1 = (b_0)_{p+2}^p(-1)^{z+1}(b_0)_1^m,\\ (T_{2,1})_p^1 = 0 \iff (b_0)_1^m = 0\,.\end{gathered}$$

In the same manner as above we obtain: If $(T_{2,0})^1_p = \dots = (T_{2,z-1})^1_p = 0$, then

$$(b_0)^m_1 = \dots = (b_0)^{m-z+2}_1 = 0\,.$$

(3) We generalize easily the above process to obtain: If $(b_0)^p_{p+1} = \dots = (b_0)^p_{p+k-1} = 0$, $(b_0)^p_{p+v} \neq 0$, $v \neq z-1$, and $(T_{2,0})^1_p = \dots = (T_{2,z-1})^1_p = 0$, then

$$(b_0)^m_1 = 0 \quad \dots \quad (b_0)^{m-z+v}_1 = 0\,.$$

If

$$\begin{gathered}(b_0)^p_{p+1} = \dots = (b_0)^p_{p+z-1} = 0, \quad (b_0)^p_{p+z} \neq 0,\\ (T_{2,0})^1_p = \dots = (T_{2,z-1})^1_p,\end{gathered}$$

then $(b_0)^m_1 = 0$.

(4) Assume $(b_0)^m_1 \neq 0$ and $(T_{2,0})^1_p = 0 \iff (b_0)^p_{p+1} = 0$. Then

$$(-1)^{p-1}(T_{2,1})^1_p = (-1)^{z+1}(b_0)^p_{p+2}(b_0)^m_1;$$

so that

$$(T_{2,1})^1_p = 0 \iff (b_0)^p_{p+2} = 0\,.$$

Assume $(b_0)^m_1 \neq 0$ and $(T_{2,0})^1_p = \dots = (T_{2,u-1})^1_p = 0$. Then

$$(T_{2,u})^1_p = (b_0)^p_{p+u+1}(b_0)^m_1\,.$$

If $(b_0)^m_1 \neq 0$ and $(T_{2,0})^1_p = \dots = (T_{2,z-1})^1_p = 0$, then

$$(b_0)^p_{p+1} = \dots = (b_0)^p_{p+z} = 0\,.$$

(5) If $(b_0)^m_1 = 0$ and $(b_0)^{m-1}_1 \neq 0$, then

$$(T_{2,2})^1_p \iff (b_0)^p_{z+3} = 0\,.$$

(6) In general, if $(b_0)^m_1 = \dots = (b_0)^{m-z+1}_1 = 0$, then $(b_0)^{m-z}_1 \neq 0$ and

$$(T_{2,z-1})^1_p = 0 \iff (b_0)^p_{p+z} = 0\,.$$

Lemma 4.2. By Remark 4.1-(1), using (3.17) and (3.18): if

$$(T_{2,0})^1_p = \dots = (T_{2,z-1})^1_p = 0, \quad (b_0)^p_{p+1} \neq 0,$$

and $z \leq u < 2z$, then $(T_{2u})^{p+1}_p = 0$.

Subcase (I-2) $z \leq u < q$: We have

$$\begin{aligned}(-1)^{p-1}(T_{2,u})^1_p = (-1)^z&\Big[(b_0)^p_{z+1}(\Lambda_{1,u-z})^1_p + \dots + (-1)^{u+z}(b_0)^p_{u+1}(\Lambda_{1,0})^1_p\Big]\\ &+ (b_0)^p_{p+1}(\Lambda_{1,u})^{p+1}_p\\ &+ \dots + (-1)^{k+1}(b_0)^p_{p+k}(\Lambda_{1,u-k+1})^{p+1}_p + \dots\\ &+ (-1)^u(b_0)^p_{p+u+1}(\Lambda_{1,0})^{p+1}_p\,.\end{aligned} \tag{4.2}$$

Example 4.3. $(-1)^{p-1}(T_{2,z})^1_p = \big[(b_0)^p_{z+1}\big]^2 + (b_0)^p_{p+1}(b_0)^{m-z}_1 + \dots + (b_0)^p_{p+z+1}(b_0)^m_1$.

Assume $(b_0)^p_{p+1} \neq 0$ and $(T_{2,0})^1_p = \dots = (T_{2,z-1})^1_p = 0$. Then

$$\begin{gathered}(-1)^{p-1}(T_{2,z})^1_p = \big((b_0)^p_{z+1}\big)^2 + (b_0)^p_{p+1}(b_0)^{m-z}_1\\ (-1)^{p-1}(T_{2,z+1})^1_p = -\big[2\,(b_0)^p_{z+1}(b_0)^p_{z+2} + (b_0)^p_{p+1}(b_0)^{m-z-1}_1 + (b_0)^p_{p+2}(b_0)^{m-z}_1\big]\,.\end{gathered}$$

Remark 4.4. Assume $(b_0)^p_{p+1} \neq 0$, $(T_{2,0})^1_p = \dots = (T_{2,z-1})^1_p = 0$, and $v < u$. Then $(T_{2,v})^1_m = 0$.

Using (3.14) and (3.18), if $u \geq 2z$, then

$$(b_0)^p_{p+1}(T_{2u+z})^{p+1}_p = (-1)^z(b_0)^{m-z}_1(T_{2u})^1_m\,.$$

Subcase (I-3) $q \leq u \leq p-2$: We have

$$\begin{aligned}(-1)^{p-1}(T_{2,q})^1_p &= (-1)^z\big[(b_0)^p_{z+1}(\Lambda_{1,q-z})^1_p + \dots + (-1)^p(b_0)^p_{u+1}(\Lambda_{1,0})^1_p\big] \\ &\quad + (-1)^q(b_1)^p_1 - (b_0)^p_{p+2}(\Lambda_{1,q-1})^{p+1}_p + \dots \\ &\quad - (-1)^{q+1}(b_0)^p_m(\Lambda_{1,1})^{p+1}_p,\end{aligned} \tag{4.3}$$

$$\begin{aligned}(-1)^{p-1}(T_{2,u})^1_p &= (-1)^z\big[(b_0)^p_{z+1}(\Lambda_{1,u-z})^1_p + \dots + (-1)^{u+z}(b_0)^p_{u+1}(\Lambda_{1,0})^1_p\big] \\ &\quad + (-1)^{z+u}(b_1)^p_{u-q+1} + (b_0)^p_{p+1}(\Lambda_{1,u})^{p+1}_p, + \dots \\ &\quad + (-1)^{q+1}(b_0)^p_m(\Lambda_{1,u-q+1})^{p+1}_p,\end{aligned} \tag{4.4}$$

$$\begin{aligned}(-1)^{p-1}(T_{2,p-2})^1_p &= (-1)^z\Big[(b_0)^p_{z+1}(\Lambda_{1,q-2})^1_p + \dots + (-1)^q(b_0)^p_{p-1}(\Lambda_{1,0})^1_p\Big] \\ &\quad + (-1)^q(b_1)^p_{z-1} + (b_0)^p_{p+2}(\Lambda_{1,p-2})^{p+2}_p + \dots \\ &\quad + (-1)^{q+1}(b_0)^p_m(\Lambda_{1,z-1})^{p+1}_p\,.\end{aligned} \tag{4.5}$$

4.1.2. *Case (II): $q \leq z$.* Subcase (II-1) $u < q$: We have

$$(-1)^{p-1}(T_{2,u})^1_p = (b_0)^p_{p+1}(\Lambda_{1,u})^{p+1}_p + \dots + (-1)^u(b_0)^p_{p+u+1}(\Lambda_{1,0})^{p+1}_p\,. \tag{4.6}$$

Subcase (II-2) $q \leq u < z$: We have

$$\begin{aligned}(-1)^{p-1}(T_{2,u})^1_p &= (b_0)^p_{p+1}(\Lambda_{1,u})^{p+1}_p + \dots \\ &\quad + (-1)^{q+1}(b_0)^p_m(\Lambda_{1,u-q+1})^{p+1}_p + (-1)^{z+u}(b_1)^p_{u-q+1}\,.\end{aligned} \tag{4.7}$$

Subcase (II-3) $q \leq z \leq u \leq p-2$: We have

$$\begin{aligned}(-1)^{p-1}(T_{2,u})^1_p &= (-1)^z\big[(b_0)^p_{z+1}(\Lambda_{1,u-z})^1_p \\ &\quad + \dots + (b_0)^p_{p+1}(\Lambda_{1,u})^{p+1}_p + \dots \\ &\quad + (-1)^{q+1}(b_0)^p_m(\Lambda_{1,u-q+1})^{p+1}_p \\ &\quad + (-1)^{u+z}(b_0)^p_{u+1}(\Lambda_{1,0})^1_p\big] + (-1)^{z+u}(b_1)^p_{u-q+1}\,.\end{aligned} \tag{4.8}$$

Note that for case (II), the previous Remark 4.4 is valuable.

Now we consider $(T_{2,u})^{p+1}_m$. We have previously obtained (3.15) and (3.16).

If $z \leq u \leq p-2$. First, we have

$$(-1)^{p-1}(T_{2,z})^{p+1}_m = (-1)^z(b_0)^m(\Lambda_{1,0})_m = (-1)^z(b_0)^m_1(\Lambda_{1,0})^1_m = (b_0)^m_1(b_0)^p_{p+1}\,.$$

So that

$$(T_{2,z})^{p+1}_m = (-1)^z(T_{2,0})^1_p \tag{4.9}$$

We replace u by $v+z$, with $1 \leq v \leq q-2$. Then

$$\begin{aligned}&(-1)^{p-1}(T_{2,v+z})^{p+1}_m \\ &= (-1)^z\Big((b_0)^m(\Lambda_{1,v})_m + \dots + (-1)^v(b_0)^{m-v}_1(\Lambda_{1,0})^1_m\Big) \\ &= (-1)^z\Big[(b_0)^m_1(\Lambda_{1,v})^1_m - (b_0)^{m-1}_1(\Lambda_{1,v-1})^1_m + \dots + (-1)^v(b_0)^{m-v}_1(\Lambda_{1,0})^1_m \\ &\quad + (b_0)^m_{p+1}(\Lambda_{1,v})^{p+1}_m + \dots + (-1)^{v+z}(b_0)^m_{v+q+1}(\Lambda_{1,z})^{p+1}_m\Big]\end{aligned}$$

By (4.1) and (4.6), we have the value of $(T_{2,u})^1_p$. Then we compare to obtain

$$\begin{aligned}(-1)^{p-1}(T_{2,v+z})^{p+1}_m &= (-1)^z\big[(-1)^{p-1}(T_{2,v})^1_p\big]\\ &\quad + (-1)^v\big[(b_0)^m_{p+1}(b_0)^m_{q+v+1} + \dots + (b_0)^m_{v+q+1}(b_0)^m_{p+1}\big]\end{aligned}$$

Using (3.41),

$$\begin{aligned}(-1)^{p-1}(T_{2v+z})^{p+1}_m &= (-1)^z\big[(-1)^{p-1}(T_{2v})^1_p\big]\\ &\quad + (-1)^{v+q}K_{1,p}\big[(b_0)^p_{v+1} - (b_0)^m_{v+q+1}\big] + \dots\\ &\quad + (-1)^p K_{1,v+q}\big[(b_0)^p_{z+1} - (b_0)^m_{p+1}\big]\end{aligned}$$

If we assume $K_{1,p} = \dots = K_{1,v+q-1} = 0$, then

$$(-1)^{p-1}(T_{2v+z})^{p+1}_m = (-1)^z\big[(-1)^{p-1}(T_{2v})^1_p\big] + (-1)^p K_{1,v+q}\big[(b_0)^p_{z+1} - (b_0)^m_{p+1}\big]\,.$$

Lemma 4.5. *Assume that $v < u$ implies $(T_{1,v})^1_m = 0$.*

(1) *If $(b_0)^p_{p+1} \neq 0$, using (3.45), for $1 \le v \le q-2$ we have*

$$(T_{2v+z})^{v+1}_m = (-1)^z\Big[(T_{2,v})^1_p - \frac{(b_0)^p_{z+1} - (b_0)^m_{p+1}}{(b_0)^p_{p+1}}\,(T_{2,v})^1_m\Big],$$

$$(T_{2v+z})^{v+1}_m = (-1)^z\Big[(T_{2,v})^1_p - \frac{2\,(b_0)^p_{z+1}}{(b_0)^p_{p+1}}\,(T_{2,v})^1_m\Big]\,.$$

If $(T_{2,z})^1_m = 0$, then $(T_{2,2z})^{p+1}_m = (-1)^z(T_{2,z})^1_p$.

(2) *If $(b_0)^p_{p+1} = \dots = (b_0)^p_{p+k} = 0$, and $(b_0)^p_{p+k+1} \neq 0$, then for $k \le q-3$ we have*

$$(T_{2,v+z})^{p+1}_m = (-1)^z\Big[(T_{2,v})^1_p - (-1)^k\,\frac{2\,(b_0)^p_{z+k+1}}{(b_0)^p_{p+k+1}}\,(T_{2,v})^{p+1}_p\Big]$$

(3) *When $(b_0)^m_{p+1} = (b_0)^m_1 = 0$, we can extend easily the above results to the cases $(b_0)^p_{p+2} \neq 0,\ \dots,\ (b_0)^{m-1}_1 \neq 0,\ \dots$.*

4.2. **Study of $K_{2,u}$, $u \le p-2$.** We identify the homogeneous part of the symbols of order $m-2$,

$$a\mathscr{A}_2 + b_0\mathscr{A}_1 + D_0\mathscr{A}_1 + JD_1\mathscr{A}_1 + b_1\Lambda_0\xi_0^q = K_2\,.$$

We order this by the powers of ξ_0, $0 \le u \le m-2$:

$$J\mathscr{A}_{2,u} + \mathscr{A}_{2,u-1} + b_0\mathscr{A}_{1,u} + D_0\mathscr{A}_{1,u} + JD_1\mathscr{A}_{1,u} + b_1\Lambda_{0,u-q} = K_{2,u}\,,$$

line p:

$$(\mathscr{A}_{2,u-1})^p + (b_0)^p\mathscr{A}_{1,u} + D_0(\mathscr{A}_{1,u})^p + (b_1)^p\Lambda_{0,u-q} = (K_{2,u})^p\,, \tag{4.10}$$

line $k-1$, $1 \le k-1 \le p-1$:

$$(\mathscr{A}_{2,u})^k + (\mathscr{A}_{2,u-1})^{k-1} + D_0(\mathscr{A}_{1,u})^{k-1} + D_1(\mathscr{A}_{1,u})^k = (K_{2,u})^{k-1}\,, \tag{4.11}$$

line m:

$$(\mathscr{A}_{2,u-1})^m + (b_0)^m\mathscr{A}_{1,u} + D_0(\mathscr{A}_{1,u})^m + (b_1)^m\Lambda_{0,u-q} = (K_{2,u})^m\,, \tag{4.12}$$

line $k-1$, $p+1 \le k-1 \le m-1$:

$$\begin{aligned}(\mathscr{A}_{2,u})^k + (\mathscr{A}_{2,u-1})^{k-1} + (b_0)^{k-1}_1\mathscr{A}_{1,u} + D_0(\mathscr{A}_{1,u})^k\\ + D_1(\mathscr{A}_{1,u})^k + (b_1)^{k-1}_1(\Lambda_{0,u-q})^1 = (K_{2,u})^{k-1}\,.\end{aligned} \tag{4.13}$$

We distinguish two cases

Case: $0 < u \le p-2$. By (3.6), (3.7), (3.8), (3.26), (3.27), (3.29), we have

$$-(b_0)^p_{p+1}(\Lambda_{1,0})^{p+1}_p = (K_{2,0})^p_p,$$
$$(K_{2,0})^p_p = (-1)^{z+1}(b_0)^p_{p+1}(b_0)^m_1 = (-1)^p (T_{2,0})^1_p\,,$$
$$(K_{2,0})^p_m = 0, \quad (K_{2,0})^p_j = 0;$$

if $j \ne p, m$, then

$$(K_{2,0})^m = (b_0)^m \mathscr{A}_{1,0} = (b_0)^m_1(\mathscr{A}_{1,0})^1 + (b_0)^m_{p+1}(\mathscr{A}_{1,0})^{p+1},$$
$$(K_{2,0})^m_m = -(b_0)^m_1(\Lambda_{1,0})^1_m = (K_{2,0})^p_p,$$
$$\begin{aligned}(K_{2,0})^m_p &= (-1)^z(b_0)^m_1(\Lambda_{1,z})^{p+1}_m - (b_0)^m_{p+1}(\Lambda_{1,0})^{p+1}_p\\ &= (-1)^z\big[(b_0)^m_1(b_0)^m_{p+1} - (b_0)^m_{p+1}(b_0)^m_1\big]; = 0\end{aligned}$$

for $j \ne p, m$, we have $(K_{2,0})^m_j = 0$. Finally

$$K_{2,0} = (-1)^p (T_{2,0})^1_p I\,.$$

Case: $z \le q$. If $1 \le u \le q-1$, then

$$\begin{aligned}(K_{2,u})^p_p = (-1)^z\Big[&(b_0)^p_{z+1}(\mathscr{A}_{1,u-z})^1_p - (b_0)^p_{z+2}(\mathscr{A}_{1,u-z-1})^1_p\\ &+ (-1)^{u+j}(b_0)^p_{u+1}(\mathscr{A}_{1,0})^1_p\Big] + (b_0)^p_{p+1}(\mathscr{A}_{1,u})^{p+1}_p\\ &- (b_0)^p_{p+2}(\mathscr{A}_{1,u-1})^{p+1}_p + \dots + (-1)^u(b_0)^p_{u+p+1}(\mathscr{A}_{1,0})^{p+1}_p\,.\end{aligned}$$

If $u < z$, then

$$(K_{2,u})^p_p = (b_0)^p_{p+1}(\mathscr{A}_{1,u})^{p+1}_p - (b_0)^p_{p+2}(\mathscr{A}_{1,u-1})^{p+1}_p + \dots + (-1)^u(b_0)^p_{u+p+1}(\mathscr{A}_{1,0})^{p+1}_p\,.$$

Using definitions (3.38), (3.40), (3.42), (3.37), (3.39), and (3.12), we have

$$(K_{2,u})^p_p = -\Big[(b_0)^p_{p+1}(\Lambda_{1,u})^{p+1}_p + \dots + (-1)^u(b_0)^p_{u+p+1}(\Lambda_{1,0})^{p+1}_p\Big] = (-1)^p(T_{2,u})^1_p,$$
$$(K_{2,u})^p_m = 0, \quad (K_{2,u})^p_k = 0 \quad \text{for } k \ne p,$$
$$(K_{2,u})^m = (b_0)^m\mathscr{A}_{1,u} - (b_0)^{m-1}_1(\mathscr{A}_{1,u-1})^1_m + \dots + (b_0)^{m-u}_1(\mathscr{A}_{1,0})^1_m$$
$$\begin{aligned}(K_{2,u})^m_m &= -(b_0)^m_1(\Lambda_{1,u})^1_m + (b_0)^{m-1}_1(\Lambda_{1,u-1})^1_m + \dots + (-1)^{u+1}(b_0)^{m-u}_1(\Lambda_{1,0})^1_m\\ &= (K_{2,u})^p_p\end{aligned}$$
$$\begin{aligned}(K_{2,u})^m_p = (-1)^z\Big[&(b_0)^m_1(\Lambda_{1,u+z})^{p+1}_m + \dots + (-1)^u(b_0)^{m-u}_1(\Lambda_{1,z})^{p+1}_m\Big]\\ &- (b_0)^m_{p+1}(\Lambda_{1,u})^{p+1}_p + \dots + (-1)^{u+1}(b_0)^m_{p+u+1}(\Lambda_{1,0})^{p+1}_p\Big] = 0\end{aligned}$$
$$(K_{2,u})^m_k = 0 \quad \text{for } k \ne m\,.$$

Therefore,

$$K_{2,u} = (-1)^p(T_{2,u})^1_p I\,. \tag{4.14}$$

We set

$$\widehat{K}_{2,u} = (-1)^p(T_{2,u})^1_p\,.$$

If $z \le u \le q-1$, using definitions (3.38) and (3.42), we have

$$\begin{aligned}(K_{2,u})^p_p &= (b_0)^p_{z+1}(\Lambda_{1,u})^{p+1}_m + \dots + (-1)^{u-z}(b_0)^p_{u+1}(\Lambda_{1,z})^{p+1}_m\\ &\quad - (b_0)^p_{p+1}(\Lambda_{1,u})^{p+1}_p + \dots + (-1)^{u+1}(b_0)^p_{u+p+1}(\Lambda_{1,0})^{p+1}_p\end{aligned} \tag{4.15}$$

Since

$$\widehat{K}_{1,q+u} = (-1)^{p-1}\big[(\Lambda_{1,v-z})^1_p + (-1)^z(\Lambda_{1,u})^{p+1}_m\big],$$

it follows that

$$\begin{aligned}(K_{2,u})_p^p &= (-1)^p (T_{2,u})_p^1 + (-1)^{q+1}\Big[(b_0)_{z+1}^p \widehat{K}_{1,q+u} \\ &\quad - (b_0)_{z+2}^p \widehat{K}_{1,q+u-1} + \dots + (-1)^{u+z}(b_0)_{u+1}^p \widehat{K}_{1,p}\Big],\\ (K_{2,u})_m^p &= (-1)^{z+1}\Big[(b_0)_{z+1}^p (\Lambda_{1,u-z})_m^1 + \dots + (-1)^{u+z}(b_0)_{u+1}^p(\Lambda_{1,0})_m^1 \\ &\quad - (b_0)_{p+1}^p(\Lambda_{1,u-z})_m^1 + \dots + (-1)^{u+z+1}(b_0)_{u+q+1}^p(\Lambda_{1,0})_p^1\Big] = 0\,,\\ (K_{2,u})_m^m &= (K_{2,u})_p^p, \quad (K_{2,u})_p^m = 0,\end{aligned}$$

and finally

$$K_{2,u} = (K_{2,u})_p^p I\,. \tag{4.16}$$

In general for $q \le u \le p-2$, by (4.10) and (4.4), we have

$$(K_{2,u})^p = (b_0)^p \mathscr{A}_{1,u} + (b_1)^p \Lambda_{0,u-q}\,.$$

As previously

$$\begin{aligned}(K_{2,u})_p^p &= (-1)^p (T_{2,u})_p^1 + (-1)^{q+1}\Big[(b_0)_{z+1}^p \widehat{K}_{1,q+u} \\ &\quad - (b_0)_{z+2}^p \widehat{K}_{1,q+u-1} + \dots + (-1)^{u+z}(b_0)_{u+1}^p \widehat{K}_{1,p}\Big]\\ (K_{2,u})_m^p &= (-1)^{z}\Big[-(b_0)_{z+1}^p (\Lambda_{1,u-z})_m^1 + (b_0)_{z+2}^p(\Lambda_{1,u-z-1})_m^1 \\ &\quad + (-1)^{z+u}(b_0)_{u+1}^p(\Lambda_{1,0})_m^1 - (b_0)_{p+1}^p(\Lambda_{1,u})_m^{p+1} \\ &\quad + (b_0)_{p+2}^p(\Lambda_{1,u-1})_m^{u+1} + \dots + (-1)^{u+z+1}(b_0)_{p+u-z+1}^p(\Lambda_{1,z})_m^{p+1}\Big] = 0\,.\end{aligned}$$

By (4.12), we have

$$\begin{aligned}(K_{2,u})^m &= (\mathscr{A}_{2,u-1})^m + (b_0)^m \mathscr{A}_{1,u} + D_0(\mathscr{A}_{1,u})^m + (b_1)^m \Lambda_{0,u-q}\\ &= (-1)^{q-1}(\mathscr{A}_{2,u-q})^{p+1} + (b_0)^m \mathscr{A}_{1,u} - (b_0)_1^{m-1}\mathscr{A}_{1,u-1} + \dots\\ &\quad + D_0(\mathscr{A}_{1,u})^m - D_0(\mathscr{A}_{1,u-1})^{m-1} + \dots + (b_1)^m\Lambda_{0,u-q}\\ &\quad - (b_1)_1^{m-1}(\Lambda_{0,u-q-1})^1 + \dots\end{aligned}$$

By definition 3.40, we have $(\mathscr{A}_{1,u})_m^1 = -(\Lambda_{1,u})_m^1$, and by (3.41),

$$(\mathscr{A}_{1,u})_m^{p+1} = (-1)^z (\Lambda_{1,u-z})_p^1\,.$$

We set

$$(\mathscr{A}_{2,u-q})_m^{p+1} = (-1)^{p+u+1}(b_1)_{u-q+1}^p\,,$$

and for $0 \le v \le z-2$, we define

$$(\mathscr{A}_{2,v})_m^{p+1} = (-1)^{z+u}(b_1)_{u+1}^p\,. \tag{4.17}$$

Then we obtain

$$(\mathscr{A}_{2,u})_m^m = (\mathscr{A}_{2,u})_p^p, \tag{4.18}$$

$$(K_{2,u})_p^m = (-1)^{q-1}(\mathscr{A}_{2,u-q})_p^{p+1} + (b_1)^m(\Lambda_{0,u-q})_p\,. \tag{4.19}$$

We set

$$(\mathscr{A}_{2,u-q})_p^{p+1} = (-1)^q (b_1)^m (\Lambda_{0,u-q})_p\,,$$

and for $0 \le v \le z-2$, we define

$$(\mathscr{A}_{2,v})_p^{p+1} = (-1)^q (b_1)^m (\Lambda_{0,v})_p\,.$$

The we obtain $(K_{2,u})^m_p = 0$ and

$$\begin{aligned} K_{2,u} &= (-1)^p (T_{2,u})^1_p I + (-1)^{q+1} \Big[(b_0)^p_{z+1} \widehat{K}_{1,q+u} \\ &\quad + \dots + (-1)^{u+z} (b_0)^p_{u+1} \widehat{K}_{1,p} \Big] I \end{aligned} \tag{4.20}$$

Example 4.6.

$$K_{2,p-2} = (-1)^p (T_{2,p-2})^1_p I + (-1)^{q+1} \Big[(b_0)^p_{z+1} \widehat{K}_{1,m-2} + \dots + (-1)^q (b_0)^p_{p+1} \widehat{K}_{1,p} \Big] I\,.$$

Case (2): $z > q$. In the same manner as above we obtain (4.20).

Proposition 4.7. *Assuming the necessity of* (L_1)*, if*

$$\begin{gathered} K_{1,u} = 0 \quad \text{for } 0 \le u \le m-2, \\ K_{2,u} = 0 \quad \text{for } 0 \le u \le p-2, \end{gathered}$$

then (L_2) *is satisfied.*

4.3. **Study of $(T_{3,u})^1_m$.** In the following, the expression

$$A = B \mod L_{1,2}$$

means that $A - B$ is equal to a linear combination of coefficients of (L_1) and (L_2).

4.3.1. *Study of* $\Lambda_{2,u}$. For $0 \le u \le m-2$, by the definition of microlocalized S_1 (cf. Proposition 2.7):

$$\begin{aligned} &(-1)^{p-1} \Lambda_{2,u} \\ &= J^{p-1}(b_0 + D_0 + JD_1)\Lambda_{1,u+p-1} - J^{p-2}(b_0 + D_0 + JD_1)\Lambda_{1,u+p-2} + \dots \\ &\quad + (-1)^{k-1} J^{p-k}(b_0 + D_0 + JD_1)\Lambda_{1,u+p-k} + \dots \\ &\quad + (-1)^{p-1}(b_0 + D_0 + JD_1)\Lambda_{1,u} \\ &\quad - J^{p-1} b_1 \Lambda_{0,u+z-1} + \dots + (-1)^k J^{p-k} b_1 \Lambda_{0,u+z-k} + \dots + (-1)^p b_1 \Lambda_{0,u-q} \end{aligned} \tag{4.21}$$

For $u \ge 1$, we obtain easily

$$J\Lambda_{2,u} + \Lambda_{2,u-1} = (b_0 + D_0 + JD_1)\Lambda_{1,u-1} - b_1 \Lambda_{0,u-q-1}, \tag{4.22}$$

$$\Lambda_{2,u} J + \Lambda_{2,u-1} = 0\,. \tag{4.23}$$

By (3.9),

$$J\Lambda_{2,0} = \Lambda_{2,0} J = 0 \mod L_{1,2}$$

So for $1 \le k-1 \le p-1$, we have

$$(\Lambda_{2,u})^k = -(\Lambda_{2,u-1})^{k-1} + D_0(\Lambda_{1,u-1})^{k-1} + D_1(\Lambda_{1,u-1})^k$$

and for $p+1 \le k-1 \le m-1$ we have

$$\begin{aligned} (\Lambda_{2,u})^k &= -(\Lambda_{2,u-1})^{k-1} + (b_0)^{k-1}_1 (\Lambda_{1,u-1})^1 + D_0(\Lambda_{1,u-1})^{k-1} \\ &\quad + D_1(\Lambda_{1,u-1})^k - (b_1)^{k-1}_1 (\Lambda_{0,u-q-1})^1\,, \end{aligned} \tag{4.24}$$

and

$$\begin{gathered} (\Lambda_{2,u-1})^p = (b_0)^p \Lambda_{1,u-1} - (b_1)^p \Lambda_{0,u-q-1} + D_0(\Lambda_{1,u-1})^p \\ (\Lambda_{2,u-1})^m = (b_0)^m \Lambda_{1,k-1} - (b_1)^m \Lambda_{0,u-q-1} + (-1)^{q-1} D_0(\Lambda_{1,u-q})^{p+1} \end{gathered}$$

Using that $(\Lambda_{1,u})^1_p = 0$ for $u > q-2$, for $u \le q-2$ we have

$$\begin{aligned}(-1)^{p-1}(\Lambda_{2,u})^1_p &= (-1)^{u+z+1}\Big[(b_0)^p_{z+u+2}(\Lambda_{1,q-2})^1_p + \dots \\ &\quad + (-1)^{k+q}(b_0)^p_{z+k}(\Lambda_{1,u+q-k})^1_p + \dots \\ &\quad + (-1)^{u+q-1}(b_0)^p_{p-1}(\Lambda_{1,u+1})^1_p\Big] \\ &\quad + (-1)^{p+1}pD_0(\Lambda_{1,u})^1_p + (-1)^p D_1(\Lambda_{1,u-1})^1_p \\ &\quad + (-1)^{u+q+1}(b_1)^p_{u+z} \mod L_{1,2}\end{aligned} \tag{4.25}$$

Example 4.8.

$$\begin{aligned}(\Lambda_{2,0})^1_p = (-1)^q\Big[(b_0)^p_{z+2}(\Lambda_{1,q-2})^1_p + \dots + (-1)^{q+1}(b_0)^p_{p-1}(\Lambda_{1,1})^1_p\Big] \\ + pD_0(\Lambda_{1,0})^1_p + (-1)^z(b_1)^p_z \mod L_{1,2},\end{aligned} \tag{4.26}$$

$$u = q-1, \quad (\Lambda_{2,q-1})^1_p = (-1)^{p-1}\,(b_1)^p_{p-1}\,. \tag{4.27}$$

Remark 4.9. If $z \le u \le p-1$, then

$$\begin{aligned}(\Lambda_{2,u-z})^1_p = (-1)^{u+p}\Big[(b_0)^p_{u+2}(\Lambda_{q-2})^1_p + \dots + (-1)^q(b_0)^p_{u+q}(\Lambda_{1,0})^1_p\Big] \\ + pD_0(\Lambda_{1,u-z})^1_p - pD_1(\Lambda_{1,u-z-1})^1_p + (-1)^u(b_1)^p_u \mod L_{1,2}.\end{aligned} \tag{4.28}$$

Note that

$$\begin{aligned}&(-1)^{q-1}(\Lambda_{2,u})^1_m \\ &= (b_0)^p_{z+1}(\Lambda_{1,u+q-1})^1_m - (b_0)^p_{z+2}(\Lambda_{1,u+q-2})^1_m + \dots \\ &\quad + (-1)^{k-1}(b_0)^p_{z+k}(\Lambda_{1,u+q-k})^1_m + \dots + (-1)^q(b_0)^p_{p-1}(\Lambda_{1,u+1})^1_m \\ &\quad + (-1)^z\Big[(b_0)^p_{p+1}(\Lambda_{1,u+p-1})^{p+1}_m + (b_0)^p_{p+2}\Big(-(\Lambda_{1,u+p-2})^{p+1}_m \\ &\quad + (b_0)^{p+1}_1(\Lambda_{0,u-z-1})^1_m\Big) + \dots + (b_0)^p_{p+k}\Big((-1)^{k+1}(\Lambda_{1,u+p-k})^{p+1}_m \\ &\quad + (b_0)^{p+k-1}_1(\Lambda_{0,u+z-1})^1_m + \dots + (-1)^k(b_0)^{p+1}_1(\Lambda_{0,z-k+u+1})^1_m\Big) \\ &\quad + (b_0)^p_m\Big((-1)^{q+1}(\Lambda_{1,u+z})^{p+1}_m + (b_0)^{m-1}(\Lambda_{0,u+z-1})^1_m + \dots \\ &\quad + (-1)^{u+z+1}(b_0)^{m-u-z}_1(\Lambda_{0,0})^1_m\Big)\Big] \\ &\quad + (-1)^{q-1}pD_0(\Lambda_{1,u})^1_m + (-1)^q pD_0(\Lambda_{u-1})^1_m + (-1)^{p+u+1}(b_1)^p_{p+u} \mod L_{1,2}\,.\end{aligned}$$

For $u+p-k \ge j$ and $u \ge k-q$ $k \le u-q$, we have

$$\begin{aligned}&(-1)^{q-1}(\Lambda_{2,u})^1_m \\ &= (-1)^u\Big[(b_0)^p_{z+u+1}(\Lambda_{1.q-1})^1_m + \dots + (-1)^{q+u}(b_0)^p_{p-1}(\Lambda_{1,u+1})^1_m \\ &\quad + (b_0)^p_{p+u+2}(\Lambda_{1,q-2})^1_p + \dots + (-1)^{u+q}(b_0)^p_m(\Lambda_{1,u})^1_p\Big] \\ &\quad + (-1)^{q-1}pD_0(\Lambda_{1,u})^1_m + (-1)^q pD_1(\Lambda_{1,u-1})^1_m + (-1)^{p+u+1}(b_1)^p_{p+u}\,.\end{aligned} \tag{4.29}$$

For $u \le q-1$, we have

$$\begin{gathered}(\Lambda_{2,u})^1_m = pD_0(\Lambda_{1,u})^1_m - pD_1(\Lambda_{1,u-1})^1_m + (-1)^{z+u}(b_1)^p_{p+u} \mod L_{1,2}, \\ (\Lambda_{2,0})^1_m = pD_0(\Lambda_{1,0})^1_m \mod L_1, L_2.\end{gathered}$$

For $u=q$, we have $(\Lambda_{2,q})^1_m = -pD_1(\Lambda_{1,q-1})^1_m + (-1)^p(b_1)^p_m$; and for $u>q$, we have $(\Lambda_{2,u})^1_m = 0$.

Studying $(\Lambda_{2,u})^{p+1}$, we obtain

$$\begin{aligned}
&(-1)^{q-1}(\Lambda_{2,u})^{p+1}\\
&= (b_0)^m\Lambda_{1,u+q-1} - (b_0)^{m-1}_1(\Lambda_{1,u+q-2})^1 + \dots + (-1)^{q-1}(b_0)^{p+1}_1(\Lambda_{1,u})^1\\
&\quad - (b_1)^m\Lambda_{0,u-1} + (b_1)^{m-1}_1(\Lambda_{0,u-2})^1 + \dots + (-1)^q(b_0)^{p+1}_1(\Lambda_{0,u-q})^1\\
&\quad + D_0(\Lambda_{1,u+q-1})^m + \dots + (-1)^k D_0(\Lambda_{1,u+q-k-1})^{m-k} + \dots\\
&\quad + (-1)^{q-1}D_0(\Lambda_{1,u})^{p+1} - D_1(\Lambda_{1,u+q-2})^m + \dots + (-1)^q D_1(\Lambda_{1,u-1})^{m-1}_m\\
&= (b_0)^m_1(\Lambda_{1,u+q-1})^1 - (b_0)^{m-1}_1(\Lambda_{1,u+q-2})^1 + \dots + (-1)^{q-1}(b_0)^{p+1}_1(\Lambda_{1,u})^1\\
&\quad + (b_0)^m_{p+1}(\Lambda_{1,u+q-1})^{p+1} + (b_0)^m_{p+2}\big[-(\Lambda_{1,u+q-2})^{p+1} + (b_0)^{p+1}_1(\Lambda_{0,u-1})^1\big]\\
&\quad + (b_0)^m_{p+k}\Big[(-1)^{k-1}(\Lambda_{1,u+q-k})^{p+1} + (b_0)^{p+k-1}_1(\Lambda_{0,u-1})^1 + \dots\\
&\quad + (-1)^u(b_0)^{p+1}_1(\Lambda_{0,u-k+1})^1\Big] + \dots\\
&\quad + (b_0)^m_{m-1}\Big[(-1)^q(\Lambda_{1,u+1})^{p+1} + (b_0)^{m-2}_1(\Lambda_{0,u-1})^1 + \dots\\
&\quad + (-1)^{q+1}(b_0)^{p+1}_1(\Lambda_{0,u-q+2})^1\Big] + \dots\\
&\quad + D_0(\Lambda_{1,u+q-1})^m + \dots - D_1(\Lambda_{1,u+q-2})^m + \dots\\
&\quad - (b_1)^m\Lambda_{0,u-1} + (b_1)^{m-1}_1(\Lambda_{0,u-2})^1 + \dots .
\end{aligned}$$

Then

$$\begin{aligned}
(-1)^{q-1}(\Lambda_{2,0})^{p+1}_m &= (b_0)^m_1(\Lambda_{1,q-1})^1_m + \dots + (-1)^{q-1}(b_0)^{p+1}_1(\Lambda_{1,0})^1_m\\
&\quad + (b_0)^m_{p+1}(\Lambda_{1,q-1})^{p+1}_m + \dots + (-1)^{p+1}(b_0)^m_{m-z}(\Lambda_{1,z})^{p+1}_m .
\end{aligned}$$

By (4.2) and (4.6),

$$\begin{aligned}
&(-1)^{p-1}(T_{2,q-1})^1_p\\
&= (-1)^z\Big[(b_0)^p_{z+1}(\Lambda_{1,q-z-1})^1_p + \dots + (-1)^{p+1}(b_0)^p_q(\Lambda_{1,0})^1_p\Big]\\
&\quad + (b_0)^p_{p+1}(\Lambda_{1,q-1})^{p+1}_p + \dots + (-1)^{q-1}(b_0)^p_m(\Lambda_{1,0})^{p+1}_p .
\end{aligned}$$

So we obtain

$$\begin{aligned}
&(-1)^{q-1}(\Lambda_{2,0})^{p+1}_m\\
&= (b_0)^m_{p+1}(\Lambda_{1,q-1})^{p+1}_m + \dots + (-1)^{p+1}(b_0)^m_{m-z}(\Lambda_{1,z})^{p+1}_m\\
&\quad + (-1)^z\big[(b_0)^p_{z+1}(\Lambda_{1,q-z-1})^1_p + \dots + (-1)^{z+1}(b_0)^p_q(\Lambda_{1,0})^1_p\big] = 0 .
\end{aligned} \tag{4.30}$$

In general, for $u>0$, we have

$$\begin{aligned}
&(-1)^{q-1}(\Lambda_{2,u})^{p+1}_m\\
&= (-1)^u\Big[(b_0)^{m-u}_1(\Lambda_{1,q-1})^1_m + \dots + (-1)^{q+u+1}(b_0)^{p+1}_1(\Lambda_{1,u})^1_m\Big]\\
&\quad + (b_0)^m_{p+1}(\Lambda_{1,u+q-1})^{p+1}_m - (b_0)^m_{p+2}(\Lambda_{1,u+q-2})^{p+1}_m\\
&\quad + (-1)^{k-1}(b_0)^m_{p+k}(\Lambda_{1,u+q-k})^{p+1}_m + \dots + (-1)^{u+p+1}(b_0)^m_{u+2q}(\Lambda_{1,z})^{p+1}_m\\
&\quad + (-1)^{u+p+1}(b_1)^m_{u+q} + (-1)^{q-1}qD_0(\Lambda_{1,u})^{p+1}_m + (-1)^q qD_1(\Lambda_{1,u-1})^{p+1}_m .
\end{aligned}$$

We remark that

$$(\Lambda_{1,u+q-1-k})_m^{m-k} = (-1)^{q+k+1}(\Lambda_{1,u})_m^{p+1}\,.$$

We define $(\Lambda_{2,u})_m^{p+1}$ according to the values of q, z, u:

4.3.2. *Case (I): $q > z$.* Subcase (I-1) $u < z$:

$$\begin{aligned}(-1)^{q-1}(\Lambda_{2,u})_m^{p+1} &= (-1)^u\big[(b_0)_1^{m-u}(\Lambda_{1,q-1})_m^1 + \dots + (-1)^{u+q+1}(b_0)_1^{p+1}(\Lambda_{1,u})_m^1\big]\\ &\quad + (b_0)_{p+1}^m(\Lambda_{1,u+q-1})_m^{p+1} - (b_0)_{p+2}^m(\Lambda_{1,u+q-2})_m^{p+1} + \dots\\ &\quad + (-1)^{u+p+1}(b_0)_{u+2q}^m(\Lambda_{1,z})_m^{p+1}\,.\end{aligned}$$

For $u > 0$ we have (cf. (4.4))

$$\begin{aligned}&(-1)^{p-1}(T_{2,u+q-1})_p^1\\ &= (-1)^z\Big[(b_0)_{z+1}^p(\Lambda_{u+q-z-1})_p^1 + \dots\\ &\quad + (-1)^{u+p+1}(b_0)_{u+q}^p(\Lambda_{1,0})_p^1\Big] + (-1)^{u+p+1}(b_1)_u^p\\ &\quad + (-1)^u\Big[(b_0)_{p+u+1}^p(\Lambda_{1,q-1})_p^{p+1} + \dots + (-1)^{u+q+1}(b_0)_m^p(\Lambda_{1,u})_p^{p+1}\Big],\end{aligned}$$

and

$$(\Lambda_{2,u})_m^{p+1} = (-1)^{z+u+1}(b_1)_u^p\,. \tag{4.31}$$

Subcase (I-2) $z \le u < q$: We have

$$\begin{aligned}&(-1)^{q-1}(\Lambda_{2,u})_m^{p+1}\\ &= (-1)^u\Big[(b_0)_1^{m-u}(\Lambda_{1,q-1})_m^1 + \dots + (-1)^{u+q+1}(b_0)_1^{p+1}(\Lambda_{1,u})_m^1\Big]\\ &\quad + (-1)^{u+z+1}\Big[(b_0)_{u+q+2}^m(\Lambda_{1,p-2})_m^{p+1} + \dots + (-1)^z(b_0)_{u+q+z}^m(\Lambda_{1,p-z})_m^{p+1}\\ &\quad + (-1)^q(b_0)_{u+2q}^m(\Lambda_{1,z})_m^{p+1}\Big]\\ &\quad + (-1)^{u+p+1}(b_1)_{u+q}^m + (-1)^{q-1}qD_0(\Lambda_{1,u})_m^{p+1} + (-1)^q qD_1(\Lambda_{1,u-1})_m^{p+1}\,.\end{aligned} \tag{4.32}$$

Subcase (I-3) $u \ge q$: We have

$$\begin{aligned}&(-1)^{q-1}(\Lambda_{2,u})_m^{p+1}\\ &= (-1)^{u+z+1}\Big[(b_0)_{u+q+2}^m(\Lambda_{1,p-2})_m^{p+1} + \dots + (-1)^q(b_0)_{u+2q}^m(\Lambda_{1,z})_m^{p+1}\Big]\\ &\quad + (-1)^{u+p+1}(b_1)_{u+q}^m + (-1)^{q-1}qD_0(\Lambda_{1,u})_m^{p+1} + (-1)^q qD_1(\Lambda_{1,u-1})_m^{p+1}\,.\end{aligned} \tag{4.33}$$

4.3.3. *Case (II): $z \ge q$.* Subcase (II-1) $u < q$: As in case (I-1),

$$(-1)^{q-1}(\Lambda_{2,u})_m^{p+1} = (-1)^{z+u+1}(b_1)_u^p \mod L_{1,2}\,. \tag{4.34}$$

Subcase (II-2) $q \le u < z$: as in cases (I-1) and (I-3), we have

$$(-1)^{q-1}(\Lambda_{2,u})_m^{p+1} = (-1)^{z+u+1}(b_1)_u^p \mod L_{1,2}\,. \tag{4.35}$$

Subcase (II-3) $u \geq z$: as in case (I_3),

$$\begin{aligned}
&(-1)^{q-1}(\Lambda_{2,u})_m^{p+1} \\
&= (-1)^{u+z+1}\Big[(b_0)_{u+q+2}^m(\Lambda_{1,p})_1^{p+1} + \dots \\
&\quad + (-1)^{u+p+1}(b_0)_{u+2q}^m(\Lambda_{1,z})_m^{p+1} + (-1)^{u+p+1}(b_1)_{u+2q}^m \\
&\quad + (-1)^{q-1}qD_0(\Lambda_{1,u})_m^{p+1} + (-1)^q qD_1(\Lambda_{1,u-1})_m^{p+1}\Big] \mod L_{1,2}\,.
\end{aligned} \tag{4.36}$$

Remark 4.10. By considering (4.17) and (4.31), for $0 \leq u \leq z-2$, we obtain

$$(\mathscr{A}_{2,u})_m^{p+1} = (\Lambda_{2,u+1})_m^{p+1} + (-1)^z(T_{2,u+1})_p^1\,. \tag{4.37}$$

Lemma 4.11. *If $u \geq z$, in cases* (I-2) *and* (I-3)*, we have*

$$\begin{aligned}
(\Lambda_{2,u})_m^{p+1} &= (-1)^{u+q+1}\Big[(b_0)_1^{m-u}(\Lambda_{1,q-1})_m^1 + \dots + (-1)^{u+q+1}(b_0)_1^{p+1}(\Lambda_{1,u})_m^1\Big] \\
&\quad + (-1)^z\Big[(\Lambda_{2,u-z})_p^1 - mD_0(\Lambda_{1,u-z})_p^1 + (-1)^u\big((b_1)_{u+q}^m - (b_1)_u^p\big)\Big]
\end{aligned}$$

Proof. By (4.32),

$$\begin{aligned}
(-1)^{q-1}(\Lambda_{2,u})_m^{p+1} &= (-1)^u\Big[(b_0)_1^{m-u}(\Lambda_{1,q-1})_m^1 + \dots + (-1)^{q+u+1}(b_0)_1^{p+1}(\Lambda_{1,u})_m^1\Big] \\
&\quad + (-1)^u\Big[(b_0)_{u+q}^p(\Lambda_{1,0})_p^1 + \dots + (-1)^q(b_0)_{u+2}^p(\Lambda_{1,q-2})_p^1\Big] \\
&\quad + (-1)^{u+p+1}(b_1)_{u+q}^m + (-1)^p qD(\Lambda_{1,u-z})_p^1 \mod L_{1,2}\,.
\end{aligned}$$

Then consider (4.28) we obtain the result. ∎

To consider $(\Lambda_{2,u})_p^{p+1}$, we first consider $(\Lambda_{2,0})_p^{p+1}$. We use the results about $(\Lambda_{1,u})_p^1$ and $(\Lambda_{1,u})_p^{p+1}$ in §1 and $K_{1,u} = 0$, $u \geq p$. Then

$$\begin{aligned}
(-1)^{q-1}(\Lambda_{2,0})_p^{p+1} &= -(b_0)_1^{m-1}(\Lambda_{1,q-2})_p^1 + \dots + (-1)^{q+1}(b_0)_1^{p+1}(\Lambda_{1,0})_p^1 \\
&\quad + (b_0)_{p+1}^m(\Lambda_{1,q-1})_p^{p+1} - (b_0)_{p+2}^m(\Lambda_{1,q-2})_p^{p+1} + \dots \\
&\quad + (-1)^q(b_0)_{m-1}^m(\Lambda_{1,1})_p^{p+1} + q(-1)^{q-1}D_0(\Lambda_{1,0})_p^{p+1}\,, \\
(\Lambda_{2,0})_p^{p+1} &= qD_0(\Lambda_{1,0})_p^{p+1}\,.
\end{aligned}$$

Remark 4.12. If $(T_{2,0})_p^1 = 0$ and $(b_0)_{p+1}^p \neq 0$, then

$$\begin{aligned}
(\Lambda_{2,0})_p^{p+1} &= 0, \\
(-1)^{q-1}(\Lambda_{2,1})_p^{p+1} &= (b_0)_1^{m-2}(\Lambda_{1,q-2})_p^1 + \dots + (-1)^{q-1}(b_0)_1^{p+1}(\Lambda_{1,1})_p^1 \\
&\quad + (-1)^{q+1}q\,D_0(\Lambda_{1,1})_p^{p+1} + (-1)^p(b_1)_1^m\,.
\end{aligned}$$

For $q-2 \geq u \geq 1$, we have

$$\begin{aligned}
&(-1)^{q-1}(\Lambda_{2,u})_p^{p+1} \\
&= (-1)^{u+1}\Big[(b_0)_1^{m-u-1}(\Lambda_{1,q-2})_p^1 + \dots + (-1)^{u+q}(b_0)_1^{p+1}(\Lambda_{1,u})_p^1\Big] \\
&\quad + (-1)^{q+1}qD_0(\Lambda_{1,u})_p^{p+1} + (-1)^q qD_1(\Lambda_{1,u-1})_p^{p+1} \\
&\quad - (b_1)^m(\Lambda_{0,u-1})_p - (b_1)^{m-1}(\Lambda_{0,u-2})_p\,. + \dots
\end{aligned} \tag{4.38}$$

If $u > q+1$, then $(\Lambda_{2,u})_p^{p+1} = 0$. If $u = q+1$, then

$$(\Lambda_{2,q+1})_p^{p+1} = (-1)^{q+1}(b_1)_1^p$$

If $u = q$, then

$$(\Lambda_{2,q})_p^{p+1} = -qD_1(\Lambda_{1,q-1})_p^{p+1} + (-1)^p (b_1)_1^{p+1}$$

If $u = q-1$, then

$$(\Lambda_{2,q-1})_p^{p+1} = qD_0(\Lambda_{1,q-1})_p^{p+1} - qD_1(\Lambda_{1,q-2})_p^{p-1} + (-1)^{p+1}(b_1)_1^{p+2}\,.$$

Remark 4.13. Note that

$$(\Lambda_{2,q-2})_p^{p+1} = (b_0)_{p+1}^p(\Lambda_{1,q-2})_p^1 + (-1)^p(b_1)_1^{p+3} + qD_0(\Lambda_{1,q-2})_p^{p+1} - qD_0(\Lambda_{1,q-3})_p^{p+1}$$

and

$$\begin{aligned}
&(-1)^{q-1}(\Lambda_{2,z})_p^{p+1}\\
&= (-1)^{z+1}\Big[(b_0)_1^{m-z-1}(\Lambda_{1,q-2})_p^1 + \dots + (-1)^p(b_0)_1^{p+1}(\Lambda_{1,z})_p^1\Big]\\
&\quad + (-1)^{q+1}D_0(\Lambda_{1,z})_p^{p+1} + (-1)^q qD_1(\Lambda_{1,z-1})_p^{p+1} - (b_1)^m(\Lambda_{0,z-1})_p\,.
\end{aligned}$$

Remark 4.14. Considering (4.20) and (4.37), for $0 \le u \le z-2$ we have

$$(\mathscr{A}_{2,v})_p^{p+1} = (\Lambda_{2,v+1})_p^{p+1}\,. \tag{4.39}$$

4.3.4. *Study of the first part of* (L_3). We have

$$S_2 = \mathcal{L}_0(\Lambda_2) - \xi_0\,\mathcal{L}_1(\Lambda_1) + \xi_0^{q+1}\,\mathcal{L}_2(\Lambda_0) = T_3 + \xi_0^{p+1}\Lambda_3$$

and for $0 \le u \le p-2$,

$$\begin{aligned}
&(-1)^{p-1}T_{3,u}\\
&= J^{p-1}(b_0 + D_0 + JD_1)\Lambda_{2,u} - J^{p-2}(b_0 + D_0 + JD_1)\Lambda_{2,u-1} + \dots\\
&\quad + (-1)^{k-1}J^{p-k}(b_0 + D_0 + JD_1)\Lambda_{2,u-k+1} + \dots\\
&\quad + (-1)^p J(b_0 + D_0 + JD_1)\Lambda_{2,u-p+2}\\
&\quad - J^{p-1}b_1\Lambda_{1,u-1} + \dots + (-1)^k J^{p-k}b_1\Lambda_{1,u-k} + \dots\\
&\quad + (-1)^p J^2 b_1\Lambda_{1,u-p+2} + J^{p-1}b_2\Lambda_{0,u-q-1} + \dots + (-1)^{q+u}J^{m-u}b_2\Lambda_{0,0}\,.
\end{aligned}$$

We remark that

$$JT_{3,0} = 0 = T_{3,0}J, \quad JT_{3,u} = -T_{3,u-1} = T_{3,u}J$$

We consider

$$\begin{aligned}
(-1)^{p-1}(T_{3,u})^1 &= (b_0)^p\Lambda_{2,u} + D_0(\Lambda_{2,u})^p - D_0(\Lambda_{2,u-1})^{p-1} + \dots\\
&\quad + (-1)^p D_0(\Lambda_{2,u-p+2})^2 - D_1(\Lambda_{2,u-1})^p + \dots\\
&\quad - (b_1)^p\Lambda_{1,u-1} + (b_2)^p\Lambda_{0,u-q-1}\,,\\
(-1)^{p-1}(T_{3,0})^1 &= (b_0)^p\Lambda_{2,0} = (b_0)_{p+1}^p(\Lambda_{2,0})^{p+1}\,.
\end{aligned} \tag{4.40}$$

The terms in D_1 are related by shift in the term in D_0, so we do not write them to make the redaction lighter.

Lemma 4.15. *If* $0 \le u \le z-1$, *then* $(T_{3,u})_m^1 = 0$.

Proof. By (4.30),

$$(-1)^{p-1}(T_{3,0})_m^1 = (b_0)_{p+1}^p(\Lambda_{2,0})_m^{p+1} = 0\,.$$

For $u \ge 1$, we have two cases:

Case 1. $q \ge z$: We obtain $(T_{3,u})_m^1$ explicitly by 4.24.

$$(-1)^{p-1}(T_{3,u})_m^1$$

$$
\begin{aligned}
&= (b_0)^p_{p+1}(\Lambda_{2,u})^{p+1}_m + (b_0)^p_{p+2}\Big[-(\Lambda_{2,u-1})^{p+1}_m \\
&\quad + (b_0)^{p+1}_1(\Lambda_{1,u-1})^1_m - (b_1)^{p+1}_1(\Lambda_{0,u-q-1})^1_m\Big] + \dots \\
&\quad + (b_0)^p_{p+k}\Big[(-1)^{k-1}(\Lambda_{2,u-k+1})^{p+1}_m + (b_0)^{p+k-1}_1(\Lambda_{1,u-1})^1_m + \dots \\
&\quad + (-1)^k(b_0)^{p+1}_1(\Lambda_{1,u-k+1})^1_m\Big] + \dots + (b_0)^p_{p+u}\Big[(-1)^{u-1}(\Lambda_{2,1})^{p+1}_m \\
&\quad + (b_0)^{p+u-1}_1(\Lambda_{1,u-1})^1_m + \dots + (-1)^u(b_0)^{p+1}_1(\Lambda_{1,1})^1_m\Big] \\
&\quad + (b_0)^p_{p+u+1}\Big[(b_0)^{p+u}_1(\Lambda_{1,u-1})^1_m + \dots + (-1)^{u+1}(b_0)^{p+1}_1(\Lambda_{1,0})^1_m\Big] \\
&\quad + \dots + (b_0)^p_m\Big[(b_0)^{m-1}_1(\Lambda_{1,u-1})^1_m + \dots + (-1)^{u+1}(b_0)^{m-u}_1(\Lambda_{1,0})^1_m\Big] \\
&\quad - (b_1)^p_1(\Lambda_{1,u-1})^1_m + \dots + (-1)^u(b_1)^p_u(\Lambda_{1,0})^1_m \mod L_{1,2}\,.
\end{aligned}
$$

We define the terms $(\Lambda_{2,u-k+1})^{p+1}_m$ according to their values in (4.31). Then we simplify the terms $(\Lambda_{1,k})^1_m$ of §2.1.1, to obtain

$$(T_{3,u})^1_m = 0 \mod L_{1,2}\,.$$

Case 2. $q < z$: The calculations are quite similar to the ones above, and they are omitted. □

Lemma 4.16. *If $z \le u \le p-2$, then*

$$
\begin{aligned}
&(-1)^{p-1}(T_{3,u})^1_m \\
&= (b_0)^p_{p+1}\Big[(-1)^z\Big((\Lambda_{2,u-z})^1_p - mD_0(\Lambda_{1,u-z})^1_p + (-1)^u(b_1)^m_{q+u}\Big)\Big] \\
&\quad + (b_0)^p_{p+2}\Big[(-1)^{z+1}\Big((\Lambda_{2,u-z-1})^1_p - mD_0(\Lambda_{1,u-z-1})^1_p \\
&\quad + (-1)^{u+1}(b_1)^m_{q+u-1} + D_0(\Lambda_{1,u-1})^{p+1}_m\Big)\Big] + \dots \\
&\quad + (b_0)^p_{u+q+1}\Big[(-1)^u\Big((\Lambda_{2,0})^1_p - mD_0(\Lambda_{1,0})^1_p \\
&\quad + (-1)^{u+z+1}(u-z)D_0(\Lambda_{1,z})^{p+1}_m\Big)\Big] \\
&\quad + (-1)^z\Big[q(b_0)^p_{z+1}D_0(\Lambda_{1,u-z})^1_m - (q-1)(b_0)^p_{z+2}D_0(\Lambda_{1,u-z-1})^1_m + \dots \\
&\quad + (-1)^{z+u}(p-u)(b_0)^p_{u+1}D_0(\Lambda_{1,0})^1_m\Big] \mod L_{1,2}
\end{aligned}
$$

Consequence 4.17. *If $(b_0)^p_{p+1} = \dots (b_0)^p_{u+q+1} = 0$, then*

$$(T_{3,u})^1_m = 0\,, \mod L_{1,2}\,.$$

Proof. First note that

$$(-1)^{p-1}(T_{3,z})^1_m = (b_0)^p\,(\Lambda_{2,z})_m - (b_1)^p\,(\Lambda_{1,z-1})_m\,.$$

We consider two cases:

Case (1) $q \ge z$: From (4.24), it follows that

$$
\begin{aligned}
&(-1)^{p-1}(T_{3,z})^1_m \\
&= (-1)^z(b_0)^p_{z+1}\Big((\Lambda_{2,0})^1_m - zD_0(\Lambda_{1,0})^1_m\Big) \\
&\quad + (b_0)^p_{p+1}(\Lambda_{2,z})^{p+1}_m + (b_0)^p_{p+2}\Big[-(\Lambda_{2,z-1})^{p+1}_m + (b_0)^{p+1}_1(\Lambda_{1,z-1})^1_m\Big] + \dots
\end{aligned}
$$

$$
\begin{aligned}
&+ (b_0)^p_{p+z}\Big[(-1)^{z-1}(\Lambda_{2,1})^{p+1}_m + (b_0)^{p+z-1}_1(\Lambda_{1,z-1})^1_m + \dots\\
&+ (-1)^z(b_0)^{p+1}_1(\Lambda_{1,1})^1_m\Big]\\
&+ (b_0)^p_{p+z+1}\Big[(b_0)^{p+z}_1(\Lambda_{1,z-1})^1_m + \dots + (-1)^{z+1}(b_0)^{p+1}_1(\Lambda_{1,0})^1_m\Big]\\
&+ (b_0)^p_{p+z+2}\Big[(b_0)^{p+z+1}_1(\Lambda_{1,z-1})^1_m + \dots + (-1)^{p-1}(b_0)^{p+2}_1(\Lambda_{1,0})^1_m\Big] + \dots\\
&+ (b_0)^p_m\Big[(b_0)^{m-1}_1(\Lambda_{1,z-1})^1_m + \dots + (-1)^{z+1}(b_0)^{m-z}_1(\Lambda_{1,0})^1_m\Big]\\
&+ (b_1)^p_1(b_0)^p_{p+z} + \dots + (b_1)^p_z(b_0)^p_{p+1}
\end{aligned}
$$

In the second part (from $(b_0)^p_{p+z+1}$) we exchange the summations, and obtain

$$
\begin{aligned}
&(-1)^{p-1}(T_{3,z})^1_m\\
&= (-1)^z(b_0)^p_{z+1}\Big((\Lambda_{2,0})^1_m - zD_0(\Lambda_{1,0})^1_m\Big)\\
&\quad + (b_0)^p_{p+1}\Big[(\Lambda_{2,z})^{p+1}_m - \Big((b_0)^p_{p+z+1}(b_0)^{p+1}_1 + \dots + (b_0)^p_m(b_0)^{m-z}_1 - (b_1)^p_z\Big)\Big]\\
&\quad + (b_0)^p_{p+2}\Big[-(\Lambda_{2,z-1})^{p+1}_m - \Big((b_0)^p_{p+z}(b_0)^{p+1}_1 + \dots\\
&\quad + (b_0)^p_m(b_0)^{m-z+1}_1 - (b_1)^p_{z-1}\Big)\Big] + \dots + (b_0)^p_{p+z}\Big[(-1)^{z+1}(\Lambda_{2,1})^{p+1}_m\\
&\quad - \Big((b_0)^p_m(b_0)^{m-1}_1 + \dots + (b_0)^p_{p+1}(b_0)^{p+1}_1 - (b_1)^p_1\Big)\Big].
\end{aligned}
$$

We use (4.31) to obtain

$$
\begin{aligned}
(-1)^{p-1}(T_{3,z})^1_m &= (-1)^z(b_0)^p_{z+1}\Big((\Lambda_{2,0})^1_m - zD_0(\Lambda_{1,0})^1_m\Big)\\
&\quad + (b_0)^p_{p+1}\Big[(\Lambda_{2,z})^{p+1}_m - \Big((b_0)^p_{p+z+1}(b_0)^{p+1}_1 + \dots\\
&\quad + (b_0)^p_m(b_0)^{m-z}_1 - (b_1)^p_z\Big)\Big] \mod L_{1,2}\,.
\end{aligned}
$$

by (4.29) and (4.33).

$$
\begin{aligned}
(-1)^{p-1}(T_{3,z})^1_m = (-1)^z\Big[&(b_0)^p_{p+1}\Big((\Lambda_{2,0})^1_p - mD_0(\Lambda_{1,0})^1_p\Big)\\
&+ q(b_0)^p_{z+1}D_0(\Lambda_{1,0})^1_m\Big] \mod L_{1,2}\,.
\end{aligned}
\tag{4.41}
$$

Case (2) $q < z$: We adapt the calculus and we obtain the same results. $\square$

Consequence 4.18. *(1) If $(b_0)^p_{p+1} = 0$ then $(T_{3,z})^1_m = 0$.*

(2) If $(b_0)^p_{p+1} \neq 0$, using the reduction in Proposition 2.5,

$$
(-1)^{p-1}(T_{3,z})^1_m = (-1)^z\left[(\Lambda_{2,0})^1_p - mD_0(\Lambda_{1,0})^1_p\right].
$$

Now we consider $z < u \le p-2$, under the assumption that $q \ge z$ and $u \le q$. We have

$$
\begin{aligned}
&(-1)^{p-1}(T_{3,u})^1_m\\
&= (-1)^z(b_0)^p_{z+1}\Big((\Lambda_{2,u-z})^1_m - zD_0(\Lambda_{1,u-z})^1_m\Big)\\
&\quad + (-1)^{z+1}(b_0)^p_{z+2}\Big((\Lambda_{2,u-z-1})^1_m - (z+1)D_0(\Lambda_{1,u-z-1})^1_m\Big) + \dots\\
&\quad + (-1)^u(b_0)^p_{u+1}\Big((\Lambda_{2,0})^1_m - uD_0(\Lambda_{1,0})^1_m\Big) + (b_0)^p_{p+1}(\Lambda_{2,u})^{p+1}_m
\end{aligned}
$$

$$
\begin{aligned}
&+ (b_0)^p_{p+2}\Big[-(\Lambda_{2,u-1})^{p+1}_m + (b_0)^{p+1}_1(\Lambda_{1,u-1})^1_m + D_0(\Lambda_{1,u-1})^{p+1}_m\Big] + \dots \\
&+ (b_0)^p_{u+q+1}\Big[(-1)^{u+z}(\Lambda_{2,z})^{p+1}_m + (b_0)^{u+q}_1(\Lambda_{1,u-1})^1_m + \dots \\
&+ (-1)^{u+z+1}(b_0)^{p+1}_1(\Lambda_{1,z})^1_m - (b_1)^p_z(-1)^{u+z+1}(u-z)D_0(\Lambda_{1,z})^{p+1}_m\Big] \\
&+ (b_0)^p_{u+q+2}\Big[(-1)^{u+z+1}(\Lambda_{2,z-1})^{p+1}_m + (b_0)^{u+q+1}_z(\Lambda_{1,u-1})^1_m \\
&+ (-1)^{u+z}(b_0)^{p+1}_1(\Lambda_{1,z-1})^1_u - (b_1)^p_{z-1}\Big] \\
&+ (b_0)^p_{p+u}\Big[(-1)^{u+1}(\Lambda_{2,1})^{p+1}_m + (b_0)^{p+u-1}_1(\Lambda_{1,u-1})^1_m \\
&+ (-1)^u(b_0)^{p+1}_1(\Lambda_{1,1})^1_m - (b_1)^p_1\Big] \\
&+ (b_0)^p_{p+u+1}\Big[(b_0)^{p+u}_1(\Lambda_{1,u-1})^1_m + \dots + (-1)^{z+1}(b_0)^{p+u-z+1}_1(\Lambda_{1,u-z})^1_m \\
&+ (-1)^{u+1}(b_0)^{p+1}_1(\Lambda_{1,0})^1_m\Big] + \dots \\
&+ (b_0)^p_m\Big[(b_0)^{m-1}_1(\Lambda_{1,u-1})^1_m + (-1)^{z+1}(b_1)^{m-z}_1(\Lambda_{1,u-z})^1_m \\
&+ (-1)^{u+1}(b_0)^{m-u}_1(\Lambda_{1,0})^1_m\Big] \\
&- (b_1)^p_1(\Lambda_{1,u-1})^1_m + \dots + (-1)^z(b_1)^p_z(\Lambda_{1,u-z})^1_m + \dots + (-1)^u(b_1)^p_u(\Lambda_{1,0})^1_m\Big] \\
&- (b_1)^p_{p+1}(\Lambda_{1,u-1})^{p+1}_m + \dots + (-1)^u(b_1)^p_{q+u}(\Lambda_{1,z})^{p+1}_m\Big]\,.
\end{aligned}
$$

As previously, we exchange the sums in the part of the previous equality form $(b_0)^p_{p+u+1}$ and we obtain

$$
\begin{aligned}
&(-1)^{p+1}(T_{3,u})^1_m \\
&= (-1)^z(b_0)^p_{z+1}\Big((\Lambda_{2,u-z})^1_m - zD_0(\Lambda_{1,u-z})^1_m\Big) \\
&\quad + (-1)^{z+1}(b_0)^p_{z+2}\Big((\Lambda_{2,u-z-1})^1_m - (z+1)D_0(\Lambda_{1,u-z-1})^1_m\Big) + \dots \\
&\quad + (-1)^u(b_0)^p_{u+1}\Big((\Lambda_{2,0})^1_m - uD_0(\Lambda_{1,0})^1_m\Big) \\
&\quad + (b_0)^p_{p+1}\Big[(\Lambda_{2,u})^{p+1}_m + (-1)^{z+u+1}\Big((b_0)^p_m(b_0)^{m-u}_1 + \dots \\
&\quad + (b_0)^p_{p+u+1}(b_0)^{p+1}_1 - (b_1)^p_u\Big)\Big] \\
&\quad + (b_0)^p_{p+2}\Big[-(\Lambda_{2,u-1})^{p+1}_m + (-1)^{z+u+1}\Big((b_0)^p_m(b_0)^{m-u+1}_1 + \dots \\
&\quad + (b_0)^p_{p+u}(b_0)^m_1 - (b_1)^p_{u-1}\Big) + D_0(\Lambda_{1,u-1})^{p+1}_m\Big] + \dots \\
&\quad + (b_0)^p_{u+q+1}\Big[(-1)^{u+z}(\Lambda_{2,z})^{p+1}_m + (-1)^{u+z+1}\Big((b_0)^p_m(b_0)^{m-z}_1 + \dots \\
&\quad + (b_0)^p_{p+z+1}(b_0)^{p+1}_1 - (b_1)^p_z\Big) \\
&\quad + (-1)^{u+z+1}(u-z)D_0(\Lambda_{1,z})^{p+1}_m\Big] \\
&\quad + (b_0)^p_{u+q+2}\Big[(-1)^{u+z+1}(\Lambda_{2,z-1})^{p+1}_m + (-1)^{u+z+1}\Big((b_0)^p_m(b_0)^{m-z+1}_1 + \dots
\end{aligned}
$$

$$
\begin{aligned}
&+ (b_0)^p_{p+z}(b_0)^{p+1}_1 - (b_1)^p_{z-1}\Big)\Big] + \dots \\
&+ (b_0)^p_{p+u}\Big[(-1)^{u+1}(\Lambda_{2,1})^{p+1}_m + (-1)^{z+u+1}\Big((b_0)^p_m(b_0)^{m-1}_1 + \dots \\
&+ (b_0)^p_{p+2}(b_0)^{p+1}_1 - (b_1)^p_1\Big)\Big] \\
&- (b_1)^p_{p+1}(\Lambda_{1,u-1})^{p+1}_m + \dots + (-1)^{u+z+1}(b_1)^p_{u+q}(\Lambda_{1,z})^{p+1}_m .
\end{aligned}
$$

We replace $(\Lambda_{2,z-1})^{p+1}_m, \dots, (\Lambda_{2,1})^{p+1}_m$ by their values in (4.31) and we combine the corresponding terms, to obtain

$$
\begin{aligned}
&(-1)^{p+1}(T_{3,u})^1_m \\
&= (-1)^z(b_0)^p_{z+1}\Big((\Lambda_{2,u-z})^1_m - zD_0(\Lambda_{1,u-z})^1_m \\
&\quad + (-1)^{z+1}(b_0)^p_{z+2}\Big((\Lambda_{2,z-1})^1_m - (z+1)D_0(\Lambda_{1,u-z-1})^1_m\Big) + \dots \\
&\quad + (-1)^u(b_0)^p_{u+1}\Big((\Lambda_{2,0})^1_m - uD_0(\Lambda_{1,0})^1_m\Big) \\
&\quad + (b_0)^p_{p+1}\Big[(\Lambda_{2,u})^{p+1}_m + (-1)^{z+u+1}\Big((b_0)^p_m(b_0)^{m-u}_1 + \dots \\
&\quad + (b_0)^p_{p+u+1}(b_0)^{p+1}_1 - (b_1)^p_u\Big)\Big] \\
&\quad + (b_0)^p_{p+2}\Big[-(\Lambda_{2,u-1})^{p+1}_m + (-1)^{z+u+1}\Big((b_0)^p_m(b_0)^{m-u}_1 + \dots \\
&\quad + (b_0)^p_{p+u}(b_0)^{p+1}_1 - (b_1)^p_{u-1}\Big) + D_0(\Lambda_{1,u-1})^{p+1}_m\Big] + \dots \\
&\quad + (b_0)^p_{u+q+1}\Big[(-1)^{u+z}(\Lambda_{2,z})^{p+1}_m + (-1)^{z+u+1}\Big((b_0)^p_m(b_0)^{m-z}_1 + \dots \\
&\quad + (b_0)^p_{p+z+1}(b_0)^{p+1}_1 - (b_1)^p_z\Big) \\
&\quad + (-1)^{u+z+1}(u-z)D_0(\Lambda_{1,z})^{p+1}_m\Big] .
\end{aligned}
$$

Considering (4.29) and Lemma 4.15 we obtain Lemma 4.16. If $z > q$, we obtain analogous results.

4.3.5. *Study of* $(T_{3,u})^{p+1}_p$. For $0 \le u \le p-2$ we have

$$
\begin{aligned}
&(-1)^{p-1}(T_{3,u})^{p+1} \\
&= (-1)^z\Big[(b_0)^m\Lambda_{2,u-z} - (b_0)^{m-1}_1\Lambda_{2,u-z-1} + \dots + (-1)^{u+z}(b_0)^{m-u+z}_1(\Lambda_{2,0})^1 \\
&\quad - (b_1)^m\Lambda_{1,u-z-1} + (b_1)^{m-1}_1\Lambda_{1,u-z-2} + \dots \\
&\quad + (-1)^{u+z}(b_1)^{m-u+z+1}(\Lambda_{1,0})^1 \\
&\quad + D_0(\Lambda_{2,u-z})^m - D_0(\Lambda_{2,u-z-1})^{m-1} + \dots \\
&\quad + (-1)^{u+z+1}D_0(\Lambda_{2,1})^{m-u+z+1} + (-1)^{u+z}D_0(\Lambda_{2,0})^{m-u+z}\Big] .
\end{aligned}
$$

Remark 4.19. We also define the symbols $(T_{3,p-1})^{p+1}$ and $(T_{3,p})^{p+1}_p$. They are not conditions (L), but will be useful in Lemma 4.21.

Lemma 4.20. *If $u < z$, then* $(T_{3,u})^{p+1}_p = 0$.

Now if $u \ge z$, we replace u by $v + z$. For $0 \le v \le q-2$ we have

$$
(-1)^{p-1}(T_{3,v+z})^{p+1}_p
$$

$$
\begin{aligned}
&= (-1)^z\Big\{\Big[(b_0)_1^m(\Lambda_{2,v})_p^1+\dots+(-1)^v(b_0)_1^{m-v}(\Lambda_{2,0})_p^1\Big]\\
&\quad+(b_0)_{p+1}^m(\Lambda_{2,v})_p^{p+1}+(b_0)_{p+2}^m\Big[-(\Lambda_{2,v-1})_p^{p+1}\\
&\quad+(b_0)_1^{p+1}(\Lambda_{1,v-1})_p^1+D_0(\Lambda_{1,v-1})_p^{p+1}\Big]+\dots\\
&\quad+(b_0)_{p+v}^m\Big[(-1)^{v+1}(\Lambda_{2,1})_p^{p+1}+(b_0)_1^{p+v-1}(\Lambda_{1,v-1})_p^1+\dots\\
&\quad+(-1)^v(b_0)_1^{p+1}(\Lambda_{1,1})_p^1+(-1)^v(v-1)D_0(\Lambda_{1,1})_p^{p+1}\Big]\\
&\quad+(b_0)_{p+v+1}^m\Big[(-1)^v(\Lambda_{2,0})_p^{p+1}+(b_0)_1^{p+v}(\Lambda_{1,v-1})_p^1+\dots\\
&\quad+(-1)^{v+1}(b_0)_1^{p+1}(\Lambda_{1,0})_p^1+(-1)^{v+1}vD_0(\Lambda_{1,0})_p^{p+1}\Big]\\
&\quad+(b_0)_{p+v+2}^m\Big[(b_0)_1^{p+v+1}(\Lambda_{1,v-1})_p^1+\dots+(-1)^{v+1}(b_0)_1^{p+2}(\Lambda_{1,0})_p^1\Big]+\dots\\
&\quad+(b_0)_{m-1}^m\Big[(b_0)_1^{m-2}(\Lambda_{1,v-1})_p^1+\dots+(-1)^{v+1}(b_0)_1^{m-v-1}(\Lambda_{1,0})_p^1\Big]\\
&\quad-(b_1)^m(\Lambda_{1,v-1})_p+(b_1)_1^{m-1}(\Lambda_{1,v-2})_p^1+\dots+(-1)^v(b_1)_1^{m-u+1}(\Lambda_{1,0})_p^1\\
&\quad+(-1)^{v+1}\Big[vD_0\big((b_0)_1^{m-v}(\Lambda_{1,0})_p^1\big)-(v-1)D_0\big((b_0)_1^{m-v+1}(\Lambda_{1,1})_p^1\big)\\
&\quad+\dots+(-1)^{v+z}zD_0\Big((b_0)_1^{m-z}(\Lambda_{1,v-z})_p^1\Big)\Big]\\
&\quad+\dots+(-1)^{v+1}D_0\big((b_0)_1^{m-1}(\Lambda_{1,v-1})_p^1\big)\Big]\Big\}.
\end{aligned}
$$

Gathering the terms $(b_0)_{z+k}^p$, we obtain

$$
\begin{aligned}
&(-1)^{p-1}(T_{3,v+z})_p^{p+1}\\
&= (-1)^z\Big\{(b_0)_1^m(\Lambda_{2,v})_p^1+\dots+(-1)^v(b_0)_1^{m-v}(\Lambda_{2,0})_p^1\\
&\quad+(b_0)_{z+1}^p\Big[-(\Lambda_{2,v})_p^{p+1}+(-1)^{v+z}\Big((b_0)_{p+v+1}^m(b_0)_1^{p+1}+\dots\\
&\quad+(b_0)_{m-1}^m(b_0)_1^{m-v-1}+(b_1)_1^{m-v+1}\Big)\Big]\\
&\quad+(b_0)_{z+2}^p\Big[(\Lambda_{2,v-1})_p^{p+1}+(-1)^{v+z}\Big((b_0)_{p+v}^m(b_0)_1^{p+1}+\dots\\
&\quad+(b_0)_{m-1}^m(b_0)_1^{m-v}+(b_1)_1^{m-v+2}\Big)-D_0(\Lambda_{1,v-1})_p^{p+1}\Big]+\dots\\
&\quad+(b_0)_{z+v}^p\Big[(-1)^v(\Lambda_{2,1})_p^{p+1}+(-1)^{v+z}\Big((b_0)_{p+v}^m(b_0)_1^{p+1}+\dots\\
&\quad+(b_0)_{m-1}^m(b_0)_1^{m-2}+(b_1)_1^m\Big)+(-1)^{v+1}(v-1)D_0(\Lambda_{1,1})_p^{p+1}\Big]\\
&\quad+(b_0)_{z+v+1}^p\Big[(-1)^{v+1}(\Lambda_{2,0})_p^{p+1}+(-1)^vvD_0(\Lambda_{1,0})^{p+1}\Big]\Big\}\\
&\quad+(-1)^{v+z}\Big((b_1)_{p+1}^m(b_0)_1^{m-v+1}+\dots+(b_1)_{p+v}^m(b_0)_1^m\Big)\\
&\quad+(-1)^{v+1}\Big[vD_0\Big((b_0)_1^{m-v}(\Lambda_{1,0})_p^1\Big)-(v-1)D_0\Big((b_0)^{m-v+1}(\Lambda_{1,1})_p^1\Big)+\dots\\
&\quad+(-1)^{v+z}zD_0\Big((b_0)_1^{m-z}(\Lambda_{1,v-z})_p^1\Big)+\dots+(-1)^{v+1}D_0\Big((b_0)_1^{m-1}(\Lambda_{1,v-1})_p^1\Big)\Big]\Big\}.
\end{aligned}
$$

We replace terms $(\Lambda_{2,k})_p^{p+1}$ by (4.37) and simplify to obtain

$$
\begin{aligned}
&(-1)^{p+1}(T_{3,v+z})_p^{p+1}\\
&= (-1)^z\Big\{(b_0)_1^m(\Lambda_{2,v})_p^1 + \dots + (-1)^v(b_0)_1^{m-v}(\Lambda_{2,0})_p^1\\
&\quad + (b_0)_{z+1}^p\Big[-qD_0(\Lambda_{1,v})_p^{p+1}\Big] + (b_0)_{z+2}^p\Big[(q-1)D_0(\Lambda_{1,v-1})_p^{p+1}\Big] + \dots\\
&\quad + (b_0)_{z+v+1}^p\Big[(-1)^{v+1}(q-v)D_0(\Lambda_{1,0})_p^{p+1}\Big]\\
&\quad + (-1)^{v+1}\Big[v(b_0)_1^{m-v}D_0(\Lambda_{1,0})_p^1 + (v-1)(b_0)_1^{m-v+1}D_0(\Lambda_{1,1})_p^1\\
&\quad + (-1)^{v+z}z(b_0)_1^{m-z}D_0(\Lambda_{1,v-z}) + \dots +\\
&\quad (-1)^{v+1}(b_0)_1^{m-1}D_0(\Lambda_{1,v-1})_p^1\Big] + (-1)^{v+z}\Big((b_1)_{p+1}^m(b_0)_1^{m-v+1} + \dots\\
&\quad + (b_1)_{p+v}^m(b_0)_1^m\Big)\Big\}\\
&= (-1)^z\Big\{(b_0)_1^m(\Lambda_{2,v})_p^1 + \dots + (-1)^v(b_0)_1^{m-v}(\Lambda_{2,0})_p^1\\
&\quad - \Big[q(b_0)_{z+1}^pD_0(\Lambda_{1,v})_p^{p+1} - (q-1)(b_0)_{z+2}^pD_0(\Lambda_{1,v-1})_p^{p+1} + \dots\\
&\quad + (-1)^{v+z}(p-v)(b_0)_{p+1}^pD_0(\Lambda_{1,z})_p^{p+1}\\
&\quad + (-1)^{v+1}(q-v+1)(b_0)_{z+v}^pD_0(\Lambda_{1,1})_p^{p+1}\\
&\quad + (-1)^v(q-v)(b_0)_{z+v+1}^pD_0(\Lambda_{1,0})_p^{p+1}\Big]\\
&\quad + (-1)^{v+1}\Big[v(b_0)_1^{m-v}D_0(\Lambda_{1,0})_p^1 - (v-1)(b_0)_1^{m-v+1}D_0(\Lambda_{1,1})_p^1\\
&\quad + (-1)^{z+v}z(b_0)_1^{m-z}D_0(\Lambda_{1,v-z})_p^1 + \dots + (-1)^{v+1}(b_0)_1^{m-1}D_0(\Lambda_{1,v-1})_p^1\Big]\\
&\quad + (-1)^{v+z}\Big[(b_1)_{p+1}^m(b_0)_1^{m-v+1} + \dots + (b_1)_{p+v}^m(b_0)_1^m\Big]\Big\}.
\end{aligned}
$$

Lemma 4.21. *Assumeing $(b_0)_{p+1}^p \neq 0$, we have the following:*

(1) *if $0 \le v < z$, then $(T_{3,v+z})_p^{p+1} = 0$.*

(2) *if $v = z$, then*
$$(b_0)_{p+1}^p(T_{3,2z})_p^{p+1} = (-1)^z(b_0)_1^{m-z}(T_{3,z})_m^1\,.$$

(3) *if $z < v \le q-2$ and $(T_{3,z})_m^1 = \dots = (T_{3,v-1})_m^1$, then*
$$(b_0)_{p+1}^p(T_{3,z+v})_p^{p+1} = (-1)^z(b_0)_1^{m-z}(T_{3,v})_m^1\,.$$

(4) *Using the notation in Remark 4.19: for $v = q-1$,*
$$(b_0)_{p+1}^p(T_{3,p+1})_p^{p+1} = (-1)^z(b_0)_1^{m-z}(T_{3,q-1})_m^1$$
and for $v = q$,
$$(b_0)_{p+1}^p(T_{3,p})_p^{p+1} = (-1)^z(b_0)_1^{m-z}(T_{3,q})_m^1$$

Proof. By Remark 4.1 and conditions (L_1), (L_2), we have
$$(b_0)_1^m = \dots = (b_0)_1^{m-z+1} = 0\,.$$

(1) If $0 \le v < z$, the proof is immediate.

(2) If $v = z$, by (4.41), we have

$(b_0)_{p+1}^p(T_{3,2z})_p^{p+1}$

$$
\begin{aligned}
&= (-1)^{p+1}(b_0)^p_{p+1}\Big[(b_0)^{m-z}_1\Big((\Lambda_{2,0})^1_p - zD_0(\Lambda_{1,0})^1_p\Big) + (-1)^{z+1}q(b_0)^p_{z+1}D_0(\Lambda_{1,z})^{p+1}_p\Big]\\
&= (b_0)^{m-z}_1\Big[(-1)^z(T_{3,z})^1_m + (-1)^{p+1}\Big(2q(b_0)^p_{p+1}D_0(\Lambda_{1,0})^1_p - q(b_0)^p_{z+1}D_0(\Lambda_{1,0})^1_m\Big)\Big]\\
&\quad + (-1)^q q(b_0)^p_{z+1}(b_0)^p_{p+1}D_0(\Lambda_{1,z})^{p+1}_p\,.
\end{aligned}
$$

We replace $(b_0)^p_{p+1}(b_0)^{m-z}_1$ by $-\big((b_0)^p_{z+1}\big)^2$, and by (4.2) we obtain

$$
\begin{aligned}
(b_0)^p_{p+1}(T_{3,2z})^{p+1}_p &= (-1)^z(b_0)^{m-z}_1(T_{3,z})^1_m + (-1)^q q(b_0)^p_{z+1}D_0\Big((b_0)^{m-z}_1(b_0)^p_{p+1}\Big)\\
&\quad + (-1)^q 2q\big((b_0)^p_{z+1}\big)^2 D_0(b_0)^p_{z+1} = 0\,.
\end{aligned}
$$

(3) Let $z < v \le q-2$. By induction we assume $(T_{3,z+v-1})^{p+1}_p = 0$. We use Proposition 2.5 and replace $(b_0)^p_{p+1}$ by 1. For $u > z$,

$$
(-1)^{p+1}(T_{3,u+z})^{p+1}_p = A + B + C + D\,,
$$

where

$$
\begin{aligned}
A &= \Big[(b_0)^{m-z}_1(\Lambda_{2,u-z})^1_p - (b_0)^{m-z-1}_1(\Lambda_{2,u-z-1})^1_p + \dots + (-1)^{u+z}(b_0)^{m-u}_1(\Lambda_{2,0})^1_p\Big],\\
B &= (-1)^{z+1}\Big[q(b_0)^p_{z+1}D_0(\Lambda_{1,u})^{p+1}_p + \dots + (-1)^{u+z}(p-u)(b_0)^p_{u+1}D_0(\Lambda_{1,z})^{p+1}_p\Big],\\
C &= (-1)^{u+1+z}\Big[u(b_0)^{m-u}_1D_0(\Lambda_{1,0})^1_p + \dots + (-1)^{u+z}z(b_0)^{m-z}_1D_0(\Lambda_{1,u-z})^1_p\Big],\\
D &= (-1)^u\Big[(b_1)^m_{p+1}(b_0)^{m-u+1}_1 + \dots + (b_1)^m_{u+q}(b_0)^{m-z}_1\Big]\,.
\end{aligned}
$$

We use the assumption in Lemma 4.16 to obtain

$$
\begin{aligned}
&A + D\\
&= (-1)^z(b_0)^{m-z}_1(-1)^{p+1}(T_{3,u})^1_m + (b_0)^{m-z}_1 mD_0(\Lambda_{1,u-z})^1_p\\
&\quad + (b_0)^{m-z}_1(b_0)^p_{p+2}\Big[(\Lambda_{2,u-z-1})^1_p - mD_0(\Lambda_{1,u-z-1})^1_p + D_0(\Lambda_{1,u-1})^{p+1}_m\Big] + \dots\\
&\quad + (b_0)^{m-z}_1(b_0)^p_{u+q+1}\Big[(-1)^{z+u+1}\Big((\Lambda_{2,0})^1_p - mD_0(\Lambda_{1,0})^1_p\\
&\quad + (u-z)D_0(\Lambda_{1,z})^{p+1}_m\Big)\Big]\\
&\quad - (b_0)^{m-z}_1\Big[q(b_0)^p_{z+1}D_0(\Lambda_{1,u-z})^1_m - (q-1)(b_0)^p_{z+2}D_0(\Lambda_{1,u-z-1})^1_m\\
&\quad + (-1)^{z+u}(p-u)(b_0)^p_{u+1}D_0(\Lambda_{1,0})^1_m\Big] - (b_0)^{m-z-1}_1 mD_0(\Lambda_{1,u-z-1})^1_p\\
&\quad - (b_0)^{m-z-1}_1(b_0)^p_{p+2}\Big[(\Lambda_{2,u-z-2})^1_p - mD_0(\Lambda_{1,u-z-2})^1_p + \dots\\
&\quad + D_0(\Lambda_{1,u-2})^{p+1}_m\Big] + \dots - (b_0)^{m-z-1}_1(b_0)^p_{u+q}\Big[(-1)^{z+u}(\Lambda_{2,0})^1_p - mD_0(\Lambda_{1,0})^1_p\\
&\quad + (-1)^u(u-z-1)D_0(\Lambda_{1,z})^{p+1}_m\Big]\\
&\quad - (b_0)^{m-z-1}_1\Big[-q(b_0)^p_{z+1}D_0(\Lambda_{1,u-z-1})^1_m + \dots + (p-u)(b_0)^p_{u-1}D_0(\Lambda_{1,1})^1_m\Big]\\
&\quad + \dots + (-1)^{u+z}(b_0)^{m-u+2}_1 mD_0(\Lambda_{1,2})^1_p\\
&\quad + (-1)^{u+z}(b_0)^{m-u+2}_1(b_0)^p_{p+2}\Big[\Big((\Lambda_{2,1})^1_p - mD_0(\Lambda_{1,1})^1_p + D_0(\Lambda_{1,z+1})^{p+1}_m\Big)\Big]\\
&\quad + (-1)^{u+z}(b_0)^{m-u+2}_1(b_0)^p_{p+3}\Big[-\Big((\Lambda_{2,0})^1_p - mD_0(\Lambda_{1,0})^1_p + 2D_0(\Lambda_{1,z})^{p+1}_m\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{u+z}(b_0)_1^{m-u+2}\Big[q(b_0)_{z+1}^p D_0(\Lambda_{1,2})_m^1-(q-1)(b_0)_{z+2}^p D_0(\Lambda_{1,1})_m^1\\
&+(-1)^{u+z+1}(b_0)_1^{m-u+1}mD_0(\Lambda_{1,1})_p^1\\
&+(-1)^{u+z+1}(b_0)_1^{m-u+1}(b_0)_{p+2}^p\Big[(\Lambda_{2,0})_p^1-mD_0(\Lambda_{1,0})_p^1+D_0(\Lambda_{1,z})_m^{p+1}\Big]\\
&+(-1)^{u+z}(b_0)_1^{m-u+1}q(b_0)_{z+1}^p D_0(\Lambda_{1,1})_m^1+(-1)^{u+z}(b_1)_1^{m-u}mD_0(\Lambda_{1,0})_p^1\,.
\end{aligned}
$$

We transform B as

$$
\begin{aligned}
B&=(-1)^u q(b_0)_{z+1}^p D_0\Big((b_0)_{p+2}^p(b_0)_1^{m-u}+\dots+(b_0)_{u+q+1}^p(b_0)_1^{m-z}\\
&\quad+(b_0)_{z+1}^p(b_0)_{u+1}^p+\dots+(b_0)_{u+1}^p(b_0)_{z+1}^p\Big)\\
&\quad+(-1)^u(q-1)(b_0)_{z+2}^p D_0\Big((b_0)_{p+2}^p(b_0)_1^{m-u+2}+\dots+(b_0)_{u+q}^p(b_0)_1^{m-z}\\
&\quad+(b_0)_{z+1}^p(b_0)_u^p+\dots+(b_0)_u^p(b_0)_{z+1}^p\Big)\\
&\quad+\dots\\
&\quad+(-1)^u(q-u+z+1)(b_0)_u^p D_0\Big((b_0)_{p+2}^p(b_0)_1^{m-z}+(b_0)_{z+1}^p(b_0)_{z+2}^p\\
&\quad+(b_0)_{z+2}^p(b_0)_{z+1}^p\Big)\\
&\quad+(-1)^u(q-u+z)(b_0)_{u+1}^p D_0\big[(b_0)_{z+1}^p\big]^2\\
&=(-1)^u(b_0)_{p+2}^p\Big[q(b_0)_{z+1}^p D_0(b_0)_1^{m-u+1}+(q-1)(b_0)_{z+2}^p D_0(b_0)_1^{m-u+2}\\
&\quad+\dots+(q-u+z+1)(b_0)_u^p D_0(b_0)_1^{m-z}\Big]+\dots\\
&\quad+(-1)^u(b_0)_{u+q+1}^p\Big[q(b_0)_{z+1}^p D_0(b_0)_1^{m-z-1}+(q-1)(b_0)_{z+2}^p D_0(b_0)_1^{m-z}\Big]\\
&\quad+(-1)^{u+1}(b_0)_{u+q+1}^p\Big[q(b_0)_{z+1}^p D_0(b_0)_1^{m-z}\Big]\\
&\quad+(-1)^u\Big[q(b_0)_{z+1}^p(b_0)_1^{m-u+1}+(q-1)(b_0)_{z+2}^p(b_0)_1^{m-u+2}+\dots\\
&\quad+(q-u+z+1)(b_0)_u^p(b_0)_1^{m-z}\Big]D_0(b_0)_{p+2}^p+\dots\\
&\quad+(-1)^u q(b_0)_{z+1}^p(b_0)_1^{m-z}D_0(b_0)_{u+q+1}^p\\
&\quad+(-1)^u 2\Big[q(b_0)_{z+1}^p D_0(b_0)_{u+1}^p+(q-1)(b_0)_{z+2}^p D_0(b_0)_u^p+\dots\\
&\quad+(q-u+z+1)(b_0)_u^p D_0(b_0)_{z+2}^p\\
&\quad+(q-u+z)(b_0)_{u+1}^p D_0(b_0)_{z+1}^p\Big](b_0)_{z+1}^p\\
&\quad+(-1)^u 2\Big[q(b_0)_{z+1}^p D_0(b_0)_u^p+(q-1)(b_0)_{z+2}^p D_0(b_0)_{u-1}^p+\dots\\
&\quad+(q-u+z+1)(b_0)_1^p D_0(b_0)_{z+1}^p\Big](b_0)_{z+2}^p+\dots\\
&\quad+(-1)^u 2\Big[q(b_0)_{z+1}^p D_0(b_0)_{z+1}^p\Big](b_0)_{u+1}^p\,.
\end{aligned}
$$

We use $(T_{2,u})_p^1=0$ to transform C,

$$
C=(-1)^{u+z}u\Big[(b_0)_{p+2}^p(b_0)_1^{m-u+1}+\dots+(b_0)_{u+q}^p(b_0)_1^{m-z-1}+(b_0)_{u+q+1}^p(b_0)_1^{m-z}
$$

$$
\begin{aligned}
&+ (b_0)^p_{z+1}(b_0)^p_{u+1} + \dots + (b_0)^p_{u+1}(b_0)^p_{z+1}\Big] D_0(b_0)^p_{z+1} \\
&+ (-1)^{u+z}(u-1)\Big[(b_0)^p_{p+2}(b_0)^{m-u}_1 + \dots + (b_0)^p_{u+q}(b_0)^{m-z}_1 \\
&+ (b_0)^p_{z+1}(b_0)^p_u + \dots + (b_0)^p_u(b_0)^p_{z+1}\Big] D_0(b_0)^p_{z+2} + \dots \\
&+ (-1)^{u+z}(z+1)\Big[(b_0)^p_{p+2}(b_0)^{m-z}_1 + (b_0)^p_{z+1}(b_0)^p_{z+2} \\
&+ (b_0)^p_{z+2}(b_0)^p_{z+1}\Big] D_0(b_0)^p_u + (-1)^{u+z} z\big[(b_0)^p_{z+1}\big]^2 D_0(b_0)^p_{u+1}\,.
\end{aligned}
$$

We know by the induction assumption that $(T_{3,v})^1_m = 0$, if $v \le u-1$ and we know that $(T_{2,u})^1_p = 0$.

We have obtained a linear form in $(b_0)^p_{p+k}$, $2 \le k \le q+1$, $D_0(b_0)^p_k$ $(z+1 \le k \le p)$. Then we verify that the coefficients cancel and we obtain the proof. □

4.4. **Study of $K_{2,u}$, $u \ge p-1$.** We assume now that (L_1) and (L_2) are satisfied and that

$$K_{2,0} = \dots = K_{2,p-2} = 0\,.$$

and we use Proposition 2.5.

We have $p-1 \le u \le m-z$ $(q \ge 2)$. First considering $u = p-1$, by (4.11) and (4.10), we have that for $1 \le k-1 \le p-1$,

$$(\mathscr{A}_{2,p-2})^k + (\mathscr{A}_{2,p-2})^{k-1} + D_0(\mathscr{A}_{1,p-1})^{k-1} + D_1(\mathscr{A}_{1,p-1})^k = (K_{2,p-1})^{k-1}\,.$$

and

$$
\begin{aligned}
(K_{2,p-1})^p = (b_0)^p \mathscr{A}_{1,p-1} &+ D_0(\mathscr{A}_{1,p-1})^p - D_0(\mathscr{A}_{1,p-2})^{p-1} + \dots \\
&+ (-1)^{p-1} D_0(\mathscr{A}_{1,0})^1 + (b_1)^p \Lambda_{0,z-1}\,.
\end{aligned}
$$

In this section, we will not write the details of the terms in D_1, when they are deduced automatically from the terms in D_0. Then

$$
\begin{aligned}
(K_{2,p-1})^p = (-1)^z &\Big[(b_0)^p_{z+1}(\mathscr{A}_{1,q-1})^1 - (b_0)^p_{z+2}(\mathscr{A}_{1,q-2})^1 + \dots \\
&+ (-1)^q (b_0)^p_{p-1}(\mathscr{A}_{1,1})^1\Big] + (b_0)^p_{p+1}(\mathscr{A}_{1,p-1})^{p+1} \\
&+ (b_0)^p_{p+2}\Big[-(\mathscr{A}_{1,p-2})^{p+1} + (b_0)^{p+1}_1(\Lambda_{0,z-1})^1\Big] + \dots \\
&+ (b_0)^p_m\Big[(-1)^{q+1}(\mathscr{A}_{1,z})^{p+1}_p + (b_0)^{m-1}_1(\Lambda_{0,z-1})^1 + \dots \\
&+ (-1)^{z+1}(b_0)^{m-z}_1(\Lambda_{0,0})^1\Big] + (-1)^{p-1} p D_0(\mathscr{A}_{1,0})^1 + (b_1)^p(\Lambda_{0,z-1})\,.
\end{aligned}
$$

If $(b_0)^p_{p+1} \ne 0$, we set

$$(\mathscr{A}_{1,p-1})^1_p = 0, \quad \text{and} \quad (\mathscr{A}_{1,p-1})^{p+1}_p = (-1)^{p-1} m D_0(\Lambda_{1,0})^1_p\,. \tag{4.42}$$

By Definitions 3.38 and 3.42, we obtain

$$
\begin{aligned}
(K_{2,p-1})^p_p = (-1)^z &\Big[(b_0)^p_{z+2}(\Lambda_{1,q-2})^1_p + \dots + (-1)^{q+1}(b_0)^p_{p-1}(\Lambda_{1,1})^1_p\Big] \\
&+ (\mathscr{A}_{1,p-1})^{p+1}_p + (-1)^p p D_0(\Lambda_{1,0})^1_p + (b_1)^p(\Lambda_{0,z-1})_p\,.
\end{aligned}
$$

By (4.26) and Consequence 4.18,

$$(K_{2,p-1})^p_p = (-1)^p\Big[(\Lambda_{2,0})^1_p - m D_0(\Lambda_{1,0})^1_p\Big] = (-1)^{z+1}(T_{3,z})^1_m\,. \tag{4.43}$$

If $(b_0)^p_{p+1} = 0$, then $(T_{3,z})^1_m = 0$. By Example 4.3, we obtain $(b_0)^p_{z+1} = 0$.

If $(b_0)^p_{p+2} \neq 0$, using Lemma 4.16, we obtain

$$(-1)^{z+1}(T_{3,z+1})^1_m = (-1)^{p+1}(\Lambda_{2,0})^1_p\,.$$

Therefore,

$$(K_{2,p-1})^p_p = (-1)^z(T_{3,z+1})^1_m\,.$$

In general, if $(b_0)^p_{p+1} = \dots = (b_0)^p_{p+k} = 0$, $(b_0)^p_{p+k+1} \neq 0$, and $1 \leq k \leq q-2$, then

$$(K_{2,p-1})^p_p = (-1)^{z+k-1}(T_{3,z+k})^1_m\,. \tag{4.44}$$

Now we set

$$(\mathscr{A}_{1,q-1})^1_m = -(\Lambda_{1,q-1})^1_m, \quad (\mathscr{A}_{1,p-1})^{p+1}_m = 0\,. \tag{4.45}$$

Then by definitions (3.40) and (3.41),

$$\begin{aligned}
&(K_{2,p-1})^p_m\\
&= (-1)^{z+1}\Big[(b_0)^p_{z+1}(\Lambda_{1,q-1})^1_m - (b_0)^p_{z+2}(\Lambda_{1,q-2})^1_m + \dots + (-1)^q(b_0)^p_{p-1}(\Lambda_{1,1})^1_m\Big]\\
&\quad + (b_0)^p_{p+2}(\Lambda_{1,p-2})^{p+1}_m + \dots + (-1)^q(b_0)^p_m(\Lambda_{1,z})^{p+1}_m + (-1)^p pD_0(\Lambda_{1,0})^1_m\\
&= (-1)^p pD_0(\Lambda_{1,0})^1_m\,. = 0\,.
\end{aligned}$$

Next we verify (4.13): if $k \neq p, m$, then

$$(K_{2,p-1})^p_k = (-1)^p(\Lambda_{2,0})^1_k = 0\,, \tag{4.46}$$

$$\begin{aligned}
(K_{2,p-1})^m &= (\mathscr{A}_{2,p-2})^m + (b_0)^m\mathscr{A}_{1,p-1} + D_0(\mathscr{A}_{1,p-1})^m + (b_1)^m\Lambda_{0,z-1}\,,\\
(K_{2,p-1})^m_m &= (-1)^{q+1}(\mathscr{A}_{2,z-1})^{p+1}_m + (-1)^{z+1}\Big[(b_0)^{m-z}_1(\Lambda_{1,q-1})^1_m + \dots\\
&\quad + (-1)^{q-1}(b_0)^{p+1}_1(\Lambda_{1,z})^{p+1}_m\Big]\\
&\quad + (b_0)^m_{p+2}(\Lambda_{1,p-2})^{p+1}_m + \dots + (-1)^{q+1}(b_0)^m_{m-1}(\Lambda_{1,z+1})^1_m\\
&\quad - D_0(\Lambda_{1,p-1})^m_m + \dots + (-1)^q D_0(\Lambda_{1,z})^{p+1}_m\Big] + (b_1)^m(\Lambda_{0,z-1})_m\\
&= (-1)^{q+1}\Big[(\mathscr{A}_{2,z-1})^{p+1}_m - (\Lambda_{2,z})^{p+1}_m\Big]\,.
\end{aligned}$$

If $(b_0)^p_{p+1} \neq 0$, we set

$$(\mathscr{A}_{2,z-1})^{p+1}_m = (\Lambda_{2,z})^{p+1}_m + (-1)^p(T_{3,z})^1_m.$$

So that

$$(b_0)^p_{p+1}(K_{2,p-1})^m_m = (-1)^{z+1}(T_{3,z})^1_m\,.$$

and

$$(K_{2,p-1})^m_m = (K_{2,p-1})^p_p\,. \tag{4.47}$$

If $(b_0)^p_{p+2} \neq 0$ and $(b_0)^p_{p+1} = 0$, we set

$$(\mathscr{A}_{2,z-1})^{p+1}_m - (\Lambda_{2,z})^{p+1}_m = (-1)^{p+1}(T_{3,z+1})^1_m.$$

So that

$$(b_0)^p_{p+2}(K_{2,p-1})^m_m = (-1)^z(T_{3,z-1})^1_m$$

and

$$(K_{2,p-1})^m_m = (K_{2,p-1})^p_p\,. \tag{4.48}$$

By definition (4.42), $(\mathscr{A}_{1,p-1})_p^{p+1} = (-1)^{p-1} m D_0(\Lambda_{1,0})_p^1$. Therefore,

$$\begin{aligned}
(K_{2,p-1})_p^m &= (-1)^{q+1}(\mathscr{A}_{2,z-1})_p^{p+1} \\
&\quad + (-1)^z\Big[(b_0)_1^{m-z-1}(\mathscr{A}_{1,q-2})_p^1 + \dots + (-1)^p (b_0)_1^{p+1}(\mathscr{A}_{1,z})_p^1\Big] \\
&\quad + (b_0)_{p+1}^m(\mathscr{A}_{1,p-1})_p^{p+1} - (b_0)_{p+2}^m(\mathscr{A}_{1,p-2})_p^{p+1} + \dots \\
&\quad + (-1)^q (b_0)_{m-1}^m(\mathscr{A}_{1,z+1})_p^{p+1} + (-1)^q q D_0(\mathscr{A}_{1,z})_p^{p+1} \\
&\quad + (b_1)^m(\Lambda_{0,z-1})_p\,.
\end{aligned}$$

By (4.38),

$$(K_{2,p-1})_p^m = (-1)^{q+1}\big[(\mathscr{A}_{2,z-1})_p^{p+1} - (\Lambda_{2,z})_p^{p+1}\big] + (-1)^p m (b_0)_{z+1}^p D_0(\Lambda_{1,0})_p^1\,.$$

If $(b_0)_{p+1}^p \neq 0$, we set

$$(\mathscr{A}_{2,z-1})_p^{p+1} = (\Lambda_{2,z})_p^{p+1} + (-1)^z m (b_0)_{z+1}^p D_0(\Lambda_{1,0})_p^1.$$

So that

$$(K_{2,p-1})_p^m = 0\,. \tag{4.49}$$

With the notation $(b_0)_{p+1}^p \neq 0$, $\widehat{K}_{2,p-1} = (-1)^{z+1}(T_{3,z})_m^1$, and $(K_{2,p-1})_p^p = \widehat{K}_{2,p-1}$, we obtain

$$K_{2,p-1} = \widehat{K}_{2,p-1} I\,. \tag{4.50}$$

If $(b_0)_{p+1}^p = \dots = (b_0)_{p+k}^p = 0$, $(b_0)_{p+k+1}^p \neq 0$, and $k > 0$, we obtain

$$\begin{gathered}
\widehat{K}_{2,p-1} = (-1)^{z+k+1}(T_{3,z+k})_m^1\,, \qquad (4.51)\\
K_{2,p-1} = \widehat{K}_{2,p-1} I\,.
\end{gathered}$$

Now, we consider $p \le u \le m-3$ and assume that

$$(K_{2,p-1})^p = \dots = (K_{2,u-1})^p = 0\,.$$

We set

$$(\mathscr{A}_{1,u})_p^1 = 0 \quad \text{and} \quad (\mathscr{A}_{1,u})_p^{p+1} = (-1)^{p-1} m D_0(\Lambda_{1,u-p+1})_p^1.$$

Then we obtain (cf. (4.10))

$$\begin{aligned}
(K_{2,u})_p^p &= (\mathscr{A}_{2,u-1})_p^p + (b_0)^p(\mathscr{A}_{1,u})_p + D_0(\mathscr{A}_{1,u})^p + (b_1)^p(\Lambda_{0,u-q})_p \\
&= (-1)^{p-1}(\mathscr{A}_{2,u-p})_p^1 + (b_0)^p(\mathscr{A}_{1,u})_p + D_0(\mathscr{A}_{1,u})_p^p \\
&\quad - D_0(\mathscr{A}_{1,u-1})_p^{p-1} + \dots + (-1)^{p+1} D_0(\mathscr{A}_{1,u-p+1})_p^1 \\
&\quad + (-1)^{p+u+1}(b_0)_{u-q+1}^p\,.
\end{aligned}$$

It follows from (4.25) that

$$(K_{2,u})_p^p = (-1)^{p-1}\big[(\mathscr{A}_{2,u-p})_p^1 - (\Lambda_{2,u-p+1})_p^1 + m D_0(\Lambda_{1,u-p+1})_p^1\big]\,.$$

If $(b_0)_{p+1}^p \neq 0$, we set

$$(-1)^{p-1}\Big[(\mathscr{A}_{2,u-p})_p^1 - (\Lambda_{2,u-p+1})_p^1 + m D_0(\Lambda_{1,u-p+1})_p^1\Big] = (-1)^{z+1}(T_{3,u-q+1})_m^1\,,$$

and for $0 \le u \le q-3$, we set

$$(\mathscr{A}_{2,u})_p^1 = (\Lambda_{2,u+1})_p^1 + (-1)^q (T_{3,u+z+1})_m^1 - m D_0(\Lambda_{1,u+1})_p^1\,.$$

So that

$$(K_{2,u})_p^p = (-1)^{z+1}(T_{3,u-q+1})_m^1\,. \tag{4.52}$$

For $u \geq q$, we set $(\mathscr{A}_{1,u})^1_m = 0$ and $(\mathscr{A}_{1,u})^{p+1}_m = 0$. Then by (4.29) and (4.10) we have

$$(K_{2,u})^p_m = (-1)^{p-1}\Big[(\mathscr{A}_{2,u-p})^1_m - (\Lambda_{2,u-p+1})^1_m\Big] .$$

For $0 \leq u \leq q-3$, we set $(\mathscr{A}_{2,u})^1_m = (\Lambda_{2,u+1})^1_m$. So that

$$(K_{2,u})^p_m = 0 . \tag{4.53}$$

We have

$$\begin{aligned}
(K_{2,u})^m_m &= (\mathscr{A}_{2,u-1})^m_m + (b_0)^m(\mathscr{A}_{1,u})_m + D_0(\mathscr{A}_{1,u})^m_m + (b_1)^m(\Lambda_{0,u-q})_m \\
&= (-1)^{q-1}(\mathscr{A}_{2,u-q})^{p+1}_m - (b_0)^m(\Lambda_{1,u})_m + \dots \\
&\quad + (-1)^q q D_0(\Lambda_{1,u-q+1})^{p+1}_m + (b_0)^{m-1}_1(\Lambda_{1,u-1})^1_m + \dots \\
&\quad + (-1)^{u-1}(b_0)^{m-u}_1(\Lambda_{1,0})^1_m .
\end{aligned}$$

From (4.32),

$$(K_{2,u})^m_m = (-1)^{q-1}\Big[(\mathscr{A}_{2,u-q})^{p+1}_m - (\Lambda_{2,u-q+1})^{p+1}_m\Big]$$

If $(b_0)^p_{p+1} \neq 0$ and $z \leq u \leq p-3$ we set

$$(\mathscr{A}_{2,v})^{p+1}_m = (\Lambda_{2,v+1})^{p+1}_m + (-1)^p(T_{3,v+1})^1_m .$$

So that

$$(K_{2,u})^m_m = (-1)^{z+1}(T_{3,u-q+1})^1_m = (K_{2,u})^p_p . \tag{4.54}$$

By (4.28),

$$(K_{2,u})^m_p = (-1)^{q-1}\Big[(\mathscr{A}_{2,u-q})^{p+1}_p - (\Lambda_{2,u-q+1})^{p+1}_p\Big] + (-1)^p m(b_0)^p_{z+1}D_0(\Lambda_{1,u-p+1})^1_p .$$

For $0 \leq u \leq p-3$ we set

$$(\mathscr{A}_{2,v})^{p+1}_p = (\Lambda_{2,v+1})^{p+1}_p + (-1)^z m(b_0)^p_{z+1}D_0(\Lambda_{1,v-z+1})^1_p .$$

So that

$$(K_{2,u})^m_p = 0 . \tag{4.55}$$

Also we obtain

$$\begin{gathered}
(K_{2,u})^p_k = 0 \quad \text{for } k \neq p , \\
(K_{2,u})^m_k = 0 \quad \text{for } k \neq p, m , \\
(K_{2,u})^j_{j'} = 0 \quad \text{for } j \neq p, m,\ j \neq j' , \\
(K_{2,u})^j_j = (K_{2,u})^m_m \quad \text{for } j \neq p ,
\end{gathered}$$

and

$$K_{2,u} = (-1)^{z+1}(T_{3,u-q+1})^1_m I . \tag{4.56}$$

Example 4.22. $(K_{2,m-3})^m_m = (-1)^{z+1}(T_{3,p-2})^1_m I$.

If $(b_0)^p_{p+1} = \dots = (b_0)^p_{p+k} = 0$, and $(b_0)^m_{p+k+1} \neq 0$, we proceed as before to obtain

$$K_{2,u} = (-1)^{z+u+k}(T_{3,u+k-q+1})^1_m I . \tag{4.57}$$

Example 4.23. $K_{2,m-k-3} = (-1)^{z+u+k}(T_{3,p-2})^1_m I$.

Proposition 4.24. *We assume* (L_1) *and* (L_2) *are satisfied. If* $0 \leq u \leq m-3$ *and* $K_{2,u} = 0$*, then for* $0 \leq u \leq p-2$ *we have* $(T_{3,u})^1_m = 0$.

5. Necessity of (L_1), (L_2) and $(T_{3,u})^1_m = 0$

We know the necessity of $(b_0)^p_1 = 0$ by Proposition 3.8. We state the necessity of $(T_{2,0})^1_p = 0$ when $z = 1, 2$. Let

$$k = D_0^m + K'_{1,q+1} D_0^{q+1} D_1^{p-2} + \dots + (-1)^p (T_{2,0})^1_p D_1^{m-2} + \dots + K'_2 + \dots + K'_m$$

First for $z = 1$ and $m = 2p - z = 2p - 1$, we have

$$\frac{p-2}{p-1} < \frac{m-2}{m}$$

because $p - 1 < \frac{m}{2}$ as $2p - 2 < 2p - 1$.

We proceed as in Proposition 3.8. We consider the expansion

$$u_{k_0} = \exp\Big(i\, x_1 \xi_1 + \psi(x) \xi_1^{\frac{m-2}{m}})\Big) \sum_{0 \le k \le k_0} Y_k(x) \xi_1^{-k/m}\,.$$

We apply K and we obtain the coefficients of Y_0:

$$\exp(\dots)\big[\xi_1^{m-2} (D_0 \psi)^m + K_{2,0} \xi_1^{m-2}\big] = 0\,.$$

We choose

$$(D_0 \psi)^m = (-1)^{p+1} (T_{2,0})^1_p$$

and a complex root different from 0 such that $\operatorname{Re} D_0 \psi > 0$,

$$D_0 \psi = (-1)^{p+1} \big[(T_{2,0})^1_p\big]^{1/m}\,.$$

We continue the identification to 0 of the coefficients of u_{k_0} and we obtain an expansion where Y_0 is free and an analogous expansion $\mathscr{A}' u_{k_0}$ where the first coefficient is free and can be chosen different from 0.

As before, we deduce that $(T_{2,0})^1_p$ is necessary in order that the Cauchy problem be well posed in $\mathcal{C}^\infty$.

We obtain also that if $(T_{2,0})^1_p \neq 0$, then the Cauchy problem is well posed in $\gamma^{\frac{m}{m-2}}$.

Case $z = 2$: We know that (cf. (3.32)),

$$(K_{1,q+1})^p_p = (-1)^p (T_{1,1})^1_p,$$

$$k = D_0^m + \dots + (-1)^{p+1} (T_{1,1})^1_p D_0^{p-1} D_1^{p-2} + \dots + K'_{2,0} D_1^{m-2} + \dots + K'_3 + \dots + K'_m,$$

$$\frac{p-2}{p-1} = \frac{m-2}{m} = \frac{2p-4}{2p-2}\,.$$

We choose

$$u_{k_0} = \exp\Big(i\, x_1 \xi_1 + \psi(x) \xi_1^{\frac{p-2}{p-1}})\Big) \sum_{0 \le k \le k_0} Y_k(x) \xi_1^{-\frac{k}{p-1}}\,.$$

We identify $k u_{k_0}$ and obtain

$$(D_0 \psi)^m \xi_1^{2(p-2)} + (-1)^{p-1} (T_{1,1})^1_p (D_0 \psi)^{p-1} \xi_1^{2p-4} + K_{2,0} \xi_1^{2(p-2)} = 0,$$
$$(D_0 \psi)^{2(p-2)} + (-1)^{p-1} (T_{1,1})^1_p (D_0 \psi)^{p-1} + (-1)^{p+1} (T_{2,0})^1_p = 0\,.$$

We choose $D_0 \psi$ as a complex root different from 0, $\operatorname{Re} D_0 \psi > 0$ and obtain successively that $(T_{2,0})^1_p$ and $(T_{1,1})^1_p = (-1)^p (b_0)^p_2 = 0$ are necessary in order that the Cauchy problem is well-posed in $\mathcal{C}^\infty$.

We obtain also that if $(T_{2,0})^1_p \neq 0$ or $(T_{1,1})^1_p \neq 0$, then the Cauchy problem is well-posed in $\gamma^{\frac{p-1}{p-2}}$.

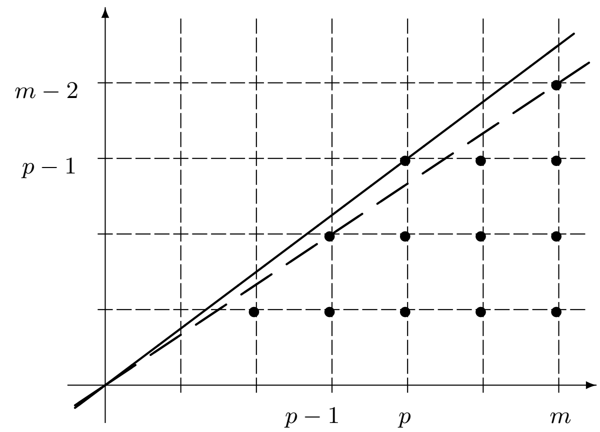

FIGURE 1. Lines with $m = 6$, $p = 4$ and $q = 2$

Recall that $z \geq 2$,

$$K_1 = \sum_{q \leq u \leq m-1} K_{1,u} \xi_0^u \xi_1^{m-u-1} ,$$

$$K_j = \sum_{0 \leq u \leq m-j} K_{j,u} \xi_0^u \xi_1^{m-j-u} \quad \text{for} \quad j > 1 .$$

Let $g_{j,u} = \frac{m-j-u}{m-u} = 1 - \frac{j}{m-u}$. These indices permit to characterize Gevrey's classes of indices $\frac{1}{g_{ju}}$. Let $i_{j,u} = \frac{m-u}{j}$.

Example 5.1. Let

$$g_{1,q} = \frac{p-1}{p}, \quad g_{1,p-1} = \frac{q}{q+1} ,$$

$$i_{1,q} = p, \quad i_{1,p-1} = q+1 ,$$

$$g_{2,0} = \frac{m-2}{m}, \quad g_{2,m-3} = \frac{1}{3} ,$$

$$i_{2,0} = \frac{m}{2}, \quad i_{2,m-3} = \frac{3}{2}.$$

We order the indices by decreasing order that corresponds to the decreasing slopes of the lines joining the origin to the point $(m-u, m-u-j)$ in the Newton's diagram.

So we consider the slopes $i_{1,u}$ such that $q+1 \leq i_{1,u} \leq p$, and the slopes $i_{2,u}$ such that $3/2 \leq i_{2,u} \leq m/2$. We remark that $q+1 \geq 3/2$. We proceed by induction on the slopes. We consider the lines (j,u) joining 0 and the point $(m-u, m-u-j)$ of the diagram.

We assume that for all the points of the diagram above this line we have corresponding $K_{j',u'} = 0$, then we obtain that for all the point of the diagram on the line (j,u), we have the corresponding $K_{k,k'} = 0$, and specially $K_{j,u} = 0$. See Figure 1.

5.1. **Index i and the corresponding K.** Let $q \leq u \leq m-1$. If $q = 1$, then

$$i_{1,q} = p \geq \dots \geq i_{1,p-1} = q+1 = i_{1,m-2} = 2 > i_{1,p} = i_{1,m-1} = 1,$$

$$\widehat{K}_{1,q} = (-1)^{p-1}(b_0)_1^p\,, \dots \widehat{K}_{1,p-1} = (b_0)_z^p\,, \dots \widehat{K}_{1,p} = K_{1,m-1} = I\,.$$

If $q \geq 2$, then

$$\begin{gathered} i_{1,q} = p \geq \dots \geq i_{1,p-1} = q+1 \geq i_{1,p} = q \geq \dots \geq i_{1,m-2} = 2 > i_{1,m-1} = 1, \\ \widehat{K}_{1,q} = (-1)^{p+1}(b_0)_1^p = (-1)^{p+1}(T_{1,0})_p^1 \\ \dots \\ \widehat{K}_{1,p-1} = (-1)^q (b_0)_z^p = (-1)^{p+1}(T_{1,z-1})_p^1 \\ \widehat{K}_{1,p} = (-1)^{q+1}\Big((b_0)_{z+1}^p + (b_0)_{p+1}^m\Big) = (-1)^z \frac{(T_{2,z})_m^1}{(b_0)_{p+1}^p}\,, \\ \widehat{K}_{1,m-2} = -\Big((b_0)_{p+1}^p + (b_0)_{m-1}^m\Big) I = (-1)^z \frac{(T_{2,p-2})_m^1}{(b_0)_{p+1}^p}\,, \\ \widehat{K}_{1,m-1} = I\,. \end{gathered}$$

If $q = 1$, then

$$\begin{gathered} i_{2,0} = \frac{m}{2} \geq \dots \quad i_{2,m-3} = i_{2,p-2} = i_{2,z-1} = \frac{3}{2} \geq i_{2,m-2} = i_{2,p-1} = 1, \\ K_{2,0} = (-1)^p (T_{2,0})_p^1 I, \quad K_{2,p-2} = (-1)^p (T_{2,p-2})_p^1 I, \quad K_{2,m-2} = I\,. \end{gathered}$$

If $q \geq 2$, then

$$\begin{gathered} i_{2,0} = \frac{m}{2} \geq \dots \geq i_{2,z-2} = q+1 \geq i_{2,z-1} = \frac{2q+1}{2} \\ \geq \dots \geq i_{2,p-2} = \frac{q+2}{2} \geq i_{2,p-1} = \frac{q+1}{2} \geq \dots \\ \geq i_{2,m-4} = 2 \geq i_{2,m-3} = \frac{3}{2} \geq i_{2,m-2} = 1, \\ K_{2,0} = (-1)^p (T_{2,0})_p^1 I, \quad K_{2,z-2} = (-1)^p (T_{2,z-2})_p^1 I, \\ K_{2,z-1} = (-1)^p (T_{2,z-1})_p^1 I, \dots, K_{2,p-2} = (-1)^p (T_{2,p-2})_p^1 I\,. \end{gathered}$$

If $(b_0)_{p+1}^p \neq 0$, then

$$K_{2,p-1} = (-1)^{z+1} \frac{(T_{3,z})_m^1}{(b_0)_{p+1}^p} I, \dots, K_{2,m-3} = (-1)^{z+1} \frac{(T_{2,p-2})_m^1}{(b_0)_{p+1}^p} I,\ K_{2,m-2} = I\,.$$

If $(b_0)_{p+1}^p = 0$, then we obtain analogous result for $(b_0)_{p+1}^p = \dots = (b_0)_{p+k}^p = 0$, $(b_0)_{p+k+1}^p \neq 0$.

We will adapt the calculus if $q = 1$. For $q \geq z-3$, we consider the following two cases:

Case (1) $m < 6$:

$$\begin{gathered} 2 > i_{3,0} = \frac{m}{3} \geq \dots \geq i_{3,p-3} = \frac{q+3}{3} \geq \dots \geq i_{3,m-5} = \frac{5}{3} \geq i_{3,m-4} = \frac{4}{3}\,, \\ K_{3,0} = (-1)^{p-1}(T_{3,1})_p^1 I, \quad K_{3,p-3} = (-1)^{p-1}(T_{3,p-2})_p^1 I\,. \end{gathered}$$

Case (2) $m \geq 6$:

$$\begin{gathered} q+1 \geq i_{3,0} = \frac{m}{3} \geq \dots \geq i_{3,p-3} = \frac{q+3}{3} \geq \dots \geq i_{3,m-6} = 2 \\ \geq i_{3,m-5} = \frac{5}{3} \geq i_{3,m-4} = \frac{4}{3}\,, \end{gathered}$$

$$K_{3,0} = (-1)^{p+1}(T_{3,1})^1_p I, \dots, K_{3,p-3} = (-1)^{p+1}(T_{3,p-2})^1_p I$$

If $q < z-3$, then

Case (1) $m < 6$:

$$\begin{gathered} i_{3,0} = \frac{m}{3} \geq \dots \geq i_{3,z-q-3} = q+1 \geq i_{3,p-3} = \frac{q+3}{3} \\ \geq \dots \geq i_{3,m-5} = \frac{5}{3} \geq i_{3,m-4} = \frac{4}{3}\,, \\ K_{3,0} = (-1)^{p-1}(T_{3,1})^1_p I, \quad K_{3,z-q-3} = (-1)^{p-1}(T_{3,z-q-2})^1_p I, \\ \dots, K_{3,p-3} = (-1)^{p-1}(T_{3,p-2})^1_p I, \dots \end{gathered}$$

Case (2) $m \geq 6$:

$$\begin{gathered} i_{3,0} = \frac{m}{3} \geq \dots \geq i_{3,z-q-3} = q+1 \geq \dots \\ \geq i_{3,m-6} = 2 \geq i_{3,m-5} = \frac{5}{3} \geq i_{3,m-4} = \frac{4}{3}\,. \end{gathered}$$

We will study $K_{4,u}$ and $K_{w,u}$ in the same manner.

Theorem 5.2. *Conditions* (L_1), (L_2) *and the conditions*

$$(T_{3,0})^1_m = \dots = (T_{3,p-2})^1_m = 0$$

are necessary for the Cauchy problem to be well-posed in $\mathcal{C}^\infty$.

The proof uses the following Lemmas.

Lemma 5.3. *If* $z \leq q+2$, *then condition* (L_1) *and the conditions*

$$(T_{2,0})^1_p = \dots = (T_{2,z-2})^1_p = 0$$

are necessary for the Cauchy problem to be well-posed in $\mathcal{C}^\infty$.

Proof. $i_{1,q} = p$ and $K_{1,q} = (b_0)^p_1 I = 0$ is necessary (cf. Proposition 3.8). Then we consider $1 \leq k \leq z-1$, $K_{1,q+k}$, $2-z \leq k' \leq z-1$, $K_{2,k'}$. If $k' = 2k - z$, then

$$i_{2,k'} = \frac{m-k'}{2} = \frac{m+z-2k}{2} = i_{1,q+k} = p-k\,. \qquad \square$$

Example 5.4. Let $i_{1,q+1} = i_{2,2-z} = p-1$ and $i_{1,p-1} = q+1 = i_{2,z-2}$.

$i_{3,0} = \frac{m}{3} < q+1$ because $z < q+3$.

If $z = 2$, then $i_{2,0} = i_{1,q+1} = p+1$ and $(T_{2,0})^1_p = (b_0)^p_2 = 0$ are necessary.

If $z = 1$, then $i_{2,0} = \frac{m}{2} = \frac{2p-1}{2} > p-1 = i_{1,q+1}$ and $(T_{2,0})^1_p = 0$ is necessary. In general, as before, by the construction of an asymptotic expansion and the method of the closed graph inequality and induction.

$$K_{1,q+1} = K_{2,0} = 0, \dots, K_{1,p-1} = K_{2,z-2} = 0\,.$$

In other words (L_1) and $(T_{2,0})^1_p = \dots = (T_{2,z-2})^1_p = 0$ are necessary.

Lemma 5.5. *If* $z \leq q+2$, *then the conditions*

$$\begin{gathered} K_{1,p} = \dots = K_{1,m-2} = 0, \\ K_{2,z-1} = \dots = K_{2,p-2} = \dots = K_{2,m-3} = 0, m \geq 6, \quad K_{3,0} = \dots = K_{3,m-6} \\ m \geq 2q-2, \quad K_{q-1,0} = \dots = K_{q-1,z+2}, \end{gathered}$$

are necessary for the Cauchy problem to be well-posed in $\mathcal{C}^\infty$.

In other words we have the necessity of (L_1), (L_2) and $(T_{3,z})^1_m = \cdots = (T_{3,p-2})^1_m = 0$. If $m \geq 6$, then we have complementary conditions that we will stated in part 2.

Proof. Let $q \geq 2$. (The case $q = 1$ was studied in the Lemma 5.3.) First, we remark that $i_{2,z-1} = \frac{2q+1}{2} > q$. We have

$$K = D_0^m + K_{2,z-1} D_0^{z-1} D_1^{2q-1} + \dots$$

we consider the corresponding asymptotic expansion of the index $i_{2,z-1}$ and we obtain the necessity of

$$K_{2,z-1} = 0\,.$$

We look at

$$\begin{gathered}
i_{2,z} = q \geq \cdots \geq i_{2,z+k'} = \frac{2q-k'}{2} \geq \cdots \geq i_{2,m-4} = 2, \quad 0 \leq k' \leq 2q-4, \\
i_{1,p} = q \geq \cdots \geq i_{1,p+k} = q-k \geq \cdots \geq i_{1,m-2} = 2, \quad 0 \leq k \leq q-2\,.
\end{gathered}$$

□

We let $k' = 2k$, so $2q - k' = 2q - 2k$, and $i_{1,p+k} = i_{2,z+k'}$. Moreover if $m < 6$, then $i_{3,0} = \frac{m}{3} < 2$; if $m \geq 6$, then $0 \leq k'' \leq m-6$ and $i_{3,0} \geq i_{3,k''} \geq i_{3,m-6} = 2$.
If $m \geq 2w$, then $i_{w,0} \geq \cdots \geq i_{w,m-2w} = 2$. If $m \geq 2q-2$, then

$$i_{q-1,0} \geq \cdots \geq i_{q-1,m-q+2} = i_{q-1,z+2} = 2\,.$$

Using asymptotic expansions and induction as before, we obtain the necessity of

$$K_{1,p} = K_{2,z} = 0, \quad K_{1,m-2} = K_{2,m-4} = 0\,.$$

and directly of $K_{2,m-3} = 0$ ($i_{2,m-3} = \frac{3}{2}$).
In other words, from (3.43), (4.20), (4.50), (4.56), we have the necessity of (L_1), (L_2), $(T_{3,z})^1_m = \cdots = (T_{3,p-2})^1_m$, and of the terms

$$\begin{gathered}
m \geq 6, \quad K_{3,0} = \cdots = K_{3,m-6} = 0 = K_{3,m-5} = 0 = K_{3,m-4}, \\
m \geq 2q-2, \quad K_{q-1,0} = \cdots = K_{q-1,z+2} = 0 = K_{q-1,p} = (-1)^z \frac{(T_{q,p-2})^1_u}{(b_0)^p_{p+1}}
\end{gathered}$$

(if $(b_0)^p_{p+1} \neq 0$).

Example 5.6. Let $k = 0$, $k' =$, $i_{1,p} = i_{2,z}$. Then the conditions $(T_{2,z})^1_p = 0$ and $K_{1,p} = 0$ are necessary in order that the Cauchy problem be well-posed in $\mathcal{C}^\infty$.

Proof. Note that

$$k = D_0^m + K_{1,p} D_0^p D_1^{q-1} + \cdots + K_{1,m-1} D_0^{m-1} + K_{2,z} D_0^z D_1^{m-z-2} + \cdots + K_{3,1} D_0 D_1^{m-4}\,.$$

Then

$$i_{1,p} = q, \quad i_{3,1} = \frac{m-1}{3} = \frac{2p-z-1}{3} < q\,.$$

On the line with slope $(q-1)/q$ in Newton's diagram. we have 2 points corresponding to $K_{1,p}$ and $K_{2,z}$.
We apply k to the expansion

$$u_k = e^{ix_1\xi_1 + \psi(x,\xi_1)\xi_1^{\frac{q-1}{q}}} \big[Y_0 + Y_1 \xi_1^{1/q} + \dots\big]$$

to obtain the first term of $k[u]$,

$$e^{ix_1\xi_1 + \psi(x,\xi_1)\xi_1^{\frac{q-1}{q}}} \xi_1^{\frac{m(q-1)}{q}} \big[(D_0\psi)^m + i^{q-1} \widehat{K}_{1,p} (D_0\psi)^p + \cdots + \widehat{K}_{2,z} (D_0\psi)^z\big]\,.$$

Then we cancel the coefficient of Y_0 $(m = p + q = 2p - 1)$,

$$(D_0\psi)^{2p-2} + i^{q-1}\widehat{K}_{1,p}(D_0\psi)^{p-1} + \cdots + \widehat{K}_{2,z} = 0$$

by the choice of $D_0\psi$, $\operatorname{Re} D_0\psi > 0$, (if $\widehat{K}_{1,p} \neq 0$ or $\widehat{K}_{2,z} \neq 0$); after we obtain the other Y_j and we can choose $Y_0 \neq 0$ satisfying an ordinary differential equation

$$D_0 Y_0 + (\dots)Y_0 = 0\,.$$

As in Proposition 3.8, considering the closed graph inequality we obtain the necessity of $\widehat{K}_{1,p}$ and $(T_{2,z})_p^1 = 0$. □

We know that

$$\widehat{K}_{1,p} = (-1)^{q-1}\big((b_0)_{z+1}^p + (b_0)_{p+1}^m\big)\,.$$

So by (3.43), we have also obtained the necessity of $(T_{2,z})_m^1 = 0$ and $(T_{2,z})_p^{p+1} = 0$. Since

$$(T_{2,z})_m^{p+1} = (-1)^z (T_{2,z})_p^1,$$

we have obtained the necessity of $T_{2,z} = 0$.

If $\widehat{K}_{2,z} \neq 0$ or $\widehat{K}_{1,p} \neq 0$, then we interpret the asymptotic expansion, by the mean of a Fourier integral operator and we obtain that the Cauchy problem is well-posed in $\gamma^{\frac{q}{q-1}}$.

Lemma 5.7. *If $z > q + 2$, then conditions* (L_1) *and* $(T_{2,0})_p^1 = \cdots = (T_{2,z-2})_p^1 = 0$ *are necessary for the Cauchy problem to be well-posed in $\mathcal{C}^\infty$. Moreover*

$$K_{3,0} = \cdots = K_{3,z-q-3=0};$$

and if $z > q + 3$, then

$$K_{3,0} = \cdots = K_{3,z-q-4} = 0\,;$$

and if $z > z - q + 1$, then

$$K_{3z-q,0} = 0\,.$$

In other words, we have also the necessity of $(T_{3,0})_p^1 = \dots (T_{3,z-q+2})_p^1 = 0$.

Proof. The scheme of the calculus is the same as for Lemma 5.3. Then we consider $i_{3,k} = \frac{m-k}{3}$ and

$$\frac{m-k}{3} \geq q + 1 \iff k \leq z - q - 3$$

As above, we obtain the necessity of

$$\begin{gathered} K_{3,0} = \cdots = K_{3,z-q-3} = 0, \\ K_{4,0} = \cdots = K_{4,z-q-4} = 0, \\ K_{z-q,0} = 0\,. \end{gathered}$$

□

Lemma 5.8. *If $z > q + 2$, then we obtain the necessity of*

$$\begin{gathered} K_{1,p} = \cdots = K_{1,m-2} = 0, \\ K_{2,z-1} = \cdots = K_{2,p-2} = \cdots = K_{2,m-3} = 0, \\ (m \geq 6) \quad K_{3,z-q-2} = \cdots = K_{3,m-6} = K_{3,m-5} = K_{3,m-4}\,. \end{gathered}$$

In other words we have the necessity of (L_1), (L_2) *and* $(T_{3,z})_m^1 = \cdots = (T_{2,p-2})_m^1 = 0$.

In other words, we have also the necessity of

$$(T_{3,0})^1_p = \dots (T_{3,z-q+2})^1_p = 0\,.$$

The proof is similar to that of Lemma 5.5. We remark that $i_{3,z-q+2} \geq 2$ and then proceed as before.

Concluding remarks. (1) In a forthcoming article we can simplify section 4. We follow the order decreasing of the indexes $i_{k,j}$ and by induction we obtain the necessity of the $K_{k,j}$ and of the conditions (L). We do the calculations easily by Newton's diagram.

(2) In this and the forthcoming article, the construction of the asymptotic expansion slightly modified gives in the Gevrey class theorem of existence and uniqueness. Some examples of this process can be found in [27].

(3) In the forthcoming article we study the case $p = q$.

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