A Defaultable HJM Modelling of the Libor Rate for Pricing Basis Swaps after the Credit Crunch

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Abstract

A great deal of recent literature discusses the major anomalies that have appeared in the interest rate market following the credit crunch in August 2007. There were major consequences with regard to the development of spreads between quantities that had remained the same until then. In particular, we consider the spread that opened up between the Libor rate and the OIS rate, and the consequent empirical evidence that FRA rates can no longer be replicated using Libor spot rates due to the presence of a Basis spread between floating legs of different tenors. We develop a credit risk model for pricing Basis Swaps in a multi-curve setup. The Libor rate is considered here as a risky rate, subject to the credit risk of a generic counterparty whose credit quality is refreshed at each fixing date. A defaultable HJM methodology is used to model the term structure of the credit spread, defined through the implied default intensity of the contributing banks of the Libor corresponding to a chosen tenor. A forward credit spread volatility function depending on the entire credit spread term structure is assumed. In this context, we implement the model and obtain the price of Basis Swaps using a numerical scheme based on the Euler-Maruyama stochastic integral approximation and the Monte Carlo method.

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1 Introduction

After the credit crunch of summer 2007 the interest rate market changed due to the appearance of Basis spreads between rates with different tenors, to the loss of the possibility of replicating swap with spot rates, and to the fact that the interest rate curve underlying of interest rate derivatives does not coincide with the discounting interest rate curve anymore. Morini (2009, 2011) gives a deep and detailed analysis on the causes and consequences of the interest rate market changes. The author designs a new approach for modelling collateralized derivatives, namely derivatives that are not affected by both the risk of default and liquidity because they are traded with a provision for liquidity. Morini (2009) shows that the gap between the Forward Rate Agreement, FRA, rates and their standard spot Libor replication can be explained by the existence of a premium associated to tenor, expressed by quoted Basis Swap spreads. Among the major anomalies that arose in the interest rate market there is the discrepancy between Libor rates and Eonia OIS rates, Overnight Indexed Swaps rates, that leads to a new definition of the Libor rate as a risky interest rate. In fact, Eonia OIS rates according to different maturities give the risk-free term structure, because the OIS rate with a generic maturity T is seen as an average of the market expectation of the overnight futures rates until T, and those rates are considered free of credit risk. On the contrary, the Libor rate is now a risky rate whose credit risk is not referred to a specific counterparty, but a generic one whose credit quality is refreshed at each fixing date. Thus, the level of Libor is provided by the fixings and assuming homogeneity and stability of Libor counterparties (banks). The fixings are trimmed averages of contribu-

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tions from a panel of the most relevant banks in the market with the highest credit quality. Among papers which propose new approaches and methodologies for building models consistent with the new interest rate market situation, we recall Mercurio (2009), Ametrano and Bianchetti (2009), Henrard (2009), Pallavicini and Tarenghi (2010), Crépey et al. (2011), Eberlein and Grbac (2013), Pallavicini and Brigo (2013), and Crépey et al. (2014). Mercurio (2009) extends the basic lognormal LMM (Brace et al. (1997), Miltersen et al. (1997)), by adding stochastic volatility, in order to obtain the dynamics of FRA rates and to price interest rate derivatives. Ametrano and Bianchetti (2009) illustrate a methodology for bootstrapping multiple interest rate yield curves from non-homogeneous plain vanilla instruments quoted on the market, obtaining that each curve is homogenous in the tenor of the underlying rate. Henrard (2009) and Pallavicini and Tarenghi (2010) propose two different frameworks to construct yield curves consistent with a multi-curve situation and derive the price of interest rate derivatives. Crépey et al. (2011) apply a defaultable HJM approach to model the term structure of multiple interest rate curves. They choose a class of non-negative multidimensional Lévy processes as driving processes combined with deterministic volatility structures, in order to obtain a flexible and efficient interest rate derivative pricing model. Eberlein and Grbac (2013) model credit risk within the LMM. They propose a rating Lévy Libor model that is arbitrage-free for defaultable forward Libor rates related to risky bonds with credit ratings. They use time-inhomogeneous Lévy processes as driving processes. Recently, Pallavicini and Brigo (2013) model multiple LIBOR and OIS based interest rate curves consistently, based only on market observables and by consistently including credit, collateral and funding effects. They develop a framework for pricing collateralized interest-rate derivatives. Crépey et al. (2014) develop a parsimonious Markovian multiple-curve model for evaluating interest rate derivatives in the post-crisis setup and they use BSDE-based numerical computations for obtaining counterparty risk and funding adjustments.

Although in this paper we develop a model for pricing Basis Swaps according to the mathematical representation of interest rate market theorized by Morini (2009, 2011), we will model the term structure of multiple interest rates in a defaultable Heath-Jorrow-Morton framework, henceforth HJM, (see Heath et al. (1992), Brigo and Mercurio (2006), and Bielecki and Rutkowski (2000)).

In Section 2 we describe the general setting of the model, namely assumptions about the probability space and the dynamics of the defaultable instantaneous forward rate. In Section 3 we derive a defaultable representation of the Libor rate and we develop a model for pricing collateralized derivatives, and in particular Basis Swaps. Section 4 deals with the specification of defaultable dynamics in a multi-curve HJM framework in compliance with no-arbitrage conditions. In Section 5 we illustrate the numerical algorithm used to simulate the Basis Swap model and we show and analyze the numerical results. Finally, Section 6 concludes.

2 The general setting

In this section we present the general setting on which the credit model for pricing Basis Swap is built.

We consider the instantaneous yield curve implicitly defined by the Libor rate. We model the dynamics of defaultable instantaneous forward interest rates within the HJM framework, but we extend it to consider Libor rates, that is the underlying of all interest rate derivatives, refer to different counterparties at different fixing times.

We assume a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ exists, $\overline{T}$ is assumed to be the finite time horizon and $\mathcal{F} = \mathcal{F}_{\overline{T}}$ is the σ-algebra at time $\overline{T}$. All statements and definitions are understood to be valid until the time horizon $\overline{T}$.

We denote by C^z the counterparty of a lending contract at time z, that defaults at time

$\tau^z > z$. The time τ^z is a stopping time, $\tau^z : \Omega \to [0, +\infty[$, defined as the first jump time of the Cox process $N(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i^z \leq t\}}$, that is

$$\tau^z = inf\{t \geq 0 | N(t) > 0\}.$$

When we consider N counterparties $C^1, C^2, ..., C^N$, $N \in \mathbb{N}$, the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is divided into two subfiltrations $\mathbf{F} = \mathbf{H} \vee \mathbf{F}^{\tau}$, which is $\mathcal{F}_t = \mathcal{H}_t \vee F_t^{\tau}$ $\forall t \geq 0$, and $\mathbf{F}^{\tau} = \mathbf{F}^{\tau^1} \vee \mathbf{F}^{\tau^2} \vee ... \vee \mathbf{F}^{\tau^N}$. The subfiltration $\mathbf{H} = (\mathcal{H}_t)_{t\geq 0} = (\sigma(X_s : 0 \leq s \leq t))_{t\geq 0}$ is generated by the background driving process X, that is an $\mathbb{R}^d$-valued right continuous stochastic process $X = \{X_t : 0 \leq t \leq \overline{T}\}$ with left limit. It represents the flow of all background information except default itself and $\mathcal{H} = \mathcal{H}_{\overline{T}}$ is the sub-σ-algebra at time $\overline{T}$. The generic subfiltration $\mathbf{F}^{\tau^z} = (\mathcal{F}_t^{\tau^z})_{t\geq 0} = (\sigma(\mathbf{1}_{\{\tau^z \leq s\}} : 0 \leq s \leq t))_{t\geq 0}$ is generated by the right-continuous default indicator process $\mathbf{1}_{\{\tau^z \leq t\}}$. Since obviously $\mathcal{F}_t^{\tau^z} \subset \mathcal{F}_t$, $\forall t \geq 0$, τ^z is a stopping time with respect to $\mathbf{F}$, but it is not necessarily a stopping time with respect to $\mathbf{H}$. The right-continuous stochastic process $\lambda^z(t)$ is the intensity of the Cox process. It is independent of $N(t)$, it is assumed to be adapted to $\mathbf{H}$ and follows the diffusion process

$$d\lambda^z(t) = \mu_\lambda^z(t)dt + \sigma_\lambda^z(t)dW_\lambda^z(t),$$

where $\mu_\lambda^z(t)$ is the drift of the intensity process, $\sigma_\lambda^z(t)$ is the volatility of the intensity process and W_λ^z is a standard Wiener process under the objective probability measure $\mathbb{P}$. Processes W_λ^z, $z = 1, ..., N$, are N independent Wiener processes.

The defaultable instantaneous forward rate, $f^z(t,T)$, $0 \leq t \leq T \leq \overline{T}$, is modeled as the sum of the risk-free instantaneous forward rate, $f(t,T)$, and the instantaneous forward credit spread $\lambda^z(t,T)$, so that we have

$$f^z(t,T) := f(t,T) + \lambda^z(t,T). \tag{1}$$

Thus the forward credit spread is obtained as difference between the two forward interest rates. If $t = T$, then we obtain the defaultable spot rate $f^z(t) := f^z(t,t) = r(t) + \lambda^z(t)$, where $r(t) := f(t,t)$ represents the risk-free spot rate and $\lambda^z(t) := \lambda^z(t,t)$ is the spot credit spread. The credit spread is referred to as the Cox intensity across maturities.

In the HJM framework the term structure of risk-free interest rates is the stochastic integral equation for the forward rate

$$f(t,T) = f(0,T) + \int_0^t \mu(v,T,\cdot)dv + \int_0^t \sigma_f(v,T,\cdot)dW(v), \tag{2}$$

where $\mu(t,T,\cdot)$ is the instantaneous forward rate drift function, $\sigma_f(t,T,\cdot)$ is the instantaneous forward rate volatility function and $W(t)$ is a standard Wiener process with respect to the objective probability measure $\mathbb{P}$. The third argument in the brackets $(t,T,\cdot)$ indicates the possible dependence of the forward rate on other path dependent quantities, such as the spot rate or the forward rate itself.

Whereas the dynamics for $\lambda^z(t,T)$ is

$$\lambda^z(t,T) = \lambda^z(0,T) + \int_0^t \mu^z_\lambda(s,T,\cdot)ds + \int_0^t \sigma^z_\lambda(s,T,\cdot)dW^z_\lambda(s). \tag{3}$$

Again, the third argument in the brackets $(t,T,\cdot)$ indicates the possible dependence of the forward rate on other path dependent quantities.

Now we apply the HJM forward rate drift restriction, that is both necessary and sufficient condition for the absence of riskless arbitrage opportunities, to the dynamics of both the risk free rate and the credit spread. So we find the following forward dynamics, respectively for the risk-free forward rate and the forward credit spread, under the risk neutral probability measure $\widetilde{\mathbb{P}}$

$$f(t,T) = f(0,T) + \int_0^t \sigma_f(v,T,\cdot)\int_v^T \sigma_f(v,s,\cdot)dsdv + \int_0^t \sigma_f(v,T,\cdot)d\widetilde{W}(v),$$

and

$$\lambda^z(t,T) \;=\; \lambda^z(0,T) + \int_0^t \sigma_\lambda^z(v,T,\cdot)\int_v^T \sigma_\lambda^z(v,s,\cdot)dsdv +$$
$$\int_0^t \rho\left[\sigma_f(v,T,\cdot)\int_v^T \sigma_\lambda^z(v,s,\cdot)ds + \sigma_\lambda^z(v,T,\cdot)\int_v^T \sigma_f(v,s,\cdot)ds\right]dv +$$
$$\int_0^t \sigma_\lambda^z(v,T,\cdot)d\widetilde{W}_\lambda^z(v), \tag{4}$$

where ρ is the correlation coefficient between the two Wiener processes $\widetilde{W}(t)$ and $\widetilde{W}_\lambda^z(t)$ under the risk neutral probability measure and that are assumed to be one-dimensional (see Chiarella et al. (2011) for further mathematical details in calculating the expression for the stochastic differential equations).

As it is well known, the main advantages of the HJM model are that in the formulation of the spot rate process and bond price process the market price of interest rate risk drops out by being incorporated into the Wiener process under the risk neutral measure; furthermore, the model is automatically calibrated to the initial yield curve and the drift term in the forward rate differential equation is a function of the volatility term. In addition, it is possible to have different HJM models choosing different volatility functions, also path dependent, giving the possibility to make the model consistent with the real market situation (see, for example, Chiarella et al. (2004, 2013)).

3 Modelling the Libor rate for pricing Swaps

The Libor rate, used as underlying of many collateralized derivatives, was considered a risk-free rate before the credit crunch. After the credit crunch it became defaultable, uncollateralized. Consequently, there arises the problem of finding a way to price collateralized derivatives as an alternative to the replication strategy that would no longer work in a defaultable market, due to the new representation of the interest rate market.

In this section we aim to address the issue of developing a model for pricing Basis Swaps, consistent with the new interest rate market situation after the credit crunch. We consider a Basis Swap, that is an interest rate swap which involves the exchange of two standard receiver fixed-for-floating swaps with the same fixed legs (usually annual frequency) and different floating legs indexed to different bases. The tenor $\alpha \in \mathbb{R}$ represents the frequency of the floating leg, that is the units of time that pass between two subsequent periodic Libor payments that one counterparty makes to the other. Furthermore, for all $i \in \mathbb{N}$, we refer to $L(i\alpha, (i+1)\,\alpha)$ as the Libor rate between successive tenors and it is given by

$$L(i\alpha, (i+1)\,\alpha) = \left(\frac{1}{P^{i\alpha}(i\alpha, (i+1)\,\alpha)} - 1 \right) \frac{1}{\alpha} \tag{5}$$

where

$$P^{i\alpha}(t,T) = \mathbf{1}_{\{\tau^{i\alpha} > t\}} e^{-\int_t^T f^{i\alpha}(t,s)ds} \tag{6}$$

is the defaultable bond of $C^{i\alpha}$, the counterparty in the Libor market at time $i\alpha$, and we are assuming zero recovery rate.

We remind the reader that since $C^{i\alpha}$ is a market counterparty at $i\alpha$, it necessarily follows that

$$\tau^{i\alpha} > i\alpha,$$

so we have

$$P^{i\alpha}(i\alpha, (i+1)\,\alpha) = e^{-\int_{i\alpha}^{(i+1)\alpha} (f(i\alpha,s) + \lambda^{i\alpha}(i\alpha,s))ds}. \tag{7}$$

Following Morini (2009), if we consider a generic $\alpha/2\alpha$ Basis Swap, meaning that we exchange two swaps with the same fixed legs and pay two floating legs with frequencies respectively of α and 2α, we can calculate the price as the expectation of the leg cash flows discounted with riskless rate. The price of the $\alpha/2\alpha$ Basis Swap with maturity 2α, $P_{Basis}(0, 2\alpha, Z)$, is given by the following formula:

$$P_{Basis}(0,2\alpha,Z) =$$

$$E_{\widetilde{\mathbb{P}}}[\underbrace{D(0,\alpha)\alpha(L(0,\alpha)+Z)+D(0,2\alpha)\alpha(L(\alpha,2\alpha)+Z)}_{\alpha-tenor\, leg}] - E_{\widetilde{\mathbb{P}}}[\underbrace{D(0,2\alpha)2\alpha L(0,2\alpha)}_{2\alpha-tenor\, leg}], \qquad (8)$$

where Z is the Basis spread, given as the difference, in basis points, between the fixed rate of the higher frequency swap and the fixed rate of the lower frequency swap. In this paper we adopt the convention that the Basis spread is added to the shorter tenor leg, and so the two fixed legs can be neglected in the Basis Swap pricing formula. $D(0,T) = e^{-\int_0^T r(s)ds}$ is the discount factor calculated using the risk-free spot rate $r(t)$. We use the risk-free spot rate since we consider interest rate derivatives that are collateralized.

If we want to price a Basis Swap with a generic maturity $n\alpha$, $n \in \mathbb{N}$, we must be able to price, at time zero, the following floating leg of a swap with tenor α,

$$F_\alpha(0,n\alpha) = \sum_{i=0}^{n-1} E_{\widetilde{\mathbb{P}}}[D(0,(i+1)\alpha)\alpha L(i\alpha,(i+1)\alpha)]. \qquad (9)$$

Let us explain our pricing procedure step by step. Assume first that $n = 2$. In this case pricing the swap is very simple:

$$\begin{aligned} F_\alpha(0,2\alpha) &= P(0,\alpha)\alpha L(0,\alpha) + E_{\widetilde{\mathbb{P}}}[D(0,2\alpha)\alpha L(\alpha,2\alpha)] \\ &= P(0,\alpha)\alpha L(0,\alpha) + E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\left(e^{\int_\alpha^{2\alpha} f^\alpha(\alpha,s)ds} - 1\right)\right] \end{aligned} \qquad (10)$$

where $P(0,T) = e^{-\int_0^T f(0,s)ds}$ is the price of a risk-free bond.

The price of the swap in (10) depends on the assumptions we make on C^α, the Libor counterparty at α, and in particular on $\lambda^\alpha(t,T)$, $0 \le t \le T \le \overline{T}$. The forward credit spread $\lambda^\alpha(t,T)$ indicates the term structure of credit spreads of C^α. At time $t = 0$ we need a term structure of defaultable bonds

$$P^\alpha(0,T), \quad 0 \le T \le \overline{T}, \tag{11}$$

and a term structure of default-free bonds

$$P(0,T), \quad 0 \le T \le \overline{T}, \tag{12}$$

in order to obtain $\lambda^\alpha(0,s)$, $0 \le s \le T \le \overline{T}$, via

$$P^\alpha(0,T) = e^{-\int_0^T (f(0,s)+\lambda^\alpha(0,s))ds}$$

and

$$P(0,T) = e^{-\int_0^T f(0,s)ds}.$$

Knowing $\lambda^\alpha(0,T)$ and the dynamics of $\lambda^\alpha(t,T)$, as further described in Section 5, we can compute via Monte Carlo simulation the following expectation used in formula (10)

$$\begin{aligned} & E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\alpha L(\alpha,2\alpha)\right] \\ = \; & E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\left(\frac{1}{P^\alpha(\alpha,2\alpha)} - 1\right)\right] \\ = \; & E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\left(e^{\int_\alpha^{2\alpha}(f(\alpha,s)+\lambda^\alpha(\alpha,s))ds} - 1\right)\right]. \end{aligned} \tag{13}$$

However, we do not know (11), that is the bond price at time zero of a counterparty in the Libor market revealing at a future time α, but we have instead (12) and $P^0(0,T)$, the prices of the bonds of C^0, which is the counterparty in the Libor market at time 0, with

$$P^0(0,T) = e^{-\int_0^T (f(0,s)+\lambda^0(0,s))ds}. \tag{14}$$

Thus from (14) we can compute only $\lambda^0(0,T)$. What is the relationship between $\lambda^0(0,T)$ and $\lambda^\alpha(0,T)$, where the latter is the quantity that we need to compute (13)?

In the market the standard assumption used to be

$$\lambda^\alpha(0,T) = \lambda^0(0,T), \tag{15}$$

but this is no longer valid after the credit crunch. In fact, we have to take into account that $\lambda^\alpha(t,T)$ is the credit spread of C^α, that has the credit quality of a Libor counterparty at time α, while $\lambda^0(t,T)$ is the credit spread of C^0, that has the credit quality of a Libor counterparty at time 0. After the credit crunch, the possibility that C^0 at α has a credit quality lower than C^α cannot be neglected.

In fact, the Libor rate is now defined as a risky rate, and its credit risk is not referred to a specific counterparty, but a generic one, whose credit quality is "refreshed" at each fixing date. Moreover, the shorter the tenor, the lower the credit risk. In order to adapt the modelling to this new market situation, we make the interest rate curve depending on the tenor. However, we know from (1) that only the credit spread depends on the tenor, that is $f^\alpha(t,s) = f(t,s) + \lambda^\alpha(t,s)$. Furthermore, if we observe the credit spread curve every day, we can notice that the first rate is always equal to zero because it coincides with the overnight credit spread that is null, and the term structure is increasing over maturities. In fact, at each fixing date the quality of the counterparty C^α is refreshed, the term structure of the credit spread is marked to market, and we obtain a new curve such that, realistically, the first credit spread value is zero and the curve increases over maturities. Following this insight, we can state that if we consider two generic tenors α and 2α, such that $\alpha = T_i - T_{i-1}$ and $2\alpha = T_{i+1} - T_{i-1}$, $\forall i \in \mathbb{N}$, over the time interval $[T_{i-1}, T_{i+1}]$ the curve $f^\alpha(T_{i-1}, s)$ overshoots the curve $f^{2\alpha}(T_{i-1}, s)$, $\forall s \in [T_{i-1}, T_{i+1}]$. Consequently, the following relationship holds

$$
\begin{aligned}
P^{\alpha}(T_{i-1},T_{i+1}) &> P^{2\alpha}(T_{i-1},T_{i+1}) \\
e^{-\int_{T_{i-1}}^{T_i} f^{\alpha}(T_{i-1},s)ds} e^{-\int_{T_i}^{T_{i+1}} f^{\alpha}(T_{i-1},s)ds} &> e^{-\int_{T_{i-1}}^{T_{i+1}} f^{2\alpha}(T_{i-1},s)ds}.
\end{aligned} \tag{16}
$$

Therefore, since C^{α} has the credit quality "refreshed" at time α after 0, in this work we assume instead of (15) that

$$
\lambda^{\alpha}(0,\alpha+T) = \lambda^{0}(0,T). \tag{17}
$$

Notice that we do not need

$$
\lambda^{\alpha}(0,T) \text{ with } T < \alpha
$$

to compute (13).

Now we consider $n = 3$, so that we have

$$
\begin{aligned}
F_{\alpha}(0,3\alpha) &= P(0,\alpha)\alpha L(0,\alpha) + E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\left(e^{\int_{\alpha}^{2\alpha} f^{\alpha}(\alpha,s)ds} - 1\right)\right] \\
&+ E_{\widetilde{\mathbb{P}}}\left[D(0,3\alpha)\left(e^{\int_{2\alpha}^{3\alpha} f^{2\alpha}(2\alpha,s)ds} - 1\right)\right].
\end{aligned}
$$

Consistently with (17), we assume

$$
\lambda^{2\alpha}(\alpha,2\alpha+T) = \lambda^{\alpha}(\alpha,\alpha+T).
$$

Finally, in order to price the general swap in (9) we need the expectations of

$$\begin{aligned}
&\lambda^{\alpha}(\alpha, \alpha+s), \\
&\lambda^{2\alpha}(2\alpha, 2\alpha+s), \\
&\text{and } \lambda^{i\alpha}(i\alpha, i\alpha+s), \quad \forall i \in \mathbb{N},
\end{aligned}$$

for $0 \leq s < \alpha$, and we assume in general, for $T \geq 0$,

$$\begin{aligned}
\lambda^{\alpha}(0, T+\alpha) &= \lambda^{0}(0, T), \\
\lambda^{2\alpha}(\alpha, T+2\alpha) &= \lambda^{\alpha}(\alpha, T+\alpha), \\
\lambda^{i\alpha}((i-1)\alpha, T+i\alpha) &= \lambda^{(i-1)\alpha}((i-1)\alpha, T+(i-1)\alpha), \quad \forall i \in \mathbb{N}.
\end{aligned}$$

4 Dynamics and No-arbitrage

The setting of the previous section corresponds to considering a generic intensity process $\lambda^{\alpha}(t, \alpha+s)$ with $0 \leqslant s \leqslant \alpha$ and with $\lambda^{\alpha}(0, \alpha+s) = \lambda^{0}(0, s)$, that is valid for every t, $0 \leqslant t \leqslant \alpha$. Then, at α we set $\lambda^{2\alpha}(\alpha, 2\alpha+s) = \lambda^{\alpha}(\alpha, \alpha+s)$, and the process $\lambda^{2\alpha}(\alpha+t, 2\alpha+s)$, $0 \leqslant t \leqslant \alpha$, is valid until 2α. And so on. In this way we can construct the term structure of credit spreads.

However, it is important to notice that we are not considering a single HJM model, but a sequence of different *credit HJM models*, since we are considering different counterparties whose default times are driven by different random variables. Furthermore, the no-arbitage conditions are also different. For example, for the initial counterparty C^0 we have that $\lambda^0(t, T)$ is given by (4) with $z = 0$. However, when $z = \alpha$, $\lambda^{\alpha}(\alpha, \alpha+k)$, $0 \leq k < \alpha$, is given by

$$\lambda^{\alpha}(\alpha,\alpha+k) = \underbrace{\lambda^{\alpha}(0,\alpha+k)}_{=\lambda^{0}(0,k)} + \int_0^{\alpha} \sigma_{\lambda}^{\alpha}(v,\alpha+k,\cdot)\int_v^{\alpha+k}\sigma_{\lambda}^{\alpha}(v,s,\cdot)ds dv +$$
$$\int_0^{\alpha} \rho\left[\sigma_f(v,\alpha+k,\cdot)\int_v^{\alpha+k}\sigma_{\lambda}^{\alpha}(v,s,\cdot)ds + \sigma_{\lambda}^{\alpha}(v,\alpha+k,\cdot)\int_v^{\alpha+k}\sigma_f(v,s,\cdot)ds\right]dv +$$
$$\int_0^{\alpha}\sigma_{\lambda}^{\alpha}(s,\alpha+k,\cdot)d\widetilde{W}_{\lambda}^{\alpha}(v), \tag{18}$$

where we are free to select any

$$\sigma_{\lambda}^{\alpha}(v,s,\cdot) \neq \sigma_{\lambda}^{0}(v,s,\cdot) \text{ for } 0 \leq s < \alpha.$$

In this paper we set, for a generic $i\alpha$, $i \in \mathbb{N}$,

$$\sigma_{\lambda}^{i\alpha}(v,s,\cdot) = 0 \text{ for } 0 \leq s < \alpha.$$

According to the HJM approach, we are free to choose a volatility function that better describes the behaviour of rate variability. In particular, we assume the following path dependent volatility[1]:

$$\sigma_u(t,T) = (\alpha_u + \gamma_u u(t,T))e^{\beta_u(T-t)}, \quad u = f, \lambda, \tag{19}$$

where α_u represents the short-term coefficient, γ_u determines the size of the hump of the volatility curve, and β_u is the time-to-maturity proportionality coefficient. At each time t, the volatility depends on the forward rate itself, namely the risk-free interest rate or the credit spread, in accordance with the exponential time-to-maturity dependent factor.

[1]Different volatility functional forms have been tested on empirical data, and the chosen volatility function is the one that, according to statistical tests, best fits historical data.

5 Basis Swap Model Implementation and Results

In this paper we will implement the model for pricing a $\alpha/2\alpha$ Basis Swap, $P_{Basis}(0, 2\alpha, Z)$, as in (8):

$$P_{Basis}(0, 2\alpha, Z) =$$
$$E_{\widetilde{\mathbb{P}}}[D(0, \alpha)\alpha(L(0, \alpha) + Z) + D(0, 2\alpha)\alpha(L(\alpha, 2\alpha) + Z)] - E_{\widetilde{\mathbb{P}}}[D(0, 2\alpha)2\alpha L(0, 2\alpha)].$$

So, we obtain

$$P_{Basis}(0, 2\alpha, Z) = E_{\widetilde{\mathbb{P}}}[D(0, 2\alpha)\alpha L(\alpha, 2\alpha)] +$$
$$P(0, \alpha)\left(\frac{1}{P^0(0, \alpha)} - 1\right) - P(0, 2\alpha)\left(\frac{1}{P^0(0, 2\alpha)} - 1\right) + [P(0, \alpha) + P(0, 2\alpha)]\alpha Z. \tag{20}$$

Pricing a Basis Swap means finding the basis spread Z that makes the quantity (20) equal to zero, namely

$$Z = \frac{P(0,2\alpha)}{\alpha[P(0,\alpha)+P(0,2\alpha)]}\left(\frac{1}{P^0(0,2\alpha)} - 1\right) +$$
$$\frac{P(0,\alpha)}{\alpha[P(0,\alpha)+P(0,2\alpha)]}\left(1 - \frac{1}{P^0(0,\alpha)}\right) - \frac{E_{\widetilde{\mathbb{P}}}[D(0,2\alpha)\alpha L(\alpha,2\alpha)]}{\alpha[P(0,\alpha)+P(0,2\alpha)]}. \tag{21}$$

We can conclude that only the last expectation in (21) must be estimated, while other quantities can be simply calculated using market data.

We will use formula (21) to find simulated basis spreads using market data and we will show as the obtained numerical results are in line with market Basis Swap quotations.

First of all we develop an algorithm to simulate the evolution of the term structure of both the risk-free interest rates and credit spreads. We proceed by discretizing integrals in all the formulas with the Euler-Maruyama approximation technique and estimating parameters of the volatility functions (19) for the risk-free interest rate and the credit spread. Finally, the price of the Basis Swap is then obtained through Monte Carlo simulations.

Given the time horizon $\overline{T}$, we can divide $[0, \overline{T}]$ into N subintervals of length $\triangle t = \frac{\overline{T}}{N}$, so that $n = \frac{t}{\triangle t}$, $m = \frac{T}{\triangle t}$ for $0 \leq t \leq T \leq \overline{T}$ and $f(t,T) + \lambda(t,T) = f(n\Delta t, m\Delta t) + \lambda(n\Delta t, m\Delta t)$. The Euler-Maruyama discretisation is used to approximate the stochastic integral equations (2) and (4) (see Kloeden and Platen (1999)).

We start by considering, at time zero, the initial defaultable forward curve with generic maturity $T = m\Delta t$, where $1 \leq m \leq N$, that is $f(0,0) + \lambda(0,0) = r(0) + \lambda(0)$, $f(0,\Delta t) + \lambda(0,\Delta t)$, $f(0,2\Delta t) + \lambda(0,2\Delta t)$, ..., $f(0,N\Delta t) + \lambda(0,N\Delta t)$. Hence we obtain the generic recursive formula for the forward curve evolution in the integral form,

$$\begin{aligned}
&f^{\alpha}((n+1)\Delta t, m\Delta t) = f(n\Delta t, m\Delta t) + \lambda^{\alpha}(n\Delta t, m\Delta t, \cdot) \\
&+\sigma_f(n\Delta t, m\Delta t, \cdot)\sum_{i=n}^{m-1}\sigma_f(n\Delta t, i\Delta t, \cdot)\Delta t + \sigma_\lambda(n\Delta t, m\Delta t, \cdot)\sum_{i=n}^{m-1}\sigma_\lambda(n\Delta t, i\Delta t, \cdot)\Delta t \\
&+\rho\left[\sigma_f(n\Delta t, m\Delta t, \cdot)\sum_{i=n}^{m-1}\sigma_\lambda(n\Delta t, i\Delta t, \cdot)\Delta t + \sigma_\lambda(n\Delta t, m\Delta t, \cdot)\sum_{i=n}^{m-1}\sigma_f(n\Delta t, i\Delta t, \cdot)\Delta t\right] \\
&+\sigma_f(n\Delta t, m\Delta t, \cdot)\Delta\widetilde{W}(n+1) + \sigma_\lambda(n\Delta t, m\Delta t, \cdot)\Delta\widetilde{W}_\lambda(n+1). \qquad (22)
\end{aligned}$$

A numerical scheme is used to calculate the expectation in (13). We consider a generic time t, $0 \leq t \leq \alpha \leq T \leq \overline{T}$, and we simulate the evolution of the function $f(t,\tau)$, $\tau \geq t$, $\tau \in [0, \overline{T}]$, with t varying in $[0,\alpha]$. For every t we obtain therefore an approximation to the curve $f(t,\tau)$, $\tau \geq t$. We simulate $\prod$ evolutions of the curve, and for the $k-th$ simulated curves at time α, $f_k(\alpha,s)$ and $\lambda_k^{\alpha}(\alpha,s)$, we calculate the value

$$e^{\int_{\alpha}^{2\alpha}(f_k(\alpha,s)+\lambda_k^{\alpha}(\alpha,s))ds}, \qquad k = 1, ..., \Pi.$$

If we consider $h = \frac{\alpha}{\Delta t}$, the quantity in (13) is approximated by the Monte Carlo method as

$$E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\left(e^{\int_{\alpha}^{2\alpha}(f(\alpha,s)+\lambda^{\alpha}(\alpha,s))ds}-1\right)\right] \simeq P(0,2\alpha)\left(\frac{\sum_{k=1}^{\Pi} e^{\sum_{i=h}^{2h}(f_k(h\Delta t,i\Delta t)+\lambda_k^{\alpha}(h\Delta t,i\Delta t))\Delta t}}{\Pi}-1\right).$$

For these calculations we simply use the Euler-Maruyama integral approximation

$$\int_{\alpha}^{2\alpha}(f_k(\alpha,s)+\lambda_k^{\alpha}(\alpha,s))ds \simeq \sum_{i=h}^{2h}(f_k(h\Delta t,i\Delta t)+\lambda_k^{\alpha}(h\Delta t,i\Delta t))\Delta t.$$

In this way all the quantities in (21) are estimated and the Basis Swap fair rate can be calculated.

Turning to the volatility functions (19), we assess the parameters using historical data. Our data set consists of daily time series, spanning from January 2007 to the half of May 2010, of the term structures of the Eonia rates and the Libor rates. From Eonia and Libor rates, through formulas (5) and (6), we obtain the term structures of the risk-free forward interest rate f, of the defaultable forward rate f^0 and, consequently, of the credit spread λ^0. We assume that the parameters estimated for counterparty C^0 are valid also for a generic counterparty C^{α}. Finally, we evaluate the volatilities and estimate the parameters of equation (19) by applying the nonlinear least squares regression technique to historical data. The results of parameter estimations for the risk-free interest rate and credit spread volatility functions are summarized in Tables 1 and 2.

Table 1: Results of risk-free rate volatility parameter estimation on Eonia rate time series (January 2007 - May 2010)

Parameter	Estimate	Std. Err.	t Statistic	p-value
α_f	-	-	-	-
β_f	0.479391	0.00127086	31.4349	0.0000
γ_f	0.0182197	0.000190438	95.6726	0.0000
Residual Std. Err. ($\hat{\sigma}$)			$8.94334\mathrm{e}^{-006}$	
R^2			0.991461	

Table 2: Results of the credit spread volatility parameter estimation on the Libor rate time series (January 2007 - May 2010)

Parameter	Estimate	Std. Err.	t Statistic	p-value
α_λ	0.000623129	1.35367e^{-05}	46.0325	0.0000
β_λ	0.434791	0.00270461	13.3966	0.0000
γ_λ	-0.0297221	0.00314946	-9.4372	0.0000
Residual Std. Err. ($\hat{\sigma}$)			8.53952e^{-006}	
R^2			0,963956	

Finally, from historical data the correlation factor ρ between f and λ^α is also estimated. It is equal to 0.67995964, meaning that the two rates are positively correlated.

We estimate the Basis spread of a 6/12 Basis Swap, receiving 6-month Libor plus the basis spread and paying 12-month Libor with 1 year maturity, according to different numbers of discretizations N, and paths Π, and according to formula (21). The results are shown in Table 3. We compare the simulated spreads with the market quotations on 17^{th} May 2010, the first day after the time series period considered for parameter estimation and model simulation. We recall the strong point of the HJM approach is that, given the market volatility and the initial market interest rate term structure, the model is implicitly calibrated on the actual market situation. We calibrate the model on 17^{th} May 2010 market data just because it is the first day after the sampling time period considered for volatility parameters' estimation, and this allows to obtain more accurate results. However, parameters could be updated according to other historical data in order to implement the model and forecast the Basis Swap spread on another day. It is impressive how the numerical results are close to market data, meaning that the model represents price dynamics in an accurate and precise way. We also notice that the accuracy of the approximation improves by increasing both the number of discretizations and the number of simulations.

Furthermore, according to the described framework we use the following formula to find the Basis spread of a 3/6 Basis Swap, receiving 3-month Libor plus the Basis spread and paying 6-month Libor with 1 year maturity:

$$
\begin{aligned}
&P_{Basis}(0, 4\alpha, Z) = \\
E_{\widetilde{\mathbb{P}}}\,[D(0,\alpha)\alpha(L(0,\alpha) + Z) + D(0,2\alpha)\alpha(L(\alpha,2\alpha) + Z) &+ \\
D(0,3\alpha)\alpha(L(2\alpha,3\alpha) + Z) + D(0,4\alpha)\alpha(L(3\alpha,4\alpha) + Z)] &- \\
E_{\widetilde{\mathbb{P}}}[D(0,2\alpha)2\alpha L(0,2\alpha) + D(0,4\alpha)2\alpha L(2\alpha,4\alpha)]&.
\end{aligned}
$$

Table 4 shows the numerical results and we can draw the same conclusion with regards to the simulation of the 6/12 Basis Swap spread. The numerical results demonstrate that the algorithm allows a very accurate estimation of the spread.

Table 3: 6/12 Basis Swap

$Z^{5/17/2010}_{market} = 17\,bps$				
N	Π	Z (bps)	St. Dev.	St. Err.
100	100	16.894800571	*0.309492755*	*0.030949275*
	1,000	16.868914852	*0.340703018*	*0.010773975*
	10,000	16.849677682	*0.356193033*	*0.003561930*
	100,000	16.852258194	*0.355172845*	*0.001123155*
200	100	17.125547933	*0.167207037*	*0.016720704*
	1,000	17.113796002	*0.178884633*	*0.005656829*
	10,000	17.105295211	*0.177064062*	*0.001770641*
	100,000	17.105755463	*0.177663635*	*0.000561822*

Table 4: 3/6 Basis Swap

$Z^{5/17/2010}_{market} = 20.5\,bps$				
N	Π	Z (bps)	St. Dev.	St. Err.
100	100	20.292426630	*0.157188121*	*0.015718810*
	1,000	20.309201701	*0.166496630*	*0.005265086*
	10,000	20.309962484	*0.163634894*	*0.001636349*
	100,000	20.310419858	*0.163712119*	*0.000517703*
200	100	20.654334885	*0.075560523*	*0.007556052*
	1,000	20.643573746	*0.082724743*	*0.002615986*
	10,000	20.644792133	*0.082780364*	*0.000827803*
	100,000	20.643669064	*0.082078129*	*0.000259553*

The main purpose of the developed model is to price a Basis Swap according to the actual market situation. In fact, the strong point of our approach is that we can estimate the Basis Swap spread every day, taking as inputs the actual interest rate and credit spread curves available in the market. This leads to obtain accurate short term estimations, as illustrated in Tables 3 and 4. In a second analysis we aim at testing the model validity and effectiveness in forecasting Basis Swap spreads after a time period longer than few days from the sampling day. Therefore, at first, we define the pricing formulas for a generic forward $\alpha/2\alpha$ Basis Swap at a future time t, with maturity $T = N\alpha$, $N \in \mathbb{N}$, as follows

$$\begin{aligned} P^{forward}_{Basis}(t,T,Z) = \\ \underbrace{\sum_{i=1}^{N} E_{\widetilde{\mathbb{P}}}[D(t,t+i\alpha)\alpha(L(t+(i-1)\alpha,t+\alpha)+Z)]}_{\alpha-tenor\,leg} - \\ \underbrace{\sum_{i=1}^{\frac{N}{2}} E_{\widetilde{\mathbb{P}}}[D(t,t+2i\alpha)2i\alpha L(t+2(i-1)\alpha,t+2i\alpha)]}_{2\alpha-tenor\,leg}. \end{aligned} \tag{23}$$

We implement the forward model according to the numerical scheme described above and we forecast the spread of the 6/12 and 3/6 Basis Swaps with one-year maturity at about one-week and one-month after the sampling date. Therefore, we compare simulated values with the real ones. We apply the Monte Carlo simulations according to 200 discretizations and to different number of paths. Results are displayed in Tables 5 and 6. We can conclude that one-week forecasts are still very accurate and reliable, giving values in line with the realized market quotations, and we find that one-month estimates slightly deviate from real market values.

Table 5: Forward 6/12 Basis Swap

$Z^{5/26/2010}_{market} = 15.5\, bps$				
N	Π	Z (bps)	St. Dev.	St. Err.
200	100	15.258237330	*0.171172626*	*0.017117263*
	1,000	15.251140553	*0.179796634*	*0.005685669*
	10,000	15.242148634	*0.177019432*	*0.001770194*
	100,000	15.241917418	*0.177825956*	*0.000562335*
$Z^{6/21/2010}_{market} = 12.2\, bps$				
N	Π	Z (bps)	St. Dev.	St. Err.
200	100	11.603643114	*0.177257268*	*0.017725726*
	1,000	11.559175154	*0.179625358*	*0.005680252*
	10,000	11.551664955	*0.177424407*	*0.177424407*
	100,000	11.551697515	*0.177838628*	*0.000562375*

Table 6: Forward 3/6 Basis Swap

$Z^{5/26/2010}_{market} = 19.4\, bps$				
N	Π	Z (bps)	St. Dev.	St. Err.
200	100	19.754351200	*0.079488409*	*0.007948841*
	1,000	19.741937433	*0.083857567*	*0.002651809*
	10,000	19.743076714	*0.083416362*	*0.000834164*
	100,000	19.741744418	*0.082574125*	*0.000261122*
$Z^{6/21/2010}_{market} = 19.3\, bps$				
N	Π	Z (bps)	St. Dev.	St. Err.
200	100	18.018747737	*0.079117839*	*0.007911784*
	1,000	18.012620280	*0.083898410*	*0.002653101*
	10,000	18.015435455	*0.084460746*	*0.000844607*
	100,000	18.014042691	*0.083645166*	*0.000264509*

Finally, the robustness of model pricing to changes in parameter values of the credit spread volatility is assessed with the sensitivity analysis. We calculate 6/12 and 3/6 Basis Swap spreads with one year maturity by using 10,000 simulations, and according to different values of credit spread parameters, that is low, medium and high. In particular, we refer to parameters estimated on historical data as medium values, $\alpha_\lambda = 0.000623129$, $\beta_\lambda = 0.434791$ and $\gamma_\lambda = -0.0297221$, and we assume other two values for each parameter, one lower than the estimated value, $\alpha_\lambda = -0.01$, $\beta_\lambda = -0.1$ and $\gamma_\lambda = -1$, and one higher, $\alpha_\lambda = 0.01$, $\beta_\lambda = 1$ and $\gamma_\lambda = 0.5$. Sensitivity analysis results are shown in Tables 7 and 8. We notice that if we perturb only one parameter, keeping the other two parameters assuming the medium values estimated on historical data, the Basis Swap spread remains almost stables. Furthermore, even if we change simultaneously two parameters, keeping the medium value only for one parameter, the model solution is slightly perturbed. The most significant impact on the model's main outcome is obtained when β_λ assumes the high value, but, overall, we can realize that the model gives robust results, which are not heavily influenced by credit spread volatility parameter value changes.

Table 7: 6/12 Basis Swap Spread sensitivity analysis with respect to credit spread volatility parameters

$Z_{market}^{5/17/2010} = 17\,bps$				
$\alpha_\lambda = -0.01$	$\beta_\lambda \setminus \gamma_\lambda$	-1	-0.0297221	0.5
	-0.1	*17.11228147*	*17.12002292*	*17.14146658*
	0.434791	*17.12102039*	*17.13907382*	*17.18942263*
	1	*16.20542772*	*16.03803382*	*15.06418298*
$\alpha_\lambda = 0.000623129$	$\beta_\lambda \setminus \gamma_\lambda$	-1	-0.0297221	0.5
	-0.1	*17.11167075*	*17.10542135*	*17.10597444*
	0.434791	*17.11970907*	*17.105295211*	*17.10685663*
	1	*17.14087112*	*17.10505065*	*17.11053497*
$\alpha_\lambda = 0.01$	$\beta_\lambda \setminus \gamma_\lambda$	-1	-0.0297221	0.5
	-0.1	*17.12079319*	*17.11291859*	*17.10575736*
	0.434791	*17.14230431*	*17.123428648*	*17.10612874*
	1	*15.69248019*	*15.53699573*	*16.83432697*

Table 8: 3/6 Basis Swap Spread sensitivity analysis with respect to credit spread volatility parameters

$Z_{market}^{5/17/2010} = 20.5\,bps$				
$\alpha_\lambda = -0.01$	$\beta_\lambda \setminus \gamma_\lambda$	-1	-0.0297221	0.5
	-0.1	*20.6320238*	*20.64337607*	*20.64650063*
	0.434791	*20.53028616*	*20.60616562*	*20.63058167*
	1	*20.94537174*	*21.354200120*	*20.78175089*
$\alpha_\lambda = 0.000623129$	$\beta_\lambda \setminus \gamma_\lambda$	-1	-0.0297221	0.5
	-0.1	*20.64014503*	*20.64470549*	*20.645688202*
	0.434791	*20.63564112*	*20.644792133*	*20.63053726*
	1	*20.613071501*	*20.644891201*	*20.55905215*
$\alpha_\lambda = 0.01$	$\beta_\lambda \setminus \gamma_\lambda$	-1	-0.0297221	0.5
	-0.1	*20.64186334*	*20.62946762*	*20.61902413*
	0.434791	*20.63976408*	*20.59923029*	*20.56045110*
	1	*20.817230701*	*21.463667780*	*21.346892070*

6 Conclusions

In this work we have developed a model, consistent with the new interest rate market situation after the credit crunch, for pricing a particular collaterized derivative that has the Libor rate as underlying: the Basis Swap. The Libor rate at a generic time t is classified as a risky rate subject to the credit risk of a generic counterparty lending at time t. From the Libor market formula we have extracted the instantaneous defaultable forward rates. The evolution of the defaultable forward curve has been modelled extending the HJM model to consider a class of credit models, each corresponding to a risky term structure of a counterparty available at a time t. A particular dynamic for the credit spread of the counterparty has been derived within the no-arbitrage conditions. So we have developed an algorithm to simulate the evolution of the risky forward interest rate curve using a stochastic, path dependent volatility. Finally, we have estimated the fair rates of Basis Swaps (3/6 and 6/12 Basis Swaps) by using Monte Carlo simulations to obtain the expected value of the floating legs. Simulation numerical results have shown that the model provides accurate and robust estimates of the Basis Swap spreads, in line with the real market values.

Future research will focus on studying the consequences of the developed model with respect to other existing models in order to confirm its effectiveness, in particular, we will investigate how the introduction of jumps in the credit spread volatility function impacts Basis Swap pricing.

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