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INFINITELY MANY SOLUTIONS FOR A PERTURBED SCHRÖDINGER EQUATION

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Abstract. We find multiple solutions for a nonlinear perturbed Schrödinger equation by means of the so–called Bolle's method.

1. **Introduction.**

This note concerns with the elliptic equation

$$-\,\Delta\, u + V(x)u \;= g(x,u) + f(x) \quad \text{ in } \mathbb{R}^N, \tag{1}$$

where $N \geq 2$, V is a potential function on $\mathbb{R}^N$, $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a superlinear, but subcritical, nonlinearity (namely, it satisfies the Ambrosetti–Rabinowitz condition) and $f : \mathbb{R}^N \to \mathbb{R}$ is a given function.

When $f = 0$ the study of equation (1) begins with Rabinowitz's paper [15] and then it has been carried out by several authors (cf. [6] and references therein): even if it has a variational structure, the main problem with classical variational tools is the lack of compactness. Thus, in [15] the existence of a nontrivial solution is shown by using the Mountain Pass Theorem but assuming that $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is positive and coercive; later on in [6], by means of the Symmetric Mountain Pass Theorem (see [1, Theorem 2.8]), Bartsch and Wang find infinitely many solutions if g is odd in u and V is a positive continuous function such that

$$\text{meas}\left(\left\{x \in \mathbb{R}^N : V(x) \leq b\right\}\right) < +\infty \qquad \text{for all } b > 0.$$

Motivated by the fact that on bounded domains, starting with the pioneer papers [2, 3, 16, 19], it is shown that multiplicity results may persist when the symmetry is destroyed by a perturbation term (see also [10, 11, 22]), we study (1) for $f \neq 0$. Our approach is based on the so–called Bolle's method (cf. [9, 10]) and on some ideas in [22]. We are only aware of a few previous contributions in this direction: indeed, in [17, Theorem 1.1] (see also [18]) it is proved a multiplicity result for a problem related to ours, provided that the eigenvalues of the involved Schrödinger operator have a suitable growth; on the other hand, in [4] a sharp result is obtained under radial assumptions.

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Hereafter, in order to have a variational formulation of the problem and to overcome the lack of compactness, we assume the following conditions:

(H_1) the potential $V \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ is such that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0 \tag{2}$$

and

$$\lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{V(y)}\,\mathrm{d}y = 0,$$

where $B_1(x) = \{y \in \mathbb{R}^N : |x-y| < 1\}$;

(H_2) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., $g(\cdot,t)$ is measurable in $\mathbb{R}^N$ for all $t \in \mathbb{R}$ and $g(x,\cdot)$ is continuous in $\mathbb{R}$ for a.e. $x \in \mathbb{R}^N$) such that there exist $a_1, a_2 > 0$, $\mu > 2$ and $\delta > 0$ small enough (cf. Remark 3.2) satisfying

- (g_1) $|g(x,s)| \le a_1|s|^{p-1} + \delta|s|$ for a.e. $x \in \mathbb{R}^N, s \in \mathbb{R}$, with $p \in]2, 2^*[$;
- (g_2) $g(x,s)s \ge \mu G(x,s) > 0$ for a.e. $x \in \mathbb{R}^N, s \in \mathbb{R} \setminus \{0\}$;
- (g_3) $G(x,s) \ge a_2|s|^\mu$ for a.e. $x \in \mathbb{R}^N, s \in \mathbb{R}$;
- (g_4) $g(x,\cdot)$ is odd for a.e. $x \in \mathbb{R}^N$,

with $G(x,s) := \int_0^s g(x,t)\,\mathrm{d}t$.

Remark 1.1. Assumption (g_3) is somehow related to (g_2): indeed, by (g_2) and direct computations it follows that for any $\varepsilon > 0$ there exists a constant $a_\varepsilon > 0$ such that

$$G(x,s) \ge a_\varepsilon |s|^\mu \quad \text{if } |s| \ge \varepsilon, \text{ for a.e. } x \in \mathbb{R}^N.$$

In what follows by a solution we mean a weak solution; classical solutions are found when all the involved functions are smooth enough (e.g, cf. [6]).

Our main result is the following.

Theorem 1.2. *Assume that $(H_1) - (H_2)$ hold. Then, for all $f \in L^{\frac{\mu}{\mu-1}}(\mathbb{R}^N)$ problem (1) has infinitely many solutions, provided that*

$$\frac{\mu}{\mu-1} < \frac{4}{N(p-2)}. \tag{3}$$

Clearly, by (g_1) and (g_3) it follows that $\mu \le p$. If $\mu = p$, in particular when $g(x,s)$ is exactly a pure power, condition (3) can be rewritten as follows.

Corollary 1.3. *Assume that (H_1) holds and $g(x,u) = |u|^{p-2}u$, with $p \in]2, 2^*[$. Then, for all $f \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$ problem (1) has infinitely many solutions, provided that $p \in]2, p_N[$, where $p_N := \dfrac{N+2+\sqrt{N^2+4}}{N}$.*

Condition (H_1) on function V is weaker than those used in [6, 15], as shown in [17, Proposition 3.1]. On the other hand, our set of conditions (H_2) is similar to the analogous in [17], even if a comparison between [17, Corollary 1.6] and our Theorem 1.2 can be carried out only when the spectrum of the Schrödinger operator is known (cf. Proposition 3.1). For example, if $N = 3$ and $V(x) = |x|^2$, the corresponding operator in $L^2(\mathbb{R}^3)$ admits the sequence of eigenvalues $(\lambda_k)_k$, with $\lambda_k = 2k+3$ (cf. [13, p. 514]). Taking the model nonlinearity, Corollary 1.3 gives infinitely many solutions for (1) if p varies in the range $]2, \frac{5+\sqrt{13}}{3}[$, while the range obtained in [17, Corollary 1.6] is smaller, being $]2, \frac{19+\sqrt{213}}{14}[$. As usual, for results concerning with problems with broken symmetry, Theorem 1.2 is far from being optimal, since we do not cover the entire subcritical range $]2, 2^*[$. In spite of this, when dealing with radial assumptions and $N \ge 3$, one finds almost optimal results (cf. [4, 5] for unbounded domains and [11, 20, 21] for bounded ones).

The paper is organized as follows: in Section 2 we recall Bolle's method, then in Section 3 we introduce the variational setting of our problem and prove some technical results; finally, in Section 4 we prove Theorem 1.2.

Notations. Throughout this paper we denote by

- $2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = +\infty$ otherwise;
- s' the conjugate exponent of $s \geq 1$, namely $s' = \frac{s}{s-1}$ if $s > 1$ and $s' = +\infty$ if $s = 1$;
- $|\cdot|_s$ the standard norm in the Lebesgue space $L^s(\mathbb{R}^N)$, $1 \leq s \leq +\infty$;
- $m^*(\bar{x}, \Psi)$ the large Morse index of a C^2 functional Ψ at a critical point $\bar{x}$;
- d_j, C_j positive real numbers, for any $j \in \mathbb{N}$.

2. **Bolle's perturbation method.** In this section we introduce the Bolle's perturbation method firstly stated in [9] but in the version presented in [10] and improved in [12], as the involved functionals are C^1 instead of C^2. The key point of this approach is dealing with a continuous path of functionals $(I_\theta)_{\theta\in[0,1]}$ which starts at a symmetric functional I_0 and ends at the "true" non–even functional I_1 associated to the given perturbed problem, so that the critical points of mini–max type of the symmetric map I_0 "shift" into critical points of I_1.

Throughout this section, let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be a Hilbert space with dual $(\mathcal{H}', \|\cdot\|_{\mathcal{H}'})$ and $I : (\theta, v) \in [0,1] \times \mathcal{H} \mapsto I(\theta, v) \in \mathbb{R}$ a C^1 functional. For simplicity, let us set $I_\theta = I(\theta, \cdot) : \mathcal{H} \to \mathbb{R}$ and $I'_\theta(\cdot) = \frac{\partial I}{\partial v}(\theta, \cdot) : \mathcal{H} \to \mathcal{H}'$, for each $\theta \in [0,1]$. Assume that $\mathcal{H}$ can be decomposed so that $\mathcal{H} = H_- \oplus H_+$, with $\dim(H_-) < +\infty$, and $(e_k)_{k\geq 1}$ is an orthonormal basis of H_+. Setting

$$H_0 = H_-, \quad H_{k+1} = H_k \oplus \mathbb{R}e_{k+1} \text{ if } k \in \mathbb{N},$$

we have that $(H_k)_k$ is an increasing sequence of finite dimensional subspaces of $\mathcal{H}$.

Furthermore, we define

$$\Gamma = \{\gamma \in C(\mathcal{H}, \mathcal{H}) : \ \gamma \text{ is odd and } \exists \rho > 0 \text{ s.t. } \gamma(v) = v \text{ if } \|v\|_{\mathcal{H}} \geq \rho\} \tag{4}$$

and

$$c_k = \inf_{\gamma\in\Gamma} \sup_{v\in H_k} I_0(\gamma(v)).$$

Let us assume that:

(A_1) I satisfies the following variant of the Palais–Smale condition: each sequence $((\theta_n, v_n))_n \subset [0,1] \times \mathcal{H}$ such that

$$(I(\theta_n, v_n))_n \text{ is bounded and } \lim_{n\to+\infty} \|I'_{\theta_n}(v_n)\|_{\mathcal{H}'} = 0 \tag{5}$$

converges, up to subsequences;

(A_2) for all $b > 0$ there exists $C_b > 0$ such that, if $(\theta, v) \in [0,1] \times \mathcal{H}$, then

$$|I_\theta(v)| \leq b \implies \left|\frac{\partial I}{\partial \theta}(\theta, v)\right| \leq C_b \ (\|I'_\theta(v)\|_{\mathcal{H}'} + 1)(\|v\|_{\mathcal{H}} + 1);$$

(A_3) there exist two continuous maps $\eta_1, \eta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$, with $\eta_1(\theta, \cdot) \leq \eta_2(\theta, \cdot)$ for all $\theta \in [0,1]$, which are Lipschitz continuous with respect to the second variable and such that, if $(\theta, v) \in [0,1] \times \mathcal{H}$, then

$$I'_\theta(v) = 0 \implies \eta_1(\theta, I_\theta(v)) \leq \frac{\partial I}{\partial \theta}(\theta, v) \leq \eta_2(\theta, I_\theta(v)); \tag{6}$$

(A_4) I_0 is even and for each finite dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ it results

$$\lim_{\substack{v\in\mathcal{V} \\ \|v\|_{\mathcal{H}}\to+\infty}} \sup_{\theta\in[0,1]} I_\theta(v) = -\infty\ .$$

Now, for $i \in \{1, 2\}$, let $\psi_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ be the (unique, global) solution of the problem

$$\begin{cases} \dfrac{\partial \psi_i}{\partial \theta}(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)) \\ \psi_i(0, s) = s, \end{cases}$$

with η_i as in (A_3).

Note that $\psi_i(\theta,\cdot)$ is continuous, non–decreasing on $\mathbb{R}$, $i \in \{1,2\}$, and $\psi_1(\theta,\cdot) \le \psi_2(\theta,\cdot)$. Moreover, we set

$$\bar\eta_1(s) = \max_{\theta\in[0,1]} |\eta_1(\theta,s)|, \qquad \bar\eta_2(s) = \max_{\theta\in[0,1]} |\eta_2(\theta,s)| \ .$$

The following result holds (cf. [9, Theorem 3], [10, Theorem 2.2] and [12, Section 2]).

Theorem 2.1. *Let $I : [0,1]\times\mathcal{H} \to \mathbb{R}$ be a C^1 path of functionals satisfying assumptions $(A_1)-(A_4)$. Then, there exists $C>0$ such that for all $k\in\mathbb{N}$ it results:*

(a) either I_1 has a critical level $\widetilde{c}_k$ with $\psi_2(1,c_k) < \psi_1(1,c_{k+1}) \le \widetilde{c}_k$,
(b) or $c_{k+1} - c_k \le C\,(\bar\eta_1(c_{k+1}) + \bar\eta_2(c_k) + 1)$.

Remark 2.2. We point out that, if $\eta_2 \ge 0$ in $[0,1]\times\mathbb{R}$, the function $\psi_2(\cdot,s)$ is non–decreasing on $[0,1]$. Hence, $c_k \le \widetilde{c}_k$ for all c_k verifying case (a).

3. Variational set–up.

In this section we present the functional framework of our problem.

Firstly, by (2) it makes sense to consider the weighted Sobolev space

$$E_V := H^1_V(\mathbb{R}^N) = \left\{u\in\ H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x < +\infty\right\}$$

endowed with the norm

$$\|u\|_V = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x\right)^{\frac12}.$$

The following proposition (cf. [7, Theorems 3.1 and 4.1] and [17, Proposition 3.3]) is crucial to overcome the lack of compactness.

Proposition 3.1. *Let $V:\mathbb{R}^N\to\mathbb{R}$ be such that (H_1) holds. Then, for all $s\in[2,2^*[$ it is $E_V \hookrightarrow\!\!\!\hookrightarrow L^s(\mathbb{R}^N)$, i.e. the embedding of $(E_V,\|\cdot\|_V)$ in $(L^s(\mathbb{R}^N),|\cdot|_s)$ is compact. Moreover, the linear Schrödinger operator*

$$u\in C_0^\infty(\mathbb{R}^N)\mapsto -\Delta u + V(x)u \in L^2(\mathbb{R}^N)$$

is essentially self–adjoint, the spectrum of its self–adjoint extension is an increasing sequence $(\lambda_n)_n$ of eigenvalues of finite multiplicity and

$$L^2\left(\mathbb{R}^N\right) = \sum_{n=1}^{+\infty} M_n \qquad \text{with } M_n\perp M_m \text{ for } n\ne m,$$

where M_n denotes the eigenspace corresponding to λ_n for every $n\in\mathbb{N}$.

Remark 3.2. From (g_1) and (g_3) it follows $2<\mu\le p<2^*$; hence, Proposition 3.1 implies $E_V \hookrightarrow\!\!\!\hookrightarrow L^\mu(\mathbb{R}^N)$ and $E_V \hookrightarrow\!\!\!\hookrightarrow L^p(\mathbb{R}^N)$. Moreover, as $E_V \hookrightarrow\!\!\!\hookrightarrow L^2(\mathbb{R}^N)$, for further use by α we denote the best embedding constant and in assumption (g_1) we choose δ such that $\delta<\frac{1}{\alpha^2}$.

As direct consequence of Proposition 3.1 and [23, Theorem 1.22] we can state the following lemma.

Lemma 3.3. *Assume that (H_1) and (g_1) hold. Then, setting $\Phi : E_V\to\mathbb{R}$ as*

$$\Phi(u) = \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x \quad \text{for all } u\in E_V,$$

it results that $\Phi\in C^1(E_V,\mathbb{R})$ with

$$\Phi'(u)[\varphi] \ = \int_{\mathbb{R}^N} g(x,u)\varphi\,\mathrm{d}x \quad \text{for all } \varphi\in E_V.$$

Moreover, $\Phi' : E_V\to(E_V)'$ is compact.

By Lemma 3.3 and standard arguments, the weak solutions of (1) are the critical points of the C^1 functional on E_V

$$I_1(u) \;=\; \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)u^2\right)\,\mathrm{d}x \;-\; \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x - \int_{\mathbb{R}^N} fu\,\mathrm{d}x,$$

with

$$I_1'(u)[\varphi] \;=\; \int_{\mathbb{R}^N}\nabla u\cdot\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)u\,\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x,u)\,\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} f\,\varphi\,\mathrm{d}x$$

for all $u,\varphi\in E_V$.

In order to apply the Bolle's perturbation method, we define the path of functionals $I:[0,1]\times E_V\to\mathbb{R}$ as follows:

$$I(\theta,u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 \;+\; V(x)u^2\right)\,\mathrm{d}x \;-\; \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x - \theta\int_{\mathbb{R}^N} fv\,\mathrm{d}x. \tag{7}$$

Now we verify that, under our main assumptions, the path introduced in (7) satisfies conditions $(A_1)-(A_4)$ in Section 2.

Proposition 3.4. *Assume that $(H_1)-(H_2)$ hold. Then, the family $(I_\theta)_{\theta\in[0,1]}$ verifies $(A_1)-(A_4)$.*

Proof. The proof is organized in four steps.
Step 1. Let $((\theta_n,u_n))_n\subset[0,1]\times E_V$ be a sequence such that (5) holds; hence,

$$I_{\theta_n}(u_n) \;=\; \frac{1}{2}\,\|u_n\|_V^2 \;-\; \int_{\mathbb{R}^N} G(x,u_n)\,\mathrm{d}x \;-\; \theta_n\int_{\mathbb{R}^N} fu_n\,\mathrm{d}x \le d_1$$

and

$$\begin{aligned}\left|I_{\theta_n}'(u_n)[u_n]\right| \;&=\; \left|\|u_n\|_V^2 - \int_{\mathbb{R}^N} g(x,u_n)\,u_n\,\mathrm{d}x \;-\; \theta_n\int_{\mathbb{R}^N} f\,u_n\,\mathrm{d}x\right|\\ &\le\; \varepsilon_n\|u_n\|_V,\end{aligned}$$

where $\varepsilon_n\searrow 0$ as $n\to+\infty$. Therefore, by (g_2), Remark 3.2 and the Hölder inequality, it follows that

$$\begin{aligned}d_1+\frac{\varepsilon_n}{\mu}\|u_n\|_V \;&\ge\; I_{\theta_n}(u_n)-\frac{1}{\mu}I_{\theta_n}'(u_n)[u_n]\\ &\ge\; \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_V^2-\left(1-\frac{1}{\mu}\right)\theta_n\int_{\mathbb{R}^N} fu_n\,\mathrm{d}x\\ &\ge\; \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_V^2-\left(1-\frac{1}{\mu}\right)\theta_n|f|_{\mu'}|u_n|_\mu\\ &\ge\; \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_V^2-d_2\|u_n\|_V,\end{aligned}$$

thus the sequence $(u_n)_n$ is bounded in E_V and (A_1) follows by Proposition 3.1 and standard arguments.
Step 2. Since

$$\frac{\partial I}{\partial\theta}(\theta,u) = -\int_{\mathbb{R}^N} f\,u\,\mathrm{d}x,$$

by using again the Hölder inequality and Remark 3.2, we get that

$$\left|\frac{\partial I}{\partial\theta}(\theta,u)\right| \;\le\; d_3\|u\|_V,$$

hence (A_2) holds.
Step 3. Taking $(\theta, u) \in [0,1] \times E_V$ such that $I'_\theta(u) = 0$, we have that

$$\begin{aligned} I_\theta(u) &= I_\theta(u) - \frac{1}{2} I'_\theta(u)[u] \\ &= \frac{1}{2}\int_{\mathbb{R}^N} g(x,u)\, u \; \mathrm{d}x - \int_{\mathbb{R}^N} G(x,u)\; \mathrm{d}x - \frac{\theta}{2}\int_{\mathbb{R}^N} fu \; \mathrm{d}x, \end{aligned}$$

then by the Hölder inequality, using (g_2) and (g_3) respectively, we get that

$$\begin{aligned} I_\theta(u) &\geq \left(\frac{\mu}{2} - 1\right)\int_{\mathbb{R}^N} G(x,u)\; \mathrm{d}x - \frac{\theta}{2}\; |f|_{\mu'}|u|_\mu \\ &\geq d_4|u|_\mu^\mu - d_5|u|_\mu. \end{aligned}$$

Since $\mu > 2$, direct computations and elementary inequalities give

$$|u|_\mu \leq d_6(I_\theta^2(u) + 1)^{\frac{1}{2\mu}}.$$

Hence,

$$\left|\frac{\partial I}{\partial \theta}(\theta, u)\right| \leq |f|_{\mu'}|u|_\mu \leq C_1(I_\theta^2(u) + 1)^{\frac{1}{2\mu}}$$

and inequality (6) holds with $\eta_1, \eta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = C_1\left(s^2 + 1\right)^{\frac{1}{2\mu}}, \tag{8}$$

therefore (A_3) is proved.
Step 4. Finally, let us remark that by (g_4) the functional I_0 is even on E_V (see (7)); moreover, by (g_3) and standard arguments we have that

$$I(\theta, u) \leq \frac{1}{2}\|u\|_V^2 - a_2|u|_\mu^\mu + |f|_{\mu'}|u|_\mu.$$

Hence, taking any finite dimensional subspace $\mathcal{V}$ of E_V, as $\mu > 2$ and all norms are equivalent on $\mathcal{V}$, property (A_4) follows. □

4. **Proof of the main results.** Our aim is to apply Theorem 2.1, therefore let us introduce a suitable class of mini–max values for the even functional I_0.

Denoting by $(e_k)_k$ the basis of eigenfunctions in E_V found in Proposition 3.1, for any $k \geq 1$ let us set

$$E_k = \mathrm{span}\{e_1, \ldots, e_k\}, \qquad E_k^\perp = \overline{\mathrm{span}\{e_{k+1}, \ldots\}} \tag{9}$$

and

$$c_k = \inf_{\gamma \in \Gamma}\sup_{u \in E_k} I_0(\gamma(u)), \tag{10}$$

where Γ is as in (4) with $\mathcal{H} = E_V$.

In order to establish a lower estimate for the sequence $(c_k)_k$, we recall two lemmas, proved in [14, Corollary 2] and [4, Lemma 4.2] respectively.

Taking any $W : \mathbb{R}^N \to \mathbb{R}$, we denote by $N_-(-\Delta + W(x))$ the number of the negative eigenvalues of the operator $-\Delta + W(x)$ and set $W_-(x) = \min\{W(x), 0\}$.

Lemma 4.1. *Let $N \geq 3$ and $W \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Then, there exists $\overline{C}_N > 0$ such that*

$$N_-(-\Delta + W(x)) \leq \overline{C}_N |W_-|_{\frac{N}{2}}^{\frac{N}{2}}.$$

If $N = 2$ and $W \in L^{1+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon > 0$, then there exists $\overline{C}_\varepsilon > 0$ such that

$$N_-(-\Delta + W(x)) \leq \overline{C}_\varepsilon |W_-|_{1+\varepsilon}^{1+\varepsilon}.$$

Lemma 4.2. *Let $p \in \left]2, 2+\frac{4}{N}\right]$. Then, for some $\bar{p} \in \left]2+\frac{4}{N}, 2^*\right[$, for all $\varepsilon > 0$ there exists $D_\varepsilon > 0$ such that*

$$\int_{\mathbb{R}^N} |u|^p \, \mathrm{d}x \le \varepsilon \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x + D_\varepsilon \int_{\mathbb{R}^N} |u|^{\bar{p}} \, \mathrm{d}x \quad \textit{for all } u \in E_V.$$

Now, we are ready to prove our main result.

Proof of Theorem 1.2. By Proposition 3.4, Theorem 2.1 applies, so the proof of our result is complete if we rule out case (b) for k large enough or better, as by (8) condition (b) implies

$$c_{k+1} - c_k \;\le\; C_2 \, \left((c_k)^{\frac{1}{\mu}} + (c_{k+1})^{\frac{1}{\mu}} + 1\right), \tag{11}$$

with c_k as in (10), it is enough to prove that (11) cannot hold for k large enough.
In fact, if we assume that (11) holds for all $k \ge k_0$ for some $k_0 \ge 1$, by [2, Lemma 5.3] it follows that there exist $\overline{C} > 0$ and $\overline{k} \in \mathbb{N}$ such that

$$c_k \;\le\; \overline{C}\; k^{\frac{\mu}{\mu-1}} \quad \text{for all } k \ge \overline{k}. \tag{12}$$

On the other hand, by (g_1) it follows that

$$|G(x,u)| \;\le\; \frac{a_1}{p}|s|^p + \frac{\delta}{2}|s|^2 \quad \text{for a.e. } x \in \mathbb{R}^N,\ s \in \mathbb{R};$$

whence, by Remark 3.2 there exists $C^* > 0$ such that

$$I_0(u) \;\ge\; C^* \left(\frac{1}{2}\|u\|_V^2 - C_3|u|_p^p\right) \quad \text{for all } u \in E_V.$$

From now on, we deal with the case $N \ge 3$, since the case $N = 2$ follows by slight modifications. We claim that for any $p \in]2, 2^*[$ it is

$$c_k \ge C_4 k^{\frac{4}{N(p-2)}} \quad \text{for all } k \ge 1. \tag{13}$$

To this aim, two different cases occur.
Case 1. Let $2 + \frac{4}{N} \le p < 2^*$. Setting

$$K(u) = \frac{1}{2}\|u\|_V^2 - C_3|u|_p^p$$

and

$$b_k = \inf_{\gamma\in\Gamma} \sup_{u\in E_k} C^* K(\gamma(u)),$$

we have that

$$c_k \ge b_k. \tag{14}$$

Now, [22, Theorem B] implies that for all $k \in \mathbb{N}$ there exists $u_k \in E_V$ such that

$$K'(u_k) = 0 \quad \text{and} \quad K(u_k) \le b_k, \tag{15}$$

with $m^*(u_k, K) \ge k$, i.e., the operator

$$K''(u_k) \;=\; -\Delta + V(x) - C_3\, p(p-1)|u_k|^{p-2}$$

has at least k non–positive eigenvalues. Therefore, by [8, Proposition S1.3.1] and Lemma 4.1 with $W(x) = -C_3 p(p-1)|u_k|^{p-2}$ we infer that

$$k \;\le\; N_-(K''(u_k)) \;\le\; N_-(-\Delta - C_3 p(p-1)|u_k|^{p-2}) \;\le\; C_5 |u_k|_{(p-2)\frac{N}{2}}^{(p-2)\frac{N}{2}}.$$

In this case, we have $(p-2)\frac{N}{2} \in [2, 2^*[$, then by Proposition 3.1 we get

$$k \le C_6 \|u_k\|_V^{(p-2)\frac{N}{2}}.$$

As (15) implies $K'(u_k)[u_k] = 0$, then

$$\|u_k\|_V^2 = C_3 p |u_k|_p^p; \tag{16}$$

hence (13) follows by (14) − (16).

Case 2. Let $2 < p < 2 + \frac{4}{N}$. By Lemma 4.2, for a suitable $\bar{p} \in]2 + \frac{4}{N}, 2^*[$ and $\varepsilon > 0$ small enough, there exist $b_\varepsilon, d_\varepsilon > 0$ such that, setting

$$K_\varepsilon(u) = b_\varepsilon \|u\|_V^2 - d_\varepsilon |u|_{\bar{p}}^{\bar{p}},$$

it results

$$I_0(u) \geq K_\varepsilon(u) \quad \text{ for all } u \in E_V.$$

Then, let us define

$$c_k^\varepsilon = \inf_{\gamma \in \Gamma} \sup_{u \in E_k} K_\varepsilon(\gamma(u)),$$

where Γ is as in (4) with $\mathcal{H} = E_V$ and E_k is as in (9). Plainly, $c_k \geq c_k^\varepsilon$. By applying the arguments developed in *Case 1*, but with p replaced by $\bar{p}$ and K by K_ε, also in this case (13) holds.

At last, by (3) inequality (13) yields to a contradiction with (12); therefore condition (a) in Theorem 2.1 holds for infinitely many $k \in \mathbb{N}$ and by Remark 2.2 the proof is complete. □

Proof of Corollary 1.3. Proposition 3.4 follows by simpler arguments, with $\eta_1, \eta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = C_7 \left(s^2 + 1\right)^{\frac{1}{2p}};$$

this implies that (12) is now replaced by

$$c_k \ \leq \ C_8 \ k^{\frac{p}{p-1}} \quad \text{ for } k \text{ large enough.}$$

Then, we can reason as in the proof of Theorem 1.2, working directly on I_0. □

REFERENCES

[1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381.

[2] A. Bahri and H. Berestycki, A perturbation method in critical point theory and applications, *Trans. Amer. Math. Soc.*, **267** (1981), 1–32.

[3] A. Bahri and P. L. Lions, Morse index of some min–max critical points. I. Applications to multiplicity results, *Comm. Pure Appl. Math.*, **41** (1988), 1027–1037.

[4] S. Barile and A. Salvatore, Radial solutions of semilinear elliptic equations with broken symmetry on unbounded domains, *Discrete Contin. Dyn. Syst. Supplement 2013*, (2013), 41–49.

[5] S. Barile and A. Salvatore, Multiplicity results for some perturbed elliptic problems in unbounded domains with non-homogeneous boundary conditions, *Nonlinear Analysis*, **110** (2014), 47–60.

[6] T. Bartsch and Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Partial Differential Equations*, **20** (1995), 1725–1741.

[7] V. Benci and D. Fortunato, Discreteness conditions of the spectrum of Schrödinger operators, *J. Math. Anal. Appl.*, **64** (1978), 695–700.

[8] F. A. Berezin and M. A. Shubin, The Schrödinger Equation, *Mathematics and its Applications*, (Soviet Series) **66**, Kluwer Academic Publishers, Dordrecht, 1991.

[9] P. Bolle, On the Bolza problem, *J. Differential Equations*, **152** (1999), 274–288.

[10] P. Bolle, N. Ghoussoub and H. Tehrani, The multiplicity of solutions in non–homogeneous boundary value problems, *Manuscripta Math.*, **101** (2000), 325–350.

[11] A. Candela, G. Palmieri and A. Salvatore, Radial solutions of semilinear elliptic equations with broken symmetry, *Topol. Methods Nonlinear Anal.*, **27** (2006), 117–132.

[12] M. Clapp, Y. Ding and S. Hernández–Linares, Strongly indefinite functionals with perturbed symmetries and multiple solutions of nonsymmetric elliptic systems, *Electron. J. Differential Equations*, **100** (2004), 18 pp.

[13] D. E. Edmunds and W.D. Evans, *Spectral Theory and Differential Operators*, Oxford Mathematical Monographs, New York, 1987.

[14] P. Li and S. T. Yau, On the Schrödinger equation and the eigenvalue problem, *Comm. Math. Phys.*, **88** (1983), 309–318.

[15] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.*, **43** (1992), 270–291.

[16] P. H. Rabinowitz, Multiple critical points of perturbed symmetric functionals, *Trans. Amer. Math. Soc.*, **272** (1982), 753–769.

[17] A. Salvatore, Multiple solutions for perturbed elliptic equations in unbounded domains, *Adv. Nonlinear Stud.*, **3** (2003), 1–23.

[18] A. Salvatore, M. Squassina, Deformation from symmetry for nonhomogeneous Schrödinger equations of higher order on unbounded domains, *Electron. J. Differential Equations*, **65** (2003), 1–15.

[19] M. Struwe, Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems, *Manuscripta Math.*, **32** (1980), 335–364.

[20] M. Struwe, Infinitely many solutions of superlinear boundary value problems with rotational symmetry, *Arch. Math.*, **36** (1981), 360–369.

[21] M. Struwe, Superlinear elliptic boundary value problems with rotational symmetry, *Arch. Math.*, **39** (1982), 233–240.

[22] K. Tanaka, Morse indices at critical points related to the symmetric mountain pass theorem and applications, *Comm. Partial Differential Equations*, **14** (1989), 99–128.

[23] W. Zou and M. Schechter, *Critical Point Theory and Its Applications*, Springer, New York, 2006.

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