SPECTRAL ζ-INVARIANTS LIFTED TO COVERINGS

SARA AZZALI AND SYLVIE PAYCHA

ABSTRACT. The canonical trace and the Wodzicki residue on classical pseudodifferential operators on a closed manifold are characterised by their locality and shown to be preserved under lifting to the universal covering as a result of their local feature. As a consequence, we lift a class of spectral ζ-invariants using lifted defect formulae which express discrepancies of ζ-regularised traces in terms of Wodzicki residues. We derive Atiyah's L^2-index theorem as an instance of the $\mathbb{Z}_2$-graded generalisation of the canonical lift of spectral ζ-invariants and we show that certain lifted spectral ζ-invariants for geometric operators are integrals of Pontryagin and Chern forms.

INTRODUCTION

A differential operator A on a closed manifold M lifts to a differential operator $\widetilde{A}$ on its universal covering $\widetilde{M} \to M$. This is due to the locality property of differential operators which preserve the support of the sections they act on. This lifting property does not extend to general pseudodifferential operators which are only pseudo-local i.e., they only preserves the singular support of the sections. Hence arises the problem of lifting complex powers Q^{-z} involved in spectral ζ-functions

$$\zeta_{A,Q}(z) := \mathrm{TR}(A\, Q^{-z})\,. \tag{1}$$

where A and Q are differential operator on M, and TR is the canonical trace. Nevertheless, we prove that spectral ζ-invariants

$$\zeta_{A,Q}(0) := \mathrm{fp}_{z=0}\left(\mathrm{TR}(A\, Q^{-z})\right) \tag{2}$$

corresponding to the constant term of the Laurent expansion of (1) canonically lift to coverings (see (76) in Theorem 2.24). This results from a detailed analysis of the intertwining between the pseudo-locality of the complex powers and the locality of the canonical trace on non-integer order operators. In our approach, the locality of spectral ζ-invariants is only an instance of the more general locality expressed by defect formulae. Another central result of the paper are L^2-counterparts of such defect formulae (10). A natural application is the locality of Atiyah's L^2-index, which is expressed as a Γ-Wodzicki residue (79). Our approach can be summarised as follows. We build:

- a holomorphic germ of pseudifferential (and hence pseudo-local) operators $A(z)$ (of holomorphic order $\alpha(z)$) which at zero is the differential (and hence *local*) operator A;

- the corresponding meromorphic germ $\mathrm{TR}(A(z))$ of functions built from the *local linear form* given by the canonical trace;
- the *local* invariant is obtained as the value at $z = 0$ of this germ of functions in terms of the 1-jet of the germ of operators

$$\lim_{z \to 0} \mathrm{TR}(A(z)) = -\frac{1}{\alpha'(0)} \mathrm{Res}(A'(0)).$$

To achieve our goal, following Shubin [Sh01], we view pseudodifferential operators as "small perturbations" of pseudodifferential operator with finite propagation, more precisely those which are ε-local for some small enough $\varepsilon > 0$. Such operators, which fall into the more general class of quasi-local operators introduced by Roe [Ro88], are properly supported and hence determined by their symbol. Quasi-local operators can roughly be viewed as operators with controlled propagation at infinity, see [En15] for a detailed discussion. They bare the advantage over finite propagation operators, that they are stable under functional calculus, a property which is not needed here. An ε-local pseudodifferential operator modifies the support of the sections it is acting on, by at most the distance ε. The fact that a differential operator preserves the supports is therefore confirmed by the fact that it is 0-local.

Choosing ε small enough, one can lift without ambiguity an ε-local operator A_0 to an ε-local operator $\widetilde{A_0}$. The lifted operator is a (uniformly) properly supported pseudodifferential operator, and hence also defined in terms of its symbol $\sigma(\widetilde{A_0})$, which is the lifted symbol $\sigma(\widetilde{A_0}) = \widetilde{\sigma(A_0)}$ of the original operator.

To go from ε-local to a general classical pseudodifferential operator A on a closed manifold, we observe that the latter differs from a ε-local classical pseudodifferential operator A_0 by an operator with smooth kernel supported outside the diagonal (see Proposition 1.27), so it lies in the equivalence class $[A_0]_{\mathrm{diag}}$ of A_0 for the equivalence relation on classical pseudodifferential operators on the base manifold M

$$A \underset{\mathrm{diag}}{\sim} B \Longrightarrow A - B \quad \text{has a smooth kernel supported outside the diagonal.}$$

Clearly, $\sigma(A) \sim \sigma(A_0)$ and $\widetilde{\sigma(A)} \sim \sigma(\widetilde{A_0})$ for any A in $[A_0]_{\mathrm{diag}}$. The following diagramme, where $\mathcal{A}$, $\mathcal{A}_0$ are Γ-invariant operators on the covering, Γ being the fundamental group, $\pi_\sharp \mathcal{A}_0$ the projected operator onto the base manifold, presents the notations in a compact form:

$$
\begin{array}{cc}
\mathcal{A} \underset{\mathrm{diag}}{\sim} \mathcal{A}_0 := \widetilde{A_0} & (\varepsilon - \mathrm{local}) \\
\pi^\sharp \Big\uparrow \Big\downarrow \pi_\sharp & \\
A \underset{\mathrm{diag}}{\sim} A_0 := \pi_\sharp \mathcal{A}_0 & (\varepsilon - \mathrm{local})
\end{array}
\tag{3}
$$

Note that the operator $\pi_\sharp \colon \mathcal{A}_0 \longrightarrow (s \longmapsto \pi_* (\mathcal{A}_0 \pi^*(s)))$ is well-defined in view of the equivariance and ε-locality of $\mathcal{A}_0$ (see Proposition 1.27)).

Alongside the pseudo-locality of pseudodifferential operators, the other essential ingredient in our approach is the use of **local** linear forms defined on a class of classical pseudodifferential operators (see Definition 2.5). These only detect the symbol of an operator and are therefore constant on equivalence classes $[A_0]_{\mathrm{diag}}$ and hence constant along

the horizontal lines of the above diagramme. More precisely, a local linear form Λ reads:

$$\Lambda([A]_{\text{diag}}) := \Lambda(A) = \int_M \Lambda_x(A)\, dx := \int_M \lambda\left(\text{tr}\left(\sigma(A)(x, \cdot)\right)\right) dx, \tag{4}$$

λ being a linear form on an appropriate class of scalar valued symbols, tr the fibrewise trace on the endomorphism bundle in which the symbol $\sigma(A)(\cdot, \xi)$ of A lies for any ξ in the cotangent bundle to the underlying manifold.

- Our first main result is Theorem 2.10, which states that any continuous local linear form on the class of classical pseudodifferential operators with integer order (resp. on the class of classical pseudodifferential operators non-integer order) is proportional to the Wodzicki residue Res (see (51)) (resp. the canonical trace TR (see (52)))

$$\text{Res}(A) = \int_M \text{Res}_x(A)\, dx; \quad \left(\text{resp.} \quad \text{TR}(A) = \int_M \text{TR}_x(A)\, dx\right),$$

acting respectively on the algebra of classical pseudodifferential operators with integer order and on the class of classical pseudodifferential operators with non-integer order. The densities $\text{Res}_x(A)\, dx$ (resp. $\text{TR}_x(A)\, dx$) are defined by means of the residue res (resp. the canonical integral $\fint_{\mathbb{R}^n}$) on integer order (resp. on non-integer order) scalar symbols.

A local linear form (4) can be lifted from a ε-local operator A_0 on M to its lift $\widetilde{A_0}$ on $\widetilde{M}$ by

$$\Lambda_\Gamma(\widetilde{A_0}) = \int_F \lambda\left(\text{tr}\left(\sigma(\widetilde{A_0})(x, \cdot)\right)\right) dx, \tag{5}$$

where F is a fundamental domain for the action of the fundamental group. It further lifts to any Γ-invariant operator $\mathcal{A}$ on the covering; indeed $\mathcal{A}$ lies in the class $[\widetilde{A_0}]_{\text{diag}}$ of some lifted ε-local operator $\widetilde{A_0}$. Since Λ_Γ is constant on such a class, we set

$$\Lambda_\Gamma(\mathcal{A}) := \int_F \lambda\left(\text{tr}\left(\sigma(\widetilde{A_0})(x, \cdot)\right)\right) dx. \tag{6}$$

Prototypes are the Γ-residue (resp. Γ-canonical trace) (see Proposition 2.18)

$$\text{Res}_\Gamma(\mathcal{A}) := \int_F \text{Res}_{\widetilde{x}}(\widetilde{A_0})\, d\widetilde{x}, \quad \left(\text{resp.}\,\text{TR}_\Gamma(\mathcal{A}) := \int_F \text{TR}_{\widetilde{x}}(\widetilde{A_0})\, d\widetilde{x}\right),$$

obtained from integrating the residue and canonical trace densities on F.

Whereas the canonical trace lifts to coverings due to its local feature, the **regularised trace** evaluated at $z = p$ of a holomorphic family $A(z)$ of classical pseudodifferential operators on M, defined as the Hadamard finite part

$$\text{fp}_{z=p}\text{TR}(A(z)) := \lim_{z \to p}\left(\text{TR}(A(z)) - \frac{\text{Res}_{z=p}\left(\text{TR}(A(z))\right)}{z - p}\right),$$

(here $\text{Res}_{z=p}$ stands for the complex residue at p) is generally non local and does not a priori lift to coverings. However, **defect formulae**, which express the discrepancies of regularised traces in terms of the Wodzicki residue and therefore also enjoy a local feature, do lift to coverings (Theorem 2.21). More precisely, if $A(p)$ has a well-defined canonical trace $\text{TR}(A(p))$, the trace defect formula (Theorem 2.12, borrowed from [KV94] and [PS07]), relates the regularised trace

$\mathrm{fp}_{z=p}\mathrm{TR}(A(z))$ with the (extended) residue $\mathrm{Res}\,(\log A'(p))$ of the derivative of the family at this pole (see (57)),

$$\mathrm{fp}_{z=p}\mathrm{TR}(A(z)) = \mathrm{TR}(A(p)) + \frac{1}{q}\,\mathrm{Res}\,(A'(p))\,, \tag{7}$$

where the operators $A(z)$ have order $a - qz$ for some given positive q.

If $A(p)$ is a differential operator, then $\mathrm{TR}(A(p)) = 0$ and (7) reduces to a local expression of the regularised trace

$$\mathrm{fp}_{z=p}\mathrm{TR}(A(z)) = \frac{1}{q}\,\mathrm{Res}\,(\log A'(p)) \tag{8}$$

in terms of the Wodzicki residue.

The trace defect formula (8) is central to our approach since it relates regularised traces (on the l.h.s. of the above formula) with Wodzicki residues (on the r.h.s. of the above formula) and yields index type theorems as an application. Wodzicki residues, which only depend on one homogeneous component of the symbol and not the whole symbol, ar local. In contrast, regularised traces built from the canonical trace, apriori depend on the whole symbol so are not expected to be local.

- Our second main result is Theorem 2.21 which yields the lifted analogue of the (more general) trace-defect formula (7)

$$\mathrm{fp}_{z=p}\mathrm{TR}_\Gamma(\mathcal{A}(z)) = \mathrm{TR}_\Gamma(\mathcal{A}(p)) + \frac{1}{q}\mathrm{Res}_\Gamma\,(\mathcal{A}'(p))\,, \tag{9}$$

for a holomorphic family $\mathcal{A}(z)$ of Γ-invariant operators on the covering, such that $\mathcal{A}(p)$ at the point p has a well-defined Γ-canonical trace $\mathrm{TR}_\Gamma(\mathcal{A}(p))$. If $\mathcal{A}(p)$ is a differential operator, then (9) reduces to

$$\mathrm{fp}_{z=p}\mathrm{TR}_\Gamma(\mathcal{A}(z)) = \frac{1}{q}\mathrm{Res}_\Gamma\,(\mathcal{A}'(p))\,. \tag{10}$$

Corollary 2.22, which is useful for applications, then says that if $\mathcal{A}(z)$ is a holomorphic family on the covering and if there exists a holomorphic family $A(z)$ of ε-local operators on M, such that the difference $\mathcal{A}(p) - \widetilde{A(p)}$ at a point p has a smooth kernel, then the map $z \mapsto \mathrm{TR}_\Gamma\,(\mathcal{A}(z)) - \mathrm{TR}_\Gamma\left(\widetilde{A(z)}\right)$ is holomorphic at point p and

$$\mathrm{fp}_{z=p}\mathrm{TR}_\Gamma\,(\mathcal{A}(z)) - \mathrm{fp}_{z=p}\mathrm{TR}\,(A(z)) = \mathrm{Tr}_\Gamma(\mathcal{A}(p) - \widetilde{A(p)}). \tag{11}$$

We apply (11) to the holomorphic families $A(z) = P(\mathbf{D})\,h(\Delta)\,Q_\varepsilon(\Delta)^{-z}$ on M and $\mathcal{A}(z) = P(\widetilde{\mathbf{D}})\,h\left(Q_\varepsilon(\widetilde{\Delta})\right)\,Q_\varepsilon(\widetilde{\Delta})^{-z}$ on $\widetilde{M}$. Here $\mathbf{D}$ is a Dirac-type operator, and consequently $\Delta := \mathbf{D}^2$ a Laplace-type operator, $Q_\varepsilon(\Delta)$ defined in (35) for some $\varepsilon > 0$ is a smooth deformation of Δ, P is a polynomial, and h some measurable function on a contour around the spectrum of $Q_\varepsilon(\Delta)$.

- This yields our third main result, Theorem 2.24, which compares the corresponding $Q_\varepsilon(\Delta)$-regularised trace of $P(\mathbf{D})\,h\,(\Delta)$ and the $Q_\varepsilon(\widetilde{\Delta})$-regularised trace of $P(\widetilde{\mathbf{D}})\,h\left(Q_\varepsilon(\widetilde{\Delta})\right)$.
    - In the $\mathbb{Z}_2$-graded case and for $P \equiv 1$, $h \equiv 1$ this gives back Atiyah's L^2-index theorem (Corollary 2.25).

– In the non-graded case and for $P(x) = x$, $h(x) = x^{-\frac{1}{2}}$, assuming both operators D and its lift $\widetilde{D}$ to be invertible, the above constructions show that the eta-invariant of the lifted Dirac operator differs from the eta-invariant of the Dirac operator D on the base manifold by an ordinary Γ-trace $\text{Tr}_\Gamma(\widetilde{A}_0 - \mathscr{A})$ of the difference of two Γ-invariant operators, one of which $\widetilde{A}_0$, is the lift of an ε-local operator $A_0 \in [A]_{\text{diag}}$ (Corollary 2.26).

Theorem 2.29 discusses the case of geometric operators showing that the lifted ζ-functions correspond to integrals of densities generated by Pontrjagin forms on the fundamental domain and Chern forms on the auxillary bundle.

One advantage of our approach is that it yields the L^2-Atiyah theorem as an instance of the much more general *lifted trace defect formulae*. Here is the general scheme of the argument. Theorem 2.21 gives the lifted trace defect formulae. Theorem 2.24 compares ζ-regularised traces of operators with the ζ-regularised traces of their lifted counterparts using the locality property of the only two *local linear forms* characterised in Theorem 2.10– the canonical trace and the Wodzicki residue. Corollary 2.25 gives the L^2-index theorem combining the two previous ingredients.

The above arguments make use of functions of pseudodifferential operators, in particular their complex power and the related logarithm. Even though the constructions are similar to the ones for operators on closed manifolds, special care is to be taken in the open manifold case. In Section 1.2, we first review various classes of pseudodifferential operators on open manifolds, soon specialising to coverings. The essential difference between the various classes lies in the smoothing part, to which we therefore dedicate the first section –Section 1.1– of the paper. We then relate pseudodifferential operators on coverings to pseudodifferential operators on the associated groupoid (Appendix C) and the associated Hilbert module bundle (Appendix B), thereby relating constructions of complex powers and logarithms on groupoids and Hilbert module bundles with the ones on coverings presented here.

CONTENTS

1. Γ-INVARIANT OPERATORS ON COVERINGS AND FUNCTIONAL CALCULUS

1.1. **Operators with smooth kernels.** Let X be an n-dimensional manifold and $F \to X$ a vector bundle over X of rank k. To a linear operator $A \colon C_c^\infty(X, F) \to C^\infty(X, F)$ we assign its Schwartz kernel denoted by K_A, which is a distributional section of the bundle $F \boxtimes F$ over $M \times M$. The support of A is the smallest subset of $X \times X$ on the complement of which K_A vanishes as a distribution.

Definition 1.1. Let $\Psi^{-\infty}(X, F)$ be the space of linear operators $A \colon C_c^\infty(X, F) \to C^\infty(X, F)$ with smooth Schwartz kernel.

Sobolev spaces will be useful to introduce another class of operators; in order to have Sobolev spaces at hand, we henceforth assume that our manifold X has *bounded geometry* [Sh92, MS89, Ko87], a property verified by covering spaces of interest to us.

1.1.1. *Smoothing operators on manifolds of bounded geometry.*

Definition 1.2. A Riemannian manifold (X, g) is said to have *bounded geometry* if
- it has positive injectivity radius (there is $r > 0$ s.t. the exponential map is a diffeomorphism on $B(0, r) \subset T_x X, \forall x \in X$);
- every covariant derivative of the Riemannian curvature tensor is bounded.

In the same way, a Hermitian vector bundle $F \to X$ has bounded geometry if every covariant derivative of the curvature is bounded.

Example 1.3. Lie groups or homogeneous spaces with invariant metrics, compact Riemannian manifolds, regular Γ-covering of compact Riemannian manifolds endowed with the induced Riemannian structure, all provide examples of manifolds of bounded geometry.

We now assume the bundle $F \to X$ to be of bounded geometry. The fundamental property is the existence of a "good" partition of unity, which allows to define Sobolev spaces $H^s(X, F)$, see for instance [Sh92, Lemmas 1.3, 3.22, and (1.3)] and [Sc96, Definition 3.23].

Remark 1.4. The Banach space structure of $H^s(X, F)$ is independent of the choices in the definition (see [Sh92, Lemma 3.24]). Just as in the case of closed manifolds, Sobolev spaces can alternatively be defined by means of a (uniformly) elliptic operator, see [Sc96, Lemma 4.29 and Corollary 4.30] for the comparison with this definition.

Let $H^\infty(X, F) = \cap_{k \in \mathbb{N}} H^k(X, F)$ denote the projective limit of $H^k(X, F), k \in \mathbb{N}$ and let $H_\iota^{-\infty}(X, F) \supset \cup_{k \in \mathbb{N}} H^{-k}(X, F)$ denote the regular inductive limit of $H^{-k}(X, F), k \in \mathbb{N}$. Then define (see e.g. [Ro88, Definition 5.3] and [En15, Lemma 2.13]

$$
\begin{aligned}
\mathcal{S}\Psi^{-\infty}(X, F) \quad &:= \quad \mathcal{L}\left(H_\iota^{-\infty}(X, F), H^\infty(X, F)\right) \quad &(12) \\
&= \quad \cap_{(k,l) \in \mathbb{N}^2} \mathcal{L}\left(H^{-k}(X, F), H^l(X, F)\right) \\
&= \quad \cap_{(s,t) \in \mathbb{R}^2} \mathcal{L}\left(H^s(X, F), H^t(X, F)\right),
\end{aligned}
$$

where $\mathcal{L}(A, B)$ stands for continuous linear operators from a topological space A to a topological space B. The notation we chose is inspired by Shubin, who calls these operators

$\mathcal{S}$-smoothing [Sh, Def. 1, Ch. 3]. By [Va08, Theorem 3.5], an operator which smoothens sections has a smooth kernel, which leads to the following inclusion

$$\mathcal{S}\Psi^{-\infty}(X, F) \subseteq \Psi^{-\infty}(X, F). \tag{13}$$

Following [Sh89] and [Ro88], we set the following definition.

Definition 1.5. Let $\mathcal{U}\Psi^{-\infty}(X, F)$ denote the space of linear operators $A : C_c^\infty(X, F) \rightarrow C^\infty(X, F)$ with smooth kernel K_A satisfying the following uniform boundedness condition: for any multiindices α, β

$$\|\partial_x^\alpha \partial_y^\beta K_A(x, y)\| \le C_{\alpha, \beta} \quad \forall (x, y) \in X \times X$$

for some positive constant $C_{\alpha, \beta}$.

Proposition 2.9 in [Ro88] yields a refinement of (13), namely

$$\mathcal{S}\Psi^{-\infty}(X, F) \subseteq \mathcal{U}\Psi^{-\infty}(X, F). \tag{14}$$

The uniformity follows from uniform estimates that naturally arise in the context of bounded geometry as they do for closed manifolds.

Remark 1.6. If X is a closed manifold, the three above spaces coincide:

$$\mathcal{S}\Psi^{-\infty}(X, F) = \mathcal{U}\Psi^{-\infty}(X, F) = \Psi^{-\infty}(X, F),$$

since the equality in (13) holds. Indeed any linear operator $A \colon C_c^\infty(X, F) \rightarrow C^\infty(X, F)$ with smooth Schwarz kernel K_A is smoothing when X is closed as can be seen on direct inspection from the formula $(Au)(x) = \int_X K_A(x, y)u(y)dy$.

The notion of properly supported operators recalled in Definition A.5, extends in a straightforward manner to linear operators $A : C_c^\infty(X, F) \rightarrow C^\infty(X, F)$. The following definition is inspired by [Sh92], and follows the terminology of [En15].

Definition 1.7. An operator $A \colon C_c^\infty(X, F) \rightarrow C^\infty(X, F)$ with Schwartz kernel K_A has **finite propagation** if it is C-local (see Definition A.10) for some positive C, i.e. if there is some $C > 0$ such that $K_A(x, y) = 0\ \forall x, y$ with $|x - y| > C$ or equivalently, if

$$\forall u \in C_c^\infty(U, \mathbb{C}^k); \quad \operatorname{supp}(Au) \subset \{x : d(x, \operatorname{supp} u) \le C\}.$$

Note that requiring finite propagation is more constraining than the assumption of quasi-locality of [Ro88].

Let $\mathcal{U}\Psi_{\mathrm{fp}}^{-\infty}(X, F)$ denote the subspace of $\mathcal{U}\Psi^{-\infty}(X, F)$ consisting of finite propagation operators with uniformly bounded smooth kernels. We have [Sh89]

$$\mathcal{U}\Psi_{\mathrm{fp}}^{-\infty}(X, F) \subsetneq \mathcal{S}\Psi^{-\infty}(X, F) \subseteq \mathcal{U}\Psi^{-\infty}(X, F) \subseteq \Psi^{-\infty}(X, F). \tag{15}$$

Remark 1.8. The class $\mathcal{U}\Psi_{\mathrm{fp}}^{-\infty}(X, F)$ is strictly contained in $\mathcal{S}\Psi^{-\infty}(X, F)$ for the heat operator e^{-tD^2} on a bounded geometry manifold, belongs to the class $\mathcal{S}\Psi^{-\infty}(X, F)$ (see for example [Va08, §3.2]) but it does not have finite propagation.

Remark 1.9. Whereas $\mathcal{S}\Psi^{-\infty}(X, F)$ is an algebra, the class $\Psi^{-\infty}(X, F)$ is not. Indeed, the composition of two operators with smooth kernels is defined only under appropriate decay conditions at infinity and when this is the case, the Schwartz kernel of the composition might not be smooth.

1.1.2. *Coverings and classes of Γ-invariant operators with smooth kernel.* Let us now specialise to covering manifolds. Let M be a (connected) closed manifold and $\widetilde{M}$ a regular covering given by a Γ-principal bundle $\pi : \widetilde{M} \to M$ with $\Gamma = \mathrm{Aut}(p)$ the discrete Lie group of deck transformations (smooth diffeomorphisms $\phi : \widetilde{M} \to \widetilde{M}$ such that $\pi \circ f = \pi$). Let $\widetilde{M}$ be the universal cover so that $\Gamma = \pi_1(M)$ is the fundamental group of M.

Example 1.10. $\mathbb{R}^n$ is a universal cover of $\mathbb{T}^n$ with group $\Gamma = \pi_1(\mathbb{T}^n) = \mathbb{Z}^n$.

If M is a Riemannian manifold, we endow the covering $\widetilde{M}$ with the Riemannian structure induced by π. Let $E \to M$ be a Hermitian vector bundle and $\widetilde{E} := \pi^* E \to \widetilde{M}$ its pullback. Both $\widetilde{M}$ and $\widetilde{E}$ are of bounded geometry [Sh92].

The action of Γ on $\widetilde{M}$ via diffeomorphisms L_γ

$$\Gamma \times \widetilde{M} \longrightarrow \widetilde{M}$$
$$(\gamma, x) \longmapsto L_\gamma(x)$$

induces an action on linear operators $A : C_c^\infty(\widetilde{M}, \widetilde{E}) \to C^\infty(\widetilde{M}, \widetilde{E})$:

$$L_\gamma^\sharp A := L_\gamma \circ A \circ L_\gamma^{-1}.$$

The operator A is said to be Γ**-invariant** whenever

$$L_\gamma^\sharp A = A \quad \forall \gamma \in \Gamma. \tag{16}$$

The action $L_\gamma^\sharp$ stabilises $\Psi^{-\infty}(\widetilde{M}, \widetilde{E})$. Imposing a Γ-invariance condition leads to the following subclass of operators.

Definition 1.11. Let $\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E})$, resp. $\mathcal{U}\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E})$, resp. $\mathcal{U}\Psi_{\mathrm{fp},\Gamma}^{-\infty}(\widetilde{M}, \widetilde{E})$, resp. $\mathcal{S}\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E})$ denote the space of Γ-invariant operators in $\Psi^{-\infty}(\tilde{M}, \tilde{E})$, resp. $\mathcal{U}\Psi^{-\infty}(\widetilde{M}, \widetilde{E})$, $\mathcal{U}\Psi_{\mathrm{fp}}^{-\infty}(\widetilde{M}, \widetilde{E})$, $\mathcal{S}\Psi^{-\infty}(X, F)$.

The following inclusions follow from (15)

$$\mathcal{U}\Psi_{\mathrm{fp},\Gamma}^{-\infty}(\widetilde{M}, \widetilde{E}) \subsetneq \mathcal{S}\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E}) \subseteq \mathcal{U}\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E}) \subseteq \Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E}). \tag{17}$$

1.2. **Classes of pseudodifferential operators.** In this section we discuss different classes of pseudodifferential operators on an open manifold of bounded geometry (we also consider very general classes, which possibly do not form algebras). This short review which brings together and compares different approaches, follows [Sh92, MS89, Ko87, Ro88, En15].

1.2.1. *Classical pseudodifferential operators on manifolds with bounded geometry.* Let X be an n-dimensional manifold and $F \to X$ a vector bundle over X of rank k. We assume X and F are both of bounded geometry.

Definition-Lemma 1.12. The following relations

- $A \sim B \Longleftrightarrow A - B$ has a smooth kernel
- $A \underset{\mathrm{diag}}{\sim} B \Longleftrightarrow A - B$ has a smooth kernel supported outside the diagonal

define equivalence relations on the space $\mathcal{L}(C_c^\infty(X, F), C^\infty(X, F))$ of linear operators acting on the space $C_c^\infty(X, F)$ of compactly supported sections of F with values in the space $C^\infty(X, F)$ of smooth sections of F. We write $[A]$ (resp. $[A]_{\mathrm{diag}}$) for the equivalence class of A with respect to $\sim$ (resp. $\underset{\mathrm{diag}}{\sim}$).

Remark 1.13.   • Clearly, we have $A \underset{\text{diag}}{\sim} B \implies A \sim B$.

• Whereas the equivalence relation $\sim$ is stable under composition of operators (whenever composable), the equivalence relation $\underset{\text{diag}}{\sim}$ is not. Indeed, if $A - A_1 =: R$ and $B - B_1 =: S$ have smooth kernels supported outside the diagonal, then $AB = A_1 S + R B_1 + RS$ has a smooth kernel but it might not be supported outside the diagonal since the supports of A_1 and B_1 intersect the diagonal.

Example 1.14. Given a linear operator $A : C_c^\infty(X, F) \to C^\infty(X, F)$, any localisation $\chi_1 A \chi_2$ induced by two smooth functions χ_1, χ_2 whose compact supports have a non void intersection in a trivialising set, is properly supported.

We shall make use of the the existence of a "good" partition of unity, [Sh92, Lemmas 1.3, 3.22] and [Sh92, A1.1] built as follows. For small enough ρ (smaller than a third of the injectivity radius), there is a countable covering of X by balls $B(x_i, \rho)$ centered at $x_i \in X$ with radius ρ such that $d(x_i, x_j) \geq \rho$ for $i \neq j$ and any point $x \in X$ lies in at most C_x such balls for some constant C_x. Moreover, there is a partition of unity $1 = \sum_i \chi_i$ with smooth functions χ_i whose supports $\operatorname{supp} \chi_i$ lie in $B(x_i, 2\rho)$ and which together with their derivatives taken in normal coordinates, are bounded independently of i.

Lemma 1.15. *Given a linear operator $A : C_c^\infty(X, F) \to C^\infty(X, F)$ there is a properly supported operator $A_0 : C_c^\infty(X, F) \to C_c^\infty(X, F)$ of finite propagation such that*

$$A \underset{\text{diag}}{\sim} A_0. \tag{18}$$

For any $\varepsilon > 0$, the operator A_0 can be chosen ε-local.

Proof. Given a "good" finite open cover $\mathcal{U} = (U_i)_{i \in I}$ of X and a "good" partition of unity $(\chi_i)_{i \in I}$ subordinated to $\mathcal{U}$, we write the operator A as

$$A = \sum_{i,j} \chi_i A \chi_j = \underbrace{\sum_{\operatorname{supp}\chi_i \cap \operatorname{supp}\chi_j \neq \emptyset} \chi_i A \chi_j}_{=: \sum_{\{i,j\} \in \mathcal{P}} A_{ij}} + \sum_{\operatorname{supp}\chi_i \cap \operatorname{supp}\chi_j = \emptyset} \chi_i A \chi_j \tag{19}$$

where $\mathcal{P}$ is the set of pairs $\{i, j\}$ satisfying $\operatorname{supp}\chi_i \cap \operatorname{supp}\chi_j \neq \emptyset$, and $A_{ij} := \chi_i A \chi_j$. Then, $A_0 := \sum_{\{i,j\} \in \mathcal{P}} A_{ij}$ is a properly supported operator of finite propagation since each $\chi_i A \chi_j$ is supported in balls with uniformly bounded radii. Moreover, by construction $S(A) := \sum_{\{i,j\} \in \mathsf{C}\mathcal{P}} \chi_i A \chi_j$ has Schwartz kernel supported outside the diagonal.
For $\varepsilon > 0$, we can choose the diameter of the partition such that $\forall i \in I$, $\operatorname{diam} U_i < \frac{\varepsilon}{2}$, in which case A_0 is an ε-local operator. $\qquad\square$

Definition 1.16. Given any real (resp. complex) number m, a linear operator $A : C_c^\infty(X, F) \to C^\infty(X, F)$ is a (resp. classical) **pseudodifferential operator** of order m if there is a properly supported operator $A_0 : C_c^\infty(X, F) \to C_c^\infty(X, F)$ of finite propagation — for any $\varepsilon > 0$, the operator A_0 can be chosen ε-local— with $A \underset{\text{diag}}{\sim} A_0$ as in (18) and such that

- the operator $S(A) := A - A_0$ lies in $\Psi^{-\infty}(X, F)$,
- the operator A_0 is a sum $A_0 = \sum_\alpha \operatorname{Op}(\sigma_\alpha)$ (When applying the operator to a compactly supported section, the sum becomes finite, due to the local finiteness of a "good" open cover.) of (classical) properly supported pseudodifferential operators $\operatorname{Op}(\sigma_\alpha)$ of order m supported on "good" open subsets of X and identified via the trivialising charts with pseudodifferential operators on open subsets of $\mathbb{R}^n$. The

symbol σ_α is interpreted as the symbol $\sigma(A)$ of A seen in the trivialising chart indexed by α.

By abuse of notation we shall set

$$\mathrm{Op}(\sigma(A)) := A_0 = \sum_\alpha \mathrm{Op}(\sigma_\alpha), \tag{20}$$

so that $A \underset{\mathrm{diag}}{\sim} \mathrm{Op}(\sigma(A))$. Let $\Psi^m(X, F)$ (resp. $\Psi^m_{\mathrm{cl}}(X, F)$) denote the class of such operators.

Remark 1.17. • Neither the class $\Psi(X, F) := \cup_{m \in \mathbb{R}} \Psi^m(X, F)$ nor
$\Psi_{\mathrm{cl}}(X, F) := \cup_{m \in \mathbb{C}} \Psi^m_{\mathrm{cl}}(X, F)$ form an algebra since two such operators do not generally compose, compare Remark 1.9.

• The equivalence relations $\sim$ and $\underset{\mathrm{diag}}{\sim}$ induce equivalence relations on $\Psi^m_{\mathrm{cl}}(X, F)$ for any $m \in \mathbb{C}$, which preserve the symbol in any trivialising chart.

1.2.2. *Uniform classical pseudodifferential operators.* We now specialise to the smaller class of **uniform** pseudodifferential operators, introduced by Shubin and Meladze on Lie groups in [MS89] and by Kordyukov [Ko87] in the general setting of a bounded geometry manifold (see for instance [Sh89, Section 3]). As we shall see later, it is an appropriate class to host pseudodifferential operators on coverings and consists of the usual Hörmander properly supported pseudo-differential operators with additional uniformity conditions.

Definition 1.18. Given (resp. $m \in \mathbb{C}$) $m \in \mathbb{R}$, let $\mathcal{U}\Psi^m(X, F)$ (resp. $\mathcal{U}\Psi^m_{\mathrm{cl}}(X, F)$) be the class of all **uniform (resp. classical) pseudodifferential operators of order** m i.e., operators $A \in \Psi^m(X, F)$ (resp. $A \in \Psi_{\mathrm{cl}}(X, F)$) which in a "good" trivialising covering $X = \cup_i B(x_i, \rho)$ of X read $A \underset{\mathrm{diag}}{\sim} A_0$ as in (18), where

• the operator $S(A) := A - A_0$ lies in $\mathcal{U}\Psi^{-\infty}(X, F)$

• and A has a **uniformly** bounded symbol $\sigma(A) = \sigma(A_0)$ i.e., for any multiindices α, β there is a constant $C_{\alpha,\beta}$ independent of i such that

$$\|\partial_x^\alpha \partial_\xi^\beta \sigma(A)(x, \xi)\| \le C_{\alpha,\beta} (1 + |\xi|)^{m-|\beta|} \quad \forall (x, \xi) \in T^* B(x_i, \rho), \tag{21}$$

• (resp. for classical operators, with an additional **uniform** bound on the remainder terms in (82), namely for any multiindices α, β, for any $N \in \mathbb{N}$, and for any excision function χ around zero, there is a constant $C_{\alpha,\beta,N}$ independent of i such that

$$\left\| \partial_x^\alpha \partial_\xi^\beta \left(\sigma(A)(x, \xi) - \sum_{j=0}^{N-1} \sigma_{m-j}(A)\chi \right)(x, \xi) \right\| \le C_{\alpha,\beta,N} (1 + |\xi|)^{m-|\beta|-N} \quad \forall (x, \xi) \in T^* B(x_i, \rho) \right). \tag{22}$$

On the grounds of (13) we can furthermore require that $S(A)$ lies in $\mathcal{S}\Psi^{-\infty}(X, F)$ which defines the following subclasses of operators:

$$\mathcal{S}\mathcal{U}\Psi^m(X, F) \subseteq \mathcal{U}\Psi^m(X, F) \quad \forall m \in \mathbb{R}, \quad \mathcal{S}\mathcal{U}\Psi^m_{\mathrm{cl}}(X, F) \subseteq \mathcal{U}\Psi^m_{\mathrm{cl}}(X, F) \quad \forall m \in \mathbb{C}. \tag{23}$$

Remark 1.19. As can be seen from (81), for a vector bundle $E \to M$ on a closed manifold M, (21) is verified by any pseudodifferential operator so that $\mathcal{U}\Psi^m(M, E) = \Psi^m(M, E)$. Similarly, (22) is satisfied by any classical pseudodifferential operator and we have $\mathcal{U}\Psi^m_{\mathrm{cl}}(M, E) = \Psi^m_{\mathrm{cl}}(M, E)$.

Similarly to pseudodifferential operators on closed manifolds, uniform pseudodifferential operators modify the degree of regularity of Sobolev spaces by the order of the operator [Sh92, Remark (c) after Def. 3.3].

Lemma 1.20. [En15, Proposition 2.20] *An operator $A \in \mathcal{U}\Psi^m(X, F)$ extends to a bounded operator*

$$\overline{A} : H^s(X, F) \to H^{s-m}(X, F) , \text{ for any } s \in \mathbb{R}. \tag{24}$$

Remark 1.21.   • The proof of [En15, Proposition 2.20], stated for quasi-local uniform pseudodifferential operators, relies on the local finiteness of the covering and does not use quasi-locality. Hence the proof extends to elements of $\mathcal{U}\Psi^m(X, F)$.
   • Consequently, the space of uniform pseudodifferential operators of real order m compares with the space $Op^m(X, F)$ used in [Va08, pag. 11] to denote the space of all "m-regularising operators", i.e. the linear operators $A: C_c^\infty(X, F) \to C_c^\infty(X, F)'$ which extend as in (24)

$$\mathcal{S}\mathcal{U}\Psi^m(X, F) \subsetneq \mathcal{U}\Psi^m(X, F) \subsetneq Op^m(X, F) .$$

This leads to the following identifications.

Proposition 1.22. *(compare with* [En15, Lemma 2.22]*)*

$$\mathcal{S}\Psi^{-\infty}(X, F) = \cap_{m \in \mathbb{R}} \mathcal{U}\Psi^m(X, F) = \mathcal{U}\Psi^{-\infty}(X, F).$$

Consequently, $\mathcal{S}\mathcal{U}\Psi^m(X, F) = \mathcal{U}\Psi^m(X, F)$ for any real number m and $\mathcal{S}\mathcal{U}\Psi_{\mathrm{cl}}^m(X, F) = \mathcal{U}\Psi_{\mathrm{cl}}^m(X, F)$ for any complex number m.

Proof. For any real number m, on the one hand we have $\cap_{m \in \mathbb{R}} \mathcal{U}\Psi^m(X, F) \subset \mathcal{S}\Psi^{-\infty}(X, F)$ as a consequence of (24). On the other hand, we know by (14) that $\mathcal{S}\Psi^{-\infty}(X, F) \subset \mathcal{U}\Psi^{-\infty}(X, F) \subset \mathcal{U}\Psi^m(X, F)$ for any real number m. Hence $\mathcal{S}\Psi^{-\infty}(X, F) \subset \cap_{m \in \mathbb{R}} \mathcal{U}\Psi^m(X, F)$ and the first identity follows.

As for the second identity, we have the straightforward inclusion $\mathcal{U}\Psi^{-\infty}(X, F) \subset \mathcal{U}\Psi^m(X, F)$ for any real number m which yields the inclusion from right to left. The inclusion from left to right follows from observing that the uniform estimates (21) imply the uniform boundedness of the derivatives of the Schwartz kernel of the operator.   □

1.3. Γ-invariant classical pseudodifferential operators on covering spaces.
Let M be a (connected) closed manifold and $\pi: \widetilde{M} \to M$ a regular Γ-covering as in Section 1.1.2.

Definition 1.23. Imposing the Γ-invariance condition (16) leads to the following subclasses $\Psi_\Gamma^m(\widetilde{M}, \widetilde{E})$, $\mathcal{U}\Psi_\Gamma^m(\widetilde{M}, \widetilde{E})$, $\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E})$, $\mathcal{U}\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E})$ of Γ-invariant operators in the corresponding classes defined in Section 1.2.1. The spaces $\Psi_{\mathrm{cl},\Gamma}^m(\widetilde{M}, \widetilde{E})$ and $\mathcal{U}\Psi_{\mathrm{cl},\Gamma}^m(\widetilde{M}, \widetilde{E})$ are defined analogously.

Remark 1.24. As consequences of cocompactness of $\widetilde{M}$, a Γ-invariant operator on $\widetilde{M}$ is properly supported if and only if its Schwartz kernel has compact support in $(\widetilde{M} \times \widetilde{M})/\Gamma$, see [Sh, Chapter 3, above Definition 3].

We equip $\widetilde{M}$ with a Γ-invariant locally finite open cover in the following way: given a finite open cover $\mathcal{U}_M = \{U_j, j = 1, \cdots, N\}$ of M, we lift it to $\widetilde{M}$ and take all the connected components to have a cover by connected open subsets. We obtain a Γ-invariant locally finite open cover $\widetilde{M} = \bigcup_{\substack{j=1,\ldots,N \\ \gamma \in \Gamma}} \gamma U_j$. We then build a Γ-invariant partition of unity

$$\widetilde{\chi}_j := \{\chi_{j,\gamma} \in C_c^\infty(\gamma U_j), \gamma \in \Gamma\}_{j=1,\cdots,N} \tag{25}$$

subordinated to this cover with $\chi_{j,\gamma}(x) = \chi_{j,e}(\gamma^{-1}x)$. This way, a partition of unity $\{\chi_j\}_{j=1,\cdots,N}$ of M subordinated to the covering $\mathcal{U}_M$ is lifted to a Γ-invariant partition of unity $\{\widetilde{\chi}_j\}_{j=1,\cdots,N}$, which is a "good" partition of unity in the sense of manifolds with bounded geometry.

Such a partition of unity combined with a subordinated trivialisation of $\widetilde{E}$ can be used to construct Sobolev spaces $H^s(\widetilde{M}, \widetilde{E})$ of sections on $\widetilde{E}$ [Sc96, Definition 1, §3.9]. As a consequence of the corresponding property on manifolds with bounded geometry, see Lemma 1.20, we have:

Remark 1.25.   • An operator $A \in \mathcal{U}\Psi_\Gamma^m(\widetilde{M}, \widetilde{E})$ extends to a bounded operator $H^s(\widetilde{M}, \widetilde{E}) \to H^{s-m}(\widetilde{M}, \widetilde{E})$, for any $s \in \mathbb{R}$.
   • Consequently, by Proposition 1.22, $\mathcal{U}\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E}) = \cap_{m\in\mathbb{R}}\mathcal{U}\Psi_\Gamma^m(\widetilde{M}, \widetilde{E})$.

1.4. **Lifted operators.** As proved in [Sh], ε-local pseudodifferential operators can be lifted from M to $\widetilde{M}$, and their lifts are uniform properly supported operators. We include the proof of this classical fact for completeness.

Lemma 1.26. [Sh, Proposition 1, §3.9] *With the notation as above, let $r_0 := \inf_{x\in X}\{d(x, \gamma x), \gamma \in \Gamma \setminus \{e\}\} > 0$, where e is the unit of Γ and let $A: C^\infty(M, E) \to C^\infty(M, E)$ be an ε-local operator with $\varepsilon < \frac{r_0}{2}$.*

   (1) *There exists a unique ε-local operator $\widetilde{A}: C_c^\infty(\widetilde{M}, \widetilde{E}) \to C^\infty(\widetilde{M}, \widetilde{E})$ such that for any lifted local section $\widetilde{s}$ of F of a local section s of E*

$$\widetilde{A}\,\widetilde{s} = \pi^*(As). \tag{26}$$

   With the notations of the introduction, we write $\pi_\sharp(\widetilde{A}) = A$; $\pi^\sharp(A) = \widetilde{A}$.
   (2) *If moreover A lies in $\Psi^m(M, E)$ for some $m \in \mathbb{R}$ (resp. $\Psi_{cl}^m(M, E)$ for some $m \in \mathbb{C}$), we have (with the notation of (83), see Appendix A)*

$$\sigma(\widetilde{A}) = \widetilde{\sigma(A)}, \quad \left(\text{resp.} \quad \sigma_{m-j}(\widetilde{A}) = \widetilde{\sigma_{m-j}(A)}, \quad \forall j \geq 0\right). \tag{27}$$

   This is to be understood as a local identity in appropriate local trivialisations around a point x. In particular, $\widetilde{A}$ lies in $\mathcal{U}\Psi_\Gamma^m(\widetilde{M}, \widetilde{E})$ (resp. $\mathcal{U}\Psi_{cl,\Gamma}^m(\widetilde{M}, \widetilde{E})$).

Proof. We have that if $d(x, y) < \varepsilon$, then $d(\gamma x, y) > \varepsilon$ for all $e \neq \gamma \in \Gamma$. If K_A denotes the Schwartz kernel of A, define $\widetilde{A}$ by constructing the operator with Schwartz kernel

$$K_{\widetilde{A}} = \begin{cases} K_A(\pi(x), \pi(y)), & d(x, y) < \varepsilon \\ 0 \quad, & \text{elsewhere.} \end{cases}$$

To show (2), let (V, Φ) be a local trivialisation of E where V is an evenly covered open set. Recall that the symbol of A on this local chart, denoted by $\sigma_V(A)(x, \xi)$, is by definition the symbol of the operator $\Phi^\sharp A_V$ acting on matrix valued functions on $\phi(V) \subset \mathbb{R}^n$, where A_V is the localization of A on V. The symbol of the lifted operator $\widetilde{A}$ is described as follows. Let $\pi^{-1}(V) = \bigsqcup_{\gamma\in\Gamma} U_\gamma$; on each local chart $(U_\gamma, \phi \circ \pi)$ the symbol is defined as the symbol of $(\Phi \circ \pi)^\sharp \widetilde{A}_{U_\gamma}$. It follows that $\sigma_{U_\gamma}(\widetilde{A})(x, \xi) = \sigma_V(A)(\pi(x), \xi)$.
The fact that $\widetilde{A}$ lies in $\mathcal{U}\Psi_\Gamma^m(\widetilde{M}, \widetilde{E})$ (resp. $\mathcal{U}\Psi_{cl,\Gamma}^m(\widetilde{M}, \widetilde{E})$) then follows from the fact that $\mathcal{U}\Psi^m(M, E) = \Psi_\Gamma^m(\widetilde{M}, \widetilde{E})$ (resp. $\mathcal{U}\Psi_{cl}^m(M, E)) = \Psi_{cl,\Gamma}^m(M, E)$). $\square$

We now combine Proposition 1.26 with the partition of the unity (25) to lift operators *modulo* $\underset{\text{diag}}{\sim}$, and have therefore a "lifted" analogue of Proposition A.7.

Proposition 1.27. *Let $\varepsilon > 0$ and $\mathcal{A}$ be an operator in $\Psi_\Gamma(\widetilde{M}, \widetilde{E})$.*

(1) *If $\mathcal{A}$ is ε-local, with the notation of (26), there exists a unique A such that*

$$\mathcal{A} = \pi^{\sharp}A. \tag{28}$$

(2) *In general, there exists an ε-local operator $A \in \Psi(M, E)$ such that*

$$\mathcal{A} \underset{\mathrm{diag}}{\sim} \widetilde{A}. \tag{29}$$

Consequently, the symbols of the two operators relate by

$$\sigma(\mathcal{A}) \sim \widetilde{\sigma(A)}, \tag{30}$$

independently of the choice of $\widetilde{A} \in [\mathcal{A}]_{\mathrm{diag}}$. If $\mathcal{A}$ is ε-local, in particular if it is a differential operator, then $\mathcal{A} = \widetilde{\pi_* \mathcal{A}}$.

Proof. Let $\{\widetilde{\chi}_j\}_{j=1,\cdots,N}$ be a Γ-invariant partition of unity subordinated to a cover $X = \bigcup_{\substack{j=1,\dots,N \\ \gamma \in \Gamma}} \gamma\, U_j$ with open sets U_j of diameter smaller than ε. As in the proof of Lemma 1.15 we write a Γ-invariant operator $\mathcal{A} \in \Psi_{\Gamma}(\widetilde{M}, \widetilde{E})$ as

$$\mathcal{A} = \sum_{i,j} \widetilde{\chi}_i \mathcal{A} \widetilde{\chi}_j = \sum_{\mathrm{supp}\,\widetilde{\chi}_i \cap \mathrm{supp}\,\widetilde{\chi}_j \neq \emptyset} \widetilde{\chi}_i \mathcal{A} \widetilde{\chi}_j + \sum_{\mathrm{supp}\,\widetilde{\chi}_i \cap \mathrm{supp}\,\widetilde{\chi}_j = \emptyset} \widetilde{\chi}_i \mathcal{A} \widetilde{\chi}_j,$$

where with a slight abuse of notation, using the notations of (25), we have set $\mathrm{supp}\,\widetilde{\chi}_i = \cup_{\gamma \in \Gamma} \mathrm{supp}\,\widetilde{\chi}_{i,\gamma}$.

Choosing the diameter of the partition small enough and applying Proposition 1.26 to the ε-local operators $\chi_i \mathcal{A} \chi_j$, we have

$$\chi_i \mathcal{A} \chi_j = \widetilde{\pi_{\sharp}(\widetilde{\chi_i \mathcal{A} \widetilde{\chi}_j})},$$

which yields

$$\mathcal{A} = \sum_{\mathrm{supp}\,\widetilde{\chi}_i \cap \mathrm{supp}\,\widetilde{\chi}_j \neq \emptyset} \widetilde{A} + S(\mathcal{A}) = \widetilde{A} + S(\mathcal{A}) \tag{31}$$

with

$$A := \sum_{\mathrm{supp}\,\widetilde{\chi}_i \cap \mathrm{supp}\,\widetilde{\chi}_j \neq \emptyset} \pi_{\sharp}\left(\chi_i \mathcal{A} \chi_j\right) \tag{32}$$

and

$$S(\mathcal{A}) := \mathcal{A} - \widetilde{A} = \sum_{\mathrm{supp}\,\widetilde{\chi}_i \cap \overline{\mathrm{supp}}\chi_j = \emptyset} \widetilde{\chi}_i \mathcal{A} \widetilde{\chi}_j\,.$$

a linear operator with smooth kernel supported outside the diagonal.

If $\mathcal{A}$ is ε-local, then the above construction reduces to

$$\mathcal{A} = \overbrace{\sum_{\mathrm{supp}\,\widetilde{\chi}_i \cap \mathrm{supp}\,\widetilde{\chi}_j \neq \emptyset} \widetilde{\chi}_i(\pi_* \mathcal{A}) \widetilde{\chi}_j} = \widetilde{\pi_* \mathcal{A}}.$$

This proves (29) from which (30) then follows. $\square$

Remark 1.28. In view of (30), properties of pseudodifferential operators such as being classical, the order, invertibility of the principal symbol can be lifted without ambiguity.

On the grounds of the above Remark, we set the following

Definition 1.29. With the notations of Proposition 1.27, an operator $\mathcal{A}$ in $\mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$ is **elliptic** whenever A is elliptic, i.e. whenever its principal symbol is invertible.

On the grounds of the above proposition, we set the following

Definition 1.30. Let $A \in \Psi_{cl}(M, E)$, let A_0 be ε-local such that $A \underset{\text{diag}}{\sim} A_0$ as in (18). We define the lift of the class $[A]_{\text{diag}}$ to

$$\widetilde{[A]_{\text{diag}}} := [\widetilde{A_0}]_{\text{diag}}. \tag{33}$$

With this defintion at hand, for any $\mathcal{A} \in \widetilde{[A]_{\text{diag}}}$ we have

$$\sigma(\mathcal{A}) \sim \widetilde{\sigma(A_0)}. \tag{34}$$

1.5. **Lifting functions of operators.** Let E be a hermitian vector bundle over the closed Riemannian manifold M.

We borrow the following definition from [ALNP15].

Definition 1.31. We call a **weight** in $\Psi_{cl}(M, E)$, an operator $Q \in \Psi_{cl}(M, E)$ such that
   (1) Q is invertible, namely its kernel is non trivial or equivalently, it admits an inverse defined on $L^2(M, E)$,
   (2) Q has positive order q,
   (3) Q has a principal angle θ, which means that there exists a ray $R_\theta = \{re^{i\theta}, \ r \geq 0\}$, called **spectral cut** , which is disjoint from the spectrum of the $\text{End}(E_x)$-valued leading symbol $\sigma_L(Q)(x, \xi)$ for any $x \in M$, $\xi \in T_x^* M \setminus \{0\}$.

This last condition implies that the spectrum of the operator Q lies outside a cone Λ_θ containing the ray R_θ, [ALNP15, Lemma 1.6].

Example 1.32. Let D in $\Psi_{cl}^d(M, E)$ be an essentially self-adjoint elliptic differential operator of positive order d. Then $\Delta := D^2$ is a non-negative elliptic differential operator on M of positive order $q := 2d$ and the operator $\Delta + 1$ is a differential operator which defines a weight.

Example 1.33. With the same notations as in the above example, we can instead add to Δ a smoothing operator $\chi_{[-\varepsilon, \varepsilon]}(\Delta)$ with $\varepsilon > 0$ chosen small enough so that it coincides with the orthogonal projection $\chi_0(\Delta)$ onto the kernel of Δ. Then

$$Q_\varepsilon(\Delta) := \Delta + \chi_{[-\varepsilon, \varepsilon]}(\Delta) \tag{35}$$

defines a weight with spectral cut $R_\pi = \mathbb{R}_{\leq 0}$.

A weight $Q \in \Psi_{cl}(M, E)$ satisfies a resolvent estimate, see [Sh01, (9.30)] and [Se66, Cor. 1, p. 298]:

$$\|(Q - \lambda)^{-1}\|_{s, s+l} \leq C_{s,l} |\lambda|^{-1 + \frac{l}{q}} \quad \forall 0 \leq l \leq q \ \ \forall \lambda \in \Lambda_\theta \cap \{|\lambda| > R > 0\}. \tag{36}$$

Let Γ_θ be a contour around the ray R_θ, then to a measurable function h on Γ_θ, such that $|h(\lambda)| \leq |\lambda|^{-\delta}$ for some positive δ, we can associate the operator

$$h(Q) := -\frac{1}{2i\pi} \int_{\Gamma_\theta} h(\lambda) (Q - \lambda)^{-1} d\lambda, \tag{37}$$

whose symbol is given by the corresponding Cauchy integral

$$\sigma(h(Q)) \sim h_\star(\sigma(Q)) := -\frac{1}{2i\pi} \int_{\Gamma_\theta} h(\lambda) (\sigma(Q) - \lambda)^{\star - 1} d\lambda, \tag{38}$$

where the exponent $\star k$ stands for the k-th $\star$-product exponent of symbols. We refer the reader to any book on pseudodifferential operatorsfor the precise definition of the $\star$-product, see e.g. [Sh01, (3.41)].

Example 1.34. For a polynomial $h(x) = \sum_{k=0}^{n} a_k x^k$, (37) yields the operator $h(Q) := \sum_{k=0}^{n} a_k Q^k$ with symbol $h_\star(\sigma(Q)) = \sum_{k=0}^{n} a_k \sigma(Q)^{\star k}$.

Example 1.35. If $Q = D^2$, with D an essentially self-adjoint operator, then the function $h(x) = \frac{1}{\sqrt{x}}$ yields the operator $|D|^{-1} := Q^{-\frac{1}{2}}$ from which we build the sign operator

$$\operatorname{sgn}(D) := D\, h(Q) = D\, |D|^{-1} \quad \text{with symbol} \quad \sigma\left(\operatorname{sgn}(D)\right) \sim h_\star(\sigma(D)) := \sigma(D) \star (\sigma(\Delta))^{\star -\frac{1}{2}}. \tag{39}$$

Definition 1.31 carries out to $\mathcal{U}\Psi_\Gamma(\widetilde{M}, \widetilde{E})$, up to the fact that, in contrast with the closed case (see [ALNP15, Def. 3.6] for details), the spectrum being not necessarily purely discrete in the noncompact case, we need an extra condition for the existence of an Agmon angle, defined as follows.

Definition 1.36. For an angle β and for $\epsilon > 0$, denote

$$V_{\beta,\epsilon} := \{z \in \mathbb{C} \; : \; |z| < \epsilon\} \cup \{z \in \mathbb{C} \setminus 0 \; : \; \operatorname{arg} z \in (\beta - \epsilon, \beta + \epsilon)\}.$$

Then β is called an **Agmon angle** for $A \in \mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$ if there is some $\epsilon > 0$ such that $\operatorname{spec}(A) \cap V_{\beta,\epsilon} = \emptyset$

Definition 1.37. We call a **weight** in $\mathcal{U}\Psi_\Gamma(\widetilde{M}, \widetilde{E})$, an operator $\mathfrak{Q} \in \mathcal{U}\Psi_\Gamma(\widetilde{M}, \widetilde{E})$ such that

(1) $\mathfrak{Q}$ is invertible in the strong sense of the term, namely that it admits an inverse defined on $L^2(\widetilde{M}, \widetilde{E})$,
(2) $\mathfrak{Q}$ has positive order q,
(3) $\mathfrak{Q}$ has a principal angle θ as in Definition 1.31,
(4) θ is an Agmon angle for $\mathfrak{Q}$.

Remark 1.38. In view of (30), for any weight $\mathfrak{Q} \in \mathcal{U}\Psi_\Gamma(\widetilde{M}, \widetilde{E})$, there exists an operator Q in $\Psi(M, E))$ such that

$$\sigma(\mathfrak{Q}) \sim \widetilde{\sigma(Q)} \tag{40}$$

so that it has the same order and same principal angle. Moreover it can be chosen invertible modulo addition of the projection onto its kernel. Hence for any weight $\mathfrak{Q}$ in $\mathcal{U}\Psi_\Gamma(\widetilde{M}, \widetilde{E})$ there is a weight Q in $\Psi(M, E)$ with the same spectral cut and such that (40) holds.

FIGURE 1. Agmon angle β
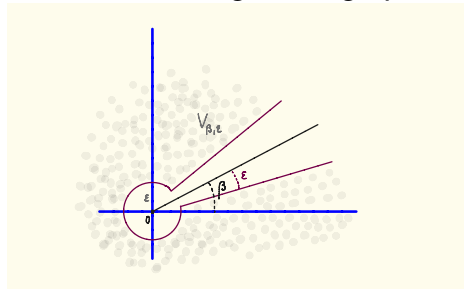
Lemma 1.39. *Let $\mathfrak{Q}$ be a weight in $\mathcal{U}\Psi_\Gamma(\widetilde{M}, \widetilde{E})$ with spectral cut R_θ and let Q be a weight in $\Psi(M, E)$ with the same spectral cut as in Remark 1.38. With the notations of (40), for every measurable function h on a contour Γ_π around the ray R_θ, such that $|h(\lambda)| \le |\lambda|^{-\delta}$ for some positive δ, we have:*

$$h_\star(\sigma(\mathfrak{Q})) \sim \widetilde{h_\star(\sigma(Q))}, \tag{41}$$

where we have used the notation of (38).

Proof. The star product $\star$, which is a local operation for it only involves derivatives, commutes with the lift. For two local symbols σ and τ, we have $\widetilde{\sigma \star \tau} \sim \widetilde{\sigma} \star \widetilde{\tau}$, which implies $\widetilde{(\sigma - \lambda)^{\star-1}} \sim (\widetilde{\sigma} - \lambda)^{\star-1}$ and hence

$$\widetilde{h_\star(\sigma)} \sim h_\star(\widetilde{\sigma}) = -\frac{1}{2i\pi} \int_\Gamma h(\lambda) (\widetilde{\sigma} - \lambda)^{\star-1}.$$

Implementing h_* therefore yields

$$\sigma(\mathfrak{Q}) \sim \widetilde{\sigma(Q)} \implies h_\star(\sigma(\mathfrak{Q})) \sim h_\star(\widetilde{\sigma(Q)}) \implies h_\star(\sigma(\mathfrak{Q})) \sim \widetilde{h_\star(\sigma(Q))}.$$

$\square$

Example 1.40. Let Q be a weight with spectral cut R_θ. For $\mathfrak{R}(z) > 0$, the function $h(x) = x_\theta^{-z}$ with the complex power determined by the angle θ, yields the complex power

$$Q_\theta^{-z} := -\frac{1}{2i\pi} \int_{\Gamma_\theta} \lambda_\theta^{-z} (Q - \lambda)^{\star-1} d\lambda, \tag{42}$$

which can be extended to any complex value z setting $Q_\theta^{-z} := Q^k Q_\theta^{-z+k}$ for $\mathfrak{R}(z) > -k$.

With the same notations as in the Example 1.33, the differential operator Δ lifts to a differential operator $\widetilde{\Delta}$ whose leading symbol is the lifted leading symbol of Δ.

With the notation introduced in Appendix B, the isomorphism Φ induces a map

$$\begin{aligned} \mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E}) &\longrightarrow \Psi(M; E \otimes \mathcal{H}) \\ A &\longmapsto \Phi^\sharp A. \end{aligned}$$

Let $\Delta_\mathcal{H} := \Phi^\sharp \widetilde{\Delta}$ be the corresponding elliptic operator in $\Psi_{\mathrm{cl}}(M, E \otimes \mathcal{H})$.

Proposition 1.41. *For any positive ε, the operator*

$$Q_\varepsilon(\widetilde{\Delta}) := \widetilde{\Delta} + \chi_{[-\varepsilon,\varepsilon]}(\widetilde{\Delta}), \tag{43}$$

resp. $Q_\varepsilon(D_\mathcal{H}) := \Delta_\mathcal{H} + \chi_{[-\varepsilon,\varepsilon]}(\Delta_\mathcal{H})$, *defines a weight in* $\mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$, *resp.* $\Psi_{\mathrm{cl}}(M, E \otimes \mathcal{H})$ *with spectral cut* $R_\pi = \mathbb{R}_{\leq 0}$ *and we have*

$$Q_\varepsilon(\Delta_\mathcal{H}) = \Phi^\sharp Q_\varepsilon(\widetilde{\Delta}). \tag{44}$$

Proof. To ensure that the operator $Q_\varepsilon(\widetilde{D})$, resp. $Q_\varepsilon(D_\mathcal{H})$ defines a weight (for the latter, see also [ALNP15]), we need to check that

(1) the operator $\chi_{[-\varepsilon,\varepsilon]}(\widetilde{\Delta})$, resp. $\chi_{[-\varepsilon,\varepsilon]}(\Delta_\mathcal{H})$ has a smooth Schwartz kernel in $\Psi_\Gamma(\widetilde{M}, \widetilde{E})$, resp. $\Psi(M, E \otimes \mathcal{H})$) which follows from [Va08, Cor 3.6] , resp. from [BFKM96].

(2) the operator $\widetilde{\Delta} + \chi_{[-\varepsilon,\varepsilon]}(\widetilde{\Delta})$, resp. $\Delta_\mathcal{H} + \chi_{[-\varepsilon,\varepsilon]}(\Delta_\mathcal{H})$ is invertible, which is an immediate consequence of its non-negativity.

The compatibility of the map Φ with functional calculus (see B.9) implies (44) . $\square$

Vassout's functional calculus on groupoids recalled in Appendix C, allows to extend the notion of weight to the groupoid $G(M) := (\widetilde{M} \times \widetilde{M})/\Gamma$ associated with $\mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$, by the isomorphism ρ defined in (94). Indeed, the map

$$\begin{aligned} \mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E}) &\longrightarrow \Psi(G, E) \\ A &\longmapsto \rho^\sharp A, \end{aligned}$$

induced preserves the properties 1)-3) of a weight and therefore transforms a weight $\mathfrak{Q} \in \mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$ with spectral cut θ to a weight $\rho^\sharp(\mathfrak{Q})$ on the associated groupoid with the

same spectral cut. The map $\rho^\sharp$ preserves the estimate (36) on weights, which enables us to transport the related functional calculus from the groupoid to $\mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$. With the notations of (37) and for a measurable function h on on a contour Γ_π around the ray $R_\pi =]-\infty, 0]$, such that $h(\lambda) \le \lambda^{-\delta}$ for some positive δ, we can define

$$h(\mathfrak{Q}) = \left(\rho^\sharp\right)^{-1}(h(\rho^\sharp(\mathfrak{Q})), \tag{45}$$

whose symbol is given by

$$\sigma(h(\mathfrak{Q})) \sim h_\star(\sigma(\mathfrak{Q})) := -\frac{1}{2i\pi} \int_{\Gamma_\theta} h(\lambda)\,(\sigma(\mathfrak{Q}) - \lambda)^{\star-1}\, d\lambda. \tag{46}$$

Example 1.42. For $\mathfrak{R}(z) > 0$, the function $h(x) = x_\theta^{-z}$ with the complex power determined by the angle θ, yields the complex power

$$\mathfrak{Q}_\theta^{-z} := -\frac{1}{2i\pi} \int_{\Gamma_\theta} \lambda_\theta^{-z}\,(\mathfrak{Q} - \lambda)^{\star-1}\, d\lambda, \tag{47}$$

where $\mathfrak{Q} \in \mathcal{U}\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$ is a weight with spectral cut R_θ.

1.6. **Lifting complex powers to coverings.** We now specialise to $h : x \mapsto x_\theta^{-z}$, where θ stands for the determination of the complex power. Applied to a weight $Q \in \Psi_{\mathrm{cl}}(M, E)$ with spectral cut R_θ, this gives rise to complex powers $Q_\theta^{-z} \in \Psi_{\mathrm{cl}}(M, E)$.

Remark 1.43. Let $m \in \mathbb{R}$, $z \in \mathbb{C}$ and $Q \in \Psi_{\mathrm{cl}}^m(G, E)$ a positive (with respect to the L^2-inner product) elliptic, invertible operator. It defines a weight with spectral cut $R_\pi = \mathbb{R}_{\le 0}$ (we drop the mention of $\theta = \pi$ in the notation). The complex power Q^{-z} is defined in [Vas06, p. 25] according to (47) and proved to belong to $\Psi_{\mathrm{cl}}^{-mz}(G, E)$ and to act as an an element of $\mathcal{L}(H^{t-m\mathfrak{R}z}(\mathcal{W}), H^t(\mathcal{W}))$ for any $t \in \mathbb{R}$. By Proposition C.4, the inverse map ρ^{-1} identifies the operator Q^{-z} with the operator $\rho^{\sharp-1}(Q^{-z})$ in $\mathcal{U}\Psi_{\mathrm{cl}}^{-mz}(\widetilde{M}, \widetilde{E})$.
Vassout's construction, which is carried out for positive operators, easily extends to any weight using an appropriate spectral cut.

Families $z \mapsto Q^{-z}$ of complex powers are holomorphic, a notion we briefly recall.

Definition 1.44. Let U be an open subset of $\mathbb{R}^n$, let V be a linear space and let W be a domain in $\mathbb{C}$. A holomorphic family of classical (also called polyhomogeneous) symbols on U with values in $\mathrm{End}(V)$ parametrized by W of order $\alpha : W \to \mathbb{C}$ is a function

$$\sigma(z)(x, \xi) := \sigma(z, x, \xi) \in C^\infty(W \times U \times \mathbb{R}^n, \mathrm{End}(V))$$

for which:

(1) $\sigma(z)(x, \xi)$ is holomorphic at $z \in W$ as an element of $C^\infty(W \times U \times \mathbb{R}^n, \mathrm{End}V)$ and

$$\sigma(z)(x, \xi) \sim \sum_{j \ge 0} \sigma_{\alpha(z)-j}(z)(x, \xi) \tag{48}$$

is a classical symbol of order $\alpha(z)$ where the function is holomorphic;

(2) for any integer $N \ge 1$ the remainder $\sigma_{(N)}(z)(x, \xi) := \sigma(z)(x, \xi) - \sum_{j=0}^{N-1} \sigma_{\alpha(z)-j}(z)(x, \xi)$

is holomorphic in $z \in W$ as an element of $C^\infty(W \times U \times \mathbb{R}^n, \mathrm{End}\, V)$ with k^{th} z-derivative

$$\sigma_{(N)}^{(k)}(z)(x, \xi) := \frac{\partial^k}{\partial z^k}(\sigma_{(N)}(z)(x, \xi)) \tag{49}$$

defining a locally uniform family of symbols of order $\mathfrak{R}(\alpha(z)) - N + \varepsilon$ in a compact neighborhood of any $z_0 \in W$, for any positive ε.

A family $z \mapsto A(z)$ in $\Psi_{\mathrm{cl}}(M, E)$ parametrised by a domain $W \subset \mathbb{C}$ is holomorphic if in each local trivialisation of E one has

$$A(z) = \mathrm{Op}(\sigma(A(z)) + S(z)$$

with $\sigma(A(z))$ a holomorphic family of classical symbols and $S(z)$ a operator with Schwartz kernel $R(z, x, y) \in C^\infty(W \times M \times M, E \boxtimes E)$ holomorphic in z.

There are at least two types of approaches to show that complex powers define holomorphic families. One by Seeley ([Se66], see also [Sh01, Thm. 11.2]) using in a central manner the calculus of the symbol of the resolvent of the operator, and a cohomological construction by Guillemin [Gu85, Thm 5.2] axiomatic in nature and therefore easily transposable to more general contexts (see e.g. [ALNV04]). Note that Seeley and Shubin consider complex powers of differential elliptic operators but their construction can be extended to classical pseudodifferential operators using the symbol of the resolvent of these operators. Guillemin applies it to classical pseudodifferential operators with leading symbols that have a unique determination of the logarithm.

These constructions, which rely on basic properties of classical pseudodifferential operators, namely

(1) an estimate of the type (36) on the resolvent of a weight, leading to the existence of Cauchy integrals for weights,
(2) the existence of a map $\mathrm{Op} : \sigma \mapsto \mathrm{Op}(\sigma)$ taking symbols to operators in the algebra, that "commutes" with Cauchy integrals,

extend to very general algebras of classical pseudodifferential operators, including classical pseudodifferential operators on groupoids and uniform classical pseudodifferential operators on coverings.

In particular, a weight $\mathfrak{Q} \in \mathcal{U}\Psi_{\Gamma,\mathrm{cl}}(\widetilde{M}, \widetilde{E})$ gives rise to a holomorphic family

$$\mathfrak{Q}^{-z} = \left(\rho^\sharp\right)^{-1} \left(\rho^\sharp(\mathfrak{Q})\right)^{-z}, \tag{50}$$

where the map $\rho : \mathcal{U}\Psi_{\Gamma,\mathrm{cl}}(\widetilde{M}, \widetilde{E}) \longrightarrow G(M, E)$ is defined in Appendix C.

2. Linear forms and trace-defect formulae on Γ-invariant operators

2.1. **Local linear forms on classical pseudodifferential operators.** Locality plays a fundamental role in the lifting procedure; in this section we single out "local" linear forms which can be lifted to coverings.

2.1.1. *The Wodzicki residue and canonical trace densities.* Let X and F be of bounded geometry as in the previous section, let $m \in \mathbb{C}$. To an operator $A = \mathrm{Op}(\sigma(A)) + R(A) \in \Psi_{\mathrm{cl}}^m(X, F)$ and $x \in M$ with symbol $\sigma(A)(x, \cdot) \sim \sum_{j=0}^\infty \sigma_{m-j}(A)(x, \cdot)$ in a local trivialisation around x, where $\sigma_{m-j}(A), \ j \in \mathbb{Z}_{\geq 0}$ are the positively homogeneous components of the symbol of degree $m - j$, we assign

- the **pointwise Wodzicki** residue

$$\mathrm{Res}_x(A) := \frac{1}{(2\pi)^n} \int_{|\xi|=1} \mathrm{tr}\, (\sigma_{-n}(A)(x, \xi))\, d\xi,$$

- the **pointwise canonical trace**

$$\mathrm{TR}_x(A) := \frac{1}{(2\pi)^n} \fint_{\mathbb{R}^n} \mathrm{tr}\left(\sigma(A)(x,\xi)\right) d\xi := \frac{1}{(2\pi)^n} \mathrm{fp}_{R\to\infty} \int_{|\xi|\le R} \mathrm{tr}\left(\sigma(A)(x,\xi)\right) d\xi,$$

where the abreviation fp for finite part means we pick the constant term in the asymptotic expansion as R tends to infinity and $\fint_{\mathbb{R}^n}$ is the corresponding cut-off integral. Here, tr stands for the fibrewise trace.

We recall well-known facts.

Lemma 2.1. [Wo84, KV94] *Given an operator $A \in \Psi_{\mathrm{cl}}(X, F)$, both $\mathrm{Res}_x(A)\, dx$ and – whenever the order of A does not lie in $[-n, +\infty[\cap \mathbb{Z}$– $\mathrm{TR}_x(A)\, dx$ define a global density on X.*

2.1.2. *The Wodzicki residue and the canonical trace on closed manifolds.* From now on in this section we assume that $X = M$ **is a closed manifold** and $F = E$ is a vector bundle over M. We adopt the notations of [Sc10] and of Lemma 2.1.

Proposition 2.2. *Let $A \in \Psi_{\mathrm{cl}}(M, E)$ have local symbol $\sigma_U(A)$ over any trivialising open subset U, and let $(U_i, \chi_i)_{i\in I}$ be a partition of unity on M subordinated to a trivialisation of E.*

(1) *The Wodzicki residue density integrates to the Wodzicki residue* [Wo84]

$$\mathrm{Res}\,(A) \quad := \quad \int_M \mathrm{Res}_x(A)\, dx \tag{51}$$

$$= \quad \sum_{\mathrm{supp}\chi_i \cap \mathrm{supp}\chi_j \ne \emptyset} \int_{U_i \cap U_j} \chi_i(x) \mathrm{Res}_x\left(\mathrm{Op}(\sigma_{U_i\cap U_j})(A)\right) \chi_j(x) dx.$$

(2) *Provided the order of the operator does not lie in $\mathbb{Z}_n := [-n, +\infty[\cap \mathbb{Z}$, the canonical trace density integrates to the canonical trace* [KV94, Le99]

$$\mathrm{TR}\,(A) \quad := \quad \int_M \mathrm{TR}_x(A)\, dx \tag{52}$$

$$= \quad \sum_{\mathrm{supp}\chi_i \cap \mathrm{supp}\chi_j \ne \emptyset} \int_{U_i \cap U_j} \chi_i(x)\, \mathrm{TR}_x\left(\mathrm{Op}(\sigma_{U_i\cap U_j}(A))\right) \chi_j(x)\, dx,$$

which are both well-defined, independently of the choice of trivialisation and subordinated partition of unity.

When A is trace-class, namely when the order of A has real part smaller than $-n$, then $\mathrm{Res}(A) = 0$ and $\mathrm{TR}(A) = \mathrm{Tr}(A)$, where Tr stands for the ordinary trace of A. Furthermore, we have

$$A \sim B \Longrightarrow \mathrm{Res}\,(A) = \mathrm{Res}\,(B) \quad and \quad A \underset{\mathrm{diag}}{\sim} B \Longrightarrow \mathrm{TR}\,(A) = \mathrm{TR}\,(B)$$

$$\mathrm{Res}(A) = \mathrm{Res}(A_0) = \mathrm{Res}([A]) \quad and \quad \mathrm{TR}(A) = \mathrm{TR}(A_0) = \mathrm{TR}([A]_{\mathrm{diag}}).$$

Proof. We refer to [Wo84, KV94] for the existence of Res and TR. The pointwise residue Res_x vanishes on operators with smooth kernels and the pointwise canonical trace TR_x vanishes on operators with smooth kernel supported outside the diagonal. Consequently, only $A_0 := \sum_{\mathrm{supp}\chi_i \cap \mathrm{supp}\chi_j \ne \emptyset} \chi_i A \chi_j$ arises in (51) and (52). $\qquad\square$

2.1.3. *Characterisation of local linear forms on operators.* As before, $\pi : E \rightarrow M$ denotes a rank k-complex vector bundle over a closed n-dimensional manifold M.

Definition 2.3. We call a Σ-**class** (Σ-for symbol) any class $\Sigma(M, E) \in \Psi_{cl}(M, E)$ of classical pseudodifferential operators such that

$$A \in \Sigma(M, E) \;\wedge\; B \underset{\text{diag}}{\sim} A \Longrightarrow B \in \Sigma(M, E),$$

so that for an operator to belong to Σ is a condition on its symbol. Let $\Sigma_{\text{symb}} \subset \text{CS}(\mathbb{R}^n, \mathfrak{gl}_k(\mathbb{C}))$ be the corresponding class of symbols so that

$$A \in \Sigma(M, E) \Longleftrightarrow A \sim_{\text{diag}} \text{Op}(\sigma(A)) \;\wedge\; \sigma(A) \in \Sigma_{\text{symb}}.$$

Example 2.4. The algebra $\Psi_{cl}^{\mathbb{Z}}(M, E)$ of integer order classical operators and the set $\Psi_{cl}^{\notin\mathbb{Z}}(M, E)$ of noninteger order classical operators form Σ-classes.

We define a class of linear forms (by linear form on a $\Sigma(M, E)$, we mean that $\Lambda(\alpha A + B) = \alpha\Lambda(A) + \Lambda(B)$ for any scalar α and any operators A and B in Σ such that $\alpha A + B$ also lies in Σ) on a Σ-class $\Sigma(M, E)$, which only detect the symbol of the operator. For this purpose, it is useful to introduce the corresponding class of scalar valued symbols

$$\Sigma_{\text{tr,symb}} := \{x \longmapsto \text{tr}_x \circ \sigma(x, \cdot), \; \sigma \in \Sigma_{\text{symb}}\} \subset \text{CS}(\mathbb{R}^n, \mathbb{C}),$$

where tr_x is the fibrewise trace on the group $\text{End}(E_x)$ of endomorphisms of the fibre E_x over the point x.

Definition 2.5. We call **local** any linear form $\Lambda : \Sigma(M, E) \rightarrow \mathbb{C}$ on a Σ-class of classical pseudodifferential operators $\Sigma(M, E)$ which

- is constant on $\underset{\text{diag}}{\sim}$ -equivalence classes:

$$A \underset{\text{diag}}{\sim} B \Longrightarrow \Lambda(A) = \Lambda(B),$$

so that

$$\Lambda(A) = \Lambda(\text{Op}(\sigma(A))), \tag{53}$$

- and such that

$$\Lambda(\text{Op}(\sigma)) = \int_M \lambda \circ \text{tr}(\sigma(x, \cdot)) \, dx$$

for some linear form $\lambda : \Sigma_{\text{tr,symb}} \rightarrow \mathbb{C}$, so that

$$\Lambda(A) = \int_M \Lambda_x(A) \, dx := \int_M \lambda \circ \text{tr}\,(\sigma(A)(x, \cdot)) \, dx. \tag{54}$$

Remark 2.6. The adjective "local" reflects the fact that the linear form Λ is the integral of a density

$$\omega_\Lambda(A)(x) := \lambda \circ \text{tr}\,(\sigma(A)(x, \cdot)) \, dx$$

over M.

Example 2.7. The Wodzicki residue on integer order classical operators (resp. the canonical trace on non-integer order classical operators) are local.

In the remaining part of this paragraph, $\Sigma(M, E) \subset \Psi_{cl}(M, E)$ is a Σ-class of operators which is **closed** for the Fréchet topology of classical pseudodifferential operators of fixed order (see e.g. [Pa12, p.117] and references therein).

Example 2.8. The Σ-classes $\Psi_{\mathrm{cl}}^{\mathbb{Z}}(M, E)$ of integer order classical pseudodifferential operators and $\Psi_{\mathrm{cl}}^{\notin\mathbb{Z}}(M, E)$ of noninteger order classical pseudodifferential operators, both determined by conditions on their order, are closed for the Fréchet topology of classical pseudodifferential operators of fixed order.

Lemma 2.9. *Given a Σ-class $\Sigma(M, E) \subset \Psi_{\mathrm{cl}}(M, E)$,*

- *the corresponding symbol class Σ_{symb} is invariant under scaling $\xi \longmapsto t\xi$ for $0 < t < 1$ and $\mathrm{O}_n(\mathbb{R})$;*
- *Given a* local and continuous *linear form $\Lambda\colon \Sigma(M, E) \to \mathbb{C}$, the associated linear form $\lambda\colon \Sigma_{\mathrm{tr,symb}} \to \mathbb{C}$ on the corresponding class of scalar valued symbols is continuous, behaves covariantly under rescaling $\xi \longmapsto t\xi$ for any $0 < t < 1$ and $\mathrm{O}_n(\mathbb{R})$-invariant.*

Proof. We first observe that λ is continuous for the Fréchet topology on symbols of constant order as a result of the continuity of Λ. Let us deduce further properties of λ from the covariance of Λ.

(1) Since $\omega_\Lambda(A)(x)$ defines a density, for any local diffeomorphism $\kappa\colon U \to U$, $\kappa^*\omega_\Lambda(\kappa^\sharp A_U) = \omega_\Lambda(A_U)$, where A_U is a localisation of A in that chart. If A_U has symbol σ, following the notations of [Pa12], let $\widetilde{\kappa_*\sigma}$ be the symbol of $\kappa^\sharp A_U$. The above covariance property for the form ω_Λ translates to

$$|\det\kappa_*|\,\lambda\left(\widetilde{\kappa_*\sigma}\right) = \lambda\left(\sigma\right) \tag{55}$$

for any local diffeomorphism $\kappa\colon U \to U$.

(2) Choosing U to be an open ball and $\kappa = tR$ for any $R \in O_n(\mathbb{R})$ with $0 < t < 1$, it follows from the invariance property (55), that the symbol class Σ_{symb} is invariant and the linear form λ behaves covariantly under rescaling $\xi \longmapsto t\xi$ for any $0 < t < 1$ as well as under isometric transformations. $\square$

In the following, continuity of local linear forms is defined with respect to the Fréchet topology of classical pseudodifferential operators of fixed order (see e.g. [Pa12, p.117] and references therein.

Theorem 2.10. *Any local continuous linear form on $\Psi_{\mathrm{cl}}^{\mathbb{Z}}(M, E)$ (resp. $\Psi_{\mathrm{cl}}^{\notin\mathbb{Z}}(M, E)$) is proportional to the Wodzicki residue* Res *(resp. the canonical trace* TR*).*

Proof. The proof relies on results of [Pa12], the basic ideas being that rescaling and isometric invariant linear forms i) on homogeneous functions are proportional to the residue (see Lemma [Pa12, 3.42]) and ii) on Schwartz functions are proportional to the ordinary integral (see Lemma [Pa12, 3.40]). Let k be the rank of E as a complex bundle over M.

(1) We first characterise Λ on $\Psi_{\mathrm{cl}}^{\notin\mathbb{Z}}(M, E)$. Theorem 3.43 in [Pa12] characterises continuous rescaling and $\mathrm{O}_n(\mathbb{R})$-invariant linear forms on $CS^{\notin\mathbb{Z}}(\mathbb{R}^n)$ which turn out to be proportional to the canonical integral. Note that the proof of [Pa12, Lemma 3.42] on which in [Pa12, Theorem 3.43] relies, only requires an invariance under rescaling by $0 < t < 1$. So there is a constant C such that $\lambda = C \fint_{\mathbb{R}^n}$ on $CS^{\notin\mathbb{Z}}(\mathbb{R}^n)$. It follows that on $\Psi_{\mathrm{cl}}^{\notin\mathbb{Z}}(M, M \times \mathbb{C})$, the linear form Λ is proportional to the canonical trace TR. Composing with the trace on matrices yields the expected characterisation on $\Psi_{\mathrm{cl}}^{\notin\mathbb{Z}}(M, E)$.

(2) Let us now characterise Λ on $\Psi_{\mathrm{cl}}^{\mathbb{Z}}(M, E)$. The covariance assumption of Theorem 4.21 in [Pa12]– which says that any covariant continuous linear form on $CS^{\mathbb{Z}}(\mathbb{R}^n)$ is proportional to the residue, the proof of which relies on Theorem 3.43– can

easily be relaxed to an invariance under rescaling and isometric transformations. Thus, any continuous linear form on $CS^{\mathbb{Z}}(\mathbb{R}^n)$ which is invariant under rescaling and isometric transformations is proportional to the residue res. It follows that Λ on $\Psi_{\text{cl}}^{\mathbb{Z}}(M, M \times \mathbb{C})$ is proportional to the Wodzicki residue Res. Composing with the trace on matrices yields the expected characterisation on $\Psi_{\text{cl}}^{\mathbb{Z}}(M, E)$. $\square$

We use this locality in an essential way to lift trace defect formulae. Here is an immediate corollary which uses the known traciality of the Wodzicki residue and the canonical trace on the appropriate classes of operators they are defined on.

Corollary 2.11. *Any local continuous linear form on the class $\Psi^{\mathbb{Z}}(M, E)$ of integer order classical pseudodifferential operators, resp. on the class $\Psi^{\notin\mathbb{Z}}(M, E)$, is a trace, resp. vanishes on brackets.*

2.2. **Trace defect formulae on closed manifolds and applications.**

2.2.1. *Trace defect formulae.* We recall (without proof) useful trace defect formulae for the canonical trace of holomorphic families of classical pseudodifferential operators [KV94, PS07].

Proposition 2.12. *For any holomorphic family $A(z) \in \Psi_{\text{cl}}(M, E)$ of classical operators parametrised by $\mathbb{C}$ with holomorphic order $-qz + a$ for some positive q and some real number a,*

(1) *the map $z \mapsto \text{TR}(A(z))$ is meromorphic with simple poles $d_j := \frac{a+n-j}{q}$, $j \in \mathbb{Z}_{\geq 0}$,*

(2) *[KV94] the complex residue at the point d_j is given by*

$$\text{Res}_{z=d_j} \text{TR}(A(z)) = \frac{1}{q} \text{Res}(A(d_j)). \tag{56}$$

(3) *[PS07] If $A(d_j)$ has a well-defined canonical trace $\text{TR}\big(A(d_j)\big)$, then $A'(d_j)$ has a well defined Wodzicki residue $\text{Res}(A'(d_j))$ and the constant term in the meromorphic expansion of $\text{TR}(A(z))$ at d_j is*

$$\text{fp}_{z=d_j} \text{TR}(A(z)) = \text{TR}\big(A(d_j)\big) + \frac{1}{q} \text{Res}(A'(d_j)). \tag{57}$$

In particular, this holds if $A(d_j)$ is a differential operator, in which case $\text{TR}\big(A(d_j)\big) = 0$ and we have

$$\text{fp}_{z=d_j} \text{TR}(A(z)) = \frac{1}{q} \text{Res}(A'(d_j)). \tag{58}$$

Remark 2.13. Actually, the derivatives $A'(d_j)$ are log-polyhomogeneous operators (see [Le99] and references therein) and formula (58) yields an extension of the Wodzicki residue to this particular operator, which differs from Lesch's graded residue.

2.2.2. *The index as a trace defect.* We specialise to a Hermitian $\mathbb{Z}_2$-bundle $E = E_+ \oplus E_-$ over a closed Riemannian manifold M and let $D =: \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ be an essentially selfadjoint elliptic differential operator of positive order d. The operator $\Delta := D_-D_+ \oplus D_+D_- = D^2$ is essentially self-adjoint and has a finite dimensional kernel.

Let $\Delta + S$ be an invertible perturbation of Δ by a smoothing operator $S \in \Psi_{cl}^{-\infty}(M, E)$; typically $S := \chi_{\{0\}}(\Delta)$, the orthogonal projection onto the kernel of Δ. The operator $\Delta + S$ is then a weight with principal angle $\theta = \pi$. Applying the $\mathbb{Z}_2$-graded version of Proposition 2.12 to the family $A(z) = \phi(\Delta + S)^{-z}$ yields the following corollary.

Corollary 2.14. *For any smooth function ϕ and any invertible perturbation $\Delta + S$ as above, the Wodzicki residue* $\mathrm{sRes}(\phi \log(\Delta + S))$ *is well-defined and we have*

$$\mathrm{fp}_{z=0}\mathrm{sTR}\left(\phi\,(\Delta + S)^{-z}\right) = -\frac{1}{2d}\mathrm{sRes}_\theta(\phi \log(\Delta + S)). \tag{59}$$

Proof. This follows from Proposition 2.12 applied to the family $A(z) = \phi\,(\Delta + S)^{-z}$ at $z = 0$, with the fibrewise trace replaced by the $\mathbb{Z}_2$-graded fibrewise trace. $\square$

Applying Corollary 2.14 to the constant 1-valued function ϕ yields the following formula for the index.

Corollary 2.15. [Sc10]

$$\mathrm{ind}D^+ = -\frac{1}{2d}\mathrm{sRes}(\log(\Delta + \chi_{\{0\}})). \tag{60}$$

Proof. Expressing the index of D_+ as the supertrace of the projection operator onto the kernel we write

$$
\begin{aligned}
\mathrm{ind}(D_+) &= \mathrm{sTr}(\chi_{\{0\}}(D)) \\
&= \mathrm{sTR}\left((\Delta + \chi_{\{0\}}(\Delta))^{-z}\right)
\end{aligned}
$$

since the nonzero eigenvalues of D_-D_+ and D_+D_- coincide. This last expression defines a constant meromorphic function, whose finite part at zero therefore coincides with the index:

$$
\begin{aligned}
\mathrm{ind}(D_+) &= \mathrm{fp}_{z=0}\mathrm{sTr}\left((\Delta + \chi_{\{0\}})^{-z}\right) \\
&\qquad \text{taking the finite part at zero} \\
&= -\frac{1}{2d}\mathrm{sRes}(\log(\Delta + \chi_{\{0\}})) \quad \text{using (59).}
\end{aligned}
$$

$\square$

2.3. **Trace defect formulae on coverings.**

Let as before, M be a (connected) closed Riemannian manifold and $\widetilde{M}$ a regular Γ-covering. Let $\pi\colon E \to M$ be a Hermitian vector bundle and $\widetilde{E} := \pi^*E \to \widetilde{M}$ its pullback.

Let $\mathcal{A} \in \Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$ be a Γ-invariant classical pseudodifferential operator. Let $x \in \widetilde{M}$, and $\sigma(\mathcal{A})(x, \cdot)$ denote the symbol in a local trivialisation around x.

Since $\mathcal{A}$ is Γ-invariant, the residue $\mathrm{Res}_x(\mathcal{A})\,dx$ and the canonical trace $\mathrm{TR}_x(\mathcal{A})\,dx$ densities introduced in Section 2.1 define Γ-invariant densities, leading to the following definitions. Let $F \subset X$ be a fundamental domain for the action of Γ on $\widetilde{M}$.

Definition 2.16. To a Γ-invariant classical pseudodifferential operator $\mathcal{A} \in \Psi_{\mathrm{cl},\Gamma}^{\mathbb{Z}}(\widetilde{M}, \widetilde{E})$, resp. $\mathcal{A} \in \Psi_{\mathrm{cl},\Gamma}^{\notin\mathbb{Z}_n}(\widetilde{M}, \widetilde{E})$ we assign the Γ-residue, resp. canonical Γ- trace

$$\mathrm{Res}_\Gamma(\mathcal{A}) := \int_F \mathrm{Res}_x(\mathcal{A})dx, \quad \text{resp.} \quad \mathrm{TR}_\Gamma(\mathcal{A}) := \int_F \mathrm{TR}_x(\mathcal{A})dx. \tag{61}$$

Remark 2.17. On operators in $\Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$ of order smaller than $-n$, which by [Va08, Satz 4.4] are Γ-trace-class, the canonical Γ-trace coincides with the ordinary L^2-trace

$$\mathrm{Tr}_\Gamma(\mathcal{A}) := \int_F \mathrm{tr}\,(\sigma(\mathcal{A})(x, \xi))\,dx\,d\xi = \int_F \mathrm{tr}\,(K_\mathcal{A}(x, x))\,dx, \tag{62}$$

where $K_\mathcal{A}$ stands for the Schwartz kernel of $\mathcal{A}$.

Definition-Proposition 2.18. (1) A Σ-class of operators $\Sigma(M, E) \subset \Psi_{\mathrm{cl}}(M, E)$ lifts to the class of Γ-invariant operators

$$\Sigma_\Gamma(\widetilde{M}, \widetilde{E}) := \{\mathcal{A} \in \Psi_{\Gamma,\mathrm{cl}}(\widetilde{M}, \widetilde{E}), \quad \mathcal{A} \underset{\mathrm{diag}}{\sim} \widetilde{A_0} \quad \text{for some } A_0 \in \Sigma(M, E)\},$$

using the notation of (20).

(2) Any local linear form $\Lambda \colon \Sigma(M, E) \to \mathbb{C}$ canonically lifts to a linear form $\Lambda_\Gamma \colon \Sigma_\Gamma(\widetilde{M}, \widetilde{E}) \to \mathbb{C}$ defined as

$$\Lambda_\Gamma(\mathcal{A}) := \Lambda(A) = \int_F \pi^* \left(\lambda\left(\mathrm{tr}_x \sigma(A)(x, \cdot)\right) dx\right) \quad \text{if } \mathcal{A} \in \widetilde{[A]_{\mathrm{diag}}}.$$

Remark 2.19. By construction local linear forms are constant on the horizontal lines of the diagramme in (3).

Proof. On an operator $\widetilde{A_0}$ for some ε-local operator $A_0 \in \Sigma(M, E)$, the Γ-invariant linear form is defined as $\Lambda_\Gamma(\mathcal{A}) := \Lambda(A_0)$. This determines Λ_Γ on $\Sigma_\Gamma(\widetilde{M}, \widetilde{E})$ since a local linear form is constant on $\sim_{\mathrm{diag}}$-equivalence classes and we can set

$$\Lambda_\Gamma(\widetilde{[A]_{\mathrm{diag}}}) = \Lambda([A]_{\mathrm{diag}}) := \Lambda(A_0) = \Lambda(A), \quad \text{for any } \varepsilon\text{-local operator } A_0 \in [A]_{\mathrm{diag}}. \quad \square$$

This applies the canonical trace and the Wodzicki residue.

Corollary 2.20. *The residue and the canonical trace densities are well-defined on lifted classes $\widetilde{[A]_{\mathrm{diag}}}$ of Definition 1.30 and are preserved under lifts.*

$$\mathcal{A} \in \widetilde{[A]_{\mathrm{diag}}} \implies \mathrm{Res}_\Gamma(\mathcal{A}) = \mathrm{Res}(A) \quad A \in \Psi_{\mathrm{cl}}^{\mathbb{Z}}(M, E),$$

and

$$\mathcal{A} \in \widetilde{[A]_{\mathrm{diag}}} \implies \mathrm{TR}_\Gamma(\mathcal{A}) = \mathrm{TR}(A) \quad A \in \Psi_{\mathrm{cl}}^{\notin \mathbb{Z}}(M, E).$$

In other words, the residue and the canonical trace are constant along the horizontal lines of the diagramme (3) in the introduction.

Consequently, trace-defect formulae on closed manifolds recalled in Proposition 2.12 induce trace-defect formula on coverings.

Theorem 2.21. *For any holomorphic family $\mathcal{A}(z) \in \Psi_{\mathrm{cl},\Gamma}(\widetilde{M}, \widetilde{E})$ of classical operators parametrised by $\mathbb{C}$ with holomorphic order $-qz + a$ for some positive q and some real number a,*

(1) *the map $z \mapsto \mathrm{TR}_\Gamma(\mathcal{A}(z))$ is meromorphic with simple poles $d_k := \frac{a+n-k}{q}$, $k \in \mathbb{Z}_{\geq 0}$,*

(2) *the complex residue at the point d_k is given by*

$$\mathrm{Res}_{z=d_k} \mathrm{TR}_\Gamma(\mathcal{A}(z)) = \frac{1}{q} \mathrm{Res}_\Gamma(\mathcal{A}(d_k)); \tag{63}$$

(3) *If $\mathcal{A}(d_k)$ has a well-defined Γ-canonical trace $\mathrm{TR}_\Gamma(\mathcal{A}(d_k))$, then $\mathcal{A}'(d_k)$ has a well defined Γ- Wodzicki residue $\mathrm{Res}_\Gamma(\mathcal{A}'(d_k))$ and the constant term in the meromorphic expansion of $\mathrm{TR}_\Gamma(\mathcal{A}(z))$ at d_k is*

$$\mathrm{fp}_{z=d_k} \mathrm{TR}_\Gamma(\mathcal{A}(z)) = \mathrm{TR}_\Gamma(\mathcal{A}(d_k)) + \frac{1}{q} \mathrm{Res}_\Gamma(\mathcal{A}'(d_j)); \tag{64}$$

(4) *If $\mathcal{A}(d_k)$ is a differential operator, this reduces to*

$$\mathrm{fp}_{z=d_k} \mathrm{TR}_\Gamma(\mathcal{A}(z)) = \frac{1}{q} \mathrm{Res}_\Gamma(\mathcal{A}'(d_j)); \tag{65}$$

(5) *All the above statements actually hold for any other representative in $[\mathcal{A}(z)]_{\mathrm{diag}}$.*

Proof. We use Proposition 1.27 to write $\mathcal{A}(z) \underset{\text{diag}}{\sim} \widetilde{A(z)}$ with $A(z)$ ε-local for some positive ε. On the grounds of Proposition 2.18, without loss of generality we can show the statements for $\mathcal{A}(z) = \widetilde{A(z)}$, and we have

$$\mathrm{TR}_\Gamma\left(\widetilde{A(z)}\right) = \mathrm{TR}\,(A(z)). \tag{66}$$

(1) then follows from the meromorphicity and the structure of the poles $\{d_k, k \in \mathbb{Z}_{\geq 0}\}$ of $\mathrm{TR}(A(z))$ discussed in Proposition 2.12. Similarly, (63) follows from (56) combined with (see Proposition 2.12)

$$\mathrm{Res}_\Gamma\left(\widetilde{A(d_j)}\right) = \mathrm{Res}\left(A(d_j)\right). \tag{67}$$

To prove (3) we assume that $A(d_k)$ has a well-defined canonical trace, in which case $\widetilde{A(d_k)}$ has a well-defined Γ-canonical trace. We apply (64) to the family $A(z)$, which combined with (66) yields

$$\begin{aligned}
\mathrm{fp}_{z=d_k}\mathrm{TR}_\Gamma(\widetilde{A(z)}) &= \mathrm{fp}_{z=d_k}\mathrm{TR}\,(A(z)) \\
&= \mathrm{TR}\,(A(d_k)) + \frac{1}{q}\mathrm{Res}\,(A'(d_k)) \\
&= \mathrm{TR}\left(\widetilde{A(d_k)}\right) + \frac{1}{q}\mathrm{Res}\,(A'(d_k)).
\end{aligned}$$

Since the operator $A'(d_k)$ has a well-defined residue, the lifted derivative $\widetilde{A'(d_k)}$ has a well-defined Γ-residue and we have $\mathrm{Res}_\Gamma\left(\widetilde{A'(d_k)}\right) = \mathrm{Res}\,(A'(d_k))$, leading to (65).

The fact that these statements depend only on the class $[\mathcal{A}(z)]_{\text{diag}}$ and not on the representative follows from the fact that TR and Res are well defined on such classes. □

Corollary 2.22. *Let $A(z)$ be a holomorphic family of operators in $\Psi_{\text{cl}}(M, E)$. For some positive number ε, there is a holomorphic family $A_0(z)$ of ε-local operators in $\Psi_{\text{cl}}(M, E)$ such that $A_0(z) \in [A(z)]_{\text{diag}}$.*

For any holomorphic family $\mathcal{A}(z)$ in $\Psi_{\text{cl},\Gamma}(\widetilde{M}, \widetilde{E})$ such that the difference $\mathcal{A}(0) - \widetilde{A_0(0)}$ at zero has a smooth kernel, the map $z \mapsto \mathrm{TR}_\Gamma\,(\mathcal{A}(z)) - \mathrm{TR}\,(A(z))$ is holomorphic at 0 with

$$\mathrm{fp}_{z=0}\mathrm{TR}_\Gamma\,(\mathcal{A}(z)) - \mathrm{fp}_{z=0}\mathrm{TR}\,(A(z)) = \mathrm{Tr}_\Gamma(\mathcal{A}(0) - \widetilde{A_0(0)}), \tag{68}$$

where as before, Tr_Γ is defined in (62).

Proof. For some given positive ε, Definition 1.30 yields a family of ε-local operators $A_0(z)$ in $[A(z)]_{\text{diag}}$. The explicit construction of these operators by means of an appropriate partition of unity shows that this family can be chosen holomorphic as a consequence of the holomorphicity of $A(z)$. It then follows from Proposition 2.18 that

$$\mathrm{TR}_\Gamma\left(\widetilde{A_0(z)}\right) = \mathrm{TR}\,(A_0(z)) = \mathrm{TR}\,(A(z)).$$

The operators $\mathcal{B}(z) := \mathcal{A}(z) - \widetilde{A_0(z)}$ define a holomorphic family in $\Psi_{\text{cl},\Gamma}(\widetilde{M}, \widetilde{E})$. It follows from the above that

$$\mathrm{fp}_{z=0}\mathrm{TR}_\Gamma\,(\mathcal{A}(z)) - \mathrm{fp}_{z=0}\mathrm{TR}\,(A(z)) = \mathrm{fp}_{z=0}\mathrm{TR}_\Gamma\,(\mathcal{A}(z)) - \mathrm{fp}_{z=0}\mathrm{TR}\,(A_0(z)) = \mathrm{fp}_{z=0}\mathrm{TR}_\Gamma\,(\mathcal{B}(z)). \tag{69}$$

Assuming that the operator $\mathcal{A}(0) - \widetilde{A_0(0)}$ has a smooth kernel, then the operator $\mathcal{B}(0)$ has a smooth kernel and hence a vanishing Wodzicki residue. It then follows from (63) that the map $z \mapsto \mathrm{TR}_\Gamma\,(\mathcal{B}(z))$ has a vanishing complex residue at zero, showing its holomorphicity

at zero. The fact that $\mathcal{B}(0)$ has a smooth kernel also implies that it has a well-defined canonical trace which coincides with the ordinary Γ-trace $\mathrm{Tr}_\Gamma\,(\mathcal{B}(0))$ and we have

$$\mathrm{fp}_{z=0}\mathrm{TR}_\Gamma\,(\mathcal{B}(z)) = \lim_{z\to 0}\mathrm{TR}_\Gamma\,(\mathcal{B}(z)) = \mathrm{Tr}_\Gamma(\mathcal{B}(0)) = \mathrm{Tr}_\Gamma(\mathcal{A}(0) - \widetilde{A_0(0)}). \qquad (70)$$

Combining equations (69) and (70) yields (68). $\qquad\square$

2.4. **Lifted spectral ζ-invariants.** This section is devoted to applications of Formula (68) in Corollary 2.22.

Let E be a hermitian vector bundle over a closed Riemannian manifold M, and let $\mathbf{D}$ be an essentially self-adjoint differential operator in $\Psi(M, E)$. Then $\mathbf{\Delta} := \mathbf{D}^2$ is a non-negative essentially self-adjoint operator.

For any $\varepsilon > 0$, let $Q_\varepsilon(\mathbf{\Delta})$ be a weight as defined in (35) and for a measurable function h on on a contour Γ_π around the ray $R_\pi =]-\infty, 0]$, such that

$$|h(\lambda)| \le |\lambda|^{-\delta} \qquad (71)$$

for some positive δ, let $h(Q_\varepsilon(\mathbf{\Delta}))$ be defined by (37). Similarly, we consider $h(Q_\varepsilon(\widetilde{\mathbf{\Delta}}))$ with $Q_\varepsilon(\widetilde{\mathbf{\Delta}})$ defined by (43).

In the specific case when h is polynomial, then $h(\mathbf{\Delta})$ is a well-defined differential operator, which lifts to $\widetilde{h(\mathbf{\Delta})} = h(\widetilde{\mathbf{\Delta}})$.

Let h be a measurable function h on a contour Γ_π around the ray $R_\pi =]-\infty, 0]$, satisfying (71) for some positive $\mathbf{\Delta}$, we set $h_\varepsilon(\mathbf{\Delta}) := h(Q_\varepsilon(\mathbf{\Delta}))$ and $h_\varepsilon(\widetilde{\mathbf{\Delta}}) := h(Q_\varepsilon(\widetilde{\mathbf{\Delta}}))$.

For any $A \in \Psi_{\mathrm{cl}}(M, E)$ and any weight $Q \in \Psi_{\mathrm{cl}}(M, E)$, the family $A(z) := A\,Q^{-z}$ is a holomorphic perturbation of A and the **spectral ζ-function** (or Q-regularised trace of A)

$$z \longmapsto \zeta_{A,Q}(z) := \mathrm{TR}(A\,Q^{-z}) \qquad (72)$$

defines a meromorphic map on $\mathbb{C}$ with a known countable set of simple poles. Similarly, for any $\mathcal{A} \in \Psi_\Gamma(\widetilde{M}, \widetilde{E})$ and any weight $\mathfrak{Q} \in \Psi_\Gamma(\widetilde{M}, \widetilde{E})$, using (50) we define the holomorphic perturbation

$$\mathcal{A}(z) := \mathcal{A}\,\mathfrak{Q}^{-z}$$

of $\mathcal{A}$ and thanks to Theorem 2.21, we know that the spectral ζ-function (or $\mathfrak{Q}$- regularised trace of $\mathcal{A}$)

$$z \longmapsto \zeta_{\mathcal{A},\mathfrak{Q}}^\Gamma(z) := \mathrm{TR}_\Gamma(\mathcal{A}\,\mathfrak{Q}^{-z}) \qquad (73)$$

is meromorphic with a known countable set of simple poles. So we can take the finite parts of the Laurent expansions at zero and build (with some abuse of notation) **spectral ζ-invariants**

$$\zeta_{A,Q}(0) := \mathrm{fp}_{z=0}\left(\zeta_{A,Q}(z)\right); \quad \zeta_{\mathcal{A},\mathfrak{Q}}^\Gamma(0) := \mathrm{fp}_{z=0}\left(\zeta_{\mathcal{A},\mathfrak{Q}}^\Gamma(z)\right). \qquad (74)$$

Take for $\varepsilon > 0$

$$A_\varepsilon(\mathbf{D}) := P(\mathbf{D})\,h_\varepsilon(\mathbf{\Delta}); \; \mathcal{A}_\varepsilon(\widetilde{\mathbf{D}}) = P(\widetilde{\mathbf{D}})\,h_\varepsilon(\widetilde{\mathbf{\Delta}}).$$

Remark 2.23. Recall from Remark 1.13 that whereas the equivalence relation $\sim$ is stable under products of operators, the equivalence relation $\underset{\mathrm{diag}}{\sim}$ is not.

In particular, we do not a priori expect the regularised trace $\zeta_{A_\varepsilon(\mathbf{D}),Q_\varepsilon(\mathbf{\Delta})}(0)$ to coincide with the regularised Γ-trace $\zeta_{\mathcal{A}_\varepsilon(\widetilde{\mathbf{D}}),\mathfrak{Q}_\varepsilon(\widetilde{\mathbf{\Delta}})}(0)$, which involves the product $P(\widetilde{D})\,h_\varepsilon(\widetilde{\mathbf{\Delta}})\,Q_\varepsilon(\widetilde{\mathbf{\Delta}})^{-z}$. The following theorem nevertheless relates the two regularised traces.

Theorem 2.24. *Let $\varepsilon > 0$. With the above notations*

- *The meromorphic map*

$$z \longmapsto \zeta_{A(\mathbf{D}), Q_\varepsilon(\Delta)}(z) - \zeta^\Gamma_{\mathcal{A}(\widetilde{\mathbf{D}}), \mathfrak{Q}_\varepsilon(\widetilde{\Delta})}(z)$$

  is holomorphic at zero.

- *There is some positive α, and an α-local operator $A_{\varepsilon,0}(\mathbf{D}) \in [A_\varepsilon(\mathbf{D})]_{\mathrm{diag}}$ such that the difference $\widetilde{A_{\varepsilon,0}(\mathbf{D})} - \mathcal{A}_\varepsilon(\widetilde{\mathbf{D}})$ has a smooth kernel and hence a well-defined trace $\mathrm{Tr}_\Gamma \left(\widetilde{A_{\varepsilon,0}(\mathbf{D})} - \mathcal{A}_\varepsilon(\widetilde{\mathbf{D}})) \right)$. We have*

$$\zeta_{A_\varepsilon(\mathbf{D}), Q_\varepsilon(\Delta)}(0) - \zeta^\Gamma_{\mathcal{A}_\varepsilon(\widetilde{\mathbf{D}}), \mathfrak{Q}_\varepsilon(\widetilde{\Delta})}(0) = \mathrm{Tr}_\Gamma(\widetilde{A_{\varepsilon,0}(\mathbf{D})} - \mathcal{A}_\varepsilon(\widetilde{\mathbf{D}})), \tag{75}$$

- *Spectral ζ-invariants canonically lift to the covering. In other words, when $h \equiv 1$, identity (75) amounts to the coincidence of the zeta-regularised trace and its lifted counterpart:*

$$\zeta_{P(\mathbf{D}), Q_\varepsilon(\Delta)}(0) = \zeta^\Gamma_{P(\widetilde{\mathbf{D}}), \mathfrak{Q}_\varepsilon(\widetilde{\Delta})}(0). \tag{76}$$

- *The above statements extend to super-regularised traces for operators acting on sections of a $\mathbb{Z}_2$-graded bundle, replacing the canonical trace TR by a $\mathbb{Z}_2$-graded canonical tace sTR and correspondingly $\zeta-$ functions by $\mathbb{Z}_2$-graded $\zeta-$ functions $s\zeta_{A,Q}(z) := \mathrm{sTR}(A\,Q^{-z})$.*

Proof. The operators $Q_\varepsilon(\Delta)^{-z}$, resp. $Q_\varepsilon(\widetilde{\Delta})^{-z}$ built from complex powers define holomorphic families on M, resp. $\widetilde{M}$ and hence so do the operators $B_\varepsilon(\mathbf{D})(z) := A_\varepsilon(\mathbf{D})\,Q_\varepsilon(\Delta)^{-z}$, resp. $\mathcal{B}_\varepsilon(\widetilde{\mathbf{D}})(z) := \mathcal{A}_\varepsilon(\widetilde{\mathbf{D}})\,Q_\varepsilon(\widetilde{\Delta})^{-z}$ define homolorphic families on M, resp. $\widetilde{M}$, to which we want to apply Corollary 2.22. To simplify notations, we drop the index ε and the operator $\mathbf{D}$ from the notation, setting $B(z) := B_\varepsilon(z)(\mathbf{D})$ and $\mathcal{B}(z) := \mathcal{B}_\varepsilon(z)(\widetilde{\mathbf{D}})$. With these notations

$$\zeta_{A_\varepsilon(\mathbf{D}), Q_\varepsilon(\Delta)}(0) - \zeta_{\mathcal{A}_\varepsilon(\widetilde{\mathbf{D}}), \mathfrak{Q}_\varepsilon(\widetilde{\Delta})}(0) = \mathrm{TR}_\Gamma\left(\mathcal{B}(z)\right) - \mathrm{TR}\left(B(z)\right). \tag{77}$$

On the one hand and as in Corollary 2.22, we build a ε-local holomorphic family $B_0(z) \in [B(z)]_{\mathrm{diag}}$. In particular, the operator $B_0 := B_0(0) = A_\varepsilon(\mathbf{D})$ has symbol $P(\sigma(\mathbf{D})) \star h_\star(\sigma(Q_\varepsilon(\Delta)))$. So, on the one hand its lift $\widetilde{B_0}$ has symbol $P(\widetilde{\sigma(\mathbf{D})}) \star h_\star(\widetilde{\sigma(Q_\varepsilon(\Delta))})$. On the other hand, the operator $\mathcal{B}(0) = \mathcal{A}_\varepsilon(\widetilde{\mathbf{D}})$ has symbol $P(\sigma(\widetilde{\mathbf{D}})) \star h_\star(\sigma(Q_\varepsilon(\widetilde{\Delta})))$. By (41) applied to $\mathfrak{Q} = Q_\varepsilon(\widetilde{\Delta})$ and $Q = Q_\varepsilon(\Delta)$, we have $h_\star(\sigma(Q_\varepsilon(\widetilde{\Delta}))) \sim h_\star(\widetilde{\sigma(Q_\varepsilon(\Delta))})$. Consequently, using again the locality of $\star$ as in Lemma 1.39, we find that the operator $\mathcal{B}(0)$ has the same symbol as $\widetilde{B_0}$. Hence the difference $\mathcal{B}(0) - \widetilde{B_0}$ lies in $\cap_{m \in \mathbb{R}} \mathcal{U}\Psi^m_\Gamma(\widetilde{M}, \widetilde{E})$ as a result of which it is Γ-trace-class and therefore has a well-defined Γ-trace $\mathrm{Tr}_\Gamma\left(\mathcal{B}(0) - \widetilde{B_0}\right)$ (see 62). Applying Corollary 2.22 yields the holomorphicity at $z = 0$ of the map in (77)

$$\zeta_{A_\varepsilon(\mathbf{D}), Q_\varepsilon(\Delta)}(0) - \zeta^\Gamma_{\mathcal{A}_\varepsilon(\widetilde{\mathbf{D}}), \mathfrak{Q}_\varepsilon(\widetilde{\Delta})}(0) = \mathrm{fp}_{z=0}\left(\mathrm{TR}_\Gamma(\mathcal{B}(z)) - \mathrm{TR}_\Gamma(\widetilde{B_0(z)})\right) = \mathrm{Tr}_\Gamma\left(\mathcal{B}(0) - \widetilde{B_0}\right),$$

which corresponds to (75).

If $h \equiv 1$, then $B_0 = B(0) = P(\mathbf{D})$ is a differential operator and we have $\mathcal{B}(0) = P(\widetilde{\mathbf{D}}) = \widetilde{P(\mathbf{D})} = \widetilde{B(0)} = \widetilde{B_0}$ so that $\mathrm{Tr}_\Gamma(\mathcal{B}(0) - \widetilde{B_0}) = 0$, from which the assertion (76) follows. Replacing TR by its $\mathbb{Z}_2$-graded analog sTR yields the last assertion. $\qquad\square$

2.4.1. *The L^2-Atiyah index theorem revisited.*

We apply the above construction to a hermitian $\mathbb{Z}_2$-bundle $E = E_+ \oplus E_-$ over a closed Riemannian manifold M, so that its pull-back $F := \widetilde{E} = \widetilde{E}_+ \oplus \widetilde{E}_-$ by π is a $\mathbb{Z}_2$ graded Γ-equivariant vector bundle over $X := \widetilde{M}$.

Let $D_+ : C^\infty(M, E_+) \to C^\infty(M, E_-)$ be an elliptic differential operator of order d with

formal adjoint $D_- : C^\infty(M, E_-) \to C^\infty(M, E_+)$. Let $\widetilde{D}_\pm : C^\infty(X, F_\pm) \to C^\infty(X, F_\mp)$ be the lifted differential operators. The operator

$$\widetilde{D} := \begin{bmatrix} 0 & \widetilde{D}_- \\ \widetilde{D}_+ & 0 \end{bmatrix} \tag{78}$$

is a Γ-invariant elliptic differential operator of positive order d to which we apply the above constructions.

Even though the kernels $\{s \in C_c^\infty(\widetilde{M}, \widetilde{E}), \ \widetilde{D}_\pm s = 0\}$ are not necessarily finite dimensional, their closures $K_{\widetilde{D}_\pm}$ are finitely generated Γ-modules and hence isometrically Γ-isomorphic to Hilbert Γ-subspaces of the Hilbert space $\ell_2(\Gamma)^n$ for some positive integer n, which can be represented by idempotent matrices $P^\pm = (p_{ij}^\pm) \in \mathrm{gl}_n(\mathcal{N}\Gamma)$. Let $\chi_{\{0\}}(\widetilde{D}_\pm) = \chi_{\{0\}}(\widetilde{\Delta})$ denote the orthogonal projections onto $K_{\widetilde{D}_+} \oplus K_{\widetilde{D}_-}$.

The Γ-dimension (resp. Γ- graded dimension) of $K_{\widetilde{D}_\pm}$ is (see Appendix B)

$$\dim_\Gamma K_{\widetilde{D}_\pm} := \sum_{i=1}^n \langle p_{ii}^\pm(e), e \rangle, \quad \text{resp.} \quad \mathrm{sdim}_\Gamma(K_{\widetilde{\Delta}}) := \dim_\Gamma K_{\widetilde{D}_+} - \dim_\Gamma K_{\widetilde{D}_-} = \mathrm{sTr}_\Gamma\left(\chi_{\{0\}}(\widetilde{\Delta})\right),$$

where $e \in \mathbb{C}\Gamma$ is the element with all components zero outside the first one which is one and the Γ-index of D is

$$\mathrm{ind}_\Gamma(\widetilde{D}) := \dim_\Gamma K_{\widetilde{D}_+} - \dim_\Gamma K_{\widetilde{D}_-}.$$

The Γ-Wodzicki residue and the Γ-canonical trace extend in a straightforward manner to a $\mathbb{Z}_2$-graded Wodzicki residue sRes_Γ and a $\mathbb{Z}_2$-graded canonical trace sTR_Γ.

Corollary 2.25. *With the notation of* (78)*,* $\log(Q_\varepsilon(\widetilde{\Delta}))$ *has a well-defined (super) Γ-residue and we have*

$$\mathrm{ind}_\Gamma(\widetilde{D}_+) = -\frac{1}{q}\mathrm{sRes}_\Gamma(\log(Q_\varepsilon(\widetilde{\Delta}))) = -\frac{1}{q}\,\mathrm{sRes}(\log(Q_\varepsilon(\Delta))) = \mathrm{ind}(D_+). \tag{79}$$

Proof. By Corollary 2.15, the index is a $\mathbb{Z}_2$-graded regularised trace of the identity: $\mathrm{ind}(D_+) = \mathrm{sTR}^{Q_\varepsilon}(Id)$ and similarly, independently of $\varepsilon > 0$, we have

$$
\begin{aligned}
\mathrm{ind}_\Gamma(\widetilde{D}_+) &= \mathrm{sTR}_\Gamma\left(\chi_{\{0\}}(\widetilde{\Delta})\right) \\
&= \mathrm{sTR}_\Gamma\left(\chi_{[-\varepsilon,\varepsilon]}(\widetilde{\Delta})\right) = \mathrm{sTR}_\Gamma(Q_\varepsilon(\widetilde{\Delta})^{-z}) \quad \text{seen as meromorphic functions} \\
&= \mathrm{fp}_{z=0}\mathrm{sTR}_\Gamma\left(\left(Q_\varepsilon(\widetilde{\Delta})\right)^{-z}\right) \quad \text{taking the finite part at zero} \\
&= \mathrm{sTR}^{Q_\varepsilon(\widetilde{\Delta})}(\widetilde{Id}).
\end{aligned}
\tag{80}
$$

Theorem 2.24 applied to $h \equiv 1$ and $\mathbf{D} \equiv 1$ then yields $\mathrm{ind}_\Gamma(\widetilde{D}_+) = \mathrm{ind}_\Gamma(D_+)$. The lifted trace-defect formula (64) derived in Theorem 2.21 applied to the family $\mathcal{A}(z) := \left(Q_\varepsilon(\widetilde{\Delta})\right)^{-z}$ further yields the expression of $\mathrm{ind}_\Gamma(\widetilde{D}_+)$ as a Wodzicki residue. $\qquad\square$

2.4.2. *The η-invariant revisited.* We now assume that both D and $\widetilde{D}$ are essentially self-adjoint and invertible, in which case $Q := D^2$ is a weight which lifts to $\widetilde{Q} = \widetilde{D}^2$. The operators $|D|^{-1} = \Delta^{-\frac{1}{2}}$ and $|\widetilde{D}|^{-1} = \widetilde{\Delta}^{-\frac{1}{2}}$ are defined as Cauchy integrals (see (37)) using $h(x) = x^{-\frac{1}{2}}$ and the η-invariant of D can be expressed in terms of regularised traces [CDP03] as $\eta(D) = \mathrm{Tr}^Q(D\,|D|^{-1}); \quad \eta_\Gamma(\widetilde{D}) = \mathrm{Tr}_\Gamma^{\widetilde{Q}}(\widetilde{D}\,|\widetilde{D}|^{-1}).$

Corollary 2.26. *There is an ε-local operator $A_0 \in \left[D \, |D|^{-1} \right]_{\mathrm{diag}}$ for some small enough positive ε (see Definition 1.30), such that the difference $\widetilde{A_0(D)} - \widetilde{D} \, |\widetilde{D}|^{-1}$ has a smooth kernel and a well-defined Γ-trace and we have*

$$\eta(D) - \eta_{\Gamma}(\widetilde{D}) = \mathrm{TR}_{\Gamma} \left(\widetilde{A_0(D)} - \widetilde{D} \, |\widetilde{D}|^{-1} \right).$$

Proof. The statement is a straightforward consequence of Theorem 2.24 applied to $P(x) = x$ and $h(x) = x^{-\frac{1}{2}}$, and the right hand side does not depend on the representative A_0 in $[D \, |D|^{-1}]_{\mathrm{diag}}$. $\qquad\square$

2.5. **Invariants built from geometric operators.** Let $F \to X$ be a vector bundle over a Riemannian manifold (X, g). We assume that X is spin and $F = S \otimes W$ where S is the spinor bundle and W an auxillary bundle equipped with a connection ∇^W. This way, F can be equipped with a connection $\nabla^F := \nabla \otimes 1 + 1 \otimes \nabla^W$, where ∇ is the Levi-Civita connection on S.

Following Gilkey's notations [Gi95, Formula (2.4.3)], for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_s)$ we introduce formal variables $g_{ij/\alpha} := \partial_\alpha g_{ij}$ for the partial derivatives of the metric tensor g on the manifold X and similarly $\omega_{i/\alpha} := \partial_\alpha \omega_i$ for the connection ω on the external bundle. Let us set $\mathrm{ord}\left(g_{ij/\alpha} \right) = |\alpha| = \alpha_1 + \cdots + \alpha_s$; $\mathrm{ord}(\omega_{i/\beta}) = |\beta|$. Inspired by Gilkey [Gi95, Formulae (1.8.18) and (1.8.19)] and following [MP] we set the following definition.

Definition 2.27. Let $A \in \Psi_{\mathrm{cl}}(X, F)$ be a classical (resp. a log-polyhomogeneous –see [Le99] and references therein–) operator of order a with symbol $\sigma(A) \sim \sum_{j=0}^\infty \sigma_{a-j}(A)(\xi)$ (resp. $\sigma(A)(\xi) \sim \sum_{\ell=0}^k \sum_{j=0}^\infty \sigma_{a-j,\ell}(A) \log^\ell |\xi|$) with $\sigma_{a-j}(A)(\xi)$ (resp. $\sigma_{a-j,\ell}(A)(\xi)$) homogeneous of degree $a - j$ for $|\xi| \geq 1$. We call A **geometric**, if in any local trivialisation, the homogeneous components $\sigma_{a-j}(A)$ (resp. $\sigma_{a-j,\ell}(A)$ for any $\ell \in \{0, \cdots, k\}$) are homogeneous of order j in the jets of the metric and of the connection.

In particular, a differential operator $A = \sum_{|\alpha| \leq a} c_\alpha(x) \partial_x^\alpha$ is geometric if $c_\alpha(x)$ is homogeneous of degree $j = a - |\alpha|$ in the metric and the connection on the auxillary bundle. The Laplace-Beltrami operator (resp. the Dirac operator) associated with the metric g (resp. and the connection on W) are geometric differential operators. More generally, the Laplace operator (resp. Dirac operator) associated with the connection ∇^F are geometric differential operators.

Lemma 2.28. *An operator $\mathcal{A} \in \Psi_{\Gamma,\mathrm{cl}}(\widetilde{M}, \widetilde{E})$ such that*

$$\mathcal{A} \sim \widetilde{A}_0 \quad \text{and} \quad A_0 \sim A$$

for some geometric operator $A \in \Psi_{\mathrm{cl}}(M, E)$, is itself geometric. In particular, with the notations of (33) we have

$$\left(A \text{ geometric and } \mathcal{A} \in \widetilde{[A]_{\mathrm{diag}}} \right) \Longrightarrow (\mathcal{A} \text{ geometric}).$$

Proof. Since $\sim$ preserves symbols and symbols lift to the covering (see (34)), we have

$$\sigma(\mathcal{A}) \sim \sigma\left(\widetilde{A}_0 \right) \sim \widetilde{\sigma(A_0)} \sim \widetilde{\sigma(A)}.$$

The fact that the operator A is geometric amounts to the components $\sigma_{a-j,\ell}(A)$ being homogeneous of degree j in the jets of the metric and the connection. Its lift $\widetilde{\sigma(A)}$ obeys the same conditions w.r.t to the metric $\widetilde{g}$ and the connection $\widetilde{\nabla}^W$ on the covering and hence so does $\sigma(\mathcal{A})$, which shows that $\mathcal{A}$ is geometric. $\qquad\square$

The results of [MP] relative to holomorphic families of the type $A(z) = A\,Q^{-z}$ generalise to any holomorphic family. The proof of this more general statement can be carried out along the same lines of the proofs of [MP, Corollary 1 and Theorem 1]. Adopting Gilkey's notations [Gi95, par. 2.4], let us denote by $\mathcal{P}^{g,\nabla^W}_{n,k,p}$ (which we write $\mathcal{P}^{g}_{n,k,p}$ in the absence of twisting) the linear space consisting of p- form valued invariant polynomials that are homogeneous of order k in the jets of the metric and of the connection ∇^W.

Theorem 2.29. *Let $A(z) \in \Psi_{\mathrm{cl}}(M,E)$ and $\mathcal{A}(z) \in \Psi_{\Gamma,\mathrm{cl}}(\widetilde{M},\widetilde{E})$ be two holomorphic families of classical pseudodifferential operators such that*

$$A(z) \text{ geometric and } \mathcal{A}(z) \in \widetilde{[A(z)]_{\mathrm{diag}}}.$$

Then $\mathcal{A}(z)$ is geometric and if both $A(0)$ and $\mathcal{A}(0)$ are differential operators,

(1) *for any $x \in \widetilde{M}$, the residue densities defined in* (61)

$$\mathrm{Res}_x(\mathcal{A}'(0)) = \widetilde{\mathrm{Res}_{\pi(x)}(A'(0))}$$

lie in $\mathcal{P}^{\widetilde{g},\widetilde{\nabla}^W}_{n,n,n}$.

(2) *Consequently,*

$$\mathrm{fp}_{z=0}\mathrm{TR}_{\Gamma}\left(\mathcal{A}(z)\right) = \frac{1}{q}\,\mathrm{Res}_{\Gamma}(\mathcal{A}'(0)) = \frac{1}{q}\int_F \mathrm{Res}_x(\mathcal{A}'(0))(x)\,dx = \frac{1}{a}\int_M \mathrm{Res}_x(\mathcal{A}'(0))(x)\,dx$$

is the integral of densities generated by Pontrjagin forms on the fundamental domain and Chern forms on the auxillary bundle.

Remark 2.30. In (61) the residue densities are defined for classical pseudodifferential operators whereas $\mathcal{A}'(0)$ is a log-polyhomogeneous operator (see e.g. [Le99] and references therein) but it follows from the previous results that they extend to derivatives $\mathcal{A}'(0)$ whenever $\mathcal{A}(0)$ is differential.

Proof. In order to get identities on the level of densities, we need to apply the above results to families $\phi\,\mathcal{A}(z)$ for any Γ-invariant function ϕ on $\widetilde{M}$.

(1) Since $\mathcal{A}(0)$ and $A(0)$ are differential operators, we know that $\phi\,\mathcal{A}'(0)$ and $\phi\,A'(0)$ (which are not classical operators) have well defined Wodzicki residue. The fact that the Wodzicki residue canonically lifts to coverings (Proposition 2.18) applied to $\phi\,\mathcal{A}'(0)$ then yields for any smooth Γ-invariant function ϕ on $\widetilde{M}$

$$\int_F \mathrm{Res}_x\left(\phi\,\mathcal{A}'(0)\right)\,dx = \mathrm{Res}_{\Gamma}\left(\phi\,\mathcal{A}'(0)\right) = \mathrm{Res}\left(\phi\,A'(0)\right) = \int_M \mathrm{Res}_x\left(\phi\,A'(0)\right)\,dx$$

from which we deduce the identity on the level of densities:

$$\mathrm{Res}_x(\mathcal{A}'(0)) = \widetilde{\mathrm{Res}_{\pi(x)}(A'(0))}.$$

(2) The second statement follows from combining (65) which relates the finite part of the canonical trace to the residue applied to $\phi\,\mathcal{A}(z)$ for any Γ-invariant function ϕ on $\widetilde{M}$, with Gilkey's theory of invariants [Gi95, Theorem 2.6.2] since $\mathcal{P}^{g,\nabla^W}_{n,n,n}$ is a polynomial in the 2-jets of the metric and the one jets of the auxiliary connection.

$\square$

Consequently, ζ-spectral invariants for geometric operators can be written as integrals of densities generated by Pontrjagin forms on the underlying manifold and Chern forms on the auxillary bundle.

Corollary 2.31. *With the notations of* (76)*, the spectral zeta invariants*

$$\zeta_{A_\varepsilon(\mathbf{D}),Q_\varepsilon(\Delta)}(0) = \zeta_{\mathscr{A}_\varepsilon(\widetilde{\mathbf{D}}),\mathfrak{Q}_\varepsilon(\widetilde{\Delta})}(0)$$

can be written as integrals of densities generated by Pontrjagin forms on the underlying manifold and Chern forms on the auxillary bundle.

Proof. This follows from applying Theorem 2.29 to the families $z \mapsto A_\varepsilon(\mathbf{D}) \, Q_\varepsilon(\Delta)^{-z} \in \Psi_{\mathrm{cl}}(M, E)$, resp. $z \mapsto \mathscr{A}_\varepsilon(\widetilde{\mathbf{D}}) \, Q_\varepsilon(\widetilde{\Delta})^{-z} \in \Psi_{\Gamma,\mathrm{cl}}(\widetilde{M}, \widetilde{E})$. $\square$

APPENDIX A. PSEUDODIFFERENTIAL OPERATORS ON OPEN SUBSETS OF $\mathbb{R}^n$

We review known results following [Sc10, Ta81], thus fixing the notation needed in the paper. Let $U \subset \mathbb{R}^n$ be an open subset and $k \in \mathbb{N}$.

Definition A.1.   • For any $m \in \mathbb{R}$, let $S^m(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$ be the set of functions $\sigma \in C^\infty(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$ such that

$$\forall \alpha, \beta \in \mathbb{Z}_{\geq 0}^n, \text{ for any compact } K \subset U, \quad \exists C_{\alpha \beta K} > 0, \quad \|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)\| \leq C_{\alpha \beta K}(1 + |\xi|)^{\Re(m) - |\beta|}, \quad (81)$$

where $\| \cdot \|$ is the supremum norm on $\mathrm{gl}_k(\mathbb{C})$,

• For any $m \in \mathbb{C}$, let $S_{cl}^m(U, \mathbb{C}^k)$ be the space of classical symbols, *i.e.* the set of symbols $\sigma \in S^{\Re(m)}(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$ such that there exists $\sigma_{m-j} = \sigma_{m-j}(x, \xi)$ in $C^\infty(U \times \mathbb{R}^n \setminus \{0\}, \mathrm{gl}_k(\mathbb{C}))$, $j = 0, 1, 2, \ldots$ with σ_{m-j} positively homogeneous of degree $m - j$ with respect to ξ obeying the following relation $\sigma(x, \xi) \sim \sum_{j=0}^\infty \sigma_{m-j}(x, \xi)$ *i.e.*, for an excision function $\chi(\xi)$ around zero (By an excision function around a point we mean a smooth function on $\mathbb{R}$ which vanishes in a neighborhood of the point and is one outside a ball of radius one centered at this point) and any positive integer N,

$$\sigma(x, \xi) - \sum_{j=0}^N \chi(\xi) \sigma_{m-j}(x, \xi) \in S^{\Re(m) - N}(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C})). \quad (82)$$

• We call $S^{-\infty}(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C})) := \cap_{m \in \mathbb{R}} S^m(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C})) = \cap_{m \in \mathbb{C}} S_{cl}^m(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$ the space of smoothing symbols on U.

Remark A.2. For any $m \in \mathbb{C}$, the relation

$$\sigma \sim \tau \iff \sigma - \tau \in S^{-\infty}(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C})) \iff \sigma_{m-j} = \tau_{m-j} \quad \forall j \in \mathbb{Z}_{\geq 0} \quad (83)$$

defines an equivalence relation on $S_{cl}^m(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$.

By Schwartz's kernel theorem, to a continuous linear operator $A : C_c^\infty(U, \mathbb{C}^k) \to C^\infty(U, \mathbb{C}^k)$ we assign its (uniquely defined) distributional Schwartz kernel $K_A \in \mathcal{D}'(U \times E) \otimes \mathrm{gl}_k(\mathbb{C})$.

Definition A.3.   (1) For any $m \in \mathbb{R}$ (resp. for any $m \in \mathbb{C}$), a continuous linear operator $A : C_c^\infty(U, \mathbb{C}^k) \to C^\infty(U, \mathbb{C}^k)$ is **pseudodifferential (resp. a classical pseudodifferential operator) of order** m if it its kernel is an oscillatory integral of the type:

$$K_A(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi \rangle} a(x, y, \xi) \, d\xi, \quad (84)$$

where the amplitude $a(x, y, \xi)$ lies in the symbol space $S^m(U \times U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$ (resp. $S_{cl}^m(U \times U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$).

(2) To a given symbol $\sigma \in S^m(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$ one assigns the pseudodifferential operator

$$(\mathrm{Op}(\sigma)u)(x) = \int_U \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \sigma(x, \xi) u(y) \, dy \, d\xi, \quad \forall u \in C_c^\infty(U, \mathbb{C}^k). \quad (85)$$

The operator $\mathrm{Op}(\sigma)$ is called **classical** if σ is.

Remark A.4.   (1) Given a symbol $\sigma \in S^m(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$, the kernel $K(x, y)$ of the operator $\mathrm{Op}(\sigma)$ defines a tempered distribution in $x - y$, if moreover $\sigma \in S^{-\infty}(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$, then $K(x, y)$ is smooth on $U \times U$.

(2) Conversely [EKS99, Comments following Th.1], any smooth kernel K on $U \times U$ is the kernel of a pseudodifferential operator with amplitude $a(x, y, \xi) := (2\pi)^n K(x, y) e^{-i\langle x-y, \xi \rangle} \psi(\xi)$, where ψ is a smooth function with compact support in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \psi(\xi) \, d\xi = 1$. The amplitude $a(x, y, \xi)$ lies in $S^{-\infty}(U \times \mathbb{R}^n, \mathrm{gl}_k(\mathbb{C}))$ since it is smooth in (x, y) and has compact support in ξ.

The operator $\text{Op}(\sigma)$ is a properly supported (see e.g. [Ta81, Def. 3.6]) pseudodifferential operator, a concept we now recall; recall that the support of a distribution is the complement of the set on which the distribution vanishes.

Definition A.5. Given an open subset U of $\mathbb{R}^n$,
   (1) a distribution $T \in \mathcal{D}'(U \times U, \text{gl}_k(\mathbb{C}))$ is **properly supported** if, for any compact $K \subset U$, its support has compact intersection with $K \times U$ and $U \times K$ or equivalently, if the restriction to the support of T of the two canonical projection maps $U \times U \to U$ mapping (x, y) to x and y respectively are proper maps (i.e., the preimage of a compact set is compact);
   (2) an operator $A \colon C_c^\infty(U, \mathbb{C}^k) \to C^\infty(U, \mathbb{C}^k)$ is **properly supported** if its Schwartz kernel is.

Remark A.6. Properly supported pseudodifferential operators stabilise $C_c^\infty(U, \mathbb{C}^k)$ and are therefore composable.

Proposition A.7. *For any $m \in \mathbb{R}$ (resp. $m \in \mathbb{C}$), the linear operator $A \colon C_c^\infty(U, \mathbb{C}^k) \to C^\infty(U, \mathbb{C}^k)$ is a (resp. classical) pseudodifferential operator of order m if and only if it is of the form*

$$A = \text{Op}(\sigma_A) + S(A), \tag{86}$$

where $\sigma(A) \in S^m(U \times \mathbb{R}^n, \text{gl}_k(\mathbb{C}))$ (resp. $\sigma(A) \in S_{\text{cl}}^m(U \times \mathbb{R}^n, \text{gl}_k(\mathbb{C}))$) and $S(A)$ is an operator with a smooth kernel supported outside the diagonal of U.

Proof. An operator of the form $\text{Op}(\sigma)$ for some $\sigma \in S^m(U \times \mathbb{R}^n, \text{gl}_k(\mathbb{C}))$, $m \in \mathbb{R}$, is a pseudodifferential operator of order m with amplitude $a(x, y, \xi) = \sigma(x, \xi)$. On the other hand, by Remark A.4 2), any smooth kernel has an amplitude which lies in $S^{-\infty}(U \times U \times \mathbb{R}^n, \text{gl}_k(\mathbb{C}))$. Combining these two facts, we conclude that any operator A of the form (86) is a psedudodifferential operator of order m; it is classical if its symbol $\sigma(A)$ is classical.
Conversely, let A be a (resp. classical) pseudodifferential operator of order m. Let $\varepsilon > 0$ and χ_ε be a smooth function with support containing the diagonal $\Delta_U := \{(x, x) \in U \times U\}$ such that χ_ε is identically one on $\{m \in U \times U \ : \ d(m, \Delta_U) \leq \frac{\varepsilon}{2}\}$ and identically zero on $\{m \in U \times U \ : \ d(m, \Delta_U) \geq \varepsilon\}$. We split the Schwartz kernel K_A of A accordingly

$$K_A = \chi_\varepsilon K_A + (1 - \chi_\varepsilon) K_A .$$

Both projection maps $\text{supp}(\chi_\varepsilon) \to U$ are proper, so $\chi_\varepsilon K_A$ is properly supported and corresponds to the kernel of some pseudodifferential operator $\text{Op}(\sigma_A)$ with symbol $\sigma(A)(x, \xi) = e^{-i\langle x, \xi \rangle} A \, e^{i\langle x, \xi \rangle}$ in $S^m(U \times \mathbb{R}^n, \text{gl}_k(\mathbb{C}))$ (see e.g. [Ta81, Thm. 3.8], [Sc10, (1.5.3.8)] and the comments that follow). The kernel $(1 - \chi_\varepsilon) K_A$ is smooth and supported outside the diagonal Δ_U. $\square$

This leads to the following definition.

Definition A.8.   • For any $m \in \mathbb{R}$, let $\Psi^m(U, \mathbb{C}^k)$ is the class of all linear operators $A \colon C_c^\infty(U, \mathbb{C}^k) \to C^\infty(U, \mathbb{C}^k)$ of the form (86), with symbol $\sigma_A \in S^m(U, \text{gl}_k(\mathbb{C}))$ and $S(A)$ an operator with a smooth Schwartz kernel $K_{S(A)} \in C^\infty(U \times U, \text{gl}_k(\mathbb{C}))$ supported outside the diagonal of U;
   • For any $m \in \mathbb{C}$, $\Psi_{cl}^m(U, \mathbb{C}^k)$ is the class of all linear operators $A \in \Psi^m(U, \mathbb{C}^k)$ whose symbol σ_A lies in $S_{cl}^m(U \times \mathbb{R}^n, \text{gl}_k(\mathbb{C}))$.
Let

$$\Psi^{-\infty}(U, \mathbb{C}^k) := \cap_{m \in \mathbb{R}} \Psi^m(U, \mathbb{C}^k) = \cap_{m \in \mathbb{C}} \Psi_{cl}^m(U, \mathbb{C}^k)$$

denote the space of smoothing operators, those whose symbols lie in $S^{-\infty}(U, \text{gl}_k(\mathbb{C}))$.

Let us state an easy yet very useful result, the proof of which we omit here, referring the reader to any classical textbook on pseudodifferential operators.

Corollary A.9. *For any $m \in \mathbb{R}$ (resp. $m \in \mathbb{C}$), the sets $\Psi^m(U, \mathbb{C}^k)$ (resp. $\Psi_{cl}^m(U, \mathbb{C}^k)$) are stable under summation $A \mapsto A + R$ with an operator R whose kernel is smooth. Moreover, the relation*

$$A \sim B \Longleftrightarrow A - B \ \text{has a smooth kernel} \tag{87}$$

defines an equivalence relation on $\Psi^m(U, \mathbb{C}^k)$ *(resp.* $\Psi^m_{cl}(U, \mathbb{C}^k)$*) and for any two elements* $A, B \in \Psi^m_{cl}(U, \mathbb{C}^k)$ *we have*

$$A \sim B \iff \sigma(A) \sim \sigma(B) \iff \sigma_{m-j}(A) = \sigma_{m-j}(B) \quad \forall j \in \mathbb{Z}_{\geq 0}. \tag{88}$$

Whereas differential operators are local, pseudo-differential operators are not local for they smear out the supports of the sections on which they act. However, they are pseudo-local in so far as they do not smear out their singular supports. The following definition captures some features of pseudo-locality.

Definition A.10. [Sh92, Def. 3 pag. 98] An operator $A \colon C^\infty_c(U, \mathbb{C}^k) \to C^\infty(U, \mathbb{C}^k)$ with Schwartz kernel K_A is C-**local** for some $C \geq 0$, if $K_A(x, y) = 0 \ \forall x, y$ with $|x - y| > C$ or equivalently, if

$$\forall u \in C^\infty_c(U, \mathbb{C}^k); \quad \operatorname{supp}(Au) \subset \{x : d(x, \operatorname{supp} u) \leq C\}.$$

Example A.11. Differential operators are 0-local operators.

Remark A.12. C-local operators are properly supported.

Appendix B. Coverings and Γ-Hilbert modules

We recall some useful definitions, referring to [Sc05].

Definition B.1. Let A be a C^*-algebra.

(1) An A-Hilbert module is a right A-module V with an A-valued inner product $\langle \cdot, \cdot \rangle$ obeying the expected properties, namely it is A-sesquilinear, and for any $v \in V$ the expression $\langle v, v \rangle$ is a non-negative self-adjoint element in A whose square root defines a norm on V for which V is complete. In particular, A^n is a Hilbert module for any $n \in \mathbb{N}$.

(2) A finitely generated projective Hilbert A-module V is a Hilbert A-module which is isomorphic as Hilbert A-modules to a (closed) orthogonal direct summand of A^n for suitable $n \in \mathbb{N}$. In other words, there is a Hilbert A-module W such that $V \oplus W \simeq A^n$. The range of the projection $P : A^n \to A^n$ with range V and kernel W is a finitely generated projective Hilbert module.

For a given A-Hilbert module V, let $\operatorname{End}_A(V)$ be the algebra of Hilbert module morphisms, namely continuous A-linear maps on V which have an adjoint. Then

$$\operatorname{End}_A(V) \simeq \mathcal{B}(V) \otimes A$$

is a C^*-algebra. If A is a von Neumann algebra, so is $\operatorname{End}_A(V)$ a von Neumann algebra [La95].

We now apply the previous construction to the von Neumann algebra of a countable discrete group Γ, which we first define. Let $\ell_2(\Gamma)$ be the completion of $\mathbb{C}\Gamma$ for the norm $\|\sum_{g \in \Gamma} a_g \, g\|_2 = \sqrt{\sum_{g \in \Gamma} |a_g|^2}$. We briefly ecall that a von Neumann algebra is a $*$-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator. The weak operator topology is the weakest topology on the set of bounded operators on a Hilbert space, such that the functional sending an operator T to the complex number $< Tx, y >$ is continuous for any vectors x and y in the Hilbert space. The group Γ acts on $\ell_2(\Gamma)$ by the right regular representation $h \cdot \left(\sum_{g \in \Gamma} a_g \, g \right) = \sum_{g \in \Gamma} a_g \, g h^{-1}$. Equipped with the inner product

$$\left\langle \sum_{g \in \Gamma} a_g \, g, \sum_{g \in \Gamma} b_g \, g \right\rangle := \sum_{g \in \Gamma} a_g \, \overline{b_g},$$

$\ell_2(\Gamma)$ is a Hilbert space and via the right regular representation, $\mathbb{C}\Gamma$ can be viewed as a subalgebra of the C^*-algebra $\mathcal{B}(\ell_2(\Gamma))$ of bounded linear operators on $\ell_2(\Gamma)$. The group von Neumann algebra $\mathcal{N}\Gamma$ is the closure of $\mathbb{C}\Gamma$ in $\mathcal{B}(\ell_2(\Gamma))$ with respect to the weak operator topology.

Example B.2. Take $\Gamma = \mathbb{Z}^n$. The Fourier transform gives an isometric $\mathbb{Z}^n$-equivariant isomorphism $\ell^2(\mathbb{Z}^n) \to L^2(\mathbb{T}^n)$, where $\mathbb{T}^n$ is the n-dimensional torus. Therefore $\mathcal{N}\mathbb{Z}^n$ coincides with the commutant $\mathcal{L}(L^2(T^n))^{\mathbb{Z}_n}$ of the $\mathbb{Z}_n$-action on $L^2(\mathbb{T}^n)$, and one obtains an isomorphism $\mathcal{N}\mathbb{Z}^n \simeq L^\infty(\mathbb{T}^n)$.

Definition B.3.   (1) The **Kaplansky trace**, also called Γ-**trace** on $\mathcal{N}\Gamma$ is the linear form $\mathrm{tr}_\Gamma : \mathcal{N}\Gamma \to \mathbb{C}$ defined as

$$\mathrm{tr}_\Gamma(a) = \langle a\, e, e \rangle = a_e,$$

where $a = \sum_{g \in \Gamma} a_g\, g$ and $e \in \mathbb{C}\Gamma$ is the element with all components zero outside the first one which is one.

(2) The Γ-**dimension** of a finitely generated Γ-module V represented by an idempotent matrix $P = (p_{ij}) \in \mathrm{gl}_n(\mathcal{N}\Gamma)$ is

$$\dim_\Gamma(V) := \sum_{i=1}^n \mathrm{tr}_\Gamma(p_{ii}) = \mathrm{tr}_\Gamma\left(\mathrm{tr}_{\mathrm{gl}_n(\mathbb{C})}(P)\right),$$

where $\mathrm{tr}_{\mathrm{gl}_n(\mathbb{C})}$ is the matrix trace given by the sum of the diagonal elements of the matrix with complex entries.

(3) More generally, the Γ-trace extends to any operator $T \in \mathrm{End}_{\mathcal{N}\Gamma}(V) \simeq \mathcal{B}(V) \otimes \mathcal{N}\Gamma$ by

$$\mathrm{tr}_\Gamma(T) := \mathrm{tr}_\Gamma\left(\mathrm{tr}_{\mathrm{gl}_n(\mathbb{C})}(TP)\right).$$

Example B.4. The Kaplansky trace on $\mathcal{N}\mathbb{Z}^n \simeq L^\infty(\mathbb{T}^n)$ is given by the integration map against the canonical volume measure on the flat torus:

$$\tau(f) = \int_{\mathbb{T}^n} f\, d\mu. \tag{89}$$

Let M be a closed manifold with $\pi_1(M) = \Gamma$, let $\pi \colon \widetilde{M} \to M$ be a universal covering of M, with Γ acting on the right by deck transformations. Then $\ell^2(\Gamma)$ is a finite type $\mathcal{N}\Gamma$-module for $\ell^2(\Gamma) \simeq (\mathcal{N}\Gamma)_2$, the L^2-closure of $\mathcal{N}\Gamma$. More generally, we have

Lemma B.5. *Let $\widetilde{M}$ be a Γ-covering of M. The bundle $\mathcal{H} \to M$ defined by*

$$\mathcal{H} = \widetilde{M} \times_\Gamma \ell^2\Gamma$$

is a flat bundle of finitely generated projective $\mathcal{N}\Gamma$-modules.

Proof. This follows from the fact that the left Γ-action and the right $\mathcal{N}\Gamma$-action on $\ell^2(\Gamma)$ commute so $\mathcal{H} := \widetilde{M} \times_\Gamma \ell^2(\Gamma)$ is a finitely generated projective bundle of (right) $\mathcal{N}\Gamma$-Hilbert modules over M. Moreover, $\mathcal{H}$ is endowed with a flat structure since the transition functions are locally constant.   □

There is a well known dictionary between the space of L^2-sections of $\widetilde{E}$ on $\widetilde{M}$ and the sections of the bundle E twisted by $\mathcal{H}$, for which we refer for example to [Sc05, 7.5] or [PS07, Prop. E.6] for full details.

Let us first recall that the space $C_c(\widetilde{M}, \widetilde{E})$ is a right $\mathbb{C}\Gamma$-module with structure given by $(\xi \cdot s)(\tilde{x}) = \sum_{g \in \Gamma} (R_g^* \xi)(\tilde{x})s(g^{-1})$, where R_g^* denotes the pullback map, s is in $\mathbb{C}\Gamma$ and $\xi \in C_c(\widetilde{M}, \widetilde{E})$. By completion, this endows $L^2(\widetilde{M}, \widetilde{E})$ with the structure of right $\mathcal{N}\Gamma$-module.

Definition B.6. Let s be a section of $\widetilde{E}$. Define the section $\Phi(s)$ of $E \otimes \mathcal{H}$ by

$$\Phi(s)(x) = \hat{s}(x) = \sum_{\gamma \in \Gamma} s(\gamma \tilde{x}) \otimes [\tilde{x}, \gamma]$$

where $\tilde{x}$ is an arbitrary lift of $x \in M$, and $\widetilde{E}_{\gamma \tilde{x}}$ is identified with E_x.

The map $s \mapsto \hat{s} = \Phi(s)$ identifies $\{s \in C_c^\infty(\widetilde{M}, \widetilde{E}) \mid \sum_\gamma |s(\gamma x)|^2 < \infty\}$ with $C^\infty(M, E \otimes \mathcal{H})$. It extends to an isometry $\Phi : L^2(\tilde{M}, \widetilde{E}) \to L^2(M, E \otimes \mathcal{H})$ of $\mathcal{N}\Gamma$-Hilbert modules.

Example B.7. To the universal covering $\pi \colon \mathbb{R}^n \to \mathbb{T}^n$ with fundamental group $\mathbb{Z}^n$ corresponds the finitely generated projective bundle $\mathcal{H} := \mathbb{R}^n \times_{\mathbb{Z}^n} \ell^2(\mathbb{Z}^n)$ of (right) $L^\infty(\mathbb{T}^n)$-Hilbert modules over $\mathbb{T}^n$. In this correspondence, L^2-functions on $\mathbb{R}^n$ are viewed as L^2-sections of the bundle $\mathcal{H}$ over $\mathbb{T}^n$ via the map which sends f to $\hat{f} \colon x \mapsto \sum_{\gamma \in \mathbb{Z}^n} f(\gamma \pi^{-1}(x)) \otimes [\pi^{-1}(x), \gamma]$. This induces an isometry

$$\Phi \colon L^2(\mathbb{R}^n) \simeq L^2(\mathbb{T}^n) \otimes \ell^2(\mathbb{Z}^n) \longrightarrow L^2(\mathbb{T}^n, \mathcal{H}) \tag{90}$$

which sends $\{ f \in C^\infty(\mathbb{R}^n) : \sum_{\gamma \in \mathbb{Z}^n} |f(\gamma x)|^2 < \infty \ \forall x \in \mathbb{R}^n \}$ to $C^\infty(\mathbb{T}^n, \mathcal{H})$.

Definition B.8. To any first-order differential operator D on $C^\infty(M, E)$, one can assign the first order differential operator $D_\mathcal{H}$ defined on a section $\hat{s}(x) := \sum_\gamma \tilde{s}(\gamma x) \otimes [\tilde{x}, \gamma]$ of $E \otimes \mathcal{H}$ as

$$D_\mathcal{H}(\hat{s})(x) := \sum_{\gamma \in \Gamma} \tilde{D} \tilde{s}(\gamma \tilde{x}) \otimes [\tilde{x}, \gamma], \tag{91}$$

locally on an evenly covered neighborhood U of $x \in M$, with lift $\widetilde{U}$ an open set such that π_U is a diffeomorphism, so that $U \ni y \mapsto [\tilde{y}, \gamma]$ is a flat section of $\mathcal{H}$ for all $\gamma \in \Gamma$.

The following proposition is proved in [Sc05] (see also [BP09, Proposition 3.12]). We sketch the proof in this appendix for the sake of completeness.

Proposition B.9. *([Sc05]) The map*

$$\Phi \colon L^2(\tilde{M}, \widetilde{E}) \to L^2(M, E \otimes \mathcal{H})$$

yields an isometry of $N\Gamma$-Hilbert modules which for any first-order differential operator D on $C^\infty(M, E)$ satifies

    (1) $\Phi^\sharp \widetilde{D} := \Phi \widetilde{D} \Phi^{-1} = D_\mathcal{H}$,
    (2) $\Phi^\sharp \big(h(\widetilde{D}) \big) = h(D_\mathcal{H})$, *for every bounded measurable function* $h \colon \mathbb{R} \to \mathbb{R}$.

Proof. Relation (1) follows from (91) and hence (2) follows observing that since the unitary equivalence of the self-adjoint unbounded operators $\widetilde{D}$ and $D_\mathcal{H}$ implies the same property for all bounded measurable functions of the latter, using functional calculus. $\qquad\square$

APPENDIX C. THE GROUPOID INTERPRETATION

We use the notations of the previous appendix. An equivalent description of the calculus of Γ-invariant operators on the covering $\widetilde{M}$ involves the fundamental groupoid G defined as the quotient space

$$G := (\widetilde{M} \times \widetilde{M})/\Gamma$$

of $\widetilde{M} \times \widetilde{M}$ by the diagonal action $\gamma \cdot (\tilde{x}, \tilde{y}) := (\gamma \cdot \tilde{x}, \gamma \cdot \tilde{y})$. The groupoid structure on G is given in terms of the space of units is $G^{(0)} = M$ and the source and range maps

$$s[(\tilde{x}, \tilde{y})] := \pi(\tilde{x}), \qquad r[(\tilde{x}, \tilde{y})] := \pi(\tilde{y}).$$

For further details on the structure of this groupoid we refer to [BP09, Section 2] in the more general setting of the monodromy groupoid of a foliated bundle.

Pseudodifferential calculus on groupoids was introduced by Connes [Co79], Nistor–Weinstein–Xu [NWX99], Monthubert–Pierrot [MP] and by Vassout [Vas06] in relation with the Wodzicki residue. We will compare it here with the calculus developed by Vassout which bears the advantage of including a large subalgebra of non properly supported smoothing operators and is therefore well suited for the construction of complex powers [Vas06].

Let G be a differentiable groupoid and $E \to G^{(0)}$ a vector bundle on its space of units, and for any $x \in G^{(0)}$ let $G_x := s^{-1}(x)$.

One considers families $A = (A_x)_{x \in M}$, where A_x is an operator acting on $C_c^\infty(G_x, r^*(E))$, satisfying the G-invariance condition

$$A_{r(g)} U_g = U_g A_{d(g)} \tag{92}$$

where $U_g : C^\infty(G_{s(g)}, r^*(E)) \to C^\infty(G_{r(g)}, r^*(E))$ is $(U_g f)(g') := f(g' g)$. The support of a family A is is a subset of $\{(g, h) \in G \times G, s(g) = s(h)\}$ defined as

$$\mathrm{supp}(A) := \bigcup_{x \in G^{(0)}} \mathrm{supp}(A_x).$$

Definition C.1. [NWX99, Definition 8] The family A is said to be

- *properly supported*, if $p_i^{-1}(K) \cap \mathrm{supp}(A)$ is compact for any compact subset $K \subset G$, where $p_1, p_2 : G \times G \to G$ are the projections on the two factors;
- *compactly supported*, if $\mathrm{supp}(A)$ is compact;
- *uniformly supported*, if $\mu_1(\mathrm{supp}(A))$ is compact (where $\mu_1(g, h) := gh^{-1}$).

The algebra of G-pseudodifferential operators (resp. classical G-pseudodifferential operators in the sense of Vassout) consists of two building blocks:

a) an algebra $\Psi_{\mathrm{cpt}}(G, E)$ whose elements are smooth (The degree of regularity can be chosen according to the needs) families $A = (A_x)_{x \in M}$ where A_x is a (resp. classical) pseudo-differential operator acting on $C_c^\infty(G_x, r^*(E))$, such that A is G-invariant and compactly supported

b) an algebra $\Psi^{-\infty}(G, E)$ of *smoothing* operators defined by means of a scale of Sobolev modules $H^s(\mathcal{W})$, $s \in \mathbb{R}$, on the groupoid C^*-algebra [Vas06, Proposition 4.2.5] as the intersection

$$\Psi^{-\infty}(G) = \cap_{s,t} \mathcal{L}\left(H^s(\mathcal{W}), H^t(\mathcal{W})\right) \tag{93}$$

of the spaces of linear maps which take a Sobolev module $H^s(\mathcal{W})$ to another Sobolev module $H^t(\mathcal{W})$.

Definition C.2. A (resp. classical) pseudodifferential operator in the sense of [Vas06] consists of a sum

$$A = A_0 + S(A), \quad A_0 \in \Psi_{\mathrm{cpt}}(G, E), \ S(A) \in \Psi^{-\infty}(G, E),$$

(resp. with A classical). The linear space generated by (resp. classical) pseudodifferential operators of order $m \in \mathbb{R}$ (resp. $m \in \mathbb{C}$) is denoted by $\Psi^m(G, E)$ (resp. $\Psi_{\mathrm{cl}}^m(G, E)$) and the whole algebra generated by such operators by $\Psi(G, E)$ (resp. $\Psi_{\mathrm{cl}}(G, E)$).

Example C.3. In the case of the pair groupoid $G = M \times M$ of a closed smooth manifold $G^{(0)} := M$, $\Psi_{\mathrm{cpt}}(G, E)$ corresponds to the set of compactly supported operators on M, which in particular are properly supported [NWX99, Example 1], so $\Psi^m(G, E) = \Psi^m(M, E)$ for any $m \in \mathbb{R}$.

We now focus on the case $G = (\widetilde{M} \times \widetilde{M})/\Gamma$, and summarize the relations between pseudodifferential operators on $\widetilde{M}$ of Section 1.2.2 with pseudodifferential operators on G given in [NWX99, Example 4], [Vas06, §6], see also [BP09, §3.2].

Each fibre $G_x := s^{-1}(x)$ can be identified to $\widetilde{M}$ via the map

$$\rho_x : \widetilde{M} \to G_x \quad , \quad \tilde{p} \mapsto [\tilde{x}_0, \tilde{p}], \tag{94}$$

where $\tilde{x}_0$ is a fixed element in $\pi^{-1}(x)$. The identification of G_x with $\widetilde{M}$ is unique up to the action of Γ.

Given a family $A = (A_x)_{x \in M}$, the G-invariance condition (92) implies that A_x is Γ-invariant. Hence the family A can be identified with a Γ-invariant operator on $\widetilde{M}$.

Proposition C.4. [NWX99, Example 4], [Vas06, 6] *Let $G = (\widetilde{M} \times \widetilde{M})/\Gamma$. One has the following identifications:*

(1) *For any $m \in \mathbb{R}$ (resp. $m \in \mathbb{C}$)*

$$\Psi_{\mathrm{cpt}}^m(G, E) \simeq \{A \in \Psi_\Gamma^m(\widetilde{M}, \widetilde{E}), \ A \ \text{is properly supported}\},$$

resp. $\quad \Psi_{\mathrm{cpt}}^m(G) \cap \Psi_{\mathrm{cl}}(G, E) \simeq \{A \in \Psi_\Gamma^m(\widetilde{M}, \widetilde{E}) \cap \Psi_{\mathrm{cl}}(\widetilde{M}, \widetilde{E}), \ A \ \text{is properly supported}\},$

(2) $\Psi^{-\infty}(G, E) \simeq \mathcal{S}\Psi_\Gamma^{-\infty}(\widetilde{M}, \widetilde{E})$.

(3) *Consequently,* $\Psi_{\mathrm{cl}}^m(G, E) \simeq \mathcal{U}\Psi_{\mathrm{cl},\Gamma}^m(\widetilde{M}, \widetilde{E})$.

References

[ALNV04] B. Ammann, R. Lauter, V. Nistor, A. Vasy, *Complex powers and non-compact manifolds*, Communications in Partial Differential Equations **29** (2004) 671–705

[ALNP15] S. Azzali, C. Lévy, C. Neira-Jiménez, S. Paycha, *Traces of holomorphic families of operators on the noncommutative torus and on Hilbert modules*, Geometric Methods in Physics XXXIII Workshop 2014, Trends in Mathematics, Springer (2015) 3–38

[BP09] M.-T. Benameur, P. Piazza, *Index, eta and rho invariants on foliated bundles*, Astérisque **327** (2009) 201–287

[BFKM96] D. Burghelea, L. Friedlander, T. Kappeler, P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules*, Geometric and Functional Analysis **6**, n. 5 (1996) 751–859

[Co79] A. Connes, *Sur la théorie non commutative de l'intégration* Algèbres d'opérateurs (Sém., Les Plans-sur-Bex) (1978) 19–143, Lecture Notes in Math., **725**, Springer, Berlin, 1979.

[CDP03] A. Cardona, C. Ducourtioux, S. Paycha, *From tracial anomalies to anomalies in quantum field theory*, Communications in Mathematical Physics **242** (2003) 31–65

[EKS99] Yu.V. Egorov, A.I. Komesh, M.A. Shubin, *Elements of the modern theory of partial differential equations*, Springer Verlag 1999 (Russian edition, Viniti, Moscow 1988)

[En15] A. Engel, *Index theory of uniform pseudodifferential operators*, http://arxiv.org/abs/1502.00494

[Gi95] P. Gilkey, *Invariance theory, The heat equation and the Atiyah-Singer index theorem*, CRC Press (1995)

[Gu85] V. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. in Math. **55**, 2 (1985) 131–160

[KV94] M. Kontsevich, S. Vishik, *Geometry of determinants of elliptic operators*, Func. Anal. on the Eve of the XXI century, Vol I, Progress in Mathematics **131** (1994) 173–197; *Determinants of elliptic pseudo-differential operators*, Max Planck Preprint (1994)

[Ko87] Y. Kordyukov, *Elliptic operators on manifolds with bounded geometry*, PhD thesis, Moscow State University (1987)

[La95] E. C. Lance, *Hilbert C*-modules – A toolkit for operator algebraists*, Cambridge University Press, Cambridge, 1995

[Le99] M. Lesch, *On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols*, Ann. Global Anal. Geom., **17**(2) (1999) 151–187

[MP] J. Mickelsson, S. Paycha, *The logarithmic residue density of a generalised Laplacian*, J. Austr. Math. Soc. **90** (2011) 53–80

[MS89] G.A. Meladze, M.A. Shubin, *Proper uniform pseudodifferential operators on unimodular Lie groups* J. Sov. Math. **45**, No.5 (1989) 1421–1439

[NWX99] V. Nistor, A. Weinstein, P. Xu, *Pseudodifferential operators on differential groupoids*, Pacific J. Math. **189** 1 (1999) 117–152

[Pa12] S. Paycha, *Regularised integrals, sums and traces. An analytic point of view*, AMS University Lecture Notes **59** (2012)

[PS07] S. Paycha, S. Scott, *A Laurent expansion for regularised integrals of holomorphic symbols*, Geom. Funct. Anal. **17** (2007) 491–536

[Ro98] J. Roe, *Elliptic operators, topology and asymptotic methods*, Second edition, 1998, Chapmann and Hall

[Ro88] J. Roe, *An index theorem on open manifolds*, J. Diff. Geom. **27** (1988) 87–113

[PS07] P. Piazza, T. Schick, *Bordism, rho-invariants and the Baum-Connes conjecture*, J. Noncommut. Geom. **1**, (2007) 27–111

[Sc05] T. Schick, *L^2-index theorems, KK-theory, and connections*, New York J. Math. **11** (2005) 387–443

[Sc96] T. Schick, *Analysis on ∂-Manifolds of Bounded Geometry, Hodge-De Rham Isomorphism and L^2-Index Theorem* Dissertation, Universität Mainz 1996

[Sc10]   S. Scott, *Traces and determinants of pseudodifferential operators*, Oxford University Press 2010

[Se66]   R.T. Seeley, *Complex powers of an elliptic operator, Singular integrals*, Proc. Symp. Pure Math., Chicago, Amer. Math. Soc., Providence (1966) 288–307

[Sh01]   M.A. Shubin, Pseudodifferential Operators and Spectral Theory, Springer-Verlag (2001)

[Sh92]   M.A. Shubin, *Spectra of elliptic operators on non-compact manifolds*, in: Méthodes semi-classiques, Vol.1, Astérisque **207** (1992) 35–108

[Sh89]   M.A. Shubin, *Weak Bloch property and weight estimates for elliptic operators*, Séminaire Equations aux dérives partielles - Ecole Polytechnique - 1998/90

[Sh]   M.A. Shubin, *Von Neumann algebras and L^2-techniques*, book www.mccme.ru/ium/postscript/f02/L2titlepages.ps.gz

[Ta81]   M. Taylor, *Pseudodifferential operators*, Princeton University Press (1981)

[Va08]   B. Vaillant, *Index Theory for Coverings*, http://arxiv.org/abs/0806.4043

[Vas06]   S. Vassout *Unbounded pseudodifferential calculus on Lie groupoids* J. Func Anal. **236**, 1, (2006) 161–200

[Vas01]   S. Vassout, *Feuillettages et résidu non commutatif longitudinal*, Thesis, Université Paris 6, 2001

[Wo84]   M. Wodzicki, *Spectral asymmetry and noncommutative residue*, PhD thesis, Steklov Mathematics Institute, Moscow 1984 (in Russian); *Non commutative residue, Chapter I. Fundamentals*. In "K-Theory, Arithmetic and Geometry", Springer Lecture Notes in Math. **1289** (1987) 320–399.

INSTITUTE OF MATHEMATICS, UNIVERSITÄT POTSDAM, CAMPUS II - GOLM, HAUS 9, KARL-LIEBKNECHT-STRASSE 24-25, D-14476 POTSDAM, GERMANY
  E-mail address: azzali@uni-potsdam.de

INSTITUTE OF MATHEMATICS, UNIVERSITÄT POTSDAM, CAMPUS II - GOLM, HAUS 9, KARL-LIEBKNECHT-STRASSE 24-25, D-14476 POTSDAM, GERMANY, (ON LEAVE FROM THE UNIVERSITÉ BLAISE PASCAL, CLERMONT-FERRAND)
  E-mail address: paycha@math.uni-potsdam.de