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Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations in the scattering case with mixed nonlinear terms

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Abstract

In this paper we consider the blow-up of solutions to a weakly coupled system of semilinear damped wave equations in the scattering case with nonlinearities of mixed type, namely, in one equation a power nonlinearity and in the other a semilinear term of derivative type. The proof of the blow-up results is based on an iteration argument. As expected, due to the assumptions on the coefficients of the damping terms, we find as critical curve in the p - q plane for the pair of exponents (p, q) in the nonlinear terms the same one found by Hidano-Yokoyama and, recently, by Ikeda-Sobajima-Wakasa for the weakly coupled system of semilinear wave equations with the same kind of nonlinearities. In the critical and not-damped case we provide a different approach from the test function method applied by Ikeda-Sobajima-Wakasa to prove the blow-up of the solution on the critical curve, improving in some cases the upper bound estimate for the lifespan. More precisely, we combine an iteration argument with the so-called slicing method to show the blow-up dynamic of a weighted version of the functionals used in the subcritical case.

Keywords: Semilinear weakly coupled system; Damped wave equation; Blow-up; Scattering producing damping; Critical curve; Mixed nonlinearities.
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1. Introduction

In this paper we consider a weakly coupled system of wave equations with time-dependent and scattering producing damping terms and with mixed kinds of power nonlinearity, namely,

$$\begin{cases} u_{tt} - \Delta u + b_1(t) u_t = |v|^q, & x \in \mathbb{R}^n, \ t > 0, \\ v_{tt} - \Delta v + b_2(t) v_t = |\partial_t u|^p, & x \in \mathbb{R}^n, \ t > 0, \\ (u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x) & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $b_1, b_2 \in \mathscr{C}([0,\infty)) \cap L^1([0,\infty))$ are nonnegative functions, ε is a positive parameter describing the size of initial data and $p, q > 1$. More precisely, we will focus on blow-up phenomena for local solutions and we will derive the corresponding upper bound for the lifespan.

In order to motivate the study of (1), let us recall some semilinear models which are strongly related to this weakly coupled system.

Let us begin with the Cauchy problem for the semilinear wave equation with power nonlinearity

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & x \in \mathbb{R}^n, \ t > 0, \\ (u, u_t,)(0, x) = (\varepsilon u_0, \varepsilon u_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{2}$$

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After John's pioneering paper [18], it was conjectured by Strauss in [42] that the critical exponent for the Cauchy problem (2) is the positive root of the quadratic equation

$$(n-1)p^2-(n+1)p-2=0,$$

which is nowadays named after him Strauss exponent and denoted in this paper by $p_{\mathrm{Str}}(n)$. In the classical works [20, 10, 9, 41, 38, 30, 7, 45, 17, 53, 57] this conjecture is proved to be true. Here, critical exponent means that for $1<p\leqslant p_{\mathrm{Str}}(n)$ local in time solutions blow up in finite times under certain sign assumptions on the initial data and regardless of the smallness of these, while for $p>p_{\mathrm{Str}}(n)$ the global in time existence of small data solutions holds in suitable function spaces. Moreover, the sharp lifespan estimate for local solutions has been derived both in the subcritical case and in the critical case, cf. [41, 29, 54, 55, 31, 44, 58, 43].

A similar situation has been studied in the case of the Cauchy for the semilinear wave equation of derivative type as well, namely,

$$\begin{cases} u_{tt}-\Delta u=|\partial_t u|^p, & x\in\mathbb{R}^n,\ t>0,\\ (u,u_t,)(0,x)=(\varepsilon u_0,\varepsilon u_1)(x), & x\in\mathbb{R}^n. \end{cases} \tag{3}$$

For (3) it has been proved that the critical exponent is the so-called Glassey exponent $p_{\mathrm{Gla}}(n)\doteq\frac{n+1}{n-1}$, although the global in time existence in the supercritical case for non radial solutions is still open for spatial dimensions $n\geqslant 4$, see also [19, 32, 39, 37, 1, 56] for the blow-up results and [40, 11, 46, 12] for the global existence results.

Concerning the weakly coupled systems of semilinear wave equations

$$\begin{cases} u_{tt}-\Delta u=G_1(v,\partial_t v), & x\in\mathbb{R}^n,\ t>0,\\ v_{tt}-\Delta v=G_2(u,\partial_t u), & x\in\mathbb{R}^n,\ t>0,\\ (u,u_t,v,v_t)(0,x)=(\varepsilon u_0,\varepsilon u_1,\varepsilon v_0,\varepsilon v_1)(x), & x\in\mathbb{R}^n, \end{cases} \tag{4}$$

the cases $G_1(v,\partial_t v)=|v|^p$, $G_2(u,\partial_t u)=|u|^q$ and $G_1(v,\partial_t v)=|\partial_t v|^p$, $G_2(u,\partial_t u)=|\partial_t u|^q$ have been studied in [5, 3, 4, 2, 23, 22, 8, 24] and in [6, 52, 21, 16], respectively. While in the case of power nonlinearities (that is, for $G_1(v,\partial_t v)=|v|^p$, $G_2(u,\partial_t u)=|u|^q$) the critical curve is given by

$$\max\left\{\frac{p+2+q^{-1}}{pq-1},\frac{q+2+p^{-1}}{pq-1}\right\}=\frac{n-1}{2},$$

the case of semilinear terms of derivative type (that is, for $G_1(v,\partial_t v)=|\partial_t v|^p$, $G_2(u,\partial_t u)=|\partial_t u|^q$) the critical curve is

$$\max\left\{\frac{p+1}{pq-1},\frac{q+1}{pq-1}\right\}=\frac{n-1}{2},$$

even though the global existence part has been studied so far only in the three dimensional and radial symmetric case. Recently, the case with mixed nonlinear terms $G_1(v,\partial_t v)=|v|^q$, $G_2(u,\partial_t u)=|\partial_t u|^p$ has been investigated for (4) in [13, 16]. In this paper we shall prove that the for same range of exponents $p,q>1$ as in [16] a blow-up result can be proved in the subcritiacal case even when we add as lower order terms in the linear part damping terms with time-dependent and scattering producing coefficients (see [49, 50, 51] for this classification of a damping term with time-dependent coefficient for wave models). Furthermore, the same upper bound for the lifespan can be derived. In the critical case, we will restrict our considerations to the not-damped case, improving in some cases the upper bound for the lifespan with respect to [16], but using a quite different method.

Recently, several results for semilinear wave equations and for weakly coupled systems of semilinear wave equations have been proved in presence of time-dependent and scattering-producing coefficients for damping terms by Lai-Takamura, Wakasa-Yordanov and Palmieri-Takamura. More precisely, the blow-up dynamic for local solutions of

$$\begin{cases} u_{tt}-\Delta u+b(t)u_t=G(u,\partial_t u), & x\in\mathbb{R}^n,\ t>0,\\ (u,u_t)(0,x)=(\varepsilon u_0,\varepsilon u_1)(x), & x\in\mathbb{R}^n, \end{cases}$$

has been considered in [25, 48] for the case of power nonlinearity $G(u, \partial_t u) = |u|^p$, in [26] for the case of derivative type $G(u, \partial_t u) = |\partial_t u|^p$ and in [27] for the case of combined nonlinearity $G(u, \partial_t u) = |\partial_t u|^p + |u|^q$. Finally, really recently the weakly coupled system of semilinear damped wave equations in the scattering case

$$\begin{cases} u_{tt} - \Delta u + b_1(t) u_t = G_1(v, \partial_t v), & x \in \mathbb{R}^n, \ t > 0, \\ v_{tt} - \Delta v + b_2(t) v_t = G_2(u, \partial_t u), & x \in \mathbb{R}^n, \ t > 0, \\ (u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x), & x \in \mathbb{R}^n, \end{cases}$$

has been considered in [34] for the case with power nonlinearities $G_1(v, \partial_t v) = |v|^p, G_2(u, \partial_t u) = |u|^q$ and in [35] for the case with semilinear terms of derivative type $G_1(v, \partial_t v) = |\partial_t v|^p, G_2(u, \partial_t u) = |\partial_t u|^q$.

In this paper our approach is based on the following methods: in the subcritical case we employ two multipliers, that are introduced in [25], in order to apply a standard iteration argument based on lower bound estimates for the spatial integrals of the nonlinear terms and on a coupled system of ordinary integral inequalities; in the critical case, we modify the approach introduced by Wakasa-Yordanov in [47, 48] and adapted to weakly coupled systems in [34] with the purpose to deal with the nonlinear term of derivative type. We underline that in the case with time-dependent coefficients for the damping terms in the scattering case, we may not apply the revised test function method which is introduced by Ikeda-Sobajima-Wakasa in [16] for semilinear wave models. Furthermore, in the critical case, where we consider the not-damped case as in Section 9 of [16], it is interesting to compare how our different approach leads to different upper bound estimates for the lifespan and in some cases to an improvement of these estimates. We refer to [15] and to [14, 16, 36, 33] for further details on this revised test function method based on a family of self similar solutions of the adjoint linear equation involving Gauss hypergeometric functions, for the study of semilinear heat, Schrödinger and damped wave equations and for the treatment of semilinear and scale-invariant model with time-dependent coefficients, respectively.

Before stating the blow-up results of this paper, let us introduce a suitable notion of energy solutions.

Definition 1.1. *Let $u_0, v_0 \in H^1(\mathbb{R}^n)$ and $u_1, v_1 \in L^2(\mathbb{R}^n)$. We say that (u, v) is an energy solution of* (1) *on $[0, T)$ if*

$$u \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u \in L^p_{\text{loc}}([0,T) \times \mathbb{R}^n),$$
$$v \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \quad \text{and} \quad v \in L^q_{\text{loc}}([0,T) \times \mathbb{R}^n)$$

satisfy $u(0, x) = \varepsilon u_0(x)$, $v(0, x) = \varepsilon v_0(x)$ in $H^1(\mathbb{R}^n)$ and the equalities

$$\begin{aligned} &\int_{\mathbb{R}^n} \partial_t u(t, x) \phi(t, x)\, dx - \int_{\mathbb{R}^n} \varepsilon u_1(x) \phi(0, x)\, dx - \int_0^t \int_{\mathbb{R}^n} \partial_t u(s, x) \phi_s(s, x)\, dx\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \nabla u(s, x) \cdot \nabla \phi(s, x)\, dx\, ds + \int_0^t \int_{\mathbb{R}^n} b_1(s) \partial_t u(s, x) \phi(s, x)\, dx\, ds \\ &\quad = \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \phi(s, x)\, dx\, ds \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} \partial_t v(t, x) \psi(t, x)\, dx - \int_{\mathbb{R}^n} \varepsilon v_1(x) \psi(0, x)\, dx - \int_0^t \int_{\mathbb{R}^n} \partial_t v(s, x) \psi_s(s, x)\, dx\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \nabla v(s, x) \cdot \nabla \psi(s, x)\, dx\, ds + \int_0^t \int_{\mathbb{R}^n} b_2(s) \partial_t v(s, x) \psi(s, x)\, dx\, ds \\ &\quad = \int_0^t \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \psi(s, x)\, dx\, ds \end{aligned} \tag{6}$$

for any test functions $\phi, \psi \in \mathscr{C}_0^\infty([0,T) \times \mathbb{R}^n)$ and any $t \in [0, T)$.

Performing a further step of integrations by parts in (5), (6) and letting $t \to T$, we find that (u, v) fulfills the definition of weak solution to (1).

Let us state the blow-up result for (1) in the subcritical case.

Theorem 1.2. *Let b_1, b_2 be continuous, nonnegative and summable functions. Let us consider $p, q > 1$ satisfying*

$$\max\left\{\frac{q+1+p^{-1}}{pq-1}, \frac{2+q^{-1}}{pq-1}\right\} > \frac{n-1}{2}. \tag{7}$$

Assume that $u_0, v_0 \in H^1(\mathbb{R}^n)$ and $u_1, v_1 \in L^2(\mathbb{R}^n)$ are nonnegative and compactly supported in B_R functions such that $u_1 \not\equiv 0$ and $v_0 \not\equiv 0$.

Let (u, v) be an energy solution of (1) with lifespan $T = T(\varepsilon)$ such that

$$\operatorname{supp} u,\ \operatorname{supp} v \subset \{(t,x) \in [0,T) \times \mathbb{R}^n : |x| \leqslant t + R\}. \tag{8}$$

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, b_1, b_2, R)$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$ the solution (u, v) blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon) \leqslant C\varepsilon^{-\max\{\Theta_1(n,p,q), \Theta_2(n,p,q)\}^{-1}} \tag{9}$$

holds, where C is an independent of ε, positive constant and

$$\Theta_1(n,p,q) \doteq \frac{q+1+p^{-1}}{pq-1} - \frac{n-1}{2} \quad \text{and} \quad \Theta_2(n,p,q) \doteq \frac{2+q^{-1}}{pq-1} - \frac{n-1}{2}. \tag{10}$$

Remark 1.3. *The upper bound estimates (9) for the lifespan coincide with the ones for the case $b_1 = b_2 = 0$, for more details see also [16, Section 9].*

The main result in the critical and not-damped case is the following theorem.

Theorem 1.4. *Let $n \geqslant 2$ and $b_1 = b_2 = 0$. Let us assume that $p, q > 1$ satisfy*

$$\max\left\{\frac{q+1+p^{-1}}{pq-1}, \frac{2+q^{-1}}{pq-1}\right\} = \frac{n-1}{2}, \tag{11}$$

Assume that $u_0, v_0 \in H^1(\mathbb{R}^n)$ and $u_1, v_1 \in L^2(\mathbb{R}^n)$ are nonnegative and compactly supported in B_R functions such that $u_1 \not\equiv 0$ and $v_0 \not\equiv 0$. Let (u, v) be a weak solution of

$$\begin{cases} u_{tt} - \Delta u = |v|^q, & x \in \mathbb{R}^n,\ t > 0, \\ v_{tt} - \Delta v = |\partial_t u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t, v, v_t)(0, x) = \varepsilon(u_0, u_1, v_0, v_1)(x) & x \in \mathbb{R}^n, \end{cases} \tag{12}$$

satisfying (8) with lifespan $T = T(\varepsilon)$ (cf. Definition 6.1).

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution (u, v) blows up in finite time. Moreover, the upper bound estimates for the lifespan

$$T(\varepsilon) \leqslant \begin{cases} \exp\left(C\varepsilon^{-p(pq-1)}\right) & \text{if } \Theta_1(n,p,q) = 0, \\ \exp\left(C\varepsilon^{-q(pq-1)}\right) & \text{if } \Theta_2(n,p,q) = 0, \\ \exp\left(C\varepsilon^{-\frac{q}{q+1}(pq-1)}\right) & \text{if } \Theta_1(n,p,q) = \Theta_2(n,p,q) = 0, \end{cases} \tag{13}$$

hold, where C is an independent of ε, positive constant.

The remaining part of this paper is organized as follows: in Section 2 we derive the coupled system of ODIs (ordinary differential inequalities) that the spatial averages of the components of a local solution has to satisfy, then, using a suitable pair of multipliers (m_1, m_1) (cf. (14) below) we derive the corresponding integral iteration frame from this system of ODIs; in Section 3 we prove suitable lower bounds for the space integrals of the nonlinearities, that is, for $\|\partial_t u(t,\cdot)\|^p_{L^p(\mathbb{R}^n)}$, $\|v(t,\cdot)\|^q_{L^q(\mathbb{R}^n)}$; hence, in Section 4 we combine the results from Sections 2-3 in an iterative procedure which allows us to determine a sequence of lower bound estimates for the above cited spatial averages; finally, in Section 5 we conclude the proof of Theorem 1.2 proving the blow-up result thanks to the sequence of lower bounds obtained via the iteration argument and deriving the upper bound for the lifespan of a local solution. Finally, in Section 6 we prove Theorem 1.4. The intermediate steps are similar to the ones for the subcritical case: derivation of the iteration frame, lower bound estimates for integrals related to the nonlinear terms, yet containing a logarithmic factor, and iteration procedure combined with the slicing method. Nevertheless, a crucial difference consists in the choice of the functionals, whose blow-up dynamic is considered. Indeed, differently from the subcritical case, we do not consider spatial averages of the components of a local solution rather weighted spatial averages of this components.

Notations

Throughout this paper we will use the following notations: B_R denotes the ball around the origin with radius R; $f \lesssim g$ means that there exists a positive constant C such that $f \leqslant Cg$ and, analogously, for $f \gtrsim g$; moreover, $f \asymp g$ means $f \lesssim g$ and $f \gtrsim g$; finally, as in the introduction, $p_{\mathrm{Str}}(n)$ and $p_{\mathrm{Gla}}(n)$ denote the Strauss exponent and the Glassey exponent, respectively.

2. Iteration frame

Let us recall the definition of some multipliers related to our model, which have been introduced in [25], and some properties of them, that we will employ throughout the remaining sections.

Definition 2.1. *Let $b_1, b_2 \in \mathscr{C}([0,\infty)) \cap L^1([0,\infty))$ be the nonnegative, time-dependent coefficients in* (1). *We define the* multipliers

$$m_j(t) \doteq \exp\left(- \int_t^\infty b_j(\tau) d\tau \right) \qquad \textit{for} \ \ t \geqslant 0 \ \ \textit{and} \ j = 1, 2. \tag{14}$$

As b_1, b_2 are nonnegative functions, then, m_1, m_2 are increasing functions. Moreover, due to the summability of b_1, b_2, the multipliers are bounded and

$$m_j(0) \leqslant m_j(t) \leqslant 1 \qquad \text{for} \ \ t \geqslant 0 \ \ \text{and} \ j = 1, 2. \tag{15}$$

Finally, a remarkable property of these multipliers is the following one:

$$m_j'(t) = b_j(t)\, m(t) \qquad \text{for} \ \ j = 1, 2. \tag{16}$$

The properties given in (15) and (16) are essential in order to handle and somehow to “neglect” the damping term.

Henceforth, we assume that u_0, u_1, v_0, v_1 satisfy the assumptions of Theorem 1.2. Let (u, v) be an energy solution of (1) on $[0, T)$ in the sense of Definition 1.1. Then, we introduce the following pair of functionals

$$U(t) \doteq \int_{\mathbb{R}^n} u(t,x)\, dx, \quad V(t) \doteq \int_{\mathbb{R}^n} v(t,x)\, dx. \tag{17}$$

Let us point out that the pair of functionals whose dynamic will investigated in Section 4 is actually (U', V) due the nonlinearity in (1).

The support condition (8) can be rewritten as

$$\operatorname{supp} u(t,\cdot), \operatorname{supp} v(t,\cdot) \subset B_{R+t} \quad \text{for any } t \geqslant 0.$$

Therefore, using Green's identity, it results that U, V satisfy

$$U''(t) + b_1(t)U'(t) = \int_{\mathbb{R}^n} |v(t,x)|^q \, dx, \tag{18}$$

$$V''(t) + b_2(t)V'(t) = \int_{\mathbb{R}^n} |\partial_t u(t,x)|^p \, dx. \tag{19}$$

Let us derive first integral lower bound estimates for V from (19). Multiplying both sides of (19) by m_2 and using (16), we get

$$m_2(t)V''(t) + m_2(t)b_2(t)V'(t) = \frac{d}{dt}\big(m_2(t)V'(t)\big) = m_2(t)\int_{\mathbb{R}^n} |\partial_t u(t,x)|^p \, dx.$$

Hence, integrating over$[0,t]$ the last relation and rearranging the resulting equation, we have

$$\begin{aligned} V'(t) &= \frac{m_2(0)}{m_2(t)}V'(0) + \int_0^t \frac{m_2(s)}{m_2(t)}\int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \, dx \, ds \\ &\geqslant m_2(0)V'(0) + m_2(0)\int_0^t \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \, dx \, ds, \end{aligned}$$

where in the last step we used (15). A further integration over $[0,t]$ provides

$$V(t) \geqslant V(0) + m_2(0)V'(0)t + m_2(0)\int_0^t \int_0^s \int_{\mathbb{R}^n} |\partial_t u(\tau,x)|^p \, dx \, d\tau \, ds \qquad \text{for any } t \geqslant 0. \tag{20}$$

Using again the support property for $u_t(t,\cdot)$ and Hölder's inequality, we find that (20) implies

$$V(t) \geqslant C\int_0^t \int_0^s (1+\tau)^{-n(p-1)}(U'(\tau))^p \, d\tau \, ds \qquad \text{for any } t \geqslant 0. \tag{21}$$

for a suitable positive constant $C = C(n, p, b_2, R)$.

Proceeding in a similar way, we derive now two lower bound estimates for U'. A multiplication by m_1 in (18) and a successive integration over $[0,t]$ lead to

$$U'(t) = \frac{m_1(0)}{m_1(t)}U'(0) + \int_0^t \frac{m_1(s)}{m_1(t)}\int_{\mathbb{R}^n} |v(s,x)|^q \, dx \, ds.$$

Employing again (15), from the last estimate we derive

$$U'(t) \geqslant m_1(0)U'(0) + m_1(0)\int_0^t \int_{\mathbb{R}^n} |v(s,x)|^q \, dx \, ds. \tag{22}$$

Finally, thanks to the support condition for $v(t,\cdot)$, by Hölder's inequality we find

$$U'(t) \geqslant K\int_0^t (1+s)^{-n(q-1)}(V(s))^q \, ds \qquad \text{for any } t \geqslant 0. \tag{23}$$

for a suitable positive constant $K = K(n, q, b_1, R)$.

In Section 4 we employ (21) and (23) as iteration scheme. However, in order to start with the iteration procedure we need to derive lower bound estimates for the integral nonlinear terms, so that, plugging these lower bounds in (20) and (22) we get the first step of the iterative procedure. We will complete this task in the next section.

3. Lower bounds for the spatial integral of the nonlinearities

As we have already announced the goal of this section is to determine lower bound estimates for the integrals of the semilinear terms. According to this purpose, we need to take into account the analysis of further auxiliary functionals related to the local solution (u, v) of (1). More specifically, we are going to estimate the functionals

$$U_1(t) \doteq \int_{\mathbb{R}^n} u(t,x)\Psi(t,x)\, dx, \tag{24}$$

$$V_1(t) \doteq \int_{\mathbb{R}^n} v(t,x)\Psi(t,x)\, dx, \tag{25}$$

$$U_2(t) \doteq \int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\, dx. \tag{26}$$

In the definition of the functionals U_1, V_1, U_2 we used the function $\Psi = \Psi(t,x) \doteq e^{-t}\Phi(x)$, where

$$\Phi = \Phi(x) \doteq \begin{cases} e^x + e^{-x} & \text{for } \ n = 1, \\ \displaystyle\int_{\mathbb{S}^{n-1}} e^{\omega\cdot x}\, dS_\omega & \text{for } \ n \geqslant 2 \end{cases} \tag{27}$$

is an eigenfunction of the Laplace operator, as $\Delta\Phi = \Phi$. The test function Ψ has been introduce for the first time in [53] in the study of the blow-up result for the semilinear classical wave equation with power nonlinearity in the critical case for high space dimension.

Lemma 3.1. *Let $(w, \widetilde{w})$ be a local energy solution of the Cauchy problem*

$$\begin{cases} w_{tt} - \Delta w + b_1(t)w_t = G_1(t, x, w, w_t, \widetilde{w}, \widetilde{w}_t), & x \in \mathbb{R}^n, \ t \in (0, T), \\ \widetilde{w}_{tt} - \Delta \widetilde{w} + b_2(t)\widetilde{w}_t = G_2(t, x, w, w_t, \widetilde{w}, \widetilde{w}_t), & x \in \mathbb{R}^n, \ t \in (0, T), \\ (w, w_t, \widetilde{w}, \widetilde{w}_t)(0, x) = (\varepsilon w_0, \varepsilon w_1, \varepsilon \widetilde{w}_0, \varepsilon \widetilde{w}_1)(x), & x \in \mathbb{R}^n, \end{cases}$$

where the time-dependent coefficients of the damping terms $b_1, b_2 \in \mathscr{C}([0,\infty)) \cap L^1([0,\infty))$ and the nonlinear terms G_1, G_2 are nonnegative. Furthermore, we assume that $w_0, w_1, \widetilde{w}_0, \widetilde{w}_1$ are nonnegative, nontrivial and compactly supported and that $w, \widetilde{w}$ satisfy a support condition as in (8)*. Let $W_1, \widetilde{W}_1$ be defined by*

$$W_1(t) \doteq \int_{\mathbb{R}^n} w(t,x)\Psi(t,x)\, dx \quad \text{and} \quad \widetilde{W}_1(t) \doteq \int_{\mathbb{R}^n} \widetilde{w}(t,x)\Psi(t,x)\, dx$$

for any $t \geqslant 0$. Then, for any $t \geqslant 0$ the following estimates hold

$$W_1(t) \geqslant \varepsilon\, \tfrac{m_1(0)}{2} \int_{\mathbb{R}^n} w_0(x)\Phi(x)\, dx \quad \text{and} \quad \widetilde{W}_1(t) \geqslant \varepsilon\, \tfrac{m_2(0)}{2} \int_{\mathbb{R}^n} \widetilde{w}_0(x)\Phi(x)\, dx.$$

Proof. See Lemma 2.2 in [35]. □

In particular, from Lemma 3.1 we get immediately the lower bound estimates

$$U_1(t) \geqslant \varepsilon I_1[u_0] \qquad \text{for any } t \geqslant 0, \tag{28}$$

$$V_1(t) \geqslant \varepsilon I_2[v_0] \qquad \text{for any } t \geqslant 0, \tag{29}$$

where $I_j[f] \doteq \frac{m_j(0)}{2} \int_{\mathbb{R}^n} f(x)\Phi(x)\, dx$ for $j = 1, 2$.

In the next step we follow the main ideas of [26, Section 3] and [27, Section 4] in order to control the functional U_2 from below.

Lemma 3.2. *Let U_2 be defined by* (26)*. Under the same assumptions of Theorem 1.2, the following estimate holds*

$$U_2(t) \geqslant \varepsilon I_1[u_1] \qquad \textit{for any } t \geqslant 0. \tag{30}$$

Proof. Let us begin pointing out that

$$\begin{aligned}
&\frac{d}{dt}\left(m_1(t)\int_{\mathbb{R}^n}\big(\partial_t u(t,x)+u(t,x)\big)\Psi(t,x)\,dx\right)\\
&\qquad = b_1(t)m_1(t)\int_{\mathbb{R}^n}\big(\partial_t u(t,x)+u(t,x)\big)\Psi(t,x)\,dx + m_1(t)\frac{d}{dt}\int_{\mathbb{R}^n}\big(\partial_t u(t,x)+u(t,x)\big)\Psi(t,x)\,dx. \qquad (31)
\end{aligned}$$

Choosing $\psi\equiv\Psi$ in (6), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx - \int_{\mathbb{R}^n}\varepsilon u_1(x)\Phi(x)\,dx - \int_0^t\int_{\mathbb{R}^n}\partial_t u(s,x)\Psi_s(s,x)\,dx\,ds\\
&\quad + \int_0^t\int_{\mathbb{R}^n}\nabla u(s,x)\cdot\nabla\Psi(s,x)\,dx\,ds + \int_0^t\int_{\mathbb{R}^n} b_1(s)\partial_t u(s,x)\Psi(s,x)\,dx\,ds = \int_0^t\int_{\mathbb{R}^n}|v(s,x)|^q\Psi(s,x)\,dx\,ds.
\end{aligned}$$

Differentiating both sides of the previous equality with respect to t, we arrive at

$$\begin{aligned}
\int_{\mathbb{R}^n}|v(t,x)|^q\Psi(t,x)\,dx &= \frac{d}{dt}\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx\\
&\quad + \int_{\mathbb{R}^n}\big(-\partial_t u(t,x)\Psi_t(t,x)+\nabla u(t,x)\cdot\nabla\Psi(t,x)+b_1(t)\partial_t u(t,x)\Psi(t,x)\big)\,dx. \qquad (32)
\end{aligned}$$

Using $\Delta\Psi=\Psi$ and $\Psi_t=-\Psi$, (32) yields

$$\int_{\mathbb{R}^n}|v(t,x)|^q\Psi(t,x)\,dx = \frac{d}{dt}\int_{\mathbb{R}^n}\big(\partial_t u(t,x)+u(t,x)\big)\Psi(t,x)\,dx + b_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx. \qquad (33)$$

If we combine (31) and (33), we obtain

$$\frac{d}{dt}\left(m_1(t)\int_{\mathbb{R}^n}\big(\partial_t u(t,x)+u(t,x)\big)\Psi(t,x)\,dx\right) = b_1(t)m_1(t)\,U_1(t) + m_1(t)\int_{\mathbb{R}^n}|v(t,x)|^q\Psi(t,x)\,dx, \qquad (34)$$

where U_1 is defined by (24).
Thanks to (28) we have that U_1 is nonnegative. Then, integrating (34) over $[0,t]$, we get the estimate

$$\begin{aligned}
&m_1(t)\int_{\mathbb{R}^n}\big(\partial_t u(t,x)+u(t,x)\big)\Psi(t,x)\,dx\\
&\qquad\geqslant \varepsilon\, m_1(t)\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\Phi(x)\,dx + \int_0^t m_1(s)\int_{\mathbb{R}^n}|v(s,x)|^q\Psi(s,x)\,dx. \qquad (35)
\end{aligned}$$

Furthermore, we may rewrite (32) as follows

$$\begin{aligned}
\int_{\mathbb{R}^n}|v(t,x)|^q\Psi(t,x)\,dx &= \frac{d}{dt}\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx + b_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx\\
&\quad + \int_{\mathbb{R}^n}\big(\partial_t u(t,x)-u(t,x)\big)\Psi(t,x)\,dx. \qquad (36)
\end{aligned}$$

If we multiply both sides of (36) by $m_1(t)$, we find

$$\begin{aligned}
&\frac{d}{dt}\left(m_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\right) + m_1(t)\int_{\mathbb{R}^n}\big(\partial_t u(t,x)-u(t,x)\big)\Psi(t,x)\,dx\\
&\qquad = m_1(t)\int_{\mathbb{R}^n}|v(t,x)|^q\Psi(t,x)\,dx. \qquad (37)
\end{aligned}$$

Adding (35) and (37), we find

$$\frac{d}{dt}\left(m_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx\right)+2m_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx$$
$$\geqslant \varepsilon\, m_1(0)\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\Phi(x)\,dx+m_1(t)\int_{\mathbb{R}^n}|v(t,x)|^q\Psi(t,x)\,dx$$
$$+\int_0^t m_1(s)\int_{\mathbb{R}^n}|v(s,x)|^q\Psi(s,x)\,dx. \tag{38}$$

Let us set the auxiliary functional

$$U_3(t)\doteq m_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx-\varepsilon\,\frac{m_1(0)}{2}\int_{\mathbb{R}^n}u_1(x)\Phi(x)\,dx-\frac{1}{2}\int_0^t m_1(s)\int_{\mathbb{R}^n}|v(s,x)|^q\Psi(t,x)\,dx\,ds\,.$$

Clearly, $U_3(0)=\varepsilon I_1[u_1]$. Besides, (38) implies

$$U_3'(t)+2U_3(t)\geqslant \varepsilon\, m_1(0)\int_{\mathbb{R}^n}u_0(x)\Phi(x)\,dx+\frac{1}{2}\,m_1(t)\int_{\mathbb{R}^n}|v(t,x)|^q\Psi(t,x)\,dx\geqslant 0. \tag{39}$$

Hence, multiplying (39) by e^{2t} and integrating over $[0,t]$, we get $U_3(t)\geqslant e^{-2t}U_3(0)\geqslant 0$. Therefore, as U_3 is nonnegative we may write

$$m_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx\geqslant \varepsilon\,\frac{m_1(0)}{2}\int_{\mathbb{R}^n}u_1(x)\Phi(x)\,dx+\frac{1}{2}\int_0^t m_1(s)\int_{\mathbb{R}^n}|v(s,x)|^q\Psi(t,x)\,dx\,ds$$
$$\geqslant \varepsilon\,\frac{m_1(0)}{2}\int_{\mathbb{R}^n}u_1(x)\Phi(x)\,dx$$

which implies immediately (30) due to (15). □

Using (28) and (30), we may finally derive the lower bounds for the integrals with respect to the spatial variables of the semilinear terms.

Proposition 3.3. *Let (u,v) be an energy solution of (1) on $[0,T)$ with nonnegative, continuous and summable coefficients of the damping terms b_1,b_2. Furthermore, we require the same assumptions on u_0,u_1,v_0,v_1 as in Theorem 1.2. Then, the following estimates hold*

$$\int_{\mathbb{R}^n}|v(t,x)|^q\,dx\geqslant \widetilde{C}\varepsilon^q\,(1+t)^{n-1-\frac{n-1}{2}q}, \tag{40}$$
$$\int_{\mathbb{R}^n}|\partial_t u(t,x)|^p\,dx\geqslant \widetilde{K}\varepsilon^p\,(1+t)^{n-1-\frac{n-1}{2}p} \tag{41}$$

for any $t\geqslant 0$, where $\widetilde{C},\widetilde{K}$ are positive constants depending on n,p,q,b_1,b_2,R,u_1,v_0.

Remark 3.4. *Let us underline explicitly that the conditions $u_1\not\equiv 0$ and $v_0\not\equiv 0$ guarantee that the multiplicative constants in (29) and (30) are positive. This fact will play a fundamental role in the proof of Proposition 3.3.*

Proof. Let us prove (40). By Hölder's inequality and $\operatorname{supp} v(t,\cdot)\subset B_{R+t}$ it follows

$$\int_{\mathbb{R}^n}|v(t,x)|^q\,dx\geqslant \big(V_1(t)\big)^q\left(\int_{B_{R+t}}\big(\Psi(t,x)\big)^{q'}\,dx\right)^{-(q-1)}$$
$$\gtrsim \Big(\varepsilon I_2[v_0]\Big)^q(1+t)^{n-1-\frac{n-1}{2}q},$$

where in the second inequality we used (29) and the following estimate (cf. [53, estimate (2.5)]):

$$\int_{B_{R+t}}\big(\Psi(t,x)\big)^{q'}\,dx\lesssim (1+t)^{n-1-\frac{n-1}{2}q'}.$$

Using (30), we can prove (41) in a completely analogous way. □

4. Iteration argument

In this section we combine the results from Sections 2-3 by using an iteration procedure to get a sequence of lower bound estimates for the functionals V and U' (for the definition of U and V see (17) in Section 2).

More precisely, we want to prove that

$$V(t) \geqslant C_j(1+t)^{-b_j}t^{a_j} \qquad \text{for any } t \geqslant 0, \tag{42}$$

$$U'(t) \geqslant K_j(1+t)^{-\beta_j}t^{\alpha_j} \qquad \text{for any } t \geqslant 0, \tag{43}$$

where $\{C_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$, $\{b_j\}_{j\in\mathbb{N}}$, $\{K_j\}_{j\in\mathbb{N}}$, $\{\alpha_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$ are suitable sequences of nonnegative numbers that we will determine afterwards.

Our strategy is to prove (42) and (43) by induction.

Let us begin with the base case $j = 0$. Plugging (40) in (22), it results

$$U'(t) \geqslant m_1(0)\widetilde{C}\varepsilon^q \int_0^t (1+s)^{n-1-\frac{n-1}{2}q}\, ds \geqslant \tfrac{m_1(0)\widetilde{C}}{n}\,\varepsilon^q(1+t)^{-\frac{n-1}{2}q}t^n$$

which is (43) for $j=0$ provided that $K_0 \doteq \frac{m_1(0)\widetilde{C}}{n}\varepsilon^q$, $\alpha_0 \doteq n$, $\beta_0 \doteq \frac{n-1}{2}q$. Analogously, combining (41) and (20), we find

$$V(t) \geqslant m_2(0)\widetilde{K}\varepsilon^p \int_0^t\int_0^s (1+\tau)^{n-1-\frac{n-1}{2}p}\, d\tau\, ds \geqslant \tfrac{m_2(0)\widetilde{K}}{n(n+1)}\,\varepsilon^p(1+t)^{-\frac{n-1}{2}p}t^{n+1}.$$

So, we proved also (42) for $j=0$ provided that $C_0 \doteq \frac{m_2(0)\widetilde{K}}{n(n+1)}\varepsilon^p$, $a_0 \doteq n+1$, $b_0 \doteq \frac{n-1}{2}p$.

Let us proceed now with the inductive step. If we plug (42) in (23), then, for any $t \geqslant 0$ we have

$$\begin{aligned} U'(t) &\geqslant KC_j^q \int_0^t (1+s)^{-n(q-1)-b_jq}s^{a_jq}\, ds \geqslant KC_j^q(1+t)^{-n(q-1)-b_jq}\int_0^t s^{a_jq}\, ds \\ &= KC_j^q(a_jq+1)^{-1}(1+t)^{-n(q-1)-b_jq}t^{a_jq+1}. \end{aligned}$$

Thus, using the last lower bound in (21), we obtain for $t \geqslant 0$

$$\begin{aligned} V(t) &\geqslant CK^pC_j^{pq}(a_jq+1)^{-p}\int_0^t\int_0^s (1+\tau)^{-n(pq-1)-b_jpq}\tau^{a_jpq+p}\, d\tau\, ds \\ &\geqslant CK^pC_j^{pq}(a_jq+1)^{-p}(1+t)^{-n(pq-1)-b_jpq}\int_0^t\int_0^s \tau^{a_jpq+p}\, d\tau\, ds \\ &= CK^pC_j^{pq}(a_jq+1)^{-p}(a_jpq+p+1)^{-1}(a_jpq+p+2)^{-1}(1+t)^{-n(pq-1)-b_jpq}t^{a_jpq+p+2}. \end{aligned}$$

Also, we proved (42) for $j+1$ provided that $C_{j+1} \doteq CK^pC_j^{pq}(a_jq+1)^{-p}(a_jpq+p+1)^{-1}(a_jpq+p+2)^{-1}$, $a_{j+1} \doteq pqa_j + p + 2$ and $b_{j+1} \doteq b_j + n(pq-1)$.

Similarly, if we plug (43) in (21), then, for any $t \geqslant 0$ we get

$$\begin{aligned} V(t) &\geqslant CK_j^p\int_0^t\int_0^s (1+\tau)^{-n(p-1)-\beta_jp}\tau^{\alpha_jp}\, d\tau\, ds \\ &\geqslant CK_j^p(1+t)^{-n(p-1)-\beta_jp}\int_0^t\int_0^s \tau^{\alpha_jp}\, d\tau\, ds \\ &= CK_j^p(\alpha_jp+1)^{-1}(\alpha_jp+2)^{-1}(1+t)^{-n(p-1)-\beta_jp}t^{\alpha_jp+2}. \end{aligned}$$

Consequently, a combination of the last lower bound with (23) yields

$$\begin{aligned} U'(t) &\geqslant KC^qK_j^{pq}(\alpha_jp+1)^{-q}(\alpha_jp+2)^{-q}\int_0^t (1+s)^{-n(qp-1)-\beta_jpq}s^{\alpha_jpq+2q}\, ds \\ &\geqslant KC^qK_j^{pq}(\alpha_jp+1)^{-q}(\alpha_jp+2)^{-q}(\alpha_jpq+2q+1)^{-1}(1+t)^{-n(qp-1)-\beta_jpq}t^{\alpha_jpq+2q+1}\, ds \end{aligned}$$

for any $t \geqslant 0$. Hence, we proved (43) for $j+1$ provided that $\alpha_{j+1} \doteq pq\alpha_j + 2q + 1$, $\beta_{j+1} \doteq \beta_j + n(pq-1)$ and $K_{j+1} \doteq KC^q K_j^{pq} (\alpha_j p + 1)^{-q} (\alpha_j p + 2)^{-q} (\alpha_j pq + 2q + 1)^{-1}$.

It is clear, from the recursive relations and from the nonnegative values of the initial constants C_0, K_0, a_0, b_0, α_0, β_0, that $C_j, K_j, a_j, b_j, \alpha_j, \beta_j$ are nonnegative real numbers for all $j \in \mathbb{N}$. Next we determine the explicit expressions for $a_j, b_j, \alpha_j, \beta_j$ and lower bound estimates for C_j, K_j. As $a_j = pqa_{j-1} + p + 2$, employing iteratively this condition and the value $a_0 = n+1$, we find

$$a_j = pqa_{j-1} + p + 2 = \cdots = a_0 (pq)^j + (p+2) \sum_{k=0}^{j-1} (pq)^k = \Big(n + 1 + \tfrac{p+2}{pq-1}\Big)(pq)^j - \tfrac{p+2}{pq-1}.$$

Analogously,

$$\begin{aligned}
\alpha_j &= \alpha_0 (pq)^j + (2q+1) \sum_{k=0}^{j-1} (pq)^k = \Big(n + \tfrac{2q+1}{pq-1}\Big)(pq)^j - \tfrac{2q+1}{pq-1}, \\
b_j &= b_0 (pq)^j + n(pq-1) \sum_{k=0}^{j-1} (pq)^k = \Big(\tfrac{n-1}{2} p + n\Big)(pq)^j - n, \\
\beta_j &= \beta_0 (pq)^j + n(pq-1) \sum_{k=0}^{j-1} (pq)^k = \Big(\tfrac{n-1}{2} q + n\Big)(pq)^j - n.
\end{aligned}$$

In particular, using the representation formulas for a_j and α_j, we may derive lower bounds for C_j and K_j. Indeed, due to

$$\begin{aligned}
a_{j-1} pq + p + 2 = a_j &\leqslant \Big(n + 1 + \tfrac{p+2}{pq-1}\Big)(pq)^j, \\
\alpha_{j-1} pq + 2q + 1 = \alpha_j &\leqslant \Big(n + \tfrac{2q+1}{pq-1}\Big)(pq)^j,
\end{aligned}$$

we have

$$\begin{aligned}
C_j &= CK^p C_{j-1}^{pq} (a_{j-1} q + 1)^{-p} (a_{j-1} qp + p + 1)^{-1} (a_{j-1} pq + p + 2)^{-1} \\
&\geqslant CK^p C_{j-1}^{pq} (a_{j-1} pq + p + 2)^{-(p+2)} \geqslant M (pq)^{-(p+2)j} C_{j-1}^{pq} \qquad (44)
\end{aligned}$$

and

$$\begin{aligned}
K_j &= KC^q K_{j-1}^{pq} (\alpha_j p + 1)^{-q} (\alpha_j p + 2)^{-q} (\alpha_j pq + 2q + 1)^{-1} \\
&\geqslant KC^q K_{j-1}^{pq} (\alpha_j pq + 2q + 1)^{-(2q+1)} \geqslant \widetilde{M} (pq)^{-(2q+1)j} K_{j-1}^{pq}, \qquad (45)
\end{aligned}$$

where $M \doteq CK^p \big(n + 1 + \frac{p+2}{pq-1}\big)^{-(p+2)}$ and $\widetilde{M} \doteq KC^q \big(n + \frac{2q+1}{pq-1}\big)^{-(2q+1)}$.

Applying the logarithmic function to both sides of (44) and using in an iterative way the resulting estimate, we arrive at

$$\begin{aligned}
\log C_j &\geqslant pq \log C_{j-1} - j \log((pq)^{p+2}) + \log M \\
&\geqslant (pq)^2 \log C_{j-2} - (j + (j-1)pq) \log((pq)^{p+2}) + (1 + pq) \log M \\
&\geqslant \cdots \geqslant (pq)^j \log C_0 - \sum_{k=0}^{j-1} (j-k)(pq)^k \log((pq)^{p+2}) + \sum_{k=0}^{j-1} (pq)^k \log M \\
&= (pq)^j \Big(\log C_0 - \tfrac{pq}{(pq-1)^2} \log((pq)^{p+2}) + \tfrac{\log M}{pq-1} \Big) + (j+1) \tfrac{\log((pq)^{p+2})}{pq-1} + \tfrac{\log((pq)^{p+2})}{(pq-1)^2} - \tfrac{\log M}{pq-1}, \qquad (46)
\end{aligned}$$

where we used the formulas

$$\sum_{k=0}^{j-1} (pq)^k = \frac{(pq)^j - 1}{pq - 1}, \qquad \sum_{k=0}^{j-1} (j-k)(pq)^k = \frac{1}{pq-1} \bigg(\frac{(pq)^{j+1} - 1}{pq - 1} - (j+1) \bigg), \qquad (47)$$

that can be proved via an inductive argument.

Therefore, for $j \geqslant j_1 \doteq \lceil \frac{\log M}{\log((pq)^{p+2})} - 1 - \frac{1}{pq-1} \rceil$ by (46) we get

$$\log C_j \geqslant (pq)^j \Big(\log C_0 - \tfrac{pq}{(pq-1)^2} \log((pq)^{p+2}) + \tfrac{\log M}{pq-1} \Big) = (pq)^j \log(N\varepsilon^p), \tag{48}$$

where $N \doteq \frac{m_2(0)\widetilde{K}}{n(n+1)}((pq)^{p+2})^{-\frac{pq}{(pq-1)^2}} M^{\frac{1}{pq-1}}$.

Analogously, from (45) we derive the estimate

$$\log K_j \geqslant (pq)^j \Big(\log K_0 - \tfrac{pq}{(pq-1)^2} \log((pq)^{2q+1}) + \tfrac{\log \widetilde{M}}{pq-1} \Big) = (pq)^j \log(\widetilde{N}\varepsilon^q) \tag{49}$$

for $j \geqslant j_2 \doteq \lceil \frac{\log \widetilde{M}}{\log((pq)^{2q+1})} - 1 - \frac{1}{pq-1} \rceil$, where $\widetilde{N} \doteq \frac{m_1(0)\widetilde{C}}{n}((pq)^{2q+1})^{-\frac{pq}{(pq-1)^2}} \widetilde{M}^{\frac{1}{pq-1}}$.

In the next section we will combine (42), (48) and (43), (49) to complete the proof of Theorem 1.2 in the case $\Theta_1(n,p,q) > 0$ and in the case $\Theta_2(n,p,q) > 0$, respectively.

5. conclusion of the proof of Theorem 1.2

Let us start with the case $\Theta_1(n,p,q) > 0$. Combining (42) and (48), we have for $t \geqslant 0$ and $j \geqslant j_1$

$$\begin{aligned} V(t) &\geqslant \exp\big((pq)^j \log(N\varepsilon^p)\big)(1+t)^{-b_j} t^{a_j} \\ &= \exp\Big((pq)^j \log\Big(N\varepsilon^p (1+t)^{-(\frac{n-1}{2}p+n)} t^{n+1+\frac{p+2}{pq-1}}\Big)\Big)(1+t)^n t^{-\frac{p+2}{pq-1}}. \end{aligned}$$

As for $t \geqslant 1$ it holds $(1+t) \leqslant 2t$, the previous estimate yields

$$\begin{aligned} V(t) &\geqslant \exp\Big((pq)^j \log\Big(2^{-(\frac{n-1}{2}p+n)} N\varepsilon^p t^{\frac{pq+p+1}{pq-1} - \frac{n-1}{2}p}\Big)\Big)(1+t)^n t^{-\frac{p+2}{pq-1}} \\ &= \exp\Big((pq)^j \log\Big(2^{-(\frac{n-1}{2}p+n)} N\varepsilon^p t^{p\Theta_1(n,p,q)}\Big)\Big)(1+t)^n t^{-\frac{p+2}{pq-1}} \\ &= \exp\big((pq)^j \log\big(\varepsilon^p J(t)\big)\big)(1+t)^n t^{-\frac{p+2}{pq-1}} \end{aligned} \tag{50}$$

for $t \geqslant 1$, where $J(t) \doteq 2^{-(\frac{n-1}{2}p+n)} N t^{p\Theta_1(n,p,q)}$. Consequently, we may choose ε_0 sufficiently small such that

$$2^{(\frac{n-1}{2}+\frac{n}{p})\Theta_1(n,p,q)^{-1}} N^{-(p\Theta_1(n,p,q))^{-1}} \varepsilon_0^{\Theta_1(n,p,q)^{-1}} \geqslant 1.$$

So, for $\varepsilon \in (0,\varepsilon_0]$ and for $t \geqslant 2^{(\frac{n-1}{2}+\frac{n}{p})\Theta_1(n,p,q)^{-1}} N^{-(p\Theta_1(n,p,q))^{-1}} \varepsilon^{\Theta_1(n,p,q)^{-1}}$ it holds $J(t) > 0$. Consequently, letting $j \to \infty$ in (50), the lower bound of $V(t)$ blows up and, then, $V(t)$ cannot be finite. Also, we proved that V may be definite only for $t \lesssim \varepsilon^{\Theta_1(n,p,q)^{-1}}$.

Now, we prove the result in the case $\Theta_2(n,p,q) > 0$. Combining (43) and (49), we have for $t \geqslant 0$ and $j \geqslant j_2$

$$\begin{aligned} U'(t) &\geqslant \exp\big((pq)^j \log(\widetilde{N}\varepsilon^q)\big)(1+t)^{-\beta_j} t^{\alpha_j} \\ &= \exp\Big((pq)^j \log\Big(\widetilde{N}\varepsilon^q (1+t)^{-(\frac{n-1}{2}q+n)} t^{n+\frac{2q+1}{pq-1}}\Big)\Big)(1+t)^n t^{-\frac{2q+1}{pq-1}}. \end{aligned}$$

Then, for $t \geqslant 1$ it holds

$$\begin{aligned} U'(t) &\geqslant \exp\Big((pq)^j \log\Big(2^{-(\frac{n-1}{2}q+n)} \widetilde{N}\varepsilon^q t^{q\Theta_2(n,p,q)}\Big)\Big)(1+t)^n t^{-\frac{2q+1}{pq-1}} \\ &= \exp\big((pq)^j \log\big(\varepsilon^q \widetilde{J}(t)\big)\big)(1+t)^n t^{-\frac{p+2}{pq-1}} \end{aligned} \tag{51}$$

for $t \geqslant 1$, where $\widetilde{J}(t) \doteq 2^{-(\frac{n-1}{2}q+n)} \widetilde{N} t^{q\Theta_2(n,p,q)}$. Hence, we can take ε_0 so small that

$$2^{(\frac{n-1}{2}+\frac{n}{q})\Theta_2(n,p,q)^{-1}} \widetilde{N}^{-(q\Theta_2(n,p,q))^{-1}} \varepsilon_0^{\Theta_2(n,p,q)^{-1}} \geqslant 1.$$

Thus, for $\varepsilon \in (0,\varepsilon_0]$ and for $t \geqslant 2^{(\frac{n-1}{2}+\frac{n}{q})\Theta_2(n,p,q)^{-1}} \widetilde{N}^{-(q\Theta_2(n,p,q))^{-1}} \varepsilon^{\Theta_2(n,p,q)^{-1}}$ it holds $\widetilde{J}(t) > 0$. Also, as $j \to \infty$ in (51), the lower bound of $U'(t)$ diverges and $U'(t)$ is not finite. In this second case, we proved that U' can be finite only for $t \lesssim \varepsilon^{\Theta_2(n,p,q)^{-1}}$. Combining the two possible cases, we proved the result and the upper bound estimate for the lifespan (9).

6. Critical case

In the critical case, we restrict our considerations to the not-damped case. Therefore, we shall consider the weakly coupled system of semilinear wave equations (12) in the critical case $\max\{\Theta_1(n,p,q), \Theta_2(n,p,q)\} = 0$. We will generalize the approach from [47, 48] for a single semilinear equation and from [34] for a weakly coupled system with power nonlinearities, in order to deal with the mixed type of nonlinear terms.

For the sake of readability, we recall the definition of weak solution to (12).

Definition 6.1. *Let $u_0, v_0 \in H^1(\mathbb{R}^n)$ and $u_1, v_1 \in L^2(\mathbb{R}^n)$. We say that (u,v) is a weak solution of (12) on $[0,T)$ if*

$$u \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u \in L^p_{\mathrm{loc}}([0,T) \times \mathbb{R}^n),$$
$$v \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \quad \text{and} \quad v \in L^q_{\mathrm{loc}}([0,T) \times \mathbb{R}^n)$$

satisfy the equalities

$$\int_{\mathbb{R}^n} \Big(\partial_t u(t,x)\phi(t,x) - u(t,x)\phi_s(t,x)\Big)\,dx - \varepsilon \int_{\mathbb{R}^n} \Big(u_1(x)\phi(0,x) - u_0(x)\phi_s(0,x)\Big)\,dx$$
$$+ \int_0^t \int_{\mathbb{R}^n} u(s,x)\Big(\phi_{ss}(s,x) - \Delta\phi(s,x)\Big)\,dx\,ds = \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^q \phi(s,x)\,dx\,ds \tag{52}$$

and

$$\int_{\mathbb{R}^n} \Big(\partial_t v(t,x)\psi(t,x) - v(t,x)\psi_s(t,x)\Big)\,dx - \varepsilon \int_{\mathbb{R}^n} \Big(v_1(x)\psi(0,x) - v_0(x)\psi_s(0,x)\Big)\,dx$$
$$+ \int_0^t \int_{\mathbb{R}^n} v(s,x)\Big(\psi_{ss}(s,x) - \Delta\psi(s,x)\Big)\,dx\,ds = \int_0^t \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \phi(s,x)\,dx\,ds \tag{53}$$

for any test functions $\phi, \psi \in \mathscr{C}_0^\infty([0,T) \times \mathbb{R}^n)$ and any $t \in [0,T)$.

The remaining part of this section is organized as follows: first, in Section 6.1 we recall some auxiliary functions from [47] and we use them to introduce the functionals for the critical case; in Section 6.2 we derive the iteration frame for these functionals, that is, a coupled system of nonlinear ordinary integral inequalities; in Section 6.3 lower bound estimates for the functionals, that allow to start with the iteration procedure, are derived; then, in Section 6.4 we combine the iteration frame from Section 6.2 and the lower bounds from Section 6.3 with a slicing method; hence, in Section 6.5 we use the sequences of lower bounds for the functionals from Section 6.4 to prove the blow-up result and to establish the upper bound for the lifespan; finally, in Section 6.6 we compare our results with those proved in Section 9 of [16] and we provide the analytic expression of the coordinates of the cusp point for the critical curve in the p - q plane.

6.1. Introduction of the functionals for the critical case

Throughout the treatment of the critical case we will employ the auxiliary functions

$$\eta_r(t,s,x) \doteq \int_0^{\lambda_0} e^{-\lambda(R+t)} \frac{\sinh \lambda(t-s)}{\lambda(t-s)}\, \Phi(\lambda x)\, \lambda^r\, d\lambda,$$
$$\xi_r(t,s,x) \doteq \int_0^{\lambda_0} e^{-\lambda(R+t)} \cosh \lambda(t-s)\, \Phi(\lambda x)\, \lambda^r\, d\lambda,$$

where $r > -1$, λ_0 is a fixed positive constant and Φ is defined by (27). These auxiliary functions have been introduced in [47] as generalizations of the test function considered by Zhou (see [57, equation (3.2)]) in the treatment of the critical case for the semilinear wave equation with power nonlinearity in the higher dimensional case. Let us underline that the assumption on r is done in order to guarantee the integrability of the function λ^r in a neighborhood of 0.

As functionals to study the blow-up dynamic we will consider

$$\mathcal{U}(t) \doteq \int_{\mathbb{R}^n} \partial_t u(t,x)\, \eta_{r_1}(t,t,x)\, dx, \tag{54}$$

$$\mathcal{V}(t) \doteq \int_{\mathbb{R}^n} v(t,x)\, \eta_{r_2}(t,t,x)\, dx. \tag{55}$$

We point out that the choice of the conditions for the pair (r_1, r_2) depends on the critical case we deal with. More specifically, we have to distinguish among the three possible subcases $\Theta_1(n,p,q) = 0 > \Theta_2(n,p,q)$, $\Theta_1(n,p,q) < 0 = \Theta_2(n,p,q)$ and $\Theta_1(n,p,q) = \Theta_2(n,p,q) = 0$.

First, we derive two fundamental identities for $\mathcal{U}$ and $\mathcal{V}$, which involve the initial data and the nonlinear terms.

Proposition 6.2. *Let (u,v) be a weak solution of (12) on $[0,T)$ and let $\mathcal{U},\mathcal{V}$ denote the functionals defined by (54), (55). Then, the following identities are satisfied for any $t \geqslant 0$:*

$$\mathcal{U}(t) = \varepsilon t \int_{\mathbb{R}^n} u_0(x)\, \eta_{r_1+2}(t,0,x)\, dx + \varepsilon \int_{\mathbb{R}^n} u_1(x)\, \xi_{r_1}(t,0,x)\, dx + \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^q\, \xi_{r_1}(t,s,x)\, dx\, ds, \tag{56}$$

$$\mathcal{V}(t) = \varepsilon \int_{\mathbb{R}^n} v_0(x)\, \xi_{r_2}(t,0,x)\, dx + \varepsilon t \int_{\mathbb{R}^n} v_1(x)\, \eta_{r_2}(t,0,x)\, dx + \int_0^t (t-s) \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p\, \eta_{r_2}(t,s,x)\, dx\, ds. \tag{57}$$

Proof. In order to show the validity of (56) and (57) we will employ the definition of weak solution for (12) with a suitable choice of the test functions (ϕ,ψ) in (52) and (53). If we assume that (u,v) satisfies (8), then, $\operatorname{supp} u(t,\cdot), \operatorname{supp} v(t,\cdot) \subset B_{R+t}$ for any $t \geqslant 0$. Therefore, we may remove the assumption of compactness for the supports of the test functions in Definition 6.1. Hence, it is possible to consider

$$\phi = \phi(t;s,x) = \cosh \lambda(t-s)\, \Phi(\lambda x),$$
$$\psi = \psi(t;s,x) = \frac{\sinh \lambda(t-s)}{\lambda}\, \Phi(\lambda x).$$

Since $\Delta\Phi(\lambda x) = \lambda^2 \Phi(\lambda x)$, then, ϕ, ψ are solutions of the homogeneous free wave equation. Moreover,

$$\begin{aligned} &\phi(t;t,x) = \Phi(\lambda x), && \phi(t;0,x) = \cosh \lambda t\, \Phi(\lambda x), && \phi_s(t;t,x) = 0, && \phi_s(t;0,x) = -\lambda \sinh \lambda t\, \Phi(\lambda x), \\ &\psi(t;t,x) = 0, && \psi(t;0,x) = \lambda^{-1} \sinh \lambda t\, \Phi(\lambda x), && \psi_s(t;t,x) = -\Phi(\lambda x), && \psi_s(t;0,x) = -\cosh \lambda x\, \Phi(\lambda x). \end{aligned}$$

Consequently, from (52) and (53) we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \partial_t u(t,x) \Phi(\lambda x)\, dx &= \varepsilon \lambda \sinh \lambda t \int_{\mathbb{R}^n} u_0(x) \Phi(\lambda x)\, dx + \varepsilon \cosh \lambda t \int_{\mathbb{R}^n} u_1(x) \Phi(\lambda x)\, dx \\ &\quad + \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^q \cosh \lambda(t-s) \Phi(\lambda x)\, dx\, ds \end{aligned} \tag{58}$$

$$\begin{aligned} \int_{\mathbb{R}^n} v(t,x) \Phi(\lambda x)\, dx &= \varepsilon \cosh \lambda t \int_{\mathbb{R}^n} v_0(x) \Phi(\lambda x)\, dx + \varepsilon \lambda^{-1} \sinh \lambda t \int_{\mathbb{R}^n} v_1(x) \Phi(\lambda x)\, dx \\ &\quad + \int_0^t \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \lambda^{-1} \sinh \lambda(t-s) \Phi(\lambda x)\, dx\, ds. \end{aligned} \tag{59}$$

Multiplying both sides of (58) by $e^{-\lambda(R+t)} \lambda^{r_1}$, integrating the resulting relation with respect to λ over $[0,\lambda_0]$ and, finally, applying Fubini's theorem, we get (56). Similarly, from (59) we find (57). This concludes the proof. □

The next step is to derive from (56) and (57) the iteration frame. In order to do so, we need to estimate sharply the auxiliary functions η_r and ξ_r.

Lemma 6.3. *Let $n \geqslant 2$. There exist $\lambda_0 > 0$ such that the following properties hold:*

(i) *if $r > -1$, $|x| \leqslant R$ and $t \geqslant 0$, then,*

$$\begin{aligned}\xi_r(t,0,x) &\geqslant A_0,\\ \eta_r(t,0,x) &\geqslant B_0\langle t\rangle^{-1};\end{aligned}$$

(ii) *if $r > -1$, $|x| \leqslant s+R$ and $t > s \geqslant 0$, then,*

$$\begin{aligned}\xi_r(t,s,x) &\geqslant A_1\langle s\rangle^{-r-1},\\ \eta_r(t,s,x) &\geqslant B_1\langle t\rangle^{-1}\langle s\rangle^{-r};\end{aligned}$$

(iii) *if $r > \frac{n-3}{2}$, $|x| \leqslant t+R$ and $t > 0$, then,*

$$\eta_r(t,t,x) \leqslant B_2\langle t\rangle^{-\frac{n-1}{2}}\langle t-|x|\rangle^{\frac{n-3}{2}-r}.$$

Here A_0 and B_k, $k = 0,1,2$, are positive constants depending only on λ_0, r and R and we denote $\langle y\rangle \doteq 3+|y|$.

Remark 6.4. *Let us stress that differently from [47, Lemma 3.1] we require in the statement of (i) and (ii) the condition of $r > -1$ instead of $r > 0$. Nonetheless, the proofs from [47] of (i) and of the lower bound for $\eta_r(t,s,x)$ in (ii) are still valid even for $r > -1$.*

Proof. We can restrict our considerations to the lower bound estimate for $\xi(t,s,x)$ in (ii), as the other properties are already proved in [47, Lemma 3.1]. Since $\langle s\rangle \geqslant 2$, we may shrink the domain of integration in the definition of $\xi_r(t,s,x)$ as follows

$$\xi_r(t,s,x) \geqslant \int_{\lambda_0/\langle s\rangle}^{2\lambda_0/\langle s\rangle} e^{-\lambda(R+t)}\cosh\lambda(t-s)\,\Phi(\lambda x)\,\lambda^r\,d\lambda.$$

We remark that the condition

$$\Phi(x) \asymp \langle x\rangle^{-\frac{n-1}{2}}e^{|x|} \quad \text{for any} \quad x\in\mathbb{R}^n$$

implies that the infimum

$$\inf_{\lambda\in\left[\frac{\lambda_0}{\langle s\rangle},\frac{2\lambda_0}{\langle s\rangle}\right]}\ \inf_{|x|\leqslant s+R} e^{-\lambda(s+R)}\Phi(\lambda x)$$

can be estimate from below by a constant $A = A(\lambda_0,R) > 0$ that does not depend on λ, s and x. Therefore, we may estimate

$$\begin{aligned}\xi_r(t,s,x) &\geqslant \int_{\lambda_0/\langle s\rangle}^{2\lambda_0/\langle s\rangle} e^{-\lambda(t-s)}\cosh\lambda(t-s)\,e^{-\lambda(R+s)}\,\Phi(\lambda x)\,\lambda^r\,d\lambda\\ &= \int_{\lambda_0/\langle s\rangle}^{2\lambda_0/\langle s\rangle} \frac{1}{2}\left(1+e^{-2\lambda(t-s)}\right)e^{-\lambda(R+s)}\,\Phi(\lambda x)\,\lambda^r\,d\lambda \geqslant A\int_{\lambda_0/\langle s\rangle}^{2\lambda_0/\langle s\rangle}\frac{1}{2}\left(1+e^{-2\lambda(t-s)}\right)\lambda^r\,d\lambda\\ &\geqslant \frac{A}{2}\int_{\lambda_0/\langle s\rangle}^{2\lambda_0/\langle s\rangle}\lambda^r\,d\lambda = \frac{A}{2}\frac{\lambda_0^{r+1}}{(r+1)}(2^{r+1}-1)\langle s\rangle^{-r-1},\end{aligned}$$

which is the desired lower bound estimate for $\xi_r(t,s,x)$. ◻

6.2. Derivation of the iteration frame in the critical case

In order to derive the iteration scheme, we have to consider separately the three critical cases. In each case we will fix suitable conditions on the pair (r_1, r_2), that will influence, on the one hand, the structure of the scheme itself with the possible presence of a logarithmic factor in the integral inequalities and, on the other hand, the functional $\mathcal{U}$ and/or $\mathcal{V}$ for which we can derive a lower bound containing a logarithmic factor.

6.2.1. Case $\Theta_1(n,p,q)=0$

In this case we consider $r_1 = \frac{n-1}{2} - \frac{1}{p}$ and $r_2 > \frac{n-1}{2} - \frac{1}{q}$. The purpose of this section is to derive the frame for the iteration argument, which is a coupled system of integral inequalities for the functionals $\mathcal{U}, \mathcal{V}$. In order to get this system we will combine the fundamental identities (56), (57) and the estimates for the auxiliary functions in Lemma 6.3. Combining (8), (54) and Hölder's inequality, we find

$$\mathcal{U}(s) \leqslant \left(\int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \eta_{r_2}(t,s,x)\, dx \right)^{\frac{1}{p}} \left(\int_{B_{R+s}} \frac{\eta_{r_1}(s,s,x)^{p'}}{\eta_{r_2}(t,s,x)^{\frac{p'}{p}}}\, dx \right)^{\frac{1}{p'}}. \tag{60}$$

Using Lemma 6.3 (ii)-(iii) and the condition $r_1 = \frac{n-1}{2} - \frac{1}{p}$, we may estimate

$$\begin{aligned} \int_{B_{R+s}} \frac{\eta_{r_1}(s,s,x)^{p'}}{\eta_{r_2}(t,s,x)^{\frac{p'}{p}}}\, dx &\lesssim \langle t\rangle^{\frac{p'}{p}} \langle s\rangle^{r_2 \frac{p'}{p} - \frac{n-1}{2}p'} \int_{B_{R+s}} \langle s-|x|\rangle^{\left(\frac{n-3}{2}-r_1\right)p'}\, dx \\ &\lesssim \langle t\rangle^{\frac{p'}{p}} \langle s\rangle^{r_2 \frac{p'}{p} - \frac{n-1}{2}p' + n-1} \log\langle s\rangle. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \eta_{r_2}(t,s,x)\, dx &\gtrsim \big(\mathcal{U}(s)\big)^p \left(\int_{B_{R+s}} \frac{\eta_{r_1}(s,s,x)^{p'}}{\eta_{r_2}(t,s,x)^{\frac{p'}{p}}}\, dx \right)^{-\frac{p}{p'}} \\ &\gtrsim \langle t\rangle^{-1} \langle s\rangle^{-r_2 + \frac{n-1}{2}p - (n-1)(p-1)} (\log\langle s\rangle)^{-(p-1)} \big(\mathcal{U}(s)\big)^p. \end{aligned}$$

Consequently, from (57) we obtain

$$\mathcal{V}(t) \gtrsim \langle t\rangle^{-1} \int_0^t (t-s) \langle s\rangle^{-r_2 + \frac{n-1}{2}p - (n-1)(p-1)} (\log\langle s\rangle)^{-(p-1)} \big(\mathcal{U}(s)\big)^p\, ds. \tag{61}$$

Now we will derive an analogous integral lower bound for $\mathcal{U}$. By (55) and Hölder's inequality we have

$$\mathcal{V}(s) \leqslant \left(\int_{\mathbb{R}^n} |v(s,x)|^q \xi_{r_1}(t,s,x)\, dx \right)^{\frac{1}{q}} \left(\int_{B_{R+s}} \frac{\eta_{r_2}(s,s,x)^{q'}}{\xi_{r_1}(t,s,x)^{\frac{q'}{q}}}\, dx \right)^{\frac{1}{q'}}. \tag{62}$$

Employing again Lemma 6.3 and the condition $r_2 > \frac{n-1}{2} - \frac{1}{q}$, we arrive at

$$\begin{aligned} \int_{B_{R+s}} \frac{\eta_{r_2}(s,s,x)^{q'}}{\xi_{r_1}(t,s,x)^{\frac{q'}{q}}}\, dx &\lesssim \langle s\rangle^{(r_1+1)\frac{q'}{q} - \frac{n-1}{2}q'} \int_{B_{R+s}} \langle s-|x|\rangle^{\left(\frac{n-3}{2}-r_2\right)q'}\, dx \\ &\lesssim \langle s\rangle^{(r_1+1)\frac{q'}{q} - \frac{n-1}{2}q' + n + \left(\frac{n-3}{2}-r_2\right)q'}, \end{aligned}$$

which implies in turn

$$\begin{aligned} \int_{\mathbb{R}^n} |v(s,x)|^q \xi_{r_1}(t,s,x)\, dx\, ds &\gtrsim \big(\mathcal{V}(s)\big)^q \left(\int_{B_{R+s}} \frac{\eta_{r_2}(s,s,x)^{q'}}{\xi_{r_1}(t,s,x)^{\frac{q'}{q}}}\, dx \right)^{-\frac{q}{q'}} \\ &\gtrsim \langle s\rangle^{-(r_1+1) + \frac{n-1}{2}q - n(q-1) - \left(\frac{n-3}{2}-r_2\right)q} \big(\mathcal{V}(s)\big)^q. \end{aligned}$$

Finally, (56) and the previous inequality yield

$$\mathcal{U}(t) \gtrsim \int_0^t \langle s\rangle^{-r_1+n-1-(n-1)q+r_2 q}\big(\mathcal{V}(s)\big)^q\, ds. \tag{63}$$

6.2.2. Case $\Theta_2(n,p,q)=0$

For this critical case we assume $r_1 > \frac{n-1}{2}-\frac{1}{p}$ and $r_2 = \frac{n-1}{2}-\frac{1}{q}$. Due to the fact that we switch in some sense the role of r_1 and r_2 with respect to the previous critical case $\Theta_1(n,p,q)=0$, somehow also the structure of the iteration frame is reversed with respect to the previous section.

By Lemma 6.3 (ii)-(iii) and the condition $r_1 > \frac{n-1}{2}-\frac{1}{q}$ it follows

$$\begin{aligned}\int_{B_{R+s}} \frac{\eta_{r_1}(s,s,x)^{p'}}{\eta_{r_2}(t,s,x)^{\frac{p'}{p}}}\, dx &\lesssim \langle t\rangle^{\frac{p'}{p}}\langle s\rangle^{r_2\frac{p'}{p}-\frac{n-1}{2}p'}\int_{B_{R+s}}\langle s-|x|\rangle^{\left(\frac{n-3}{2}-r_1\right)p'}\, dx \\ &\lesssim \langle t\rangle^{\frac{p'}{p}}\langle s\rangle^{r_2\frac{p'}{p}-\frac{n-1}{2}p'+n+\left(\frac{n-3}{2}-r_1\right)p'}.\end{aligned}$$

Then, from (60) we get

$$\begin{aligned}\int_{\mathbb{R}^n}|\partial_t u(s,x)|^p\eta_{r_2}(t,s,x)\, dx &\gtrsim \langle t\rangle^{-1}\langle s\rangle^{-r_2+\frac{n-1}{2}p-n(p-1)-\left(\frac{n-3}{2}-r_1\right)p}\big(\mathcal{U}(s)\big)^p \\ &\gtrsim \langle t\rangle^{-1}\langle s\rangle^{-r_2-(n-1)p+n+r_1 p}\big(\mathcal{U}(s)\big)^p.\end{aligned}$$

Also, (57) yields

$$\mathcal{V}(t) \gtrsim \langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-r_2-(n-1)p+n+r_1 p}\big(\mathcal{U}(s)\big)^p\, ds. \tag{64}$$

We determine now the integral lower bound for $\mathcal{U}$. By using Lemma 6.3 and the condition $r_2 = \frac{n-1}{2}-\frac{1}{q}$, we arrive at

$$\begin{aligned}\int_{B_{R+s}} \frac{\eta_{r_2}(s,s,x)^{q'}}{\xi_{r_1}(t,s,x)^{\frac{q'}{q}}}\, dx &\lesssim \langle s\rangle^{(r_1+1)\frac{q'}{q}-\frac{n-1}{2}q'}\int_{B_{R+s}}\langle s-|x|\rangle^{\left(\frac{n-3}{2}-r_2\right)q'}\, dx \\ &\lesssim \langle s\rangle^{(r_1+1)\frac{q'}{q}-\frac{n-1}{2}q'+n-1}\log\langle s\rangle.\end{aligned}$$

The last estimate together with (62) provides

$$\int_{\mathbb{R}^n}|v(s,x)|^q\xi_{r_1}(t,s,x)\, dx \gtrsim \langle s\rangle^{-(r_1+1)+\frac{n-1}{2}q-(n-1)(q-1)}(\log\langle s\rangle)^{-(q-1)}\big(\mathcal{V}(s)\big)^q.$$

Thus, (56) and the last estimate imply

$$\mathcal{U}(t) \gtrsim \int_0^t \langle s\rangle^{-(r_1+1)+\frac{n-1}{2}q-(n-1)(q-1)}(\log\langle s\rangle)^{-(q-1)}\big(\mathcal{V}(s)\big)^q\, ds. \tag{65}$$

6.2.3. Case $\Theta_1(n,p,q)=\Theta_2(n,p,q)=0$

In this case we choose $r_1 = \frac{n-1}{2}-\frac{1}{p}$ and $r_2 = \frac{n-1}{2}-\frac{1}{q}$. In particular, one can prove the identities

$$\tfrac{n-1}{2}-\tfrac{1}{p} = n-1-\tfrac{n-1}{2}q, \tag{66}$$

$$\tfrac{n-1}{2}-\tfrac{1}{q} = n-\tfrac{n-1}{2}p, \tag{67}$$

due to the fact that the pair (p,q) satisfies both the critical conditions $\Theta_1(n,p,q)=\Theta_2(n,p,q)=0$. Indeed, if we denote $\kappa_1 = n-1-\frac{n-1}{2}q-\frac{n-1}{2}+\frac{1}{p}$ and $\kappa_1 = n-\frac{n-1}{2}p-\frac{n-1}{2}+\frac{1}{q}$, then

$$\begin{aligned}\kappa_1+q\kappa_2 &= (pq-1)\Theta_1(n,p,q)=0, \\ p\kappa_1+\kappa_2 &= (pq-1)\Theta_2(n,p,q)=0.\end{aligned}$$

As $pq \neq 1$, then, trivially $\kappa_1 = \kappa_2 = 0$, but this means exactly the validity of (66)-(67).

Since $r_1 = \frac{n-1}{2} - \frac{1}{p}$ as in Section 6.2.1, we can prove (61). However, thanks to (67) we see that the power of $\langle s \rangle$ in the right hand side of (61) is exactly -1, that is,

$$\mathscr{V}(t) \gtrsim \langle t \rangle^{-1} \int_0^t (t-s)\langle s \rangle^{-1} (\log\langle s \rangle)^{-(p-1)} \big(\mathscr{U}(s)\big)^p \, ds. \tag{68}$$

Similarly, since $r_2 = \frac{n-1}{2} - \frac{1}{q}$ as in Section 6.2.2 it holds (65). Yet, due to (66) we find again that the power of $\langle s \rangle$ in the right hand side of (65) is exactly -1, that is,

$$\mathscr{U}(t) \gtrsim \int_0^t \langle s \rangle^{-1} (\log\langle s \rangle)^{-(q-1)} \big(\mathscr{V}(s)\big)^q \, ds. \tag{69}$$

6.3. Lower bound estimates for the functionals containing a logarithmic factor

Purpose of this section is to derive lower bounds for $\mathscr{U}$ and/or $\mathscr{V}$ of logarithmic type. As in the previous section, we shall consider separately the three critical cases. We point out that the assumptions on the pair (r_1, r_2) are the same as in Section 6.2 and they depend on the critical case that we consider.

6.3.1. Case $\Theta_1(n,p,q) = 0$

In this case we will derive a lower bound for the functional $\mathscr{U}$ in two step. From (57), Lemma 6.3 (ii) and Proposition 3.3, we get for $t \geqslant 0$

$$\begin{aligned} \mathscr{V}(t) &\geqslant \int_0^t (t-s) \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \eta_{r_2}(t,s,x)\, dx\, ds \gtrsim \langle t \rangle^{-1} \int_0^t (t-s)\langle s \rangle^{-r_2} \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \, dx\, ds \\ &\gtrsim \varepsilon^p \langle t \rangle^{-1} \int_0^t (t-s) \langle s \rangle^{-r_2+n-1-\frac{n-1}{2}p} \, ds. \end{aligned} \tag{70}$$

Consequently, for $t \geqslant 1$

$$\begin{aligned} \mathscr{V}(t) &\gtrsim \varepsilon^p \langle t \rangle^{-1-r_2-\frac{n-1}{2}p} \int_0^t (t-s)\langle s \rangle^{n-1} \, ds \gtrsim \varepsilon^p \langle t \rangle^{-1-r_2-\frac{n-1}{2}p} \int_{\frac{t}{2}}^t (t-s)\langle s \rangle^{n-1} \, ds \\ &\gtrsim \varepsilon^p \langle t \rangle^{-1-r_2-\frac{n-1}{2}p} \langle \tfrac{t}{2} \rangle^{n-1} \int_{\frac{t}{2}}^t (t-s)\, ds \gtrsim \varepsilon^p \langle t \rangle^{-r_2-\frac{n-1}{2}p+n}. \end{aligned}$$

Plugging the last lower bound for $\mathscr{V}$ in (63), we have for $t \geqslant 1$

$$\begin{aligned} \mathscr{U}(t) &\gtrsim \varepsilon^{pq} \int_1^t \langle s \rangle^{-r_1+n-1-(n-1)q+r_2 q+\left(-r_2-\frac{n-1}{2}p+n\right)q} \, ds \\ &\gtrsim \varepsilon^{pq} \int_1^t \langle s \rangle^{-r_1+n-1+q-\frac{n-1}{2}pq} \, ds \gtrsim \varepsilon^{pq} \int_1^t \langle s \rangle^{q+p^{-1}-\frac{n-1}{2}(pq-1)} \, ds \\ &\gtrsim \varepsilon^{pq} \int_1^t \langle s \rangle^{-1} ds \gtrsim \varepsilon^{pq} \int_1^t s^{-1} ds \gtrsim \varepsilon^{pq} \log t, \end{aligned} \tag{71}$$

where we used in the third inequality the actual value of r_1 and in the fourth one the critical condition $\Theta_1(n,p,q) = 0$.

6.3.2. Case $\Theta_2(n,p,q) = 0$

Let us determine a lower bound for $\mathscr{V}$ in two step. From (56), Lemma 6.3 (ii) and Proposition 3.3 we obtain for $t \geqslant 0$

$$\begin{aligned} \mathscr{U}(t) &\geqslant \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^q \xi_{r_1}(t,s,x)\, dx\, ds \gtrsim \int_0^t \langle s \rangle^{-(r_1+1)} \int_{\mathbb{R}^n} |v(s,x)|^q \, dx\, ds \\ &\gtrsim \varepsilon^q \int_0^t \langle s \rangle^{-(r_1+1)+n-1-\frac{n-1}{2}q} \, ds. \end{aligned} \tag{72}$$

Also, for $t \geqslant 0$

$$\begin{aligned}\mathcal{U}(t) &\gtrsim \varepsilon^q \int_0^t \langle s\rangle^{-(r_1+1)+n-1-\frac{n-1}{2}q}\, ds \gtrsim \varepsilon^q \langle t\rangle^{-(r_1+1)-\frac{n-1}{2}q} \int_0^t \langle s\rangle^{n-1}\, ds \\ &\gtrsim \varepsilon^q \langle t\rangle^{-(r_1+1)-\frac{n-1}{2}q+n}.\end{aligned}$$

Plugging the last lower bound for $\mathcal{U}$ in (64), we have for $t \geqslant \frac{3}{2}$

$$\begin{aligned}\mathcal{V}(t) &\gtrsim \varepsilon^{pq} \langle t\rangle^{-1} \int_0^t (t-s)\langle s\rangle^{-r_2-(n-1)p+n+r_1p+\left(-(r_1+1)-\frac{n-1}{2}q+n\right)p} ds \\ &= \varepsilon^{pq} \langle t\rangle^{-1} \int_0^t (t-s)\langle s\rangle^{-r_2+n-\frac{n-1}{2}pq} ds = \varepsilon^{pq} \langle t\rangle^{-1} \int_0^t (t-s)\langle s\rangle^{1+q^{-1}-\frac{n-1}{2}(pq-1)} ds \\ &= \varepsilon^{pq} \langle t\rangle^{-1} \int_0^t (t-s)\langle s\rangle^{-1} ds \gtrsim \varepsilon^{pq} \langle t\rangle^{-1} \int_1^t \frac{t-s}{s}\, ds \gtrsim \varepsilon^{pq} \langle t\rangle^{-1} \int_1^t \log s\, ds \\ &\gtrsim \varepsilon^{pq} \langle t\rangle^{-1} \int_{\frac{2t}{3}}^t \log s\, ds \gtrsim \varepsilon^{pq} \log\Big(\frac{2t}{3}\Big), \end{aligned} \tag{73}$$

where we employed in the third step the actual value of r_2 and in the fourth one the critical condition $\Theta_2(n,p,q) = 0$.

6.3.3. Case $\Theta_1(n,p,q) = \Theta_2(n,p,q) = 0$

In this last case we can improve both (71) and (73) thanks to (66), (67). Indeed, combining (72) and (66), for any $t \geqslant 0$ we obtain

$$\mathcal{U}(t) \gtrsim \varepsilon^q \int_0^t \langle s\rangle^{-1} ds \gtrsim \varepsilon^q \log t. \tag{74}$$

Analogously, using (70) and (67), for any $t \geqslant \frac{3}{2}$ we have

$$\mathcal{V}(t) \gtrsim \varepsilon^p \langle t\rangle^{-1} \int_0^t (t-s)\langle s\rangle^{-1} ds \gtrsim \varepsilon^p \log\Big(\frac{2t}{3}\Big). \tag{75}$$

6.4. Iterated lower bound estimates: slicing method

In this section we derive iteratively a sequence of lower bound estimates for $\mathcal{U}$ or $\mathcal{V}$. Then, in Section 6.5 we will employ these iterated lower bounds to prove the blow-up and to derive the upper bound for the lifespan of the local solution (u,v).

However, before starting with this iterative procedure, we summarize the estimates that we proved in Sections 6.2 and 6.3.

In Section 6.2 we proved the coupled system of integral inequalities

$$\begin{cases} \mathcal{U}(t) \geqslant C \displaystyle\int_0^t \langle s\rangle^{-r_1+n-1-(n-1)q+r_2q} \big(\mathcal{V}(s)\big)^q\, ds & \text{if } \Theta_1 = 0, \\ \mathcal{U}(t) \geqslant C \displaystyle\int_0^t \langle s\rangle^{-(r_1+1)+\frac{n-1}{2}q-(n-1)(q-1)} (\log\langle s\rangle)^{-(q-1)} \big(\mathcal{V}(s)\big)^q\, ds & \text{if } \Theta_2 = 0, \\ \mathcal{U}(t) \geqslant C \displaystyle\int_0^t \langle s\rangle^{-1} (\log\langle s\rangle)^{-(q-1)} \big(\mathcal{V}(s)\big)^q\, ds & \text{if } \Theta_1 = \Theta_2 = 0, \end{cases} \tag{76}$$

$$\begin{cases} \mathcal{V}(t) \geqslant K\langle t\rangle^{-1} \displaystyle\int_0^t (t-s)\langle s\rangle^{-r_2+\frac{n-1}{2}p-(n-1)(p-1)} (\log\langle s\rangle)^{-(p-1)} \big(\mathcal{U}(s)\big)^p\, ds & \text{if } \Theta_1 = 0, \\ \mathcal{V}(t) \geqslant K\langle t\rangle^{-1} \displaystyle\int_0^t (t-s)\langle s\rangle^{-r_2-(n-1)p+n+r_1p} \big(\mathcal{U}(s)\big)^p\, ds & \text{if } \Theta_2 = 0, \\ \mathcal{V}(t) \geqslant K\langle t\rangle^{-1} \displaystyle\int_0^t (t-s)\langle s\rangle^{-1} (\log\langle s\rangle)^{-(p-1)} \big(\mathcal{U}(s)\big)^p\, ds & \text{if } \Theta_1 = \Theta_2 = 0, \end{cases} \tag{77}$$

for any $t \geqslant 0$, where C, K are positive constants depending on n, p, q, R. Let us underline that the range for the pair (r_1, r_1) is implicitly fixed by the corresponding critical case according to Section 6.2.

On the other hand, the lower bound estimates (71), (73), (74) and (75) from Section 6.3 can be summarized as follows

$$\begin{cases} \mathcal{U}(t) \geqslant \widetilde{C}\varepsilon^{pq} \log t & \text{if } \Theta_1 = 0, \\ \mathcal{U}(t) \geqslant \widetilde{C}\varepsilon^{q} \log t & \text{if } \Theta_1 = \Theta_2 = 0, \end{cases} \tag{78}$$

for any $t \geqslant 1$ and

$$\begin{cases} \mathcal{V}(t) \geqslant \widetilde{K}\varepsilon^{pq} \log\left(\frac{2t}{3}\right) & \text{if } \Theta_2 = 0, \\ \mathcal{V}(t) \geqslant \widetilde{K}\varepsilon^{p} \log\left(\frac{2t}{3}\right) & \text{if } \Theta_1 = \Theta_2 = 0, \end{cases} \tag{79}$$

for any $t \geqslant \frac{3}{2}$, where $\widetilde{C}, \widetilde{K}$ are positive constants depending on $n, p, q, R, u_0, u_1, v_0, v_1$.

Now we can start with the iteration argument. As in the previous sections, we consider separately the three critical cases.

6.4.1. Case $\Theta_1(n,p,q) = 0$

Let us introduce the sequence of positive real numbers $\{\ell_j\}_{j\in\mathbb{N}}$, where $\ell_j \doteq 2 - 2^{-j}$, that will be use to split the time interval in the slicing method. In this case the goal is to prove that

$$\mathcal{U}(t) \geqslant C_j \big(\log\langle t\rangle\big)^{-b_j} \left(\log\left(\frac{t}{\ell_j}\right)\right)^{a_j} \qquad \text{for } \ t \geqslant \ell_j \ \text{ and for any } \ j \in \mathbb{N}, \tag{80}$$

where $\{C_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative real numbers that we shall determine throughout the iteration argument. Thanks to (78) we see that (80) is satisfied for $j = 0$, provided that the initial values of the sequences are given by $a_0 \doteq 1, b_0 \doteq 0$ and $C_0 \doteq \widetilde{C}\varepsilon^{pq}$. Hence, we employ an inductive argument to prove the validity of (80) for any $j \in \mathbb{N}$. We proceed now with the inductive step. Let us plug (80) in (77), after shrinking the domain of integration, then, for $s \geqslant \ell_{j+1}$ we obtain

$$\begin{aligned}
\mathcal{V}(s) &\geqslant KC_j^p \langle s\rangle^{-1} \int_{\ell_j}^{s} (s-\tau)\langle\tau\rangle^{-r_2+\frac{n-1}{2}p-(n-1)(p-1)} \big(\log\langle\tau\rangle\big)^{-(p-1)-b_jp} \left(\log\left(\tfrac{\tau}{\ell_j}\right)\right)^{a_jp} d\tau \\
&\geqslant KC_j^p \langle s\rangle^{-1-r_2-\frac{n-1}{2}p} \big(\log\langle s\rangle\big)^{-(p-1)-b_jp} \int_{\ell_j}^{s} (s-\tau)\langle\tau\rangle^{n-1} \left(\log\left(\tfrac{\tau}{\ell_j}\right)\right)^{a_jp} d\tau \\
&\geqslant KC_j^p \langle s\rangle^{-1-r_2-\frac{n-1}{2}p} \big(\log\langle s\rangle\big)^{-(p-1)-b_jp} \int_{\frac{\ell_j s}{\ell_{j+1}}}^{s} (s-\tau)\tau^{n-1} \left(\log\left(\tfrac{\tau}{\ell_j}\right)\right)^{a_jp} d\tau \\
&\geqslant KC_j^p \left(\tfrac{\ell_j}{\ell_{j+1}}\right)^{n-1} \langle s\rangle^{-1-r_2-\frac{n-1}{2}p} s^{n-1} \big(\log\langle s\rangle\big)^{-(p-1)-b_jp} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{a_jp} \int_{\frac{\ell_j s}{\ell_{j+1}}}^{s} (s-\tau)\, d\tau \\
&\geqslant 2^{-1}KC_j^p \left(\tfrac{\ell_j}{\ell_{j+1}}\right)^{n-1} \left(1-\tfrac{\ell_j}{\ell_{j+1}}\right)^{2} \langle s\rangle^{-1-r_2-\frac{n-1}{2}p} s^{n+1} \big(\log\langle s\rangle\big)^{-(p-1)-b_jp} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{a_jp} \\
&\geqslant 2^{-2j-3n-6}KC_j^p \langle s\rangle^{-r_2-\frac{n-1}{2}p+n} \big(\log\langle s\rangle\big)^{-(p-1)-b_jp} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{a_jp},
\end{aligned}$$

where in the last step we used the inequalities $2\ell_j \geqslant \ell_{j+1}$, $1-\frac{\ell_j}{\ell_{j+1}} \geqslant 2^{-(j+2)}$ and $4s \geqslant \langle s\rangle$ for any $s \geqslant 1$. Using this lower bound for $\mathscr{V}(s)$ in (76) and the critical relation $\Theta_1(n,p,q)=0$, for $t \geqslant \ell_{j+1}$ we get

$$\begin{aligned}\mathscr{U}(t) &\geqslant 2^{-2qj-3q(n+2)} C K^q C_j^{pq} \int_{\ell_{j+1}}^t \langle s\rangle^{-r_1+n-1+q-\frac{n-1}{2}pq} \big(\log\langle s\rangle\big)^{-q(p-1)-b_j pq} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{a_j pq} ds \\ &\geqslant 2^{-2qj-3q(n+2)} C K^q C_j^{pq} \big(\log\langle t\rangle\big)^{-q(p-1)-b_j pq} \int_{\ell_{j+1}}^t \langle s\rangle^{q+\frac{1}{p}-\frac{n-1}{2}(pq-1)} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{a_j pq} ds \\ &= 2^{-2qj-3q(n+2)} C K^q C_j^{pq} \big(\log\langle t\rangle\big)^{-q(p-1)-b_j pq} \int_{\ell_{j+1}}^t \langle s\rangle^{-1} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{a_j pq} ds \\ &= 2^{-2qj-3q(n+2)} C K^q C_j^{pq} (a_j pq+1)^{-1} \big(\log\langle t\rangle\big)^{-q(p-1)-b_j pq} \left(\log\left(\tfrac{t}{\ell_{j+1}}\right)\right)^{a_j pq+1},\end{aligned}$$

which is exactly (80) for $j+1$, if we define

$$C_{j+1} \doteq 2^{-2qj-3q(n+2)} C K^q C_j^{pq} (a_j pq+1)^{-1}, \quad a_{j+1} \doteq a_j pq+1, \quad b_{j+1} \doteq q(p-1)+b_j pq.$$

In order to derive the upper bound estimate for the life span of the solution, it is convenient to derive an estimate from below of C_j, where the dependence on j in the lower bound is more explicit than the one in the definition of C_j itself. But first, let us derive the explicit expression for a_j and b_j. Using iteratively the recursive relations between two successive elements that we just proved, we find

$$\begin{aligned} a_j &= a_{j-1}pq+1 = \cdots = a_0(pq)^j + \sum_{k=0}^{j-1}(pq)^k = (pq)^j + \tfrac{(pq)^j-1}{pq-1} = \tfrac{(pq)^{j+1}-1}{pq-1}, \\ b_j &= b_{j-1}pq + q(p-1) = \cdots = b_0(pq)^j + q(p-1)\sum_{k=0}^{j-1}(pq)^k = \tfrac{q(p-1)}{pq-1}\big((pq)^j-1\big). \end{aligned} \tag{81}$$

Therefore,

$$a_{j-1}pq+1 \leqslant \tfrac{pq}{pq-1}(pq)^j.$$

In particular, the previous inequality implies

$$C_j \geqslant M N^{-j} C_{j-1}^{pq}, \tag{82}$$

where $M \doteq 2^{-q(3n+4)} C K^q \frac{(pq-1)}{pq}$ and $N \doteq 2^{2q} pq$. Applying the logarithmic function to both sides of (82) and using iteratively the resulting inequality, we get

$$\begin{aligned} \log C_j &\geqslant (pq)\log C_{j-1} - j\log N + \log M \\ &\geqslant (pq)^2 \log C_{j-2} - \big(j+(j-1)(pq)\big)\log N + \big(1+pq\big)\log M \\ &\geqslant \cdots \geqslant (pq)^j \log C_0 - \sum_{k=0}^{j-1}(j-k)(pq)^k \log N + \sum_{k=0}^{j-1}(pq)^k \log M \\ &= (pq)^j \log C_0 - (pq)^j \sum_{k=1}^{j} k(pq)^{-k}\log N + \tfrac{(pq)^j-1}{pq-1}\log M \\ &= (pq)^j\left(\log C_0 - S_j \log N + \tfrac{\log M}{pq-1}\right) - \tfrac{\log M}{pq-1}, \end{aligned}$$

where $S_j \doteq \sum_{k=1}^{j} k(pq)^{-k}$. As $\{S_j\}_{j\geqslant 1}$ is a sequence of the partial sums of a convergent series, if we denote by S the limit of this sequence, because of $S_j \uparrow S$ as $j\to\infty$, then, we may estimate

$$C_j \geqslant M^{-(pq-1)} \exp\left((pq)^j \log\left(C_0 N^{-S} M^{pq-1}\right)\right). \tag{83}$$

6.4.2. Case $\Theta_2(n,p,q)=0$

In this second critical case we shall prove that

$$\mathscr{V}(t)\geqslant K_j\big(\log\langle t\rangle\big)^{-\beta_j}\left(\log\left(\frac{t}{\ell_{2j+1}}\right)\right)^{\alpha_j}\qquad\text{for }\ t\geqslant \ell_{2j+1}\ \text{ and for any }\ j\in\mathbb{N},\tag{84}$$

where $\{K_j\}_{j\in\mathbb{N}}$, $\{\alpha_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative real numbers that we will be fixed during the iterative procedure. Due to (79) we see that (84) is satisfied for $j=0$, supposed that the initial values of the sequences are given by $\alpha_0\doteq 1,\beta_0\doteq 0$ and $K_0\doteq \widetilde{K}\varepsilon^{pq}$. Also in this case it remains to prove the inductive step in order to show the validity of (84) for any $j\in\mathbb{N}$. For this purpose we plug in (84) in (76), so that, after a restriction of the domain of integration, for $s\geqslant \ell_{2j+2}$ we have

$$\begin{aligned}
\mathscr{U}(s)&\geqslant CK_j^q\int_{\ell_{2j+1}}^s\langle\tau\rangle^{-(r_1+1)+\frac{n-1}{2}q-(n-1)(q-1)}\big(\log\langle\tau\rangle\big)^{-(q-1)-\beta_j q}\left(\log\left(\tfrac{\tau}{\ell_{2j+1}}\right)\right)^{\alpha_j q}d\tau\\
&\geqslant CK_j^q\langle s\rangle^{-(r_1+1)-\frac{n-1}{2}q}\big(\log\langle s\rangle\big)^{-(q-1)-\beta_j q}\int_{\ell_{2j+1}}^s\langle\tau\rangle^{n-1}\left(\log\left(\tfrac{\tau}{\ell_{2j+1}}\right)\right)^{\alpha_j q}d\tau\\
&\geqslant CK_j^q\langle s\rangle^{-(r_1+1)-\frac{n-1}{2}q}\big(\log\langle s\rangle\big)^{-(q-1)-\beta_j q}\int_{\frac{\ell_{2j+1}s}{\ell_{2j+2}}}^s\tau^{n-1}\left(\log\left(\tfrac{\tau}{\ell_{2j+1}}\right)\right)^{\alpha_j q}d\tau\\
&\geqslant CK_j^q\left(\tfrac{\ell_{2j+1}}{\ell_{2j+2}}\right)^{n-1}\left(1-\tfrac{\ell_{2j+1}}{\ell_{2j+2}}\right)\langle s\rangle^{-(r_1+1)-\frac{n-1}{2}q}s^n\big(\log\langle s\rangle\big)^{-(q-1)-\beta_j q}\left(\log\left(\tfrac{s}{\ell_{2j+2}}\right)\right)^{\alpha_j q}\\
&\geqslant 2^{-2j-3n-2}CK_j^q\langle s\rangle^{-(r_1+1)-\frac{n-1}{2}q+n}\big(\log\langle s\rangle\big)^{-(q-1)-\beta_j q}\left(\log\left(\tfrac{s}{\ell_{2j+2}}\right)\right)^{\alpha_j q}.
\end{aligned}$$

Combining this lower bound for $\mathscr{U}(s)$ and (77) and using the critical relation $\Theta_2(n,p,q)=0$, for $t\geqslant\ell_{2j+3}$ we arrive at

$$\begin{aligned}
\mathscr{V}(t)&\geqslant 2^{-2pj-(3n+2)p}KC^pK_j^{pq}\langle t\rangle^{-1}\int_{\ell_{2j+2}}^t(t-s)\langle s\rangle^{-r_2+n-\frac{n-1}{2}pq}\big(\log\langle s\rangle\big)^{-p(q-1)-\beta_j q}\left(\log\left(\tfrac{s}{\ell_{2j+2}}\right)\right)^{\alpha_j pq}ds\\
&\geqslant 2^{-2pj-(3n+2)p}KC^pK_j^{pq}\big(\log\langle t\rangle\big)^{-p(q-1)-\beta_j q}\langle t\rangle^{-1}\int_{\ell_{2j+2}}^t(t-s)\langle s\rangle^{1+\frac{1}{q}-\frac{n-1}{2}(pq-1)}\left(\log\left(\tfrac{s}{\ell_{2j+2}}\right)\right)^{\alpha_j pq}ds\\
&=2^{-2pj-(3n+2)p}KC^pK_j^{pq}\big(\log\langle t\rangle\big)^{-p(q-1)-\beta_j q}\langle t\rangle^{-1}\int_{\ell_{2j+2}}^t(t-s)\langle s\rangle^{-1}\left(\log\left(\tfrac{s}{\ell_{2j+2}}\right)\right)^{\alpha_j pq}ds\\
&\geqslant 2^{-2pj-(3n+2)p-2}KC^pK_j^{pq}\big(\log\langle t\rangle\big)^{-p(q-1)-\beta_j q}\langle t\rangle^{-1}\int_{\ell_{2j+2}}^t\tfrac{t-s}{s}\left(\log\left(\tfrac{s}{\ell_{2j+2}}\right)\right)^{\alpha_j pq}ds\\
&=2^{-2pj-(3n+2)p-2}KC^pK_j^{pq}(\alpha_j pq+1)^{-1}\big(\log\langle t\rangle\big)^{-p(q-1)-\beta_j q}\langle t\rangle^{-1}\int_{\ell_{2j+2}}^t\left(\log\left(\tfrac{s}{\ell_{2j+2}}\right)\right)^{\alpha_j pq+1}ds\\
&\geqslant 2^{-2pj-(3n+2)p-2}KC^pK_j^{pq}(\alpha_j pq+1)^{-1}\big(\log\langle t\rangle\big)^{-p(q-1)-\beta_j q}\langle t\rangle^{-1}\int_{\frac{\ell_{2j+2}t}{\ell_{2j+3}}}^t\left(\log\left(\tfrac{s}{\ell_{2j+2}}\right)\right)^{\alpha_j pq+1}ds\\
&\geqslant 2^{-2pj-(3n+2)p-4}KC^pK_j^{pq}(\alpha_j pq+1)^{-1}\left(1-\tfrac{\ell_{2j+2}}{\ell_{2j+3}}\right)\big(\log\langle t\rangle\big)^{-p(q-1)-\beta_j q}\left(\log\left(\tfrac{t}{\ell_{2j+3}}\right)\right)^{\alpha_j pq+1}\\
&\geqslant 2^{-2(p+1)j-(3n+2)p-8}KC^pK_j^{pq}(\alpha_j pq+1)^{-1}\big(\log\langle t\rangle\big)^{-p(q-1)-\beta_j q}\left(\log\left(\tfrac{t}{\ell_{2j+3}}\right)\right)^{\alpha_j pq+1},
\end{aligned}$$

that is (84) for $j+1$, provided that

$$K_{j+1}\doteq 2^{-2(p+1)j-(3n+2)p-8}KC^pK_j^{pq}(\alpha_j pq+1)^{-1},\quad \alpha_{j+1}\doteq\alpha_j pq+1,\quad \beta_{j+1}\doteq\beta_j q+p(q-1).$$

Analogously to what we did in the first critical case, we derive now a lower bound for K_j. Let us find first the expression of α_j and β_j. Applying iteratively the definitions of α_j and β_j, we end up with the representation formulas

$$\begin{aligned} \alpha_j &= \alpha_{j-1}pq+1 = \cdots = \alpha_0 (pq)^j + \tfrac{(pq)^j-1}{pq-1} = \tfrac{(pq)^{j+1}-1}{pq-1}, \\ \beta_j &= \beta_{j-1}pq + p(q-1) = \cdots = \beta_0 (pq)^j + \tfrac{p(q-1)}{pq-1}\big((pq)^j-1\big) = \tfrac{p(q-1)}{pq-1}\big((pq)^j-1\big). \end{aligned} \tag{85}$$

In particular, it holds the inequality $\alpha_{j-1}pq+1 \leqslant \frac{pq}{pq-1}(pq)^j$, that implies in turn

$$K_j \geqslant M_1 N_1^{-j} K_{j-1}^{pq}, \tag{86}$$

where $M_1 \doteq 2^{-3np-6} K C^{p\,\frac{(pq-1)}{pq}}$ and $N_1 \doteq 2^{2(p+1)}pq$. Analogously to the derivation of (83) via (82), from (86) we obtain

$$K_j \geqslant M_1^{-(pq-1)} \exp\Big((pq)^j \log\Big(K_0 N_1^{-S} M_1^{pq-1}\Big)\Big). \tag{87}$$

6.4.3. Case $\Theta_1(n,p,q)=\Theta_2(n,p,q)=0$

In this third critical case the goal is to prove that

$$\mathscr{U}(t) \geqslant D_j \big(\log\langle t\rangle\big)^{-h_j} \left(\log\left(\frac{t}{\ell_j}\right)\right)^{g_j} \qquad \text{for } \ t\geqslant \ell_j \ \text{ and for any } \ j\in\mathbb{N}, \tag{88}$$

where $\{D_j\}_{j\in\mathbb{N}}$, $\{g_j\}_{j\in\mathbb{N}}$ and $\{h_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative real numbers that we shall determine throughout the iteration argument. Due to the different iteration scheme for $\Theta_1(n,p,q)=\Theta_2(n,p,q)=0$ in (76) and (77), the proof of the inductive step will have somehow a more symmetric behavior than the ones in the previous cases. We point out that (78) implies the validity of (88) in the base case $j=0$ if we consider

$$D_0 \doteq \widetilde{C}\varepsilon^q, \quad g_0 \doteq 1, \quad h_0 \doteq 0.$$

Let us proceed with the inductive step. If we plug (88) in (77), then for $s\geqslant \ell_{j+1}$ we get

$$\begin{aligned} \mathscr{V}(s) &\geqslant K D_j^p \langle s\rangle^{-1} \int_{\ell_j}^s (s-\tau)\langle \tau\rangle^{-1} \big(\log\langle\tau\rangle\big)^{-(p-1)-h_j p} \left(\log\left(\tfrac{\tau}{\ell_j}\right)\right)^{g_j p} d\tau \\ &\geqslant 2^{-2} K D_j^p \big(\log\langle s\rangle\big)^{-(p-1)-h_j p} \langle s\rangle^{-1} \int_{\ell_j}^s \tfrac{s-\tau}{\tau} \left(\log\left(\tfrac{\tau}{\ell_j}\right)\right)^{g_j p} d\tau \\ &= 2^{-2} K D_j^p (g_j p+1)^{-1} \big(\log\langle s\rangle\big)^{-(p-1)-h_j p} \langle s\rangle^{-1} \int_{\ell_j}^s \left(\log\left(\tfrac{\tau}{\ell_j}\right)\right)^{g_j p+1} d\tau \\ &\geqslant 2^{-2} K D_j^p (g_j p+1)^{-1} \big(\log\langle s\rangle\big)^{-(p-1)-h_j p} \langle s\rangle^{-1} \int_{\frac{\ell_j s}{\ell_{j+1}}}^s \left(\log\left(\tfrac{\tau}{\ell_j}\right)\right)^{g_j p+1} d\tau \\ &\geqslant 2^{-4} K D_j^p (g_j p+1)^{-1} \left(1-\tfrac{\ell_j}{\ell_{j+1}}\right) \big(\log\langle s\rangle\big)^{-(p-1)-h_j p} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{g_j p+1} \\ &\geqslant 2^{-(j+6)} K D_j^p (g_j p+1)^{-1} \big(\log\langle s\rangle\big)^{-(p-1)-h_j p} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{g_j p+1}. \end{aligned}$$

A combination of the previous lower bound for $\mathscr{V}(s)$ and (76) yields for $t\geqslant \ell_{j+1}$

$$\begin{aligned} \mathscr{U}(t) &\geqslant 2^{-(j+6)q} C K^q D_j^{pq} (g_j p+1)^{-q} \int_{\ell_{j+1}}^t \langle s\rangle^{-1} \big(\log\langle s\rangle\big)^{-(pq-1)-h_j pq} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{g_j pq+q} ds \\ &\geqslant 2^{-(j+6)q-2} C K^q D_j^{pq} (g_j p+1)^{-q} \big(\log\langle t\rangle\big)^{-(pq-1)-h_j pq} \int_{\ell_{j+1}}^t s^{-1} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{g_j pq+q} ds \\ &= 2^{-(j+6)q-2} C K^q D_j^{pq} (g_j p+1)^{-q} (g_j pq+q+1)^{-1} \big(\log\langle t\rangle\big)^{-(pq-1)-h_j pq} \left(\log\left(\tfrac{t}{\ell_{j+1}}\right)\right)^{g_j pq+q+1}. \end{aligned}$$

The last inequality is (88) in the case $j+1$ with

$$D_{j+1} \doteq 2^{-(j+6)q-2} C K^q D_j^{pq} (g_j p+1)^{-q} (g_j pq+q+1)^{-1}, \quad g_{j+1} \doteq g_j pq + q + 1, \quad h_{j+1} \doteq h_j pq + pq - 1.$$

Finally, we find a lower bound for the coefficient D_j. First, we have

$$\begin{aligned} g_j &= g_{j-1}pq + q + 1 = \cdots = g_0 (pq)^j + (q+1) \sum_{k=0}^{j-1} (pq)^k = \left(1 + \tfrac{q+1}{pq-1}\right)(pq)^j - \tfrac{q+1}{pq-1}, \\ h_j &= h_{j-1}pq + pq - 1 = \cdots = h_0 (pq)^j + (pq-1) \sum_{k=0}^{j-1} (pq)^k = (pq)^j - 1. \end{aligned} \tag{89}$$

Hence,

$$g_{j-1}p + 1 \leqslant g_{j-1}pq + q + 1 \leqslant \tfrac{q(p+1)}{pq-1} (pq)^j$$

implies

$$D_j \geqslant M_2 N_2^{-j} D_{j-1}^{pq}, \tag{90}$$

where $M_2 \doteq 2^{-5q-2} C K^q \big(\frac{pq-1}{q(p+1)}\big)^{q+1}$ and $N_2 \doteq 2^q (pq)^{q+1}$. In an analogous way as in the derivation of (83) through (82), by (90) we have

$$D_j \geqslant M_2^{-(pq-1)} \exp\Big((pq)^j \log\Big(D_0 N_2^{-S} M_2^{pq-1}\Big)\Big). \tag{91}$$

6.5. Upper bound for the lifespan of local solutions

In this section we finally prove that a local solution (u,v) of (12) blows up in finite time under the assumption of Theorem 1.4. As in the previous sections we will consider separately the three critical cases.

6.5.1. Case $\Theta_1(n,p,q) = 0$

If we combine (80), (81) and (83), then, for $t \geqslant 2 \geqslant \ell_j$ we have

$$\begin{aligned} \mathcal{U}(t) &\geqslant M^{-(pq-1)} \exp\Big((pq)^j \log\big(C_0 N^{-S} M^{pq-1}\big)\Big) \big(\log\langle t\rangle\big)^{-b_j} \Big(\log\big(\tfrac{t}{\ell_j}\big)\Big)^{a_j} \\ &\geqslant M^{-(pq-1)} \exp\Big((pq)^j \log\big(C_0 N^{-S} M^{pq-1}\big)\Big) \big(\log\langle t\rangle\big)^{-b_j} \Big(\log\big(\tfrac{t}{2}\big)\Big)^{a_j} \\ &\geqslant M^{-(pq-1)} \left(\frac{(\log\langle t\rangle)^{q(p-1)}}{\log\big(\frac{t}{2}\big)}\right)^{\frac{1}{pq-1}} \exp\Big((pq)^j \log\Big(\big(C_0 N^{-S} M^{pq-1}\big) - \tfrac{q(p-1)}{pq-1} \log\langle t\rangle + \tfrac{pq}{pq-1} \log\big(\tfrac{t}{2}\big)\Big)\Big). \end{aligned}$$

Because for $t \geqslant 4$ the inequalities $\log\langle t\rangle \leqslant \log(2t) \leqslant 2\log t$ and $\log\big(\frac{t}{2}\big) \geqslant \frac{1}{2}\log t$ hold, then for $t \geqslant 4$ the last estimate from below for $\mathcal{U}(t)$ implies

$$\begin{aligned} \mathcal{U}(t) &\geqslant M^{-(pq-1)} \left(\frac{(\log\langle t\rangle)^{q(p-1)}}{\log\big(\frac{t}{2}\big)}\right)^{\frac{1}{pq-1}} \exp\Big((pq)^j \log\Big(2^{-\frac{q(2p-1)}{pq-1}} C_0 N^{-S} M^{pq-1} (\log t)^{\frac{q}{pq-1}}\Big)\Big) \\ &\geqslant M^{-(pq-1)} \left(\frac{(\log\langle t\rangle)^{q(p-1)}}{\log\big(\frac{t}{2}\big)}\right)^{\frac{1}{pq-1}} \exp\Big((pq)^j \log\Big(E \varepsilon^{pq} (\log t)^{\frac{q}{pq-1}}\Big)\Big), \end{aligned} \tag{92}$$

where $E \doteq 2^{-\frac{q(2p-1)}{pq-1}} \widetilde{C} N^{-S} M^{pq-1}$.

Let us point out that $H(t,\varepsilon) \doteq E\varepsilon^{pq} (\log t)^{\frac{q}{pq-1}} > 1$ if and only if $t > \exp\big(E^{-\frac{pq-1}{q}} \varepsilon^{-p(pq-1)}\big)$. Consequently, we can fix a sufficiently small ε_0 such that

$$\exp\big(E^{-\frac{pq-1}{q}} \varepsilon_0^{-p(pq-1)}\big) \geqslant 4.$$

Then, for any $\varepsilon \in (0, \varepsilon_0]$ and $t > \exp\big(E^{-\frac{pq-1}{q}} \varepsilon^{-p(pq-1)}\big) \geqslant 4$ it holds $H(t, \varepsilon) > 1$; so, letting $j \to \infty$ in (92), we find that the lower bound for $\mathscr{U}(t)$ blows up. Thus, we proved that $\mathscr{U}(t)$ can be finite only for

$$t \leqslant \exp\big(E^{-\frac{pq-1}{q}} \varepsilon^{-p(pq-1)}\big), \tag{93}$$

which is the upper bound estimate for the lifespan in (13) when $\Theta_1(n, p, q) = 0$.

6.5.2. Case $\Theta_2(n, p, q) = 0$

Combining (84), (85) and (87), then, for $t \geqslant 2 \geqslant \ell_{2j+1}$ we have

$$\begin{aligned}
\mathscr{V}(t) &\geqslant M_1^{-(pq-1)} \exp\Big((pq)^j \log\big(K_0 N_1^{-S} M_1^{pq-1}\big)\Big)\big(\log\langle t\rangle\big)^{-\beta_j}\Big(\log\big(\tfrac{t}{\ell_{2j+1}}\big)\Big)^{\alpha_j} \\
&\geqslant M_1^{-(pq-1)} \exp\Big((pq)^j \log\big(K_0 N_1^{-S} M_1^{pq-1}\big)\Big)\big(\log\langle t\rangle\big)^{-\beta_j}\Big(\log\big(\tfrac{t}{2}\big)\Big)^{\alpha_j} \\
&\geqslant M_1^{-(pq-1)} \left(\frac{(\log\langle t\rangle)^{p(q-1)}}{\log\big(\frac{t}{2}\big)}\right)^{\frac{1}{pq-1}} \exp\Big((pq)^j \log\Big(\big(K_0 N_1^{-S} M_1^{pq-1}\big) - \tfrac{p(q-1)}{pq-1}\log\langle t\rangle + \tfrac{pq}{pq-1}\log\big(\tfrac{t}{2}\big)\Big)\Big).
\end{aligned}$$

Analogously as in the last section, for $t \geqslant 4$ this estimate from below for $\mathscr{V}(t)$ provides

$$\begin{aligned}
\mathscr{V}(t) &\geqslant M_1^{-(pq-1)} \left(\frac{(\log\langle t\rangle)^{p(q-1)}}{\log\big(\frac{t}{2}\big)}\right)^{\frac{1}{pq-1}} \exp\Big((pq)^j \log\Big(2^{-\frac{p(2q-1)}{pq-1}} K_0 N_1^{-S} M_1^{pq-1} (\log t)^{\frac{p}{pq-1}}\Big)\Big) \\
&\geqslant M_1^{-(pq-1)} \left(\frac{(\log\langle t\rangle)^{p(q-1)}}{\log\big(\frac{t}{2}\big)}\right)^{\frac{1}{pq-1}} \exp\Big((pq)^j \log\Big(E_1 \varepsilon^{pq} (\log t)^{\frac{p}{pq-1}}\Big)\Big),
\end{aligned} \tag{94}$$

where $E_1 \doteq 2^{-\frac{p(2q-1)}{pq-1}} \widetilde{K} N_1^{-S} M_1^{pq-1}$. If we denote $H_1(t, \varepsilon) \doteq E_1 \varepsilon^{pq} (\log t)^{\frac{p}{pq-1}}$, then, $H_1(t, \varepsilon) > 1$ if and only if $t > \exp\big(E_1^{-\frac{pq-1}{p}} \varepsilon^{-q(pq-1)}\big)$. Therefore, we can choose a sufficiently small ε_0 such that $\exp\big(E_1^{-\frac{pq-1}{p}} \varepsilon_0^{-q(pq-1)}\big) \geqslant 4$. Also, for any $\varepsilon \in (0, \varepsilon_0]$ and $t > \exp\big(E_1^{-\frac{pq-1}{p}} \varepsilon^{-q(pq-1)}\big)$ we have $H_1(t, \varepsilon) > 1$; thus, taking the limit as $j \to \infty$ in (94), we find that the lower bound for $\mathscr{V}(t)$ blows up. Hence, we showed that $\mathscr{V}(t)$ may be finite just for

$$t \leqslant \exp\big(E_1^{-\frac{pq-1}{p}} \varepsilon^{-q(pq-1)}\big), \tag{95}$$

which is exactly the upper bound estimate for the lifespan in (13) for $\Theta_2(n, p, q) = 0$.

6.5.3. Case $\Theta_1(n, p, q) = \Theta_2(n, p, q) = 0$

For $t \geqslant 2 \geqslant \ell_{2j+1}$ the combination of (88), (89) and (91) leads to

$$\begin{aligned}
\mathscr{U}(t) &\geqslant M_2^{-(pq-1)} \exp\Big((pq)^j \log\big(D_0 N_2^{-S} M_2^{pq-1}\big)\Big)\big(\log\langle t\rangle\big)^{-h_j}\Big(\log\big(\tfrac{t}{\ell_j}\big)\Big)^{g_j} \\
&\geqslant M_2^{-(pq-1)} \exp\Big((pq)^j \log\big(D_0 N_2^{-S} M_2^{pq-1}\big)\Big)\big(\log\langle t\rangle\big)^{-h_j}\Big(\log\big(\tfrac{t}{2}\big)\Big)^{g_j} \\
&\geqslant M_2^{-(pq-1)} \log\langle t\rangle \big(\log\big(\tfrac{t}{2}\big)\big)^{-\frac{q+1}{pq-1}} \exp\Big((pq)^j \log\Big(\big(D_0 N_2^{-S} M_2^{pq-1}\big) - \log\langle t\rangle + \Big(1 + \tfrac{q+1}{pq-1}\Big)\log\big(\tfrac{t}{2}\big)\Big)\Big).
\end{aligned}$$

Similarly to the last sections, for $t \geqslant 4$ the above estimate from below for $\mathscr{U}(t)$ yields

$$\begin{aligned}
\mathscr{V}(t) &\geqslant M_2^{-(pq-1)} \log\langle t\rangle \big(\log\big(\tfrac{t}{2}\big)\big)^{-\frac{q+1}{pq-1}} \exp\Big((pq)^j \log\Big(2^{-2-\frac{q+1}{pq-1}} D_0 N_2^{-S} M_2^{pq-1} (\log t)^{\frac{q+1}{pq-1}}\Big)\Big) \\
&\geqslant M_2^{-(pq-1)} \log\langle t\rangle \big(\log\big(\tfrac{t}{2}\big)\big)^{-\frac{q+1}{pq-1}} \exp\Big((pq)^j \log\Big(E_2 \varepsilon^{q} (\log t)^{\frac{q+1}{pq-1}}\Big)\Big),
\end{aligned} \tag{96}$$

where $E_2 \doteq 2^{-2-\frac{q+1}{pq-1}} \widetilde{C} N_2^{-S} M_2^{pq-1}$. Let us denote $H_2(t, \varepsilon) \doteq E_2 \varepsilon^{q} (\log t)^{\frac{q+1}{pq-1}}$. Then, $H_2(t, \varepsilon) > 1$ if and only if $t > \exp\big(E_2^{-\frac{pq-1}{q+1}} \varepsilon^{-\frac{q}{q+1}(pq-1)}\big)$. Therefore, as before we can choose a sufficiently small ε_0 such that $\exp\left(E_2^{-\frac{pq-1}{q+1}}\varepsilon_0^{-\frac{q}{q+1}(pq-1)}\right) \geqslant 4$. Also, for any $\varepsilon \in (0,\varepsilon_0]$ and $t > \exp\left(E_2^{-\frac{pq-1}{q+1}}\varepsilon^{-\frac{q}{q+1}(pq-1)}\right)$ we have $H_2(t,\varepsilon) > 1$; thus, taking the limit as $j \to \infty$ in (96), we see that the lower bound for $\mathcal{U}(t)$ diverges. So, we proved that if $\mathcal{U}(t)$ is finite, then,

$$t \leqslant \exp\left(E_2^{-\frac{pq-1}{q+1}}\varepsilon^{-\frac{q}{q+1}(pq-1)}\right), \tag{97}$$

that is, we proved (13) in the critical case $\Theta_1(n,p,q) = \Theta_2(n,p,q) = 0$.

6.6. Final remarks on the critical case

6.6.1. Comparison with other results

As we have already mentioned in the introduction, Ikeda-Sobajima-Wakasa very recently proved a blow-up result for the semilinear weakly coupled system (12) both in the subcritical case and in the critical case, by using a revised test function method. While in the subcritical case we obtained exactly the same result (but including damping terms in the scattering case), in the critical case we got quite different estimates for the lifespan in all three subcases. Let us compare our results with theirs.

In the first critical case $\Theta_1(n,p,q) = 0$ we proved the estimate (93), while in [16] the upper bound estimate

$$T(\varepsilon) \leqslant \exp\left(C\varepsilon^{-q(pq-1)}\right) \tag{98}$$

is proved. Let us point out that in the critical case $\Theta_1(n,p,q) = 0 > \Theta_1(n,p,q)$ it is not possible to determine, in general, which exponent among p and q is the biggest one. So far, the best estimate for the lifespan that we can get is the one obtained combining (93) and (98), that is,

$$T(\varepsilon) \leqslant \exp\left(C\varepsilon^{-\min\{p(pq,1),q(pq-1)\}}\right) \quad \text{if } \ \Theta_1(n,p,q) = 0.$$

On the contrary, in the case $\Theta_2(n,p,q) = 0$ we obtained (95), which is an improvement of the estimate $T(\varepsilon) \leqslant \exp\left(C\varepsilon^{-p(pq-1)}\right)$ proved in [16] in the same critical case. Indeed, in this case we have

$$\tfrac{q+1+p^{-1}}{pq-1} - \tfrac{n-1}{2} < 0 = \tfrac{2+q^{-1}}{pq-1} - \tfrac{n-1}{2}$$

which provides $q - q^{-1} < 1 - p^{-1} < p - p^{-1}$, that implies in turn $q < p$.

We consider now the case $\Theta_1(n,p,q) = \Theta_2(n,p,q) = 0$. We point out explicitly that in this critical case we could employ an iteration argument for the functional $\mathcal{V}$ as well in the last section. Nevertheless, we would find as upper bound for the lifespan

$$T(\varepsilon) \leqslant \exp\left(C\varepsilon^{-\frac{p}{p+1}(pq-1)}\right)$$

which is weaker than the one that we derived by working with $\mathcal{U}$, namely, (97). This is due to the comparison of the two critical conditions $\Theta_1(n,p,q) = 0$ and $\Theta_2(n,p,q) = 0$ that lead to $q - q^{-1} = 1 - p^{-1} < p - p^{-1}$, which implies as above $q < p$. Morever, we emphasize that we have improved the estimate for this case in comparison to the on in [16] for the corresponding case, namely, $T(\varepsilon) \leqslant \exp\left(C\varepsilon^{-(pq-1)}\right)$.

6.6.2. The intersection point of the critical curves

Finally, we remark that in the critical case $\Theta_1(n,p,q) = \Theta_2(n,p,q) = 0$, we can determine the expression of p and q, that is, we determine the coordinates of the intersection point of the critical curves in the p - q plane. By straightforward calculations, we get that $\Theta_1(n,p,q) = \Theta_2(n,p,q)$ implies

$$p = (1 + q^{-1} - q)^{-1}. \tag{99}$$

We underline that we should require $1 < q < \frac{1+\sqrt{5}}{2}$, in order to get an admissible p. If we plug in (99) in $\Theta_2(n,p,q) = 0$, we find that q satisfies the cubic equation

$$\begin{aligned} 0 &= (n+1)q^3 - \tfrac{n+1}{2}q^2 - \tfrac{n+5}{2}q - 1 \\ &= (2q+1)\Big(\tfrac{n+1}{2}q^2 - \tfrac{n+1}{2}q - 1\Big). \end{aligned} \tag{100}$$

Therefore, the only admissible solution of (100) is

$$q_{\mathrm{mix}}(n) \doteq \frac{1}{2}\left(1 + \sqrt{\frac{n+9}{n+1}}\right).$$

It is easy to check that $q_{\mathrm{mix}}(n) < \frac{1+\sqrt{5}}{2}$ for any $n \geqslant 2$. Plugging this expression for $q_{\mathrm{mix}}(n)$ in (99), we get

$$\begin{aligned} p_{\mathrm{mix}}(n) &\doteq \frac{q_{\mathrm{mix}}(n)}{1 + q_{\mathrm{mix}}(n) - (q_{\mathrm{mix}}(n))^2} \\ &= \frac{n+1+\sqrt{(n+9)(n+1)}}{2(n-1)}. \end{aligned}$$

It is interesting to compare these exponents, $p_{\mathrm{mix}}(n)$ and $q_{\mathrm{mix}}(n)$, with the critical exponent for the semilinear wave equation with power nonlinearity, i.e., the Strauss exponent

$$p_{\mathrm{Str}}(n) = \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$$

and with the exponent for the semilinear wave equation of derivative type, i.e., the Glassey exponent

$$p_{\mathrm{Gla}}(n) = \frac{n+1}{n-1}.$$

Elementary computations show that

$$q_{\mathrm{mix}}(n) < p_{\mathrm{Gla}}(n) < p_{\mathrm{Str}}(n) < p_{\mathrm{mix}}(n)$$

for any $n \geqslant 2$. Therefore, we may conclude that for the cusp point of the critical curve for (12) the power of the nonlinear term $|\partial_t u|^p$ is bigger than the critical power for the semilinear wave equation of derivative type, while the power of the nonlinear term $|v|^q$ is smaller than the critical power for the semilinear wave equation with power nonlinearity. In this sense, we have a balance between p and q for the cusp point of the critical curve for the weakly coupled system of semilinear wave equations with mixed nonlinear terms, in comparison to the cases with power nonlinearities and of derivative type.

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References

[1] R. Agemi, Blow-up of solutions to nonlinear wave equations in two space dimensions, Manuscripta Math. **73** (1991), 153–162.

[2] R. Agemi, Y. Kurokawa, H. Takamura, Critical curve for p-q systems of nonlinear wave equations in three space dimensions, J. Differential Equations **167**(1) (2000), 87–133.

[3] D. Del Santo, Global existence and blow-up for a hyperbolic system in three space dimensions, Rend. Istit. Mat. Univ. Trieste **29**(1/2) (1997), 115–140.

[4] D. Del Santo, E. Mitidieri, Blow-up of solutions of a hyperbolic system: The critical case, Differential Equations, **34**(9) (1998), 1157–1163.

[5] D. Del Santo, V. Georgiev, E. Mitidieri, Global existence of the solutions and formation of singularities for a class of hyperbolic systems, In: Colombini F., Lerner N. (eds) Geometrical Optics and Related Topics. Progress in Nonlinear Differential Equations and Their Applications, vol 32. Birkhäuser, Boston, MA, (1997), https://doi.org/10.1007/978-1-4612-2014-5_7.

[6] K. Deng, Blow-up of solutions of some nonlinear hyperbolic systems, Rocky Mountain J. Math. **29** (1999), 807–820.

[7] V. Georgiev, H. Lindblad, C.D. Sogge, Weighted Strichartz estimates and global existence for semi-linear wave equations, Amer. J. Math. **119**(6) (1997), 1291–1319.

[8] V. Georgiev, H. Takamura, Y. Zhou, The lifespan of solutions to nonlinear systems of a high-dimensional wave equation, Nonlinear Anal. **64**(10) (2006), 2215–2250.

[9] R.T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, Math Z. **177**(3) (1981), 323–340.

[10] R.T. Glassey, Existence in the large for $\Box u = F(u)$ in two space dimensions, Math Z. **178**(2) (1981), 233–261.

[11] K. Hidano, K. Tsutaya, Global existence and asymptotic behavior of solutions for nonlinear wave equations, Indiana Univ. Math. J. **44** (1995), 1273–1305.

[12] K. Hidano, C. Wang, K. Yokoyama, The Glassey conjecture with radially symmetric data, J. Math. Pures Appl. **98**(9) (2012), 518–541.

[13] K. Hidano, K. Yokoyama, Life span of small solutions to a system of wave equations, Nonlinear Anal. **139** (2016), 106–130.

[14] M. Ikeda, M. Sobajima, Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data, Math. Ann. (2018), https://doi.org/10.1007/s00208-018-1664-1.

[15] M. Ikeda, M. Sobajima, Upper bound for lifespan of solutions to certain semilinear parabolic, dispersive and hyperbolic equations via a unified test function method, preprint, arXiv:1710.06780v2, 2018.

[16] M. Ikeda, M. Sobajima, K. Wakasa, Blow-up phenomena of semilinear wave equations and their weakly coupled systems, preprint, arXiv:1807.03937v1, 2018.

[17] H. Jiao, Z. Zhou, An elementary proof of the blow-up for semilinear wave equation in high space dimensions, J. Differential Equations **189**(2) (2003), 355–365.

[18] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. **28**(1-3) (1979), 235–268.

[19] F. John, Blow-up of solutions for quasi-linear wave equations in three space dimensions, Comm. Pure Appl. Math. **34** (1981), 29–51.

[20] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, Comm. Pure Appl. Math. **33**(4) (1980), 501–505.

[21] H. Kubo, K. Kubota, H. Sunagawa, Large time behavior of solutions to semilinear systems of wave equations, Math. Ann. **335** (2006), 435–478.

[22] Y. Kurokawa, The lifespan of radially symmetric solutions to nonlinear systems of odd dimensional wave equations, Tsukuba J. Math. **60**(7) (2005), 1239–1275.

[23] Y. Kurokawa, H. Takamura, A weighted pointwise estimate for two dimensional wave equations and its applications to nonlinear systems, Tsukuba J. Math. **27**(2) (2003), 417–448.

[24] Y. Kurokawa, H. Takamura, K. Wakasa, The blow-up and lifespan of solutions to systems of semilinear wave equation with critical exponents in high dimensions, Differential Integral Equations **25**(3/4) (2012), 363–382.

[25] N.A. Lai, H. Takamura, Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case, Nonlinear Anal. **168** (2018), 222–237.

[26] N.A. Lai, H. Takamura, Nonexistence of global solutions of nonlinear wave equations with weak time-dependent damping related to Glassey's conjecture, Differential and Integral Equations **32**(1-2) (2019), 37–48.

[27] N.A. Lai, H. Takamura, Nonexistence of global solutions of wave equations with weak time-dependent damping and combined nonlinearity, Nonlinear Anal. Real World Appl. **45** (2019), 83–96.

[28] N.A. Lai, H. Takamura, K. Wakasa, Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent, J. Differential Equations **263**(9) (2017), 5377–5394.

[29] H. Lindblad, Blow-up for solutions of $\Box u = |u|^p$ with small initial data, Comm. Partial Differential Equations **15**(6) (1990), 757–821.

[30] H. Lindblad, C. Sogge, On existence and scattering with minimal regularity for semilinear wave equations, J. Funct. Anal. **130**(2) (1995), 357–426.

[31] H. Lindblad, C. Sogge, Long-time existence for small amplitude semilinear wave equations, Amer. J. Math. **118**(5) (1996), 1047–1135.

[32] K. Masuda, Blow-up solutions for quasi-linear wave equations in two space dimensions, Lecture Notes in Num. Appl. Anal. **6** (1983), 87–91.

[33] A. Palmieri, A note on a conjecture for the critical curve of a weakly coupled system of semilinear wave equations with scale-invariant lower order terms, preprint, arXiv:1812.06588v1, 2018.

[34] A. Palmieri, H. Takamura, Blow-up for a weakly coupled system of semilinear damped wave equations in the scattering case with power nonlinearities, preprint, arXiv:1812.10086v1, 2018.

[35] A. Palmieri, H. Takamura, Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations of derivative type in the scattering case, preprint, arXiv:1812.10653v1, 2018.

[36] A. Palmieri, Z. Tu, Lifespan of semilinear wave equation with scale invariant dissipation and mass and sub-Strauss power nonlinearity, J. Math. Anal. Appl. (2018), https://doi: 10.1016/j.jmaa.2018.10.015.

[37] M.A. Rammaha, Finite-time blow-up for nonlinear wave equations in high dimensions, Comm. Partial Differential Equations **12** (1987), 677–700.

[38] J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p, Proc. Roy. Soc. Edinburgh Sect. A. **101**(1-2) (1985), 31–44.

[39] J. Schaeffer, Finite-time blow-up for $u_{tt} - \Delta u = H(u_r, u_t)$ in two space dimensions, Comm. Partial Differential Equations **11** (1986), 513–543.

[40] T.C. Sideris, Global behavior of solutions to nonlinear wave equations in three space dimensions, Comm. Partial Differential Equations **8** (1983), 1219–1323.

[41] T.C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, J. Differential Equations **52**(3) (1984), 378–406.

[42] W.A. Strauss, Nonlinear scattering theory at low energy, J. Funct. Anal. **41**(1) (1981), 110–133.

[43] H. Takamura, Improved Kato's lemma on ordinary differential inequality and its application to semilinear wave equations, Nonlinear Anal. **125** (2015), 227–240.

[44] H. Takamura, K. Wakasa, The sharp upper bound of the lifespan of solutions to critical semilinear wave equations in high dimensions, J. Differential Equations **251**(4-5) (2011), 1157–1171.

[45] D. Tataru, Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation, Trans. Amer. Math. Soc. **353**(2) (2001), 795–807.

[46] N. Tzvetkov, Existence of global solutions to nonlinear massless Dirac system and wave equations with small data, Tsukuba Math. J. **22** (1998), 198–211.

[47] K. Wakasa B. Yordanov, Blow-up of solutions to critical semilinear wave equations with variable coefficients, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.10.028

[48] K. Wakasa B. Yordanov, On the blow-up for critical semilinear wave equations with damping in the scattering case, Nonlinear Anal. **180** (2019) 67–74.

[49] J. Wirth, Solution representations for a wave equation with weak dissipation, Math. Meth. Appl. Sci. **27**(1) (2004), 101-124.

[50] J. Wirth, Wave equations with time-dependent dissipation I. Non-Effective Dissipation, J. Differential Equations **222** (2006), 487-514.

[51] J. Wirth, Wave equations with time-dependent dissipation II. Effective dissipation, J. Differential Equations **232** (2007), 74-103.

[52] W. Xu, Blowup for systems of semilinear wave equations with small initial data, J. Partial Differential Equations **17**(3) (2004), 198–206.

[53] B.T. Yordanov, Q.S. Zhang, Finite time blow up for critical wave equations in high dimensions, J. Funct. Anal. **231**(2) (2006), 361–374.

[54] Y. Zhou, Blow up of classical solutions to $\Box u = |u|^{1+\alpha}$ in three space dimensions, J. Partial Differential Equations **5**(3) (1992), 21–32.

[55] Y. Zhou, Life span of classical solutions to $\Box u = |u|^p$ in two space dimensions, Chinese Ann. Math. Ser. B **14**(2) (1993), 225–236.

[56] Y. Zhou, Blow up of solutions to the Cauchy problem for nonlinear wave equations, Chin. Ann. Math. Ser. B **22**(3) (2001), 275–280.

[57] Y. Zhou, Blow up of solutions to semilinear wave equations with critical exponent in high dimensions, Chin. Ann. Math. Ser. B **28**(2) (2007), 205–212.

[58] Y. Zhou, W. Han, Life-span of solutions to critical semilinear wave equations, Comm. Partial Differential Equations **39**(3) (2014), 439-451.