ON PI-ALGEBRAS WITH ADDITIONAL STRUCTURES: RATIONALITY OF HILBERT SERIES AND SPECHT'S PROBLEM.

LUCIO CENTRONE, ALEJANDRO ESTRADA, AND ANTONIO IOPPOLO

Abstract. One of the main problems in PI-theory is to prove the rationality of the Hilbert series of the relatively free algebra of a given PI-algebra. In this paper we consider a field F of characteristic 0 and we prove the rationality of the Hilbert series of the PI-algebra A over F both in the case A is a superalgebra with superinvolution and when a finite dimensional semisimple Hopf algebra acts on A. Along the way, we give a proof of the Specht's problem in case A is a superalgebra with superinvolution.

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1. Introduction

One of the classical and most famous problems in algebra was posed by Burnside in 1902 (see [16]): is it true that every finitely generated torsion group is finite? Almost 40 years later, Kurosh in 1941 asked the analogous question in the setting of algebras (see [43]): is every finitely generated algebraic algebra finite dimensional? It is well-known that both these problems have no positive answer in general, as it was first showed by Golod and Shafarevich.

In the present paper, we deal with the so-called PI-algebras, that is associative algebras satisfying at least one non-trivial polynomial identity. Given a field F and a countable set of variables $X = \{x_1, x_2, \ldots\}$, the free algebra $F\langle X\rangle$ is the set of all polynomials in the variables of X. A *polynomial identity* of an algebra A is a polynomial of $F\langle X\rangle$ which vanishes under all substitutions in elements of A. The importance of the theory of polynomial identities (PI-theory for short) was highlighted around the 1950s, when it was showed that the Kurosh Problem has a positive solution in this class of algebras (see [37, 54]).

In the recent decades different classes of algebras with additional structure, such as group graded algebras or algebras with involution, have been studied in the context of PI-theory. In this paper we focus our attention on H-module algebras, where H is a finite dimensional semisimple Hopf algebra and on the setting of superalgebras with superinvolution.

We would like to point out that the structure of H-module algebra generalizes several notions such as gradings by finite abelian groups and involutions whereas superalgebras with superinvolution cannot be seen as H-module algebras. Nevertheless, also superalgebras with superinvolution are a natural generalization of algebras with involution and they play a prominent role in the setting of Lie and Jordan algebras (see, for instance, [35, 51]). In recent years, superalgebras with superinvolution has been extensively studied by several mathematicians and their importance has been highlighted in 2017 by Aljadeff, Giambruno and Karasik. In [2], they showed that any algebra with involution has the same identities of the Grassmann envelope of a finite dimensional superalgebras with superinvolution. The last result is a generalization of a classical result in PI-theory, known as the Representability theorem.

In the present paper we study the so-called Hilbert series for both the setting of H-module algebras and of superalgebras with superinvolution. Hilbert polynomials, Hilbert series or Hilbert-Poincaré series of graded (in a classical meaning) algebras are strongly related notions which attracted several mathematicians in the last century. The Hilbert series of an algebra represents a crucial algebraic tool in computational algebraic geometry, as it is the easiest known way for computing the dimension and the degree of an algebraic variety defined by explicit polynomial equations. We recall that the question of whether the Hilbert series of an algebra is the Taylor expansion of a rational function is fundamental in the commutative setting because of its relations with other invariants related to the

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growth of an algebra such as the Gelfand-Kirillov dimension or the Krull dimension of algebras. As will be highlighted later in this paper, the broad range of applications of Hilbert series reaches the non-commutative environment, too.

At this point we think it could be fruitful building up a timeline through the results obtained on this research argument. Let F be a field and consider a graded (in a classical meaning) F-algebra A with finite generating set S. If we define the non-negative integer c_n as the dimension over F of the vector space generated by the monomials of degree n in the elements from S, then the Hilbert series of A is

$$\mathrm{Hilb}(A,t)=\sum_{n=0}^{\infty}c_nt^n.$$

If A is a finitely generated (affine) commutative algebra, then the *Hilbert-Serre Theorem* says that the Hilbert series of A is rational (see [7] for the proof). Nevertheless, such a theorem is not true in the case the algebra is non-commutative and, in this regard, we cite the work [6] by Anick where the author showed a famous counterexample. Anyway, there is a big class of non-commutative affine algebras whose Hilbert series is rational. We are talking about the class of finitely generated relatively free algebras, i.e., algebras isomorphic to the quotient of a finitely generated free algebra by a T-ideal. Notice that T-ideals are ideals of a free algebra invariant under all the endomorphisms of the same free algebra. Moreover, any T-ideal is the set of (ordinary) *polynomial identities* of a certain algebra. The analog of Hilbert-Serre Theorem for relatively free algebras carried a lot of results in PI-theory and we would like to cite among them the paper [15] by Berele and Regev in which the authors showed the exact asymptotic behaviour of the codimension sequence of a PI-algebra satisfying the Capelli identity.

The analog of Hilbert-Serre Theorem also holds for classes of free algebras with additional structure, such as the class of finitely generated G-graded relatively free algebras, where G is a finite group and the underlying graded T-ideal is the ideal of G-graded polynomial identities of a G-graded algebra satisfying an ordinary polynomial identity (Aljadeff and Kanel-Belov in [5]).

Now it is time to state more precisely the goal of this paper. We present a proof of the Hilbert-Serre Theorem in the case of relatively free algebras of H-module algebras and in the case of relatively free algebras of superalgebras with superinvolution. In both case we have to assume that the algebra satisfies an ordinary polynomial identity. It is worth mentioning that representable algebras, in general, can have transcendental (so non-rational) Hilbert series as showed by Belov et al. in [11].

In the setting of finite dimensional semisimple Hopf algebras, our proofs are given in the language of the action of a ring on a given algebra that is something new. We use the tools appearing in the proof of *representability* of relatively free H-module algebras (as in [38]). In the case of superalgebras with superinvolution, we show an explicit form of the so-called *Kemer polynomials* which are crucial in the proof of the rationality of the Hilbert series of any relatively free algebra. Moreover, we also get another relevant result, the so-called *Specht's property*, for the two classes of algebras above. Recall that the classical Specht's problem (ordinary associative algebras) asks whether a T-ideal can be generated as a T-ideal by a finite number of polynomials. It has been a leading light in PI-theory for many years and in the ordinary context of associative algebras (over a field of characteristic zero) it has been proved by Kemer in 1987 (see [40]).

In the final part of the paper, we also get that the Gelfand-Kirillov dimension of finitely generated relatively free algebras of an H-module algebra (or of a superalgebra with superinvolution) satisfying an ordinary polynomial identity is an integer.

2. **Preliminaries**

In this section we shall recall some well-known results about PI-theory. Unless explicitly written, every field is supposed to be of characteristic 0 and any algebra is associative.

Let F be a field and consider a countable set of variables $X=\{x_1,x_2,\ldots\}$. The free algebra $F\langle X\rangle$ is the set of all polynomials in the variables of X. A *polynomial identity* of an algebra A is a polynomial of the free algebra $F\langle X\rangle$ which vanishes under all substitutions in elements of A. We refer to the books [24] by Drensky and [29] by Giambruno and Zaicev for further details about polynomial identities of associative algebras.

We start off with the classical notion of *grading*. Let $G=\{g_1,\ldots,g_s\}$ be any group of finite order s and let F be a field. If A is an F-algebra, we say that A is a *G-graded algebra* if there are subspaces

A^g for each $g \in G$ such that

$$A = \bigoplus_{g\in G} A^g \text{ and } A^g A^h \subseteq A^{gh}.$$

If $0 \neq a \in A^g$ we say that a is *homogeneous of G-degree g* or *G-graded homogeneous of G-degree g*, and we write $\deg(a) = g$.

Now we recall the Wedderburn-Malcev decomposition for G-graded algebras (see [20]).

Theorem 1 (Wedderburn-Malcev decomposition)**.** *Let A be a finite dimensional G-graded algebra over a field F of characteristic 0 and let $J(A)$ be its Jacobson radical. Then $J(A)$ is G-graded and there exists a G-graded subalgebra B such that $A = B + J(A)$, as vector spaces.*

In the previous result, the subalgebra B is a direct product of G-simple algebras, i.e., algebras with a non-trivial multiplication containing no proper G-graded ideals.

Now, let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets. We write $X = \bigcup_{g\in G} X^g$ and we denote by $F\langle X|G\rangle$ the free associative algebra freely generated by the set X. An indeterminate (or variable) $x \in X$ is said to be of *homogeneous G-degree g*, written $\deg(x) = g$, if $x \in X^g$. We always write x^g if $x \in X^g$. The homogeneous G-degree of a monomial $m = x_{i_1} x_{i_2} \cdots x_{i_k}$ is defined to be $\deg(m) = \deg(x_{i_1}) \deg(x_{i_2}) \cdots \deg(x_{i_k})$. For every $g \in G$ we denote by $F\langle X|G\rangle^g$ the subspace of $F\langle X|G\rangle$ spanned by all monomials having homogeneous G-degree g. Notice that $F\langle X|G\rangle^g F\langle X|G\rangle^{g'} \subseteq F\langle X|G\rangle^{gg'}$ for all $g, g' \in G$. Thus

$$F\langle X|G\rangle = \bigoplus_{g\in G} F\langle X|G\rangle^g$$

and $F\langle X|G\rangle$ is a G-graded algebra. We refer to the elements of $F\langle X|G\rangle$ as *G-graded polynomials* or just *graded polynomials*. An ideal I of $F\langle X|G\rangle$ is said to be a *T_G-ideal* (or *graded T-ideal*) if it is invariant under all G-graded endomorphisms $\varphi : F\langle X|G\rangle \to F\langle X|G\rangle$ such that $\varphi(F\langle X|G\rangle^g) \subseteq F\langle X|G\rangle^g$ for all $g \in G$. If A is a G-graded algebra, a G-graded polynomial $f(x_1, \ldots, x_n)$ is said to be a *graded polynomial identity* of A if $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in \bigcup_{g\in G} A^g$ such that $a_k \in A^{\deg(x_k)}$, $k = 1, \ldots, n$. We denote by $\mathrm{Id}^G(A)$ the ideal of all graded polynomial identities of A. It is a T_G-ideal of $F\langle X|G\rangle$ in the sense that it is invariant under all graded homomorphism of $F\langle X|G\rangle$. We shall call *substitution* with elements of A any graded homomorphism $F\langle X|G\rangle \to A$ and we sometimes use the notation $\overline{x} = a \in A$ in order to denote explicitly such an evaluation of the variable x.

Given a subset $S \subseteq F\langle X|G\rangle$ one can think about the least T_G-ideal of $F\langle X|G\rangle$ containing the set S. Such a T_G-ideal will be denoted by $\langle S\rangle^{T_G}$ and will be called the *T_G-ideal generated by S*. We say that the elements of $\langle S\rangle^{T_G}$ are *consequences* of elements of S, or simply that they follow from S. If $\mathrm{Id}^G(A) = \langle S\rangle^{T_G}$, we say that S is a *basis* for the graded polynomial identities of A. If the ground field F is of characteristic 0, then we are allowed to consider only the multilinear graded identities of a given algebra. Notice that, if the grading group of a given algebra is the trivial one, we are simply talking about (ordinary) polynomial identities of that algebra and any index is omitted.

Let A be a G-graded algebra. We denote by

$$\mathcal{V}^G(A) = \{B = G\text{-graded algebra} : \mathrm{Id}^G(A) \subseteq \mathrm{Id}^G(B)\},$$

the class of all associative G-graded algebras satisfying the G-graded identities of A and we shall call it the *variety of G-graded algebras generated* by A. The graded identities of $\mathcal{V}^G(A)$ are precisely the graded identities of A.

In the classical literature towards varieties of algebras one of the main problems is the so-called *Specht problem* which concerns with the existence of a finite basis of identities for any subvariety of a given variety of algebras. The following result by Kemer (see [39,41]) is definitely crucial in PI-theory.

Theorem 2 (Specht property for associative algebras (ordinary case))**.** *Every variety of associative algebras over a field of characteristic 0 has a finite basis of its identities as well as any of its subvarieties.*

The following generalization of Kemer's result for G-graded algebras, where G is finite and abelian, was achieved by Sviridova in [57].

Theorem 3 (Specht property for G-graded algebras (abelian grading case))**.** *Every variety of G-graded associative algebras over a field of characteristic 0, where G is finite and abelian, has a finite basis of its graded identities.*

The most general result for graded algebra is the one by Aljadeff and Kanel-Belov in [4].

Theorem 4 (Specht property for G-graded algebras (finite grading case))**.** *Every variety of G-graded associative algebras over a field of characteristic 0, where G is finite, has a finite basis of its graded identities.*

Now we introduce another key character of this paper, the *Hilbert series* of a relatively free algebra. We recall that given any $\mathbb{Z}$-graded algebra A, we are allowed to consider the subspace of A generated by the monomials of degree n in one of its generating set. If we denote by a_n the dimension of this subspace of A, the Hilbert series of A is defined to be the following formal power series

$$\mathrm{Hilb}(A,t)=\sum_{n=0}^{\infty}a_nt^n.$$

It is a very big deal trying to build up a complete list of papers about Hilbert series of algebras. Nevertheless we would like to mention the papers by La Scala [44,45] for the reduction of the computation of the Hilbert series of algebras to the case of a monomial algebras, i.e., an algebra generated by monomials in one of its generating sets.

In the following theorem we state a classical result by Serre (see, for instance, Section 11 of [7]).

Theorem 5. *If A is a commutative algebra, then $\mathrm{Hilb}(A,t)$ is a rational function.*

Now, let $k\geq 1$ be an integer and consider the finite set of variables $X_k=\{x_1,\dots,x_k\}$. The quotient algebra $F_k(A)=F\langle X_k\rangle/(F\langle X_k\rangle\cap \mathrm{Id}(A))$ is the relatively free algebras of A in k variables. We denote by $\mathrm{Hilb}(F_k(A),t)$ its Hilbert series. In 1997, Belov gave the following analog of Serre's result for relatively free algebras (see [9]).

Theorem 6 (Rationality of Hilbert series of relatively free algebras (ordinary case))**.** *The Hilbert series $\mathrm{Hilb}(F_k(A),t)$ of the relatively free algebra of a PI-algebra A in K variables is a rational function.*

In the setting of G-graded algebras, we have the following generalization of Belov's result obtained by Aljadeff and Kanel-Belov in [5], for any finite group G.

Theorem 7 (Rationality of Hilbert series of relatively free algebras (graded case))**.** *Let G be a finite group. Then the Hilbert series $\mathrm{Hilb}(F_k^G(A),t)$ of the relatively free G-graded algebra of a G-graded algebra A satisfying an ordinary polynomial identity is a rational function.*

Here $F_k^G(A)=F\langle X_k\rangle/(F\langle X_k\rangle\cap\mathrm{Id}^G(A))$ and $\mathrm{Hilb}(F_k^G(A),t)$ denote the relatively free G-graded algebra of A in the G-graded variables from $X_k=\{x_1^{g_1},\dots,x_k^{g_k}\}$ and its Hilbert series, respectively.

As already mentioned in the Introduction, our intent is to develop a similar theory for the class of H-module algebras, where H is a finite dimensional semisimple Hopf algebra and for the class of superalgebras with superinvolution. Although many definition in the next sections could simply appear as restatements of the notions given in this preliminary section, we believe that each case deserves its own vocabulary, especially regarding the free algebras and the Wedderburn-Malcev decomposition.

3. Superalgebras with superinvolution

In what follows A is an associative algebra over a fixed field F of characteristic zero. If $\mathbb{Z}_2$ is the cyclic group of order 2, we say that the algebra A is $\mathbb{Z}_2$-graded if it can be written as the direct sum of subspaces $A=A_0\oplus A_1$ such that $A_0A_0+A_1A_1\subseteq A_0$ and $A_0A_1+A_1A_0\subseteq A_1$. The subspaces A_0 and A_1 are the homogeneous components of A and their elements are called homogeneous of degree zero (even elements) and of degree one (odd elements), respectively. If a is an homogeneous element we shall write $\deg(a)$ or $|a|$ to indicate its homogeneous degree. The $\mathbb{Z}_2$-graded algebras are simply called superalgebras. Recall that, if $A=A_0\oplus A_1$ and $B=B_0\oplus B_1$ are two superalgebras, then a linear map $\varphi\colon A\to B$ is said to be graded if $\varphi(A_i)\subseteq B_i$, $i=0,1$.

A superinvolution on a superalgebra $A=A_0\oplus A_1$ is a graded linear map $*\colon A\to A$ such that:

1. $(a^*)^*=a$, for all $a\in A$,
2. $(ab)^*=(-1)^{|a||b|}b^*a^*$, for any homogeneous elements $a,b\in A_0\cup A_1$.

Since $\mathrm{char}F=0$, we can write

$$A=A_0^+\oplus A_0^-\oplus A_1^+\oplus A_1^-,$$

where for $i=0,1$, $A_i^+=\{a\in A_i : a^*=a\}$ and $A_i^-=\{a\in A_i : a^*=-a\}$ denote the sets of symmetric and skew elements of A_i, respectively.

We shall refer to a superalgebra with superinvolution simply as a $*$-algebra.

The free algebra with superinvolution (free $*$-algebra), denoted by $F\langle Y\cup Z,*\rangle$, is generated by symmetric and skew elements of even and odd degree. We write

$$F\langle Y\cup Z,*\rangle = F\langle y_1^+,y_1^-,z_1^+,z_1^-,y_2^+,y_2^-,z_2^+,z_2^-,\ldots\rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree. In order to simplify the notation, sometimes we denote by y any even variable, by z any odd variable and by x an arbitrary variable. The elements of $F\langle Y\cup Z,*\rangle$ are called $*$-polynomials.

A $*$-polynomial $f(y_1^+,\ldots,y_n^+,y_1^-,\ldots,y_m^-,z_1^+,\ldots,z_t^+,z_1^-,\ldots,z_s^-)\in F\langle Y\cup Z,*\rangle$ is a $*$-identity of the $*$-algebra $A=A_0^+\oplus A_0^-\oplus A_1^+\oplus A_1^-$, and we write $f\equiv 0$, if, for all $u_1^+,\ldots,u_n^+\in A_0^+$, $u_1^-,\ldots,u_m^-\in A_0^-$, $v_1^+,\ldots,v_t^+\in A_1^+$ and $v_1^-,\ldots,v_s^-\in A_1^-$, we have

$$f(u_1^+,\ldots,u_n^+,u_1^-,\ldots,u_m^-,v_1^+,\ldots,v_t^+,v_1^-,\ldots,v_s^-)=0.$$

We denote by $\mathrm{Id}^*(A)=\{f\in F\langle Y\cup Z,*\rangle : f\equiv 0 \text{ on } A\}$ the ideal of $*$-identities of A. Notice that $\mathrm{Id}^*(A)$ is a T_2^*-ideal of $F\langle Y\cup Z,*\rangle$, i.e., an ideal that is invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra $F\langle Y\cup Z\rangle$ commuting with the superinvolution $*$.

Given two $*$-algebras A and B, we say that A is T_2^*-equivalent to B, and we write $A\sim_{T_2^*}B$, in case $\mathrm{Id}^*(A)=\mathrm{Id}^*(B)$. Moreover, we denote by $\langle f_1,\ldots,f_n\rangle_{T_2^*}$ the T_2^*-ideal generated by the $*$-polynomials $f_1,\ldots,f_n\in F\langle Y\cup Z,*\rangle$.

Because we are in characteristic 0, as in the ordinary case, it is easily seen that every $*$-identity is equivalent to a system of multilinear $*$-identities. Hence if we denote by

$$P_n^*=\mathrm{span}_F\Big\{w_{\sigma(1)}\cdots w_{\sigma(n)}\mid \sigma\in S_n,\ w_i\in\{y_i^+,y_i^-,z_i^+,z_i^-\},\ i=1,\ldots,n\Big\}$$

the space of multilinear polynomials of degree n in $y_1^+,y_1^-,z_1^+,z_1^-,\ldots,y_n^+,y_n^-,z_n^+,z_n^-$ (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial with degree 1) the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^*\cap\mathrm{Id}^*(A)$, for all $n\geq 1$. The non-negative integer

$$c_n^*(A)=\dim_F\frac{P_n^*}{P_n^*\cap\mathrm{Id}^*(A)},\ n\geq 1,$$

is called the n-th $*$-*codimension* of A. In [26], the authors proved that the sequence of $*$-codimensions is exponentially bounded, provided that the $*$-algebra satisfies an ordinary non-trivial identity.

Let $n\geq 1$ and write $n=n_1+\cdots+n_4$ as a sum of non-negative integers. We denote by $P^*_{n_1,\ldots,n_4}\subseteq P_n^*$ the vector space of multilinear $*$-polynomials in which the first n_1 variables are even symmetric, the next n_2 variables are even skew, the next n_3 variables are odd symmetric and the last n_4 variables are odd skew. The group $S_{n_1}\times\cdots\times S_{n_4}$ acts on the left on the vector space $P^*_{n_1,\ldots,n_4}$ by permuting the variables of the same homogeneous degree which are all even or all odd at the same time. Thus S_{n_1} permutes the variables $y_1^+,\ldots,y_{n_1}^+$, S_{n_2} permutes the variables $y_{n_1+1}^-,\ldots,y_{n_1+n_2}^-$, and so on. In this way $P^*_{n_1,\ldots,n_4}$ becomes a left $(S_{n_1}\times\cdots\times S_{n_4})$-module. Now $P^*_{n_1,\ldots,n_4}\cap\mathrm{Id}^*(A)$ is invariant under this action and the vector space

$$P^*_{n_1,\ldots,n_4}(A)=\frac{P^*_{n_1,\ldots,n_4}}{P^*_{n_1,\ldots,n_4}\cap\mathrm{Id}^*(A)}$$

is a left $(S_{n_1}\times\cdots\times S_{n_4})$-module with the induced action.

Our next goal is to present a Wedderburn-Malcev decomposition for $*$-algebras. To this end, recall that an ideal (subalgebra) I of a $*$-algebra A is a $*$-ideal (subalgebra) of A if it is a graded ideal (subalgebra) and $I^*=I$. The $*$-algebra A is a simple $*$-algebra if $A^2\neq 0$ and A has no non-trivial $*$-ideals.

The Wedderburn-Malcev analog for $*$-algebras was proved in [26, Theorem 4.1].

Theorem 8. *Let A be a finite dimensional $*$-algebra over a field F of characteristic* 0. *Then there exists a semisimple $*$-subalgebra B such that*

$$A=B+J(A)$$

as vector spaces and $J(A)$ is a $*$-ideal of A. Moreover $B\cong A_1\times\cdots\times A_q$, where $A_1,\ldots,A_q$ are simple $*$-algebras.

Of course, B is $*$-semisimple and the Wedderburn-Malcev decomposition enables us to consider *semisimple* and *radical* (or nilpotent) substitutions. More precisely, since in order to check whether a given multilinear $*$-polynomial is an identity of A it is sufficient to evaluate the variables in any spanning set of even/skew homogeneous elements, we may take a basis consisting of even/skew homogeneous elements of B or of $J(A)$. We refer to such evaluations as semisimple or radical evaluations, respectively. Moreover, the semisimple substitutions may be taken from $*$-simple components. This kind of evaluations, i.e., the ones from the set

$$\bigcup_{i=1}^{q} A_i \cup J(A),$$

are called *elementary*. In what follows, whenever we evaluate a polynomial on a finite dimensional $*$-algebra, we shall only consider elementary evaluations.

Next we shall present the classification of the finite dimensional simple $*$-algebras over an algebraically closed field F. Recall that if A and B are two superalgebras endowed with superinvolutions $*$ and $\star$, respectively, then $(A,*)$ and $(B,\star)$ are isomorphic, as $*$-algebras, if there exists an isomorphism of superalgebras $\psi : A \to B$ such that $\psi(x^*) = \psi(x)^\star$, for all $x \in A$.

If $n = k + h$, the matrix algebra $M_n(F)$ becomes a superalgebra, denoted by $M_{k,h}(F)$, endowed with grading

$$(M_{k,h}(F))_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} : X \in M_k(F),\ T \in M_h(F) \right\},$$
$$(M_{k,h}(F))_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} : Y \in M_{k\times h}(F),\ Z \in M_{h\times k}(F) \right\}.$$

In [50], Racine proved that, up to isomorphism and if the field F is algebraically closed and of characteristic different from 2, it is possible to define on $M_{k,h}(F)$ only the following superinvolutions.

1. The *transpose* superinvolution, denoted by *trp* and defined for $h = k$ by
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix},$$
where t is the usual transpose.
2. The *orthosymplectic* superinvolution *osp* defined when $h = 2l$ is even by
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} X^t & Z^tQ \\ QY^t & -QT^tQ \end{pmatrix},$$
where $Q = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and I_k, I_l are the $k \times k$, $l \times l$ identity matrices, respectively.

Furthermore, if A is a superalgebra, we denote by A^{sop} the superalgebra with the same graded vector space structure of A and product given on homogeneous elements $a, b \in A^{sop}$ by

$$a \circ b = (-1)^{|a||b|} ba.$$

The direct sum $R = A \oplus A^{sop}$ is a superalgebra with $R_0 = A_0 \oplus A_0^{sop}$ and $R_1 = A_1 \oplus A_1^{sop}$. Given $x, y \in R$, $x = (a, b) = (a_0 + a_1, b_0 + b_1)$, $y = (a', b') = (a'_0 + a'_1, b'_0 + b'_1)$, the product in R is given by

$$(a_0 + a_1, b_0 + b_1) \cdot (a'_0 + a'_1, b'_0 + b'_1) = (a_0a'_0 + a_1a'_1 + a_0a_1 + a_1a'_0, b'_0b_0 - b'_1b_1 + b'_0b_1 + b'_1b_0). \tag{1}$$

Moreover R is a $*$-algebra since it is endowed with the exchange superinvolution *ex* defined by:

$$(a, b)^{ex} = (b, a).$$

For example, if we consider the superalgebra $Q(n) = M_n(F \oplus cF) = Q(n)_0 \oplus Q(n)_1$, where $Q(n)_0 = M_n(F)$ and $Q(n)_1 = cM_n(F)$, with $c^2 = 1$, then $Q(n) \oplus Q(n)^{sop}$ is a $*$-algebra with exchange superinvolution.

The following result gives the classification of the finite dimensional simple $*$-algebras over an algebraically closed field F (see [8, 31, 50]).

Theorem 9. *Let A be a finite dimensional simple $*$-algebra over an algebraically closed field F of characteristic different from 2. Then A is isomorphic to one of the following:*

(1) *$M_{k,h}(F)$ with the orthosymplectic or the transpose superinvolution,*

(2) $M_{k,h}(F) \oplus M_{k,h}(F)^{sop}$ *with the exchange superinvolution,*

(3) $Q(n) \oplus Q(n)^{sop}$ *with the exchange superinvolution.*

Remark 10. *In Theorem 9, the $*$-algebra A has always an identity element that is symmetric of homogeneous degree* 0*.*

Proof. Let I be the identity matrix of $M_n(F)$. If $A \cong M_{k,h}(F)$, $n = k + h$, then I is the identity of A. Suppose that $A \cong M_{k,h}(F) \oplus M_{k,h}(F)^{sop}$ or $A \cong Q(n) \oplus Q(n)^{sop}$, then the pair (I, I) is the identity of A. Finally it is not difficult to see that the identity of A is a symmetric even element. □

We conclude this section with the following result proved in [2, Theorem 1].

Theorem 11. *Let F be an algebraically closed field of characteristic zero. Let $\mathcal{V}$ be a variety generated by a finitely generated $*$-algebra A over F, satisfying an ordinary non-trivial identity. Then $\mathcal{V} = var^*(B)$, for some finite dimensional $*$-algebra B over F.*

4. Specht's problem for superalgebras with superinvolution

The purpose of this section is to give a positive answer to the Specht's problem in the setting of superalgebras with superinvolution ($*$-algebras). More precisely, if W is a finitely generated $*$-algebra over a field F of characteristic 0 satisfying an ordinary non-trivial identity, we shall find a finite generating set for the T_2^*-ideal of identities $\mathrm{Id}^*(W)$. We recall that this result was announced in [2]. Here we shall give an explicit construction of Kemer's polynomials that are the key ingredient in solving the Specht's problem.

4.1. Kemer points and Kemer polynomials.

Let Γ be a T_2^*-ideal. Recall that, since F is a field of characteristic zero, Γ is generated by multilinear $*$-polynomials (recall that in the setting of superalgebras with superinvolution there is no notion of strongly homogeneous polynomials since the grading group $\mathbb{Z}_2$ is abelian).

Let X be a set of variables. We can write

$$X = X_0^+ \cup X_0^- \cup X_1^+ \cup X_1^-,$$

where X_0^+ is the subset of symmetric even variables (the y_i^+'s), X_0^- is the subset of skew even variables (the y_i^-'s), X_1^+ is the subset of symmetric odd variables (the z_i^+'s) and X_1^- is the subset of skew odd variables (the z_i^-'s). Let S_0 and S_1 be subsets of $Y = X_0^+ \cup X_0^-$ and $Z = X_1^+ \cup X_1^-$ respectively, and let $R_0 = Y \setminus S_0$, $R_1 = Z \setminus S_1$. Of course, if $S_i = \{x_1, \ldots, x_m\}$, then the variables x_i's are of homogeneous degree i and symmetric or skew.

Definition 12. *Let $f = f(X)$ be a multilinear $*$-polynomial. We say that f is alternating in $S_i = \{x_1, \ldots, x_m\}$, $i \in \{0, 1\}$, if there exists a multilinear $*$-polynomial $h(S_i, R_i) := h(x_1, \ldots, x_m, R_i)$ such that*

$$f(X) = \sum_{\sigma \in S_m} (-1)^\sigma h(x_{\sigma(1)}, \ldots, x_{\sigma(m)}, R_i).$$

If $S_1, \ldots, S_p$, are p disjoint sets of variables of X (belonging to Y or Z), we say that $f(X)$ is *alternating* in $S_{i_1}, \ldots, S_{i_p}$, if it is alternating in each of them.

Now we will consider polynomials which alternate in ν disjoint sets of the form S_i, $i = 0, 1$.

Definition 13. *Let $f = f(X)$ be a multilinear $*$-polynomial alternating in $S_{i_1}, \ldots, S_{i_{2\nu}}$. If all the sets $S_{i_1}, \ldots, S_{i_{2\nu}}$ belonging to the same set (Y or Z) have the same cardinality (say $d_i, i \in \{0, 1\}$), then we will say that*

$$f(X) \text{ is } \nu\text{-fold } (d_0, d_1)\text{-alternating.}$$

In order to define the $*$-index of a T_2^*-ideal Γ we need the notion of *i-th Capelli polynomial*, $i \in \{0, 1\}$. Let $X_{n,i} = \{x_1, \ldots, x_n\}$ be a set of n variables of homogeneous degree $i \in \{0, 1\}$ and let $W = \{w_1, \ldots, w_n\}$ be a set of n ungraded variables. The i-th Capelli polynomial $c_{n,i}$ of degree $2n$ is the polynomial obtained by alternating the set of variables $x_1, \ldots, x_n$ in the monomial $x_1 w_1 \cdots x_n w_n$. Hence

$$c_{n,i} = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} w_1 \cdots x_{\sigma(n)} w_n.$$

Clearly $c_{n,i}$, $i \in \{0, 1\}$ is a multilinear $*$-polynomial alternating in $\{x_1 \ldots, x_n\}$.

Lemma 14. *For any $i \in \{0,1\}$ there exists an integer n_i such that the T_2^*-ideal Γ contains $c_{n_i,i}$.*

Proof. Let A be a finite dimensional $*$-algebra such that $\mathrm{Id}^*(A) \subseteq \Gamma$. We consider the decomposition $A = A_0 \oplus A_1$ and we take $n_i = \dim A_i + 1$, $i \in \{0,1\}$. It is clear that $c_{n_i,i} \in \mathrm{Id}^*(A)$ and the proof is complete. □

As a consequence we get the following result.

Corollary 15. *If $f = f(X)$ is a multilinear $*$-polynomial alternating on a set S_i of cardinality n_i, then $f \in \Gamma$. Consequently there exists an integer M_i which bounds (from above) the cardinality of the alternating homogeneous sets in any $*$-polynomial h which is not in Γ.*

Let Γ denote the T_2^*-ideal of a finitely generated $*$-algebra. Now we are in a position to define the $*$-index $\mathrm{Ind}^*(\Gamma)$ of Γ. Here we want to highligh that in [4] Aljadeff and Belov introduced the analogous object in the setting of G-graded algebras, where G is a finite group.

$\mathrm{Ind}^*(\Gamma)$ will consist of a finite set of points (α, s) in the lattice $L = \mathbb{N}^2 \times (\mathbb{N} \cup \infty)$. Given $\alpha = (\alpha_0, \alpha_1)$, $\beta = (\beta_0, \beta_1) \in \mathbb{N}^2$, we put $\alpha \preceq \beta$ if and only if $\alpha_i \leq \beta_i$, for $i = 0, 1$. This gives a partial order in $\mathbb{N}^2$. As a consequence, we obtain a partial order on L. Given $(\alpha, s), (\beta, s') \in L$, we write $(\alpha, s) \preceq (\beta, s')$ if and only if either

1) $\alpha \prec \beta$, or
2) $\alpha = \beta$ and $s \leq s'$ (notice that $s < \infty$ for every $s \in \mathbb{N}$).

We first determine the set $\mathrm{Ind}^*(\Gamma)_0$, namely the projection of $\mathrm{Ind}^*(\Gamma)$ into $\mathbb{N}^2$.

Definition 16. *A point $\alpha = (\alpha_0, \alpha_1)$ is in $\mathrm{Ind}^*(\Gamma)_0$ if and only if for any integer ν there exists a multilinear $*$-polynomial outside Γ with ν alternating homogeneous sets (of degree i) of cardinality α_i for every $i = 0, 1$.*

Lemma 17. *The following facts hold:*

1. *The set $\mathrm{Ind}^*(\Gamma)_0$ is bounded (finite).*
2. *If $\alpha \in \mathrm{Ind}^*(\Gamma)_0$, then $\alpha' \preceq \alpha$ is also in $\mathrm{Ind}^*(\Gamma)_0$.*

Proof. The first statement follows since $\Gamma \supseteq \mathrm{Id}^*(A)$, for some finite dimensional $*$-algebra A at light of Theorem 11. The second one is a consequence of the definition of $\mathrm{Ind}^*(\Gamma)_0$. □

Definition 18. *A point $\alpha \in \mathrm{Ind}^*(\Gamma)_0$ is said extremal if for any $\beta \in \mathrm{Ind}^*(\Gamma)_0$, $\beta \succeq \alpha$ implies $\beta = \alpha$.*

We denote by $E_0(\Gamma)$ the set of all extremal points in $\mathrm{Ind}^*(\Gamma)_0$.

For any point $\alpha = (\alpha_0, \alpha_1) \in E_0(\Gamma)$ and every integer ν, consider the set $\Omega_{\alpha,\nu}$ of all ν-folds alternating polynomials in homogeneous sets of cardinality α_i, where $i = 0, 1$, that are not in Γ. Given $f \in \Omega_{\alpha,\nu}$, we consider the number $s_\Gamma(\alpha, \nu, f)$ of alternating homogeneous sets of disjoint variables, of cardinality $\alpha_i + 1$, $i = 0, 1$. The set of integers $\{s_\Gamma(\alpha, \nu, f)\}_{f \in \Omega_{\alpha,\nu}}$ is bounded. We define $s_\Gamma(\alpha, \nu) = \max\{s_\Gamma(\alpha, \nu, f)\}_{f \in \Omega_{\alpha,\nu}}$. The sequence $s_\Gamma(\alpha, \nu)$ is monotonically decreasing as a function of ν. As a consequence, there exists an integer $\mu = \mu(\Gamma, \alpha)$ for which the sequence stabilizes, that is for $\nu \geq \mu$, the sequence $s_\Gamma(\alpha, \nu)$ is constant. We let $s(\alpha) = \lim_{\nu \to \infty} s_\Gamma(\alpha, \nu) = s_\Gamma(\alpha, \mu)$. At this point the integer μ depends on α. However, since the set $E_0(\Gamma)$ is finite by Lemma 17, we take μ to be the maximum of all μ's considered above. Keeping in mind the definition of μ, we have the following definitions.

Definition 19. *The $*$-index $\mathrm{Ind}^*(\Gamma)$ of Γ is the set of points $(\alpha, s) \in L$ such that $\alpha \in \mathrm{Ind}^*(\Gamma)_0$ and $s = s_\Gamma(\alpha)$ if $\alpha \in E_0(\Gamma)$ or $s = \infty$ otherwise.*

Definition 20. *Given a T_2^*-ideal Γ containing the $*$-identities of a finite dimensional $*$-algebra A, we let the Kemer set of Γ, denoted $K(\Gamma)$, be the set of points (α, s) in $\mathrm{Ind}^*(\Gamma)$, where α is extremal. We refer to the elements of $K(\Gamma)$ as the Kemer points of Γ.*

The next remark follows immediately.

Remark 21. *Let $\Gamma_1 \supseteq \Gamma_2$ be two T_2^*-ideals containing $\mathrm{Id}^*(A)$, where A is a finite dimensional $*$-algebra. Then:*

1. $\mathrm{Ind}^*(\Gamma_1) \subseteq \mathrm{Ind}^*(\Gamma_2)$.
2. *For every $(\alpha, s) \in K(\Gamma_1)$ there is a Kemer point $(\beta, s') \in K(\Gamma_2)$ such that $(\alpha, s) \preceq (\beta, s')$.*

We are now ready to define Kemer polynomials for a T_2^*-ideal Γ.

Definition 22. *Let (α, s) be a Kemer point of Γ. A $*$-polynomial f is said to be a Kemer $*$-polynomial for the point (α, s) if $f \notin \Gamma$ and it has at least μ-folds of alternating homogeneous sets (of degree i) of cardinality α_i (small sets), where $i = 0, 1$, and s homogeneous sets of disjoint variables V (of some homogeneous degree) of cardinality $\alpha_i + 1$ (big sets). A $*$-polynomial f is Kemer for Γ if it is Kemer for a Kemer point of Γ.*

If we choose a Kemer point (α, s), then α is extremal. Because of this, we get the next result.

Remark 23. *A $*$-polynomial f cannot be Kemer simultaneously for different Kemer points of Γ.*

4.2. **Decomposition in basic $*$-algebras.**

In this section we shall introduce the so-called basic $*$-algebras and we shall prove that every finitely generated $*$-algebra satisfying an ordinary non-trivial identity is T_2^*-equivalent to the direct product of a finite number of basic $*$-algebras.

First, let A be a finite dimensional $*$-algebra and consider its Wedderburn-Malcev decomposition:

$$A = B + J(A).$$

The semisimple part B is a $*$-algebra too and so we can consider its decomposition in symmetric and skew spaces of homogeneous degree 0 and 1, respectively:

$$B = B_0 \oplus B_1 = B_0^+ \oplus B_0^- \oplus B_1^+ \oplus B_1^-.$$

We use the following notation:

- $d(B_i) = \dim_F B_i$, $i \in \{0, 1\}$,
- $n(A)$ is the nilpotency index of $J(A)$.

We write $\mathrm{Par}^*(A)$ to indicate the 3-tuple $(d(B_0), d(B_1), n(A) - 1) \in \mathbb{N}^2 \times \mathbb{N}$.

Proposition 24. *If $(\alpha, s) = (\alpha_0, \alpha_1, s)$ is a Kemer point of A, then $(\alpha, s) \preceq \mathrm{Par}^*(A)$.*

Proof. Suppose, by contradiction, that this does not happen. Hence, $\alpha_i > d(B_i)$ for some $i = 0, 1$, or $\alpha_i = d(B_i)$ in any case and $s > n(A) - 1$. We shall see that both these possibilities cannot occur. First recall that, since (α, s) is a Kemer point of A, then there exist multilinear $*$-polynomials f which are non-identities of A with arbitrary many alternating homogeneous sets of cardinality α_i, $i = 0, 1$.

1. Suppose $\alpha_i > d(B_i)$, for some $i = 0, 1$.
   We have that in each such alternating set there must be at least one radical substitution in any non-zero evaluation of a polynomial f. This implies that we cannot have more than $n(A) - 1$ alternating homogeneous sets of cardinality α_j, contradicting our previous statement.
2. Suppose $\alpha_i = d(B_i)$ in any case and $s > n(A) - 1$.
   This means that we have s alternating sets (of a certain homogeneous degree) of cardinality $\alpha_i + 1 = d(B_i) + 1$, for some $i = 0, 1$. Again this means that f will vanish if we evaluate any of these sets by semisimple elements. It follows that in each one of these s sets at least one of the evaluations is radical. Since $s > n(A) - 1$, the polynomial f vanishes on such evaluations as well and hence it is a $*$-identity of A. We reach a contradiction in this case too.

The proof is complete. □

In order to establish a precise relation between Kemer points of a finite dimensional $*$-algebra A and its structure we need to find appropriate finite dimensional algebras which will serve as minimal models for a given Kemer point. We start with the decomposition of a finite dimensional $*$-algebra into the product of subdirectly irreducible components.

Definition 25. *A finite-dimensional $*$-algebra A is said to be subdirectly irreducible if there are no non-trivial $*$-ideals I and J of A such that $I \cap J = (0)$.*

Lemma 26. *Let A be a finite dimensional $*$-algebra over F. Then A is T_2^*-equivalent to a direct product $C_1 \times \cdots \times C_n$ of finite dimensional subdirectly irreducible $*$-algebras. Furthermore for every $i = 1, \ldots, n$, $\dim_F(C_i) \leq \dim_F(A)$ and the number of $*$-simple components in C_i is bounded by the number of such components in A.*

Proof. If A is subdirectly irreducible there is nothing to prove. If A is not subdirectly irreducible, then there exist non-trivial $*$-ideals I and J of A such that $I \cap J = (0)$. It is clear that A/I (and at the same way A/J) is a $*$-algebra with superinvolution $\bar{*}\colon A/I \to A/I$ induced from the superinvolution $*$ of A by $\bar{*}(a+I) = a^* + I$, for any $a \in A$. Moreover, it is easy to prove that A is T_2^*-equivalent to $A/I \times A/J$ (notice that in the non-trivial inclusion is fundamental the fact that $I \cap J = (0)$). This completes the first part of the proof.

The second one follows by induction by taking into account the fact that $\dim_F(A/I)$ and $\dim_F(A/J)$ are strictly smaller than $\dim_F A$. □

Definition 27. *Let A be a finite-dimensional $*$-algebra and let f be a multilinear $*$-polynomial. We say that A is full with respect to f, if there exists a non-vanishing evaluation of f such that every $*$-simple component is represented (among the semisimple substitutions).*

A finite dimensional $*$-algebra A is full if it is full with respect to some multilinear $*$-polynomial f.

Remark 28. *Let A be a $*$-algebra over a field F of characteristic zero and let $\bar{F}$ be the algebraic closure of F. Then $\bar{A} = A \otimes_F \bar{F}$ is a $*$-algebra with superinvolution $(a \otimes \alpha)^{\bar{*}} = a^* \otimes \alpha$. We have that*

- $\dim_F A = \dim_{\bar{F}} \bar{A}$,
- $\mathrm{Id}^*(A) = \mathrm{Id}^*(\bar{A})$, *viewed as $*$-algebras over F,*
- $c_n^*(A) = c_n^*(\bar{A})$ *(here $\bar{A}$ is viewed as a $\bar{F}$-algebra).*

We wish to show that any finite dimensional algebra may be decomposed (up to T_2^*-equivalence) into the direct product of full algebras. Algebras without an identity element are treated separately.

Lemma 29. *Let A be a $*$-algebra subdirectly irreducible and not full.*

1. *If A has an identity element then it is T_2^*-equivalent to a direct product of finite-dimensional $*$-algebras, each having fewer $*$-simple components.*
2. *If A has no identity element then it is T_2^*-equivalent to a direct product of finite-dimensional $*$-algebras, each having either fewer $*$-simple components than A or else it has an identity element and the same number of $*$-simple components as A.*

Proof. Suppose first that A has an identity element. By Theorem 8, A can be decomposed as

$$A = B + J \cong A_1 \times \cdots \times A_q + J$$

where J is the Jacobson radical of the algebra (a $*$-ideal) and $A_1, \ldots, A_q$ are simple $*$-algebras. For $i = 1, \ldots, q$, let e_i denote the identity element of A_i and consider the decomposition

$$A \cong \bigoplus_{i,j=1}^{q} e_i A e_j.$$

By assumption, whenever $i_1, \ldots, i_q$ are distinct, it follows that

$$e_{i_1} A e_{i_2} \cdots e_{i_{q-1}} A e_{i_q} = e_{i_1} J e_{i_2} \cdots e_{i_{q-1}} J e_{i_q} = 0.$$

Let us consider the commutative algebra $R = F[\lambda_1, \ldots, \lambda_q]/I$, where I is the ideal generated by $\lambda_i^2 - \lambda_i$ and $\lambda_1 \cdots \lambda_q$. We denote by $\tilde{e}_i$ the image of λ_i in R. It is clear that $\tilde{e}_i^2 = \tilde{e}_i$ and $\tilde{e}_1 \cdots \tilde{e}_q = 0$. By Remark 28, we have the algebra $A \otimes_F R$ is a $*$-algebra with superinvolution $\bar{*}$ induced via the superinvolution $*$ defined on A. Let $\tilde{A}$ be the $*$-subalgebra generated by all $e_i A e_j \otimes \tilde{e}_i \tilde{e}_j$, for every $1 \le i, j \le q$.

CLAIM: $A \sim_{T_2^*} \tilde{A}$.

Clearly $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(A \otimes_F R) \subseteq \mathrm{Id}^*(\tilde{A})$. Hence it suffices to prove that any non $*$-identity f of A is also a non-identity of $\tilde{A}$. Clearly, we may assume that f is multilinear. Evaluating f on A it suffices to consider maps of the form $x_l^{\pm} \mapsto v_l^{i_l,\pm}$, where $x \in \{y, z\}$ and $i_l \in \{0, 1\}$ (symmetric or skew elements of homogeneous degree 0 or 1) and $v_l^{i_l,\pm} \in e_{j_k} A e_{j_{k+1}}$, for some k. In order to have $v_1^{i_1,\pm} \cdots v_n^{i_n,\pm} \neq 0$, the set of indices $\{j_k\}$ must contain at most $q-1$ distinct elements, so $e_{j_1} \cdots e_{j_n} \neq 0$. Then

$$f(v_1^{i_1,\pm} \otimes \tilde{e}_{i_1}, \ldots, v_n^{i_n,\pm} \otimes \tilde{e}_{i_n}) = f(v_1^{i_1,\pm}, \ldots, v_n^{i_n,\pm}) \otimes \tilde{e}_{i_1} \cdots \tilde{e}_{i_n} \neq 0.$$

Hence f is not in $\mathrm{Id}^*(\tilde{A})$ and the claim is proved.

In order to complete the proof we need to show that $\tilde{A}$ can be decomposed into a direct product of $*$-algebras, each having fewer $*$-simple components. Let $I_j = \langle e_j \otimes \tilde{e}_j, e_j^* \otimes \tilde{e}_j \rangle$ be a $*$-ideal of $\tilde{A}$. Hence

$$\bigcap_{j=1}^{q} I_j = (1 \otimes \tilde{e}_1 \cdots 1 \otimes \tilde{e}_q) \left(\bigcap_{j=1}^{q} I_j \right) = (1 \otimes \tilde{e}_1 \cdots \tilde{e}_q) \left(\bigcap_{j=1}^{q} I_j \right) = (0).$$

It follows that $\tilde{A}$ is subdirectly reducible to the direct product of $\tilde{A}/I_j$. Furthermore, each component $\tilde{A}/I_j$ has less than q $*$-simple components since we eliminated the idempotent corresponding to the j-th $*$-simple component. This completes the proof of the first part of the lemma.

Consider now the case in which the algebra A has no identity element. Let $e_0 = 1 - (e_1 + \cdots + e_q)$; we consider the decomposition

$$A \cong \bigoplus_{i,j=0}^{q} e_i A e_j$$

and we carry on as in the first case but with $q+1$ idempotents, variables, and so on. As above, A/I_j will have less than q $*$-simple components if $1 \leq j \leq q$ whereas A/I_0 will have an identity element and exactly q $*$-simple components. The proof now is complete. □

By putting together Lemmas 26 and 29 we get the following result.

Corollary 30. *Every finite dimensional $*$-algebra A is T_2^*-equivalent to a direct product of full, subdirectly irreducible finite dimensional $*$-algebras.*

Remark 31. *In the decomposition above, the nilpotency index of the components in the direct product is bounded by the nilpotency index of A.*

In the following definition we introduce the so-called minimal algebras.

Definition 32. *We say that a finite dimensional $*$-algebra A is* minimal *if* $\mathrm{Par}^*(A)$ *is minimal (with respect to the partial order defined before) among all finite dimensional $*$-algebras which are T_2^*-equivalent to A.*

Definition 33. *A finite dimensional $*$-algebra A is said to be* basic *if it is minimal, full and subdirectly irreducible.*

As a consequence of the results and definitions of this section we obtain the following theorem.

Theorem 34. *Every finite dimensional $*$-algebra A is T_2^*-equivalent to the direct product of a finite number of basic $*$-algebras.*

Combining this result with Theorem 11 we obtain the following corollary.

Corollary 35. *Every finitely generated $*$-algebra W satisfying an ordinary non-trivial identity is T_2^*-equivalent to the direct product of a finite number of basic $*$-algebras.*

4.3. **Kemer's lemmas.**

The task in this section is to show that any basic $*$-algebra A has a Kemer set which consists of a unique point $(\alpha, s) = \mathrm{Par}^*(A)$. We start with some preliminaries in the framework of finite dimensional simple $*$-algebras.

Let $A = (a_{ij})$ be an $n \times n$ matrix. For $j = 2, \ldots, n$, the j-th hook of A is the set of elements:

$$\{a_{1j}, a_{2j}, \ldots, a_{jj}, a_{j1}, a_{j2}, \ldots, a_{jj-1}\}.$$

Remark 36. *There exists a product of the matrix units e_{ij}, $i, j \in \{1, \ldots, n\}$, with value e_{11}.*

Proof. Let us consider the matrix $E = (e_{ij}) \in M_n(F)$ and let $e_{1j}, e_{2j}, \ldots, e_{jj}, e_{j1}, e_{j2}, \ldots, e_{jj-1}$ be the elements in the j-th hook of E. We have

$$e_{1j} e_{j2} e_{2j} e_{j3} e_{3j} \cdots e_{jj-1} e_{j-1j} e_{jj} e_{j1} = e_{11}.$$

For any $j = 2, \ldots, n$, we denote by H_j the previous product of matrix units. The proof now runs because

$$e_{11} H_2 H_3 \cdots H_n = e_{11}.$$

□

Now let us consider the $*$-algebra $M_{k,k}(F)$ with the transpose superinvolution trp. Notice that:

$$
\begin{aligned}
(M_{k,k}(F),\ \mathrm{trp})_0^+ &= \left\{ \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix} | X \in M_k(F) \right\},\\
(M_{k,k}(F),\ \mathrm{trp})_0^- &= \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} | X \in M_k(F) \right\},\\
(M_{k,k}(F),\ \mathrm{trp})_1^+ &= \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} | Y = -Y^t,\ Z = Z^t,\ Y, Z \in M_k(F) \right\},\\
(M_{k,k}(F),\ \mathrm{trp})_1^- &= \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} | Y = Y^t,\ Z = -Z^t,\ Y, Z \in M_k(F) \right\}.
\end{aligned}
$$

The following elements form a $*$-basis (basis as a vector space with homogeneous symmetric or skew elements) of $(M_{k,k}(F), trp)$:

- $\{e_{i,j} + e_{k+j,k+i}\}$, $i, j = 1, \ldots, k$.
- $\{e_{i,j} - e_{k+j,k+i}\}$, $i, j = 1, \ldots, k$.
- $\{e_{i,k+j} - e_{j,k+i},\ \ e_{k+i,j} + e_{k+j,i},\ \ e_{k+l,l}\}$, $1 \le i < j \le k$ and $l = 1, \ldots, k$.
- $\{e_{i,k+j} + e_{j,k+i},\ \ e_{k+i,j} - e_{k+j,i},\ \ e_{l,k+l}\}$, $1 \le i < j \le k$ and $l = 1, \ldots, k$.

Lemma 37. *There exists a product of the above $*$-basis elements with value e_{11}.*

Proof. Let us consider the matrix units e_{pq}, $p, q = 1, \ldots, 2k$. It is easy to see that in the above $*$-basis there is at least one element in which e_{pq} appears with a plus, for any $p, q \in \{1, \ldots, 2k\}$. When there are two elements of this kind, we make the following choice: we fix the element of the $*$-basis corresponding to e_{pq} to be that one in which in the second part of the element appears a minus. We shall denote by $\bar{e}_{pq}$ the element of the $*$-basis corresponding to e_{pq}. For instance, $e_{1,1}$ appears in the $*$-basis both in $e_{1,1} + e_{k+1,k+1}$ and $e_{1,1} - e_{k+1,k+1}$. Hence $\bar{e}_{1,1} = e_{1,1} - e_{k+1,k+1}$. In this way we are sure that $\bar{e}_{k+1,k+1} = e_{1,1} + e_{k+1,k+1}$ (notice that $e_{k+1,k+1}$ appears with a plus, as desired).

Now we construct the following $2k \times 2k$ matrix E: in the entry (p, q) we put the element of the $*$-basis $\bar{e}_{pq}$. As in Remark 36, we denote by H_j the product of the elements in the j-th hook of the matrix E, $j = 2, \ldots, 2k$. Moreover, in any element of the $*$-basis of the form $e_{ab} \pm e_{cd}$, we have $a \neq c$. Hence, as desired, we get

$$\bar{e}_{11} H_2 \cdots H_{2k} = e_{11}. \tag{2}$$

□

Let us consider the monomial $M = w_1 \cdots w_{4k^2}$, where each variable w_i has a certain homogeneous degree and it is symmetric or skew according to the corresponding element in the product in (2).

If we border each matrix $\bar{e}_{i,j}$ in the product (2) with idempotents $e_{i,i}$ and $e_{j,j}$, then we can consider the monomial obtained by M by bordering each variable with a variable of homogeneous degree 0:

$$M' = y_1 w_1 y_2 w_2 \cdots y_{4k^2} w_{4k^2} y_{4k^2+1}.$$

Clearly, the monomial M' has the property that there exists an evaluation φ such that $\varphi(M') = e_{11}$. Moreover, we have

$$e_{j,j} = \begin{cases} \dfrac{(e_{i,i} + e_{k+i,k+i}) + (e_{i,i} - e_{k+i,k+i})}{2}, & \text{if } 1 \le i = j \le k, \\[2ex] \dfrac{(e_{i,i} + e_{k+i,k+i}) - (e_{i,i} - e_{k+i,k+i})}{2}, & \text{if } j = k+i,\ 1 \le i \le k. \end{cases} \tag{3}$$

Thus we can write each bordering element $e_{i,i}$ in terms of the $*$-basis elements. In this way, we can replace each variable y_i in the monomial M' by $(y_i^+ + y_i^-)/2$ or $(y_i^+ - y_i^-)/2$ according to (3), where y_i^+ is a symmetric variable of zero degree and y_i^- is a skew variable of degree 0. Denote by P this $*$-polynomial. Then we have the next result that is a consequence of Lemma 37.

Lemma 38. *Consider the $*$-polynomial $P = \frac{y_1^+ \pm y_1^-}{2} w_1 \frac{y_2^+ \pm y_2^-}{2} w_2 \cdots \frac{y_{4k^2}^+ \pm y_{4k^2}^-}{2} w_{4k^2} \frac{y_{4k^2+1}^+ \pm y_{4k^2+1}^-}{2}$ defined above. Then there exists an evaluation φ of P such that $\varphi(P) = e_{11}$.*

Now let us consider $A = (M_{k,2l}(F),\ \mathrm{osp})$ be the $*$-algebra of $(k+2l)\times(k+2l)$ matrices endowed with the orthosymplectic superinvolution. Recall we have the following:

$$A_0^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = X^t,\ T = -QT^tQ,\ X \in M_k(F),\ T \in M_{2l}(F) \right\},$$
$$A_0^- = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X = -X^t,\ T = QT^tQ,\ X \in M_k(F),\ T \in M_{2l}(F) \right\},$$
$$A_1^+ = \left\{ \begin{pmatrix} 0 & Z^tQ \\ Z & 0 \end{pmatrix} \mid Z \text{ is a } 2l\times k \text{ matrix} \right\},$$
$$A_1^- = \left\{ \begin{pmatrix} 0 & -Z^tQ \\ Z & 0 \end{pmatrix} \mid Z \text{ is a } 2l\times k \text{ matrix} \right\}.$$

It is easy to see that the following sets $\mathcal{B}_0^+, \mathcal{B}_0^-, \mathcal{B}_1^+, \mathcal{B}_1^-$ form a $*$-basis of $A_0^+, A_0^-, A_1^+, A_1^-$ respectively:

$$\mathcal{B}_0^+ = \left\{ \begin{matrix} e_{i.i} & 1\le i\le k, \\ e_{i,j}+e_{j,i} & 1\le i<j\le k, \\ e_{k+i,k+j}+e_{k+l+j,k+l+i} & 1\le i,j\le l, \\ e_{k+i,k+l+j}-e_{k+j,k+l+i} & 1\le i<j\le l, \\ e_{k+l+i,k+j}-e_{k+l+j,k+i} & 1\le i<j\le l \end{matrix} \right\} \qquad \mathcal{B}_0^- = \left\{ \begin{matrix} e_{i,j}-e_{j,i} & 1\le i<j\le k, \\ e_{k+i,k+j}-e_{k+l+j,k+l+i} & 1\le i,j\le l, \\ e_{k+i,k+l+j}+e_{k+j,k+l+i} & 1\le i<j\le l, \\ e_{k+l+i,k+j}+e_{k+l+j,k+i} & 1\le i<j\le l, \\ e_{k+i,k+l+i} & 1\le i\le l, \\ e_{k+l+i,k+i} & 1\le i\le l \end{matrix} \right\}$$

$$\mathcal{B}_1^+ = \left\{ \begin{matrix} e_{i,k+j}-e_{k+l+j,i} & 1\le i\le k, 1\le j\le l, \\ e_{i,k+l+j}+e_{k+j,i} & 1\le i\le k, 1\le j\le l \end{matrix} \right\} \qquad \mathcal{B}_1^- = \left\{ \begin{matrix} e_{i,k+j}+e_{k+l+j,i} & 1\le i\le k, 1\le j\le l, \\ e_{i,k+l+j}-e_{k+j,i} & 1\le i\le k, 1\le j\le l \end{matrix} \right\}.$$

With a similar construction to that of Lemma 37, it is not difficult to show that there exists a product of the above $*$-basis elements with value e_{11}. In this way, we get an analog of Lemma 38.

Lemma 39. *Let $(M_{k,2l}, osp)$ be the $*$-algebra of $(k+2l)\times(k+2l)$ matrices endowed with the orthosymplectic superinvolution. Then there exists an evaluation φ of the $*$-polynomial*

$$P = \frac{y_1^+ \pm y_1^-}{2} w_1 \frac{y_2^+ \pm y_2^-}{2} w_2 \cdots \frac{y^+_{(k+2l)^2} \pm y^-_{(k+2l)^2}}{2} w_{(k+2l)^2} \frac{y^+_{(k+2l)^2+1} \pm y^-_{(k+2l)^2+1}}{2}$$

in a $*$-basis of (M_{k2l}, osp) such that $\varphi(P) = e_{11}$.

Now let us focus our attention on the $*$-algebra $M_{k,h}(F)\oplus M_{k,h}(F)^{sop}$ endowed with the exchange superinvolution. A $*$-basis of such an algebra is the following:

$$B = \{(e_{ij}, e_{ij}), (e_{ij}, -e_{ij})\}_{i,j=1,\dots,k+h}.$$

We construct the following two matrices: A^+ is the matrix having in the entry (i,j) the element (e_{ij}, e_{ij}) whereas A^- is the matrix having in the entry (i,j) the element $(e_{ij}, -e_{ij})$. Now let H_j^+ (H_j^-, resp.) be the product of the elements in the j-th hook of the matrix A^+ (A^-, resp.). By taking into account the multiplication rule in equation (1), we get

$$Q = (e_{11}, e_{11})H_2^+\cdots H_{k+h}^+ = (e_{11}, 0) \qquad \text{and} \qquad Q' = (e_{11}, -e_{11})H_2^-\cdots H_{k+h}^- = (e_{11}, 0).$$

If we consider the same product Q' but in the opposite direction, i.e. we start with the last element and we finish with the first one of Q' (we denote such a new product by Q^*), we get

$$Q^* = \begin{cases} (0, e_{11}) & \text{if } k+h+kh \text{ is even,} \\ (0, -e_{11}) & \text{if } k+h+kh \text{ is odd.} \end{cases}$$

Lemma 40. *Let us consider the monomials $M = w_1\cdots w_{(k+h)^2}$ and $M^* = u_1\cdots u_{(k+h)^2}$, where each variable w_i and u_i has a certain homogeneous degree and it is symmetric or skew according to the corresponding element in the products Q and Q^* above, respectively. Then we can consider the monomials obtained by M and M^*, respectively, by bordering each variable with a symmetric even variable:*

$$M' = y_1^+ w_1 y_2^+ w_2 \cdots y^+_{(k+h)^2} w_{(k+h)^2} y^+_{(k+h)^2+1} \quad \textit{and} \quad (M^*)' = \mathfrak{y}_1^+ u_1 \mathfrak{y}_2^+ u_2 \cdots \mathfrak{y}^+_{(k+h)^2} u_{(k+h)^2} \mathfrak{y}^+_{(k+h)^2+1}.$$

Consider the evaluation φ of the $*$-polynomial $f = M' \pm (M^*)'$ ($+$ if $k+h+kh$ is even, $-$ otherwise):

1. *each variable w_i (u_i, resp.) is evaluated in the corresponding element of Q (Q^*, resp.),*
2. *each variable y_j^+, $\mathfrak{y}_j^+$ in the suitable idempotent element $(e_{l_jl_j}, e_{l_jl_j}) \in M_{k,h}(F)\oplus M_{k,h}(F)^{sop}$.*

We get that

$$\varphi(f) = (e_{11}, e_{11}).$$

Finally, let us focus our attention to the $*$-algebra $Q(n) \oplus Q(n)^{sop}$ endowed with the exchange superinvolution. A $*$-basis of such an algebra is the following:

$$B = \{(e_{ij}, e_{ij}), (e_{ij}, -e_{ij}), (ce_{ij}, ce_{ij}), (ce_{ij}, -ce_{ij})\}_{i,j=1,\dots,n}.$$

We construct the following four matrices: A_0^+ is the matrix having in the entry (i,j) the element (e_{ij}, e_{ij}), A_0^- is the matrix having in the entry (i,j) the element $(e_{ij}, -e_{ij})$, A_c^+ is the matrix having in the entry (i,j) the element (ce_{ij}, ce_{ij}), A_c^- is the matrix having in the entry (i,j) the element $(ce_{ij}, -ce_{ij})$. Now, let $H_j^{0,+}$, $H_j^{0,-}$, $H_j^{c,+}$, $H_j^{c,-}$ be the product of the elements in the j-th hook of the matrices A_0^+, A_0^-, A_c^+ and A_c^-, respectively. Consider the following products:

$$Q = (e_{11}, e_{11}) H_2^{0,+} \cdots H_n^{0,+} (ce_{11}, ce_{11}) H_2^{c,+} \cdots H_n^{c,+},$$

$$Q' = (e_{11}, -e_{11}) H_2^{0,-} \cdots H_n^{0,-} (ce_{11}, -ce_{11}) H_2^{c,-} \cdots H_n^{c,-}.$$

Hence we have

$$Q = Q' = \begin{cases} (e_{11}, 0) & \text{if } n \text{ is even,} \\ (ce_{11}, 0) & \text{if } n \text{ is odd.} \end{cases}$$

If we consider the same product Q' but in the opposite direction, i.e., we start with the last element and we finish with the first one of Q' (we denote such a new product by Q^*), we get

$$Q^* = \begin{cases} (0, e_{11}) & \text{if } n \text{ is even,} \\ (0, -ce_{11}) & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 41. *Let us consider the monomials $M = w_1 \cdots w_{2n^2}$ and $M^* = u_1 \cdots u_{2n^2}$, where each variable w_i and u_i has a certain homogeneous degree and it is symmetric or skew according to the corresponding element in the products Q and Q^* above, respectively. Then we can consider the monomials obtained by M and M^*, respectively, by bordering each variable with a symmetric even variable:*

$$M' = y_1^+ w_1 y_2^+ w_2 \cdots y_{2n^2}^+ w_{2n^2} y_{2n^2+1}^+, \qquad \text{and} \qquad (M^*)' = \mathfrak{y}_1^+ u_1 \mathfrak{y}_2^+ u_2 \cdots \mathfrak{y}_{2n^2}^+ u_{2n^2} \mathfrak{y}_{2n^2+1}^+.$$

Finally we construct the following $*$-polynomial f:

$$f = \begin{cases} M' + (M^*)' & \text{if } n \text{ is even,} \\ M'z^+ - (M^*)'z^+ & \text{if } n \text{ is odd.} \end{cases}$$

We consider the following evaluation φ:

1. *each variable w_i (u_i, resp.) is evaluated in the corresponding element of Q (Q^*, resp.),*
2. *each variable y_j^+ is evaluated in the suitable idempotent element $(e_{l_jl_j}, e_{l_jl_j}) \in Q(n) \oplus Q(n)^{sop}$,*
3. *the variable z^+ is evaluated in the element (ce_{11}, ce_{11}).*

We get that

$$\varphi(f) = (e_{11}, e_{11}).$$

Remark 42. *In all the results above we have considered monomial or polynomial with value e_{11} or (e_{11}, e_{11}). Of course it is possible to obtain the same result for any e_{ii} or (e_{ii}, e_{ii}).*

The following result is the $*$-algebra version of Kemer's First Lemma.

Lemma 43. *Let $A = B + J$ be a finite dimensional $*$-algebra, subdirectly irreducible and full with respect to a polynomial f. Then for any integer ν there exists a non-identity f' of A in the T_2^*-ideal generated by f with ν-folds (d_0, d_1)-alternating, where $d_i = \dim B_i$ for $i \in \{0, 1\}$.*

Proof. Consider the Wedderburn-Malcev decomposition $A = B + J = A_1 \times \cdots \times A_q + J$, where the A_i's are $*$-simple algebras. Since A is full, there is a multilinear $*$-polynomial $f(x_1^\pm, \dots, x_q^\pm, w_1, \dots, w_p)$ (where $x \in \{y, z\}$ and $w_1, \dots, w_p$ are variables disjoint from $\{x_1^\pm, \dots, x_q^\pm\}$) which does not vanish under an elementary evaluation of the form $x_j^\pm = v_j^{i_j,\pm} \in A_j$, $j = 1, \dots, q$, $i_j \in \{0, 1\}$, and the variables w_j's get elementary values in A.

Now we consider the polynomial obtained from f by multiplying on the left each one of the variables $\{x_1^\pm, \dots, x_q^\pm\}$ by symmetric variables of even degree $y_1^+, \dots, y_q^+$ respectively. Clearly such a polynomial is a non-identity since the variables y_j^+'s may be evaluated on the identity elements 1_{A_j} of A_j. By Remark 10, we may write the identity element of A_j as $1_{A_j} = e_{1,1}+\cdots+e_{n_j,n_j}$ or $1_{A_j} = (e_{1,1}, e_{1,1})+\cdots+(e_{n_j,n_j}, e_{n_j,n_j})$. Applying linearity there exists a non-zero evaluation where the variables $y_1^+, \dots, y_q^+$ take values of the form e_{i_j,i_j} or $(e_{i_j,i_j}, e_{i_j,i_j})$, with $1 \le i_j \le n_j$, $j = 1, \dots, q$.

Now we replace each variable $y_1^+, \dots, y_q^+$ by $*$-polynomials $Y_1 \dots, Y_q$ such that:

- Y_j is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating, $j = 1, \dots, q$,
- Y_j takes the value e_{i_j,i_j} or $(e_{i_j,i_j}, e_{i_j,i_j})$, $j = 1, \dots, q$.

In the construction of the $*$-polynomials Y_j we have to consider 4 distinct cases.

Case 1.1: $A_j \cong M_{k,k}(F)$ with the transpose superinvolution trp.

Fix $1 \le i_j \le k + h$ and consider the $*$-polynomial P constructed in Lemma 38:

$$P = \frac{y_1^+ \pm y_1^-}{2} w_1 \frac{y_2^+ \pm y_2^-}{2} w_2 \cdots \frac{y_{4k^2}^+ \pm y_{4k^2}^-}{2} w_{4k^2} \frac{y_{4k^2+1}^+ \pm y_{4k^2+1}^-}{2}.$$

We refer to the variables w_i's as *designated variables*. Next we consider the product of ν $*$-polynomials P (with distinct variables). We denote the long $*$-polynomial obtained in this way by P_ν. Finally, we construct the $*$-polynomial Y_j by alternating separately the variables of even/odd degree in each set of designated variables w_i of P_ν. Clearly the $*$-polynomial Y_j is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating. We only need to show that Y_ν takes the value e_{i_j,i_j}, so that it will be a non-identity of A_j.

By Lemma 38 and Remark 42 there exists a suitable evaluation φ of P such that $\varphi(P) = e_{i_j,i_j}$. We consider the following evaluation for Y_j: for each polynomial P (with distinct variables) we consider the corresponding evaluation φ giving out the value e_{i_j,i_j}.

Notice that the monomials of Y_j assuming a non-zero value under this evaluation are those corresponding to permutations that only transpose the variables corresponding to elements of type $e_{i_1,j_1} + e_{i_2,j_2}$ and $e_{i_1,j_1} - e_{i_2,j_2}$. Moreover, it is not difficult to see that each of these monomials takes the value e_{i_j,i_j} (considering it with the sign). In conclusion, the evaluation of Y_j is a scalar multiple of e_{i_j,i_j} and since $\mathrm{char} F = 0$ we are done.

Case 1.2: $A_j \cong M_{k,h}(F)$ with the orthosymplectic superinvolution.

This case can be treated as the previous one. We just need to consider the $*$-polynomial

$$P = \frac{y_1^+ \pm y_1^-}{2} w_1 \frac{y_2^+ \pm y_2^-}{2} w_2 \cdots \frac{y_{(k+2l)^2}^+ \pm y_{(k+2l)^2}^-}{2} w_{(k+2l)^2} \frac{y_{(k+2l)^2+1}^+ \pm y_{(k+2l)^2+1}^-}{2}$$

constructed in Lemma 39 and then define the $*$-polynomial Y_j as before. Such a polynomial is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating and assume the value e_{i_j,i_j} as desired.

Case 2: $A_j \cong M_{k,h}(F) \oplus M_{k,h}(F)^{sop}$.

Fix $1 \le i_j \le k + h$ and consider the $*$-polynomial f (remark it is not multilinear) constructed in Lemma 40:

$$f = M' \pm M'^* = y_1^+ w_1 y_2^+ w_2 \cdots y_{(k+h)^2}^+ w_{(k+h)^2} y_{(k+h)^2+1}^+ \pm \mathfrak{y}_1^+ u_1 \mathfrak{y}_2^+ u_2 \cdots \mathfrak{y}_{(k+h)^2}^+ u_{(k+h)^2} \mathfrak{y}_{(k+h)^2+1}^+.$$

We consider the product of ν $*$-polynomials f (with distinct variables) and we denote the long $*$-polynomial obtained in this way by P_ν. Finally, we construct the $*$-polynomial Y_j by alternating separately the variables of even/odd degree in each set of designated variables w_i of P_ν. Clearly the $*$-polynomial Y_j is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating. We need to show that Y_ν is a non-identity of A_j.

By Lemma 40 and Remark 42 there exists a suitable evaluation φ of f such that $\varphi(f) = (e_{i_j,i_j}, e_{i_j,i_j})$. Notice that the permutation that only transposes the variables corresponding to elements of the type $(e_{i_1,j_1}, e_{i_2,j_2})$ and $(e_{i_1,j_1}, -e_{i_2,j_2})$ does not vanish in the evaluation φ: in fact, the evaluations in this kind of permutations are equal to $(e_{i_j,i_j}, -e_{i_j,i_j})$. Transpositions of other types vanish (in the above evaluation) because the bordering elements are different. Therefore, the evaluation of a permutation obtained from an even number of transpositions is equal to $(e_{i_j,i_j}, e_{i_j,i_j})$ and the evaluation of a permutation obtained from an odd number of transpositions is equal to $(e_{i_j,i_j}, -e_{i_j,i_j})$. In conclusion the evaluation φ of Y_j is a scalar multiple of $(e_{i_j,i_j}, e_{i_j,i_j}) - (e_{i_j,i_j}, -e_{i_j,i_j})$.

Case 3: $A_j \cong Q(n) \oplus Q(n)^{sop}$.

Fix $1 \leq i_j \leq n$ and consider the $*$-polynomial f defined in Lemma 41. Notice that the polynomial f is not multilinear. We consider the product of ν $*$-polynomials f (with distinct variables) and we denote the long $*$-polynomial obtained in this way by P_ν. Then we construct the $*$-polynomial Y_j by alternating separately the variables of even/odd degree in each set of designated variables w_i of P_ν. Clearly the $*$-polynomial Y_j is ν-folds $(\dim_F(A_j)_0, \dim_F(A_j)_1)$-alternating. We need to show Y_ν is a non-identity of A_j.

By Lemma 41, there exists a suitable evaluation φ of f such that $\varphi(f) = (e_{i_j,i_j}, e_{i_j,i_j})$. Notice that the permutation that only transposes the variables corresponding to elements of the type $(e_{i_1,j_1}, e_{i_2,j_2})$ and $(e_{i_1,j_1}, -e_{i_2,j_2})$ or of the type $(ce_{i_1,j_1}, ce_{i_2,j_2})$ and $(ce_{i_1,j_1}, -ce_{i_2,j_2})$ does not vanish in the evaluation φ: in fact, the evaluations in this kind of permutations are equal to $(e_{i_j,i_j}, -e_{i_j,i_j})$. Moreover, transpositions of other types vanish (in the above evaluation) because the bordering elements are different. Therefore the evaluation of a permutation obtained from an even number of transpositions is equal to $(e_{i_j,i_j}, e_{i_j,i_j})$ and the evaluation of a permutation obtained from an odd number of transpositions is equal to $(e_{i_j,i_j}, -e_{i_j,i_j})$. In this way the evaluation φ of Y_j is a scalar multiple of $(e_{i_j,i_j}, e_{i_j,i_j}) - (e_{i_j,i_j}, -e_{i_j,i_j})$.

In order to complete the proof we construct a $*$-polynomial f' by alternating the (symmetric/skew of a certain homogeneous degree) sets which come from different Y_j's. Clearly $f' \notin Id^*(A)$ and f' has ν-folds (d_0, d_1)-alternating as desired and the proof follows. □

Proposition 44. *Let A be a finite dimensional $*$-algebra, full and subdirectly irreducible. Then there is an extremal point α in $E_0(A)$ with $\alpha = (d(A)_0), d(A)_1)$. In particular, this extremal point is unique.*

Proof. The existence follows immediately by Lemma 43. The uniqueness is a consequence of Proposition 11. □

The last goal of this section is to give the analog of Kemer's Lemma 2 in the setting of $*$-algebras. In order to reach this goal we need some definitions and preliminary results. Recall that, if A is a $*$-algebra, then by using the Wedderburn-Malcev decomposition, we can write $A = B + J$, where B is the semisimple part and J is the Jacobson radical of A, which is a nilpotent $*$-ideal ($n(A)$ is its nilpotency index).

Lemma 45. *If (α, s) is a Kemer point of a finite dimensional $*$-algebra A, then $s \leq n(A) - 1$.*

Proof. By the definition of the parameter s we know that for arbitrary large ν there exist multilinear $*$-polynomials, not in $\mathrm{Id}^*(A)$, being ν-folds alternating on homogeneous (small) sets of cardinality $d(A)_i$ and s (big) sets of cardinality $d(A)_i + 1$, for each $i \in \{0, 1\}$. It follows that an alternating homogeneous set of cardinality $d(A)_i + 1$ in a non-identity polynomial must have at least one radical evaluation. Consequently we cannot have more than $n(A) - 1$ of such alternating sets and we are done. □

The next construction (see [36, Example 4.50]) will enable us to take some "control" on the nilpotency index of the radical of a finite dimensional $*$-algebra.

Let $B = \bar{B} + J$ be any finite-dimensional $*$-algebra and let $B' = \bar{B} \cdot F\langle X, *\rangle$ be the $*$-algebra of $*$-polynomials in the variables $X = \{x_{i_1}^{\dagger_1}, \ldots, x_{i_m}^{\dagger_m}\}$ with coefficients in $\bar{B}$, the semisimple component of B, where $i_j \in \{0, 1\}$ and $\dagger_j \in \{+, -\}$, for $j = 1, \ldots, m$. The number of homogeneous symmetric (skew) variables that we take is at least the dimension of the homogeneous symmetric (skew) component of $J(B)$. The superinvolution in B' is induced by

$$(b \cdot x)^* = (-1)^{|b||x|} x^* b^*,$$

where $b \in \bar{B}$ and x is a variable in X. Observe that any element of B' is represented by a sum of elements of the form $b_1 f_1 b_2 f_2 \cdots b_k f_k b_{k+1}$, where $b_1, \ldots, b_{k+1} \in \bar{B}$ and $f_1, \ldots, f_k \in F\langle X, *\rangle$.

Let I_1 be the $*$-ideal of B' generated by all the evaluations of the $*$-polynomials of $\mathrm{Id}^*(B)$ on B' and let I_2 be the $*$-ideal of B' generated by the variables $\{x_{i_j}^{\dagger_j}\}_{j=1}^m$. For any $u > 1$, define $\hat{B}_u = B'/(I_1 + I_2^u)$.

Proposition 46. *The following sentences hold:*

1. *$Id^*(\hat{B}_u) = Id^*(B)$, whenever $u \geq n(B)$ (the nilpotency index of B). In particular $\hat{B}_u$ and B have the same Kemer points.*
2. *$\hat{B}_u$ is finite dimensional.*

(3) *The nilpotency index of $\hat{B}_u$ is u.*

Proof. (1) By definition of $\hat{B}_u$, $\mathrm{Id}^*(\hat{B}_u) \supseteq \mathrm{Id}^*(B)$. On the other hand, by the fact that the number of symmetric (skew) homogeneous variables that we take is at least the dimension of the symmetric (skew) homogeneous component of $J(B)$, we can construct a surjective map $\phi\colon B' \to B$ such that the variables $\{x_{i_j}^{\dagger_j}\}$ are mapped onto a spanning set of $J(B)$ and $\bar{B}$ is mapped isomorphically. Indeed, $\phi(b\cdot 1_X) = b$, where 1_X represents the empty word in $F\langle X, *\rangle$ and $b \in \bar{B}$. This map is a homomorphism of superalgebras with superinvolution. The $*$-ideal I_1 consists of all evaluations of $\mathrm{Id}^*(B)$ on B' and hence is contained in $\ker(\phi)$. Also the $*$-ideal I_2^u is contained in $\mathrm{Ker}(\phi)$ since $u \geq n(B)$ and $\phi(x_{i_j}^{\dagger_j}) \in J(B)$. By the universal property, there exists a surjective homomorphism of superalgebras with superinvolution $\hat{B}_u \to B$. Hence, $\mathrm{Id}^*(\hat{B}_u) \subseteq \mathrm{Id}^*(B)$ and we are done.

(2) Notice that any element in $\hat{B}_u$ is represented by a sum of elements of the form $b_1 w_1 b_2 w_2 \cdots b_l w_l b_{l+1}$, where $l < u$, $b_k \in \bar{B}$ and $w_k \in \{x_{i_j}^{\dagger_j}\}$ for $k = 1, \ldots, l$. Then $\hat{B}_u$ is of course finite dimensional.

(3) Notice that I_2 generates a radical ideal in $\hat{B}_u$ and since $B'/I_2 \cong \bar{B}$ we have that

$$\hat{B}_u/I_2 \cong B'/(I_1 + I_2^u + I_2) = B'/(I_1 + I_2) \cong (B'/I_2)/I_1 \cong \bar{B}/I_1 = \bar{B}.$$

We see that I_2 generates the radical of $\hat{B}_u$ and therefore its nilpotency index is bounded by u. □

Definition 47. *Let f be a multilinear $*$-polynomial which is not in $\mathrm{Id}^*(A)$. We say that f has the property K if f vanishes on every evaluation with less than $n(A) - 1$ radical substitutions. We say that a finite-dimensional $*$-algebra A has the property K if it satisfies the property with respect to some multilinear $*$-polynomial which is a non-identity of A.*

Proposition 48. *Let A be a finite dimensional $*$-algebra which is minimal (in the sense of Definition 32). Then A has the property K.*

Proof. Assume A has not the property K. This means that any multilinear $*$-polynomial which vanishes on less than $n(A) - 1$ radical evaluations is in $\mathrm{Id}^*(A)$. Consider the algebra $\hat{A}_u$ (from the proposition above). We claim that, for $u = n(A) - 1$, $\hat{A}_u$ is T_2^*-equivalent to A. Once this is accomplished, we would have that the nilpotency index of $\hat{A}_u$ is $n(A) - 1$, a contradiction to the minimality of A.

By construction we have $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(\hat{A}_u)$. For the converse take a $*$-polynomial f which is not in $\mathrm{Id}^*(A)$. Then by assumption, there is a non-zero evaluation $\tilde{f}$ of f on A with less than $n(A) - 1$ radical substitutions (say k). Following this evaluation we refer to the variables of f that get semisimple (radical) values as semisimple (radical) variables, respectively. Let $X = \{x_{i_1}, \ldots, x_{i_m}\}$ be a set of variables. Consider the evaluation $\hat{f}$ of f on $A' = \bar{A} \cdot F\langle X, *\rangle$, where semisimple variables are evaluated as in $\tilde{f}$ whereas the radical variables are evaluated on $\{x_{i_j}\}$, respecting the surjective map $\phi\colon A' \to A$. Our aim is to show that $\tilde{f} \notin I_1 + I_2^u$ because, in this case, we would have $f \notin \mathrm{Id}^*(\hat{A}_u)$ and this will complete the proof.

To show $\tilde{f} \notin I_1 + I_2^u$, notice that f is not in I_1 by definition. Moreover, an element of A' is in I_1 if and only if each one of its multihomogeneous components in the variables $\{x_{i_j}\}$ is in I_1. But by construction $\tilde{f}$ is multihomogeneous of degree $k < n(A) - 1$ in the variables $\{x_{i_j}\}$ whereas any element of $I_2^u \subseteq A'$ is the sum of multihomogeneous elements of degree $\geq n(A) - 1$. We therefore have that $\tilde{f} \in I_1 + I_2^u$ if and only if $\tilde{f} \in I_1$ and we are done. □

Let A be a basic $*$-algebra. By Proposition 48 we have A satisfies the property K with respect to a non-identity f. Moreover, we have A is full with respect to a non-identity h. Our goal now is showing A is full and has property K with respect to the same $*$-polynomial.

Now we give the definition of Phoenix property.

Definition 49. *Let Γ be a T_2^*-ideal. Let P be any property which may be satisfied by $*$-polynomials (e.g. being Kemer). We say that P is Γ-Phoenix (or in short Phoenix) if given a polynomial f having P which is not in Γ and any $f' \in \langle f\rangle_{T_2^*}$, the T_2^*-ideal generated by f, which is not in Γ as well, there exists a polynomial $f'' \in \langle f'\rangle_{T_2^*}$ which is not in Γ and satisfies P.*

We say that P is strictly Phoenix if f' itself satisfies P.

The next lemma shows that property K and the property of being full are "preserved" in a T_2^*-ideal.

Lemma 50. *Let A be a finite dimensional $*$-algebra over F.*

(1) *The property of a non-identity of A of being ν-folds alternating on homogeneous sets of cardinality $d(A)_i$, $i = 0, 1$, is Phoenix.*
(2) *Property K is strictly Phoenix.*

Proof. (1) Let f be a non-identity which is ν-fold alternating on homogeneous sets of cardinality $d(A)_i$, $i \in \{0, 1\}$ (in particular A is full with respect to f). We want to show that if $f' \in \langle f \rangle$ is a non-identity in the T_2^*-ideal generated by f, then there exists a non-identity $f'' \in \langle f' \rangle$ which is ν-fold alternating on homogeneous sets of cardinality $d(A)_i$. In view of Lemma 43, it is sufficient to show that A is full with respect to f'. Remark that, for each $i \in \{0, 1\}$, in at least one alternating set S_i, the evaluations of the corresponding variables must consist of semisimple elements of A in any non-zero evaluation of the $*$-polynomial. This is clear if f' is in the ideal (rather than in the T_2^*-ideal) generated by f. Therefore, we assume that f' is obtained from f by substituting variables x_i's by monomials Z_i's. Clearly, if one of the evaluations in any of the variables of Z_i is radical, then the value of Z_i is radical. Hence in any non-zero evaluation of f' there is an alternating set Δ_i of cardinality $d(A)_i$ in f such that the variables in monomials of f' (corresponding to the variables in Δ_i) assume only semisimple values. Furthermore, each $*$-simple component must be represented in these evaluations: in fact, otherwise we would have a $*$-simple component not represented in the evaluations of the Δ_i's and this is impossible. In conclusion we get that A is full with respect to f'.

(2) If $f' \in \langle f \rangle$ is a non-identity and has less than $n(A) - 1$ radical evaluations, then the same is true for f and hence f' vanishes. □

Finally, the following lemma can be proved following word by word the proof of [4, Proposition 6.6].

Lemma 51. *Let A be a finite-dimensional $*$-algebra, which is full, subdirectly irreducible and satisfying the property K. Let f be a non-identity which is ν-folds alternating on homogeneous sets of cardinality $d(A)_i$, $i \in \{0, 1\}$ and let h be a $*$-polynomial with respect to which A has the property K. Then there is a non-identity in $\langle f \rangle \cap \langle h \rangle$. Consequently there exists a non-identity $\hat{f}$ which is ν-folds alternating on homogeneous sets of cardinality $d(A)_i$, $i = 0, 1$, and with respect to which A has the property K.*

We are in a position to prove the $*$-algebra version of Kemer's Lemma 2.

Lemma 52. *Let $A = B + J$ be a finite dimensional basic $*$-algebra. Then for any integer ν there exists a multilinear non-identity f such which is ν-folds alternating on homogeneous sets of cardinality $d(B_i) = \dim_F(B_i)$, $i = 0, 1$, and $n(A)-1$ sets of homogeneous variables of cardinality $d(B_i)+1$, $i = 0, 1$.*

Proof. By Lemma 51, there exists a multilinear non-identity f with respect to which A is full and has property K. Let us fix a non-zero evaluation $x_i \mapsto \hat{x}_i$ realizing the "full" property. Notice that by Lemma 45, f cannot have more than $n(A) - 1$ radical evaluations, and by property K, f cannot have less than $n(A) - 1$ radical evaluation. Thus, f has precisely $n(A) - 1$ radical substitutions whereas the remaining variables only take semisimple values. Let us denote by $w_1, \ldots, w_{n(A)-1}$ the variables taking radical values (in the evaluation above) and by $\hat{w}_1, \ldots, \hat{w}_{n(A)-1}$ their corresponding values.

Suppose further $B \cong A_1 \times \cdots \times A_q$ (A_i are $*$-simple algebras). We will consider four distinct cases corresponding to whether $q = 1$ or $q > 1$ and whether A has or does not have an identity element.

Case 1: A has an identity element and $q > 1$.

Choose a monomial M in f which does not vanish upon the evaluation above. By multilinearity of f, the monomial M is full (i.e. visits every $*$-simple component of A). Notice that the variables of M which get semisimple evaluations from different $*$-simple components must be separated by radical variables. Next, we may assume that the evaluation of any radical variable w_i is of the form $1_{A_{j(i)}} \hat{w}_i 1_{A_{\tilde{j}(i)}}$, $i = 1, \ldots, n(A) - 1$, where 1_{A_j} is the identity element of the $*$-simple component A_j. Notice that the evaluation remains full.

Consider the radical evaluations which are bordered by pairs of elements $(1_{A_{j(i)}}, 1_{A_{\tilde{j}(i)}})$, where $j(i) \neq \tilde{j}(i)$ (i.e. they belong to different $*$-simple components). Then, since M is full, every $*$-simple component is represented by one of the elements in those pairs.

For $t = 1, \ldots, q$, we fix a variable w_{r_t} whose radical value is $1_{A_{j(r_t)}} \hat{w}_{r_t} 1_{A_{\tilde{j}(r_t)}}$, where

(1) $j(r_t) \neq \tilde{j}(r_t)$ (i.e. different $*$-simple components),
(2) one of the elements $1_{A_{j(r_t)}}$, $1_{A_{\tilde{j}(r_t)}}$ is the identity element of A_t.

We replace now the variables w_{r_t}, $t = 1, \ldots, q$, by the product $y_{r_t} w_{r_t}$ or $w_{r_t}\tilde{y}_{r_t}$ (according to the position of the bordering), where the variables y_{r_t}'s and $\tilde{y}_{r_t}$'s are symmetric variables of even degree. Clearly, by evaluating the variable y_{r_t} by $1_{A_{j(r_t)}}$ (or the variable $\tilde{y}_{r_t}$ by $1_{A_{\tilde{j}(r_t)}}$) the value of the $*$-polynomial remains the same and we obtain a non-identity.

Remember that by Remark 10, we may write the identity element of A_j as $1_{A_j} = e^j_{1,1} + \cdots + e^j_{n_j,n_j}$ or $1_{A_j} = (e_{1,1}, e_{1,1})^j + \cdots + (e_{n_j,n_j}, e_{n_j,n_j})^j$. Thus applying linearity, each $\hat{w}_i$ may be bordered by elements of the form $e^{j(i)}_{k_{j(i)},k_{j(i)}}$ or $(e_{k_{j(i)},k_{j(i)}}, e_{k_{j(i)},k_{j(i)}})^{j(i)}$ with $1 \leq k_{j(i)} \leq n_{j(i)}$. As in the proof of Lemma 43 we can insert in the y_{r_t}'s suitable $*$-polynomials and obtain a $*$-polynomial which is ν-folds alternating on homogeneous sets of cardinality $\dim_F(B_i)$, $i \in \{0, 1\}$.

Consider the variables with radical evaluations which are bordered by variables with evaluations from different $*$-simple components (these include the variables which are bordered by the y_{r_t}). Let χ_i be such a variable of a certain homogeneous degree (according to $i \in \{0, 1\}$). We attach it to a (small) alternating homogeneous set S_i (according with i). We claim that if we alternate this set (of cardinality $d(A)_i + 1$) we obtain a non-identity. Indeed, any non-trivial permutation of χ_i with one of the variables of S_i, keeping the evaluation above, will yield a zero value since the idempotents values in the framed variables of each variable of S_i belong to the same $*$-simple component whereas the pair of idempotents $1_{A_{j(\chi)}}\hat{\chi}_i 1_{A_{\tilde{j}(\chi)}}$ belong to different $*$-simple components. At this point we have constructed the desired number of small sets and some of the big sets.

Finally we need to attach the radical variables w_i whose evaluation is $1_{A_{j(i)}}\hat{w}_i 1_{A_{\tilde{j}(i)}}$ where $j(i) = \tilde{j}(i)$ (i.e. the same $*$-simple component) to some small set S_i. We claim that if we alternate this set (of cardinality $d(A_i) + 1$) we obtain a non-identity. Indeed, any non-trivial permutation represents an evaluation with fewer radical evaluations in the original polynomial which must vanish by property K. This completes the proof in this case.

Case 2: A has an identity element and $q = 1$.

We start with a non-identity f which satisfies property K. Clearly we may multiply f by a symmetric homogeneous variable x_0^+ of even degree and get a non-identity (since x_0^+ may be evaluated by 1). Again by Lemma 43 we may replace x_0^+ by a polynomial h which is ν-folds alternating on homogeneous sets of cardinality $d(A_i)$. Consider the polynomial hf. We attach the radical variables of f to some of the small sets in h. Any non-trivial permutation vanishes because f satisfies property K. This completes the proof in this case.

Case 3: A has no identity element and $q > 1$.

Let $e_0 = 1 - 1_{A_1} - 1_{A_2} - \cdots - 1_{A_q}$ and include e_0 to the set of elements which border the radical values $\hat{w}_j$. A similar argument shows that also here every $*$-simple component $(A_1, \ldots, A_q)$ is represented in one of the bordering pairs $(1_{A_{j(i)}}, 1_{A_{\tilde{j}(i)}})$ where the pairs are different (the point is that one of these pairs may be e_0). Now we complete the proof exactly as in Case 1.

Case 4: A has no identity element and $q = 1$.

For simplicity we write $e_1 = 1_{A_1}$ and $e_0 = 1 - e_1$. Let $f(x_{i_1}, \ldots, x_{i_n})$ be a non-identity of A satisfying property K and let $f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n})$ be a non-zero evaluation for which A is full. If $e_1 f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n}) \neq 0$ or $f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n})e_1 \neq 0$, we proceed as in Case 2. To treat the remaining case we may assume that

$$e_0 f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n}) e_0 \neq 0.$$

By linearity, each one of the radical values $\hat{w}$ may be bordered by one of the pairs $\{(e_0, e_0), (e_0, e_1), (e_1, e_0), (e_1, e_1)\}$. Hence, if we replace the evaluation $\hat{w}$ of w by the corresponding element $e_i\hat{w}e_j$, $i, j = 0, 1$, we get a non-zero value.

Now, if one of the radical values (say $\hat{w}_0$) in $f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n})$ allows a bordering by the pair (e_0, e_1) (and remains non-zero), then replacing w_0 by $w_0 y$ yields a non-identity (since we may evaluate y by e_1). Invoking Lemma 43, we may replace the variable y by a polynomial h with ν-folds alternating (small) homogeneous sets of variables of cardinality $\dim_F(B)_i = \dim_F(A_1)_i$ for every $i \in \{0, 1\}$. Then we attach the radical variable w_0 to a suitable small set. Clearly, the value of any non-trivial permutation of w_0 with any element of the small set is zero since the borderings are different. Similarly, attaching radical variables w whose radical value is $e_i\hat{w}e_j$ where $i \neq j$, to small sets yields zero for any non-trivial permutation and hence the value of the polynomial remains non-zero. The remaining possible values of radical variables are either $e_0\hat{w}e_0$ or $e_1\hat{w}e_1$. Notice that since semisimple values can be bordered only by the pair (e_1, e_1), any alternation of the radical variables whose radical value is $e_0\hat{w}e_0$ with elements of a small set vanishes and again the value of the polynomial remains unchanged. Finally (in order to complete this case, namely where the radical variable w_0 is bordered by the pair (e_0, e_1)) we attach the remaining radical variables (whose values are bordered by (e_1, e_1)) to suitable small sets in h. Here, the value of any non-trivial permutation of w_0 with elements of the small set is zero because of property K (as in Case 1). This settles this case. Obviously, the same holds if the bordering pair of $\hat{w}_0$ above is (e_1, e_0).

The outcome is that we may assume that all radical values may be bordered by either (e_0, e_0) or (e_1, e_1). Under this assumption, notice that all pairs that border radical values are equal, that is are all (e_0, e_0) or all (e_1, e_1). Indeed, if we have of both kinds, we must have a radical value which is bordered by a mixed pair since the semisimple variables can be bordered only by the pair (e_1, e_1) (and in particular they cannot be bordered by mixed pairs). This of course contradicts our assumption.

A similar argument shows that we cannot have radical variables w with values $e_0\hat{w}e_0$ since semisimple values can be bordered only by (e_1, e_1) and this will force the existence of a radical value bordered by mixed idempotents.

The remaining case is the case where all values (radical and semisimple) are bordered by the pair (e_1, e_1) and this contradicts the assumption $e_0 f(\hat{x}_{i_1}, \ldots, \hat{x}_{i_n})e_0 \neq 0$. This completes the proof of the lemma. ☐

Remark 53. *Any non-zero evaluation of such f must consist only of semisimple evaluations in the ν-folds and each one of the big sets must have exactly one radical evaluation.*

Corollary 54. *If A is a finite dimensional basic $*$-algebra, then its Kemer set consists of precisely one point $(\alpha, s) = \mathrm{Par}^*(A)$.*

4.4. Specht's problem for finitely generated $*$-algebras.

Let W be a finitely generated $*$-algebra over F satisfying an ordinary non-trivial identity. The goal of this section is to find a finite generating set for the T_2^*-ideal $\mathrm{Id}^*(W)$. By Theorem 11, there exists a field extension $\bar{F}$ of F and a finite dimensional $*$-algebra A such that

$$\mathrm{Id}^*(W) = \mathrm{Id}^*(A).$$

Let $m = \dim_{\bar{F}} A$. Then clearly W satisfies the (ordinary) Capelli identity c_{m+1} on $2(m+1)$ variables, or equivalently, the finite set of $*$-identities $c_{m+1,i}$ which follow from c_{m+1} by setting its variables to be of homogeneous degree 0 or 1.

Now, observe that any T_2^*-ideal of $*$-identities is generated by at most a countable number of $*$-identities (indeed, for each n the space of multilinear $*$-identities of degree n is finite dimensional). Hence we may take a sequence of $*$-identities $f_1, \ldots, f_n, \ldots$, which generate $\mathrm{Id}^*(W)$.

Let Γ_1 be the T_2^*-ideal generated by the polynomials $c_{m+1,i} \cup \{f_1\}$, …, Γ_n be the T_2^*-ideal generated by the polynomials $c_{m+1,i} \cup \{f_1, \ldots, f_n\}$, and so on. Clearly, since the set $c_{m+1,i}$ is finite, in order to prove the finite generation of $\mathrm{Id}^*(W)$, it is sufficient to show that the ascending chain of graded T_2^*-ideals $\Gamma_1 \subseteq \cdots \subseteq \Gamma_n \subseteq \cdots$ stabilizes.

Now, for each n, the T_2^*-ideal Γ_n corresponds to a finitely generated $*$-algebra (see [27, Theorem 5.2]). Hence, invoking Theorem 11, we may replace each Γ_n by $\mathrm{Id}^*(A_n)$, where A_n is a finite dimensional $*$-algebra over a suitable field extension K_n of F. Clearly, extending the coefficients to a sufficiently large field K, we may assume that all algebras A_n are finite dimensional over an algebraically closed field K.

Our goal is to show that the sequence $\mathrm{Id}^*(A_1) \subseteq \cdots \subseteq \mathrm{Id}^*(A_n) \subseteq \cdots$ stabilizes in $F\langle Y \cup Z, *\rangle$ or equivalently in $K\langle Y \cup Z, *\rangle$.

Consider the Kemer sets of the algebras A_n, $n \geq 1$. Since the sequence of ideals is increasing, the corresponding Kemer sets are monotonically decreasing (recall that this means that for any Kemer point (α, s) of A_{i+1} there is a Kemer point (α', s') of A_i with $(\alpha, s) \preceq (\alpha', s')$). Furthermore, since these sets are finite, there is a subsequence $\{A_{i_j}\}$ whose Kemer points (denoted by E) coincide. Clearly it is sufficient to show that the subsequence $\{\mathrm{Id}^*(A_{i_j})\}$ stabilizes and so, in order to simplify notation, we replace our original sequence $\{\mathrm{Id}^*(A_i)\}$ by the subsequence.

Choose a Kemer point (α, s) in E. By Theorem 34, for any i, we may replace the algebra A_i by a direct product of basic algebras

$$A'_{i,1} \times \cdots \times A'_{i,u_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}},$$

where the $A'_{i,j}$'s correspond to the Kemer point (α, s) and the $\widehat{A_{i,l}}$'s have Kemer index $\neq (\alpha, s)$ (notice that their index may or may not be in E).

Let A be a basic $*$-algebra corresponding to the Kemer point (α, s). Let $A = B + J(A)$ be the Wedderburn-Malcev decomposition of A into the semisimple and radical components. As shown in Section 4.3, we have that $\alpha_i = \dim(B_i)$, for every $i = 0, 1$. Hence, in particular, the dimension of B is determined by α.

By considering the $*$-algebras presented in Theorem 9, the following result is obvious.

Proposition 55. *The number of isomorphism classes of semisimple $*$-algebras of a given dimension is finite.*

Immediately, we get the following corollary.

Corollary 56. *The number of structures on the semisimple components of all basic $*$-algebras which correspond to the Kemer point (α, s) is finite.*

It follows that by passing to a subsequence $\{i_s\}$ we may assume that all basic algebras that appear in the decompositions above and correspond to the Kemer point (α, s) have $*$-isomorphic semisimple components (which we denote by C) and have the same nilpotency index s.

Let now consider the $*$-algebras

$$\widehat{C_i} = \frac{C * K\langle \bar{X} \rangle}{I_i + J},$$

where

- $\bar{X}$ is a set of $*$-variables of cardinality $4(s-1)$,
- $C * K\langle \bar{X} \rangle$ is the algebra of $*$-polynomials in the variables of $\bar{X}$ and coefficients in C,
- I_i is the ideal generated by all evaluations of $\mathrm{Id}^*(A_i)$ on $C * K\langle \bar{X} \rangle$,
- J is the ideal generated by all words in $C * K\langle \bar{X} \rangle$ with s variables from $\bar{X}$.

Proposition 57. *The following facts hold:*

1. *The ideal generated by variables from $\bar{X}$ is nilpotent.*
2. *For any i, the algebra $\widehat{C_i}$ is finite dimensional.*
3. *For any i, $\mathrm{Id}^*(A_i) = \mathrm{Id}^*(\widehat{C_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}})$.*

Proof. (1) By definition of J, the number of variables appearing in a non-zero monomial of the $*$-algebra $\widehat{C_i}$ is bounded by $s-1$, then such an ideal is nilpotent.

(2) Consider a typical non-zero monomial of the $*$-algebra $\widehat{C_i}$. It has the form

$$a_{t_1} x_{t_1} a_{t_2} x_{t_2} \cdots a_{t_r} x_{t_r} a_{t_{r+a}}.$$

Since the set of variables $\bar{X}$ is finite and the index r is bounded by $s-1$, we have that the number of different configurations of these monomials is finite. Between these variables we have the elements a_{t_j}, $j = 1, \ldots, r+1$, which are taken from the finite-dimensional $*$-algebra C. Therefore the $*$-algebra $\widehat{C_i}$ is finite-dimensional.

(3) Since $\mathrm{Id}^*(A'_{i,1} \times \cdots \times A'_{i,u_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) = \mathrm{Id}^*(A_i)$, then $\mathrm{Id}^*(\widehat{A_{i,j}}) \supseteq \mathrm{Id}^*(A_i)$ for $j = 1, \ldots, r_i$. Also, from the definition of I_i we have that $\mathrm{Id}^*(\widehat{C_i}) \supseteq \mathrm{Id}^*(A_i)$ and so $\mathrm{Id}^*(\widehat{C_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) \supseteq \mathrm{Id}^*(A_i)$. On the other hand, first let us show that $\mathrm{Id}^*(\widehat{C_i}) \subseteq \mathrm{Id}^*(A'_{i,j})$ for every $j = 1, \ldots, u_i$. This implies that $\mathrm{Id}^*(\widehat{C_i}) \subseteq \mathrm{Id}^*(A'_{i,1} \times \cdots \times A'_{i,u_i})$ and therefore $\mathrm{Id}^*(\widehat{C_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) \subseteq \mathrm{Id}^*(A'_{i,1} \times \cdots \times A'_{i,u_i} \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) = \mathrm{Id}^*(A_i)$.

To see $\mathrm{Id}^*(\widehat{C_i}) \subseteq \mathrm{Id}^*(A'_{i,j})$ let us take a multilinear $*$-polynomial $f = f(x_{i_1}, \ldots, x_{i_t})$ which is a non-identity of $A'_{i,j}$ and show that f is in fact a non-identity of $\widehat{C_i}$ (the variables x_{i_j} are homogeneous of degree zero or one). Fix a non-vanishing evaluation of f in $A'_{i,j}$ where $x_{j_1} = d_1, \ldots, x_{j_k} = d_k$ ($k \leq s-1$) are the variables with the corresponding radical evaluations and $x_{q_1} = c_1, \ldots, x_{q_l} = c_l$ are the other variables with their semisimple evaluations. Consider the following homomorphism of $*$-algebras

$$\phi \colon C * K\langle \bar{X} \rangle \to A'_{i,j}$$

where C is mapped isomorphically and a subset of k variables $\{\bar{x}_1, \ldots, \bar{x}_k\}$ of $\bar{X}$ (with appropriate $\mathbb{Z}_2$-grading) are mapped onto the set $\{d_1, \ldots, d_k\}$. The other variables from $\bar{X}$ are mapped to zero.

Notice that $(I_i + J) \subseteq \mathrm{Ker}(\phi)$ and hence we obtain a homomorphism of $*$-algebras $\bar{\phi}\colon \widehat{C}_i \to A'_{i,j}$. By construction, the evaluation of the $*$-polynomial $f(x_{i_1}, \dots, x_{i_t})$ on $\widehat{C}_i$, where $x_{q_1} = c_1, \dots, x_{q_l} = c_l$ and $x_{j_1} = \bar{x}_1, \dots, x_{j_k} = \bar{x}_k$, is non-zero and the result follows. □

The following lemma shows how to replace (for a subsequence of indices i_k) the direct product of the basic algebras corresponding to the Kemer point (α, s), $A'_{i,1} \times \cdots A'_{i,u_i}$, by a certain $*$-algebra U such that, for all i:

$$\mathrm{Id}^*(A_i) = \mathrm{Id}^*(U \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}).$$

Lemma 58. *We may replace the algebra $A'_{i,1} \times \cdots A'_{i,u_i}$ by a certain $*$-algebra U as above.*

Proof. At light of the Proposition 57 we are in the following situation. We have a sequence of T_2^*-ideals

$$\mathrm{Id}^*(\widehat{C}_1 \times \widehat{A_{1,1}} \times \cdots \times \widehat{A_{1,r_1}}) \subseteq \cdots \subseteq \mathrm{Id}^*(\widehat{C}_i \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}}) \subseteq \mathrm{Id}^*(\widehat{C}_{i+1} \times \widehat{A_{i+1,1}} \times \cdots \times \widehat{A_{i+1,r_{i+1}}}) \subseteq \cdots .$$

In order to complete the construction of the algebra U we will show that in fact that, by passing to a subsequence, all $\widehat{C}_i$ are $*$-isomorphic. Indeed, since $\mathrm{Id}^*(A_i) \subseteq \mathrm{Id}^*(A_{i+1})$, we have a surjective map φ from $\widehat{C}_i$ to $\widehat{C}_{i+1}$. Since the $*$-algebras $\widehat{C}_i$'s are finite dimensional the result follows. □

At light of Lemma 58, we can continue as follows. Replace the sequence of indices $\{i\}$ by the subsequence $\{i_k\}$. Clearly, it is sufficient to show that the subsequence of T_2^*-ideals $\{\mathrm{Id}^*(A_{i_k})\}$ stabilizes.

Let I be the T_2^*-ideal generated by Kemer polynomials of U which correspond to the Kemer point (α, s). Notice that the polynomials in I are identities of the basic algebras $\widehat{A_{i,l}}$'s. It follows that the Kemer sets of the T_2^*-ideals $\{(\mathrm{Id}^*(A_i) + I)\}$ do not contain the point (α, s) and so they are strictly smaller. By induction we obtain that the following sequence of T_2^*-ideals stabilizes:

$$(\mathrm{Id}^*(A_1) + I) \subseteq (\mathrm{Id}^*(A_2) + I) \subseteq \cdots .$$

For any i, we have that:

1. $\mathrm{Id}^*(A_i) = \mathrm{Id}^*(U \times \widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}})$.
2. $I \subseteq \mathrm{Id}^*(\widehat{A_{i,1}} \times \cdots \times \widehat{A_{i,r_i}})$.

It follows that, for any i, j,

$$I \cap \mathrm{Id}^*(A_i) = I \cap \mathrm{Id}^*(A_j).$$

Combining the last statements we get the following main result.

Theorem 59. *Let W be a finitely generated $*$-algebra. Then $\mathrm{Id}^*(W)$ is finitely generated, as a T_2^*-ideal.*

5. Rationality of the Hilbert series of relatively free $*$-algebras

Let $F\langle Y \cup Z, *\rangle$ be the free $*$-algebra on the set of countable variables $y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots$. In what follows we shall denote by $\bar{F}\langle Y \cup Z, *\rangle$ the free $*$-algebra on the set of finite variables $Y = \{y_1^+, \dots, y_p^+, y_1^-, \dots, y_q^-\}$ and $Z = \{z_1^+, \dots, z_r^+, z_1^-, \dots, z_s^-\}$. Consider a T_2^*-ideal I in $\bar{F}\langle Y \cup Z, *\rangle$ containing at least an ordinary non-trivial identity and let $\bar{F}\langle Y \cup Z, *\rangle/I$ be the corresponding relatively free $*$-algebra.

Remark 60. *Since $\mathrm{Id}^*(\bar{F}\langle Y \cup Z, *\rangle/I) = I$, the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/I$ is PI, i.e. it contains an ordinary non-trivial identity.*

Let Ω_n be the (finite) set of monomials of degree n in the variables of $Y \cup Z$ and let c_n be the dimension of the F-subspace of $\bar{F}\langle Y \cup Z, *\rangle/I$ spanned by the monomials of Ω_n.

Definition 61. *The Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/I$ is given by*

$$\mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/I, t) = \sum_n c_n t^n. \tag{4}$$

The purpose of this section is to prove that the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/I$ is a rational function. For the reader's convenience, we start by recalling some well-known facts of classical PI-theory.

Definition 62. *Let W be an affine PI-algebra over F and let $a_1, \dots a_s$ be a set of generators of W. For a fixed positive integer m, consider B to be the (finite) set of all words in $a_1, \dots a_s$ of length $\leq m$. We say that W has a Shirshov base of length m and of height h if W is spanned (over F) by elements of the form $b_1^{k_1} \cdots b_l^{k_l}$, where $b_i \in B$ and $l \leq h$.*

Moreover, we say that the set B is an *essential Shirshov base of* W (of length m and of height h) if there exists a finite set D such that the elements of the form $d_{i_1} b_{i_1}^{k_1} d_{i_2} \cdots d_{i_l} b_{i_l}^{k_l} d_{i_{l+1}}$ span W, where $d_{i_j} \in D$, $b_{i_j} \in B$ and $l \leq h$.

Theorem 63. *Let W be an affine PI-algebra over F satisfying a multilinear identity of degree m. Then W has a Shirshov base of length m and of height h, where h depends only on m and on the number of generators of W.*

The following result is given in [4, Theorem 7.9].

Theorem 64. *Let C be a commutative algebra over F and let $A = C\langle a_1, \dots a_s\rangle$ be an affine algebra over C. If A has an essential Shirshov base whose elements are integral over C, then A is a finite module over C.*

In [36, Proposition 9.33], the authors proved the following proposition.

Proposition 65. *Any finitely generated module M over a commutative affine algebra A has a rational Hilbert series.*

Finally we recall the following result given in [36, Theorem J].

Theorem 66. *Let $A \subset M_n(F)$ be an algebra and let V be a d-dimensional subalgebra of $M_n(F)$ with an F-basis $a_1, \dots, a_d$ of elements of A. Given an F-linear transformation $T\colon V \to V$, let $\lambda^d + \sum_{i=1}^{d}(-1)^i \gamma_i \lambda^{d-i}$ be the characteristic polynomial of T. For any polynomial $f(x_1, \dots, x_d, Y)$ which is alternating in the variables $x_1, \dots, x_d$, and where Y is a set of variables disjoint from $\{x_1, \dots, x_d\}$, the following equation holds:*

$$\gamma_i f(a_1, \dots, a_d, \hat{Y}) = \sum_{\substack{k_1+\cdots+k_d=i \\ k_i \in \{0,1\}}} f(T^{k_1}(a_1), \dots, T^{k_d}(a_d), \hat{Y}) \tag{5}$$

where $\hat{Y}$ is any evaluation of the variables in Y.

Now we are ready to focus our attention to $*$-algebras.

Proposition 67. *Let $A = A_0 \oplus A_1$ be an affine $*$-algebra satisfying an ordinary non-trivial identity. Then A has an essential Shirshov base of elements of A_0.*

Proof. Since A is a $\mathbb{Z}_2$-graded affine algebra, the result follows from [4, Proposition 7.10]. □

Our next goal is to prove the following lemma.

Lemma 68. *Let S be a set of multilinear $*$-polynomials in $F\langle Y \cup Z, *\rangle$ and let I be the T_2^*-ideal generated by S. Given a $*$-algebra W, we consider $\mathcal{S}$, $\mathcal{I}$ to be the sets of all evaluations on W of the polynomials of S and I, respectively. Then $\mathcal{I} = \langle \mathcal{S} \rangle$ (the $*$-ideal generated by $\mathcal{S}$).*

Proof. In order to prove the lemma, we start by showing that $\mathcal{I}$ is a $*$-ideal of W. Let $a, b \in \mathcal{I}$ and consider the $*$-polynomials p_a and p_b in I with evaluations a and b, respectively. Since I is invariant under all the endomorphism of $F\langle Y \cup Z, *\rangle$ commuting with the superinvolution $*$, we may change variables and assume that p_a and p_b have disjoint sets of variables. Then we get $a+b$ as an evaluation of the $*$-polynomial $p_a + p_b$ and so it follows that $a + b \in \mathcal{I}$. Now let $c \in W$. We may take a variable x disjoint from the variables of p_a and so we get ca and ac as evaluations of xp_a and $p_a x$, respectively. Hence ca and ac belong to $\mathcal{I}$. So far we have proved that $\mathcal{I}$ is an ideal. In order to prove that $\mathcal{I}$ is a graded ideal, we have to show that $\mathcal{I} = (\mathcal{I} \cap W_0) \oplus (\mathcal{I} \cap W_1)$, where W_0 and W_1 are the homogeneous components of W. Now let $a = w_0 + w_1 \in \mathcal{I}$, $a_0 \in W_0$ and $a_1 \in W_1$. Hence there exists a $*$-polynomial $p_a \in I$ with evaluation a. Since I is a graded ideal, we have that $p_a = (p_a)_0 + (p_a)_1$, with $(p_a)_0 \in F\langle Y \cup Z, *\rangle_0$ and $(p_a)_1 \in F\langle Y \cup Z, *\rangle_1$ (the homogeneous components of $F\langle Y \cup Z, *\rangle$). Clearly $(p_a)_i$ takes value w_i, $i = 0, 1$. In conclusion $w_i \in \mathcal{I}$, $i = 0, 1$ and we are done. Finally, let $a \in \mathcal{I}$ and consider the $*$-polynomial $p_a \in I$ with evaluation a. Since I is $*$-invariant, we have that $p_a^* \in I$ (and also $-p_a^* \in I$). It is not difficult to see that one of these polynomials takes value a^*. Therefore $a^* \in \mathcal{I}$ and this implies that $\mathcal{I}$ is a $*$-ideal of W.

In order to complete the proof, it remains to show that $\mathcal{I} = \langle \mathcal{S} \rangle$. Since $S \subset I$, then $\mathcal{S} \subset \mathcal{I}$ and so $\langle \mathcal{S} \rangle \subset \mathcal{I}$. On the other hand, consider the $*$-algebra $\bar{W} = W/\langle \mathcal{S} \rangle$. Since the $*$-polynomials of S are identities of $\bar{W}$, then $I \subset \mathrm{Id}(\bar{W})$. Therefore, all evaluations of I on W are contained in $\langle \mathcal{S} \rangle$, that is, $\mathcal{I} \subset \langle \mathcal{S} \rangle$. □

Remark 69. *Let K be a T_2^*-ideal of $F\langle Y \cup Z, *\rangle$ and let $f \in F\langle Y \cup Z, *\rangle$ be a $*$-polynomial such that $f \notin K$. Let J the T_2^*-ideal generated by f and K. Taking $S = K \cup \{f\}$ and $W = F\langle Y \cup Z, *\rangle/K$ in the previous lemma, we have that J/K is the $*$-ideal of $F\langle Y \cup Z, *\rangle/K$ generated by all the evaluations on $F\langle Y \cup Z, *\rangle/K$ of the polynomial f.*

In order to prove the main result of this section we need the following technical results.

Lemma 70. *Let K and J be T_2^*-ideals of $\bar{F}\langle Y \cup Z, *\rangle$ such that $K \subset J$. Then the following holds:*

$$\mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/K, t) = \mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/J, t) + \mathrm{Hilb}(J/K, t).$$

Proof. Let I be an ideal of an ordinary algebra A. It is well-known that A may be decomposed into the direct sum $(A/I) \oplus I$ (decomposition as vector spaces). Moreover, the ordinary Hilbert series of algebras satisfies the following relation:

$$\mathrm{Hilb}(A, t) = \mathrm{Hilb}(A/I, t) + \mathrm{Hilb}(I, t).$$

Since K is a $*$-ideal of $\bar{F}\langle Y \cup Z, *\rangle$, then K is $*$-ideal of J (here J becomes a $*$-algebra with the operation of $\bar{F}\langle Y \cup Z, *\rangle$ restricted to J). Moreover, since J is a $*$-ideal of $\bar{F}\langle Y \cup Z, *\rangle$, we have that J/K is a $*$-ideal of the $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/K$. Both $\bar{F}\langle Y \cup Z, *\rangle/J$ and J/K are $*$-algebras. Taking $A = \bar{F}\langle Y \cup Z, *\rangle/K$ and $I = J/K$ we get the decomposition

$$\frac{\bar{F}\langle Y \cup Z, *\rangle/K}{J/K} \oplus \frac{J}{K} \cong \frac{\bar{F}\langle Y \cup Z, *\rangle}{J} \oplus \frac{J}{K}.$$

Now the proof is complete since we have:

$$\mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/K, t) = \mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/J, t) + \mathrm{Hilb}(J/K, t).$$

□

Lemma 71. *Let I' and I'' be T_2^*-ideals of $\bar{F}\langle Y \cup Z, *\rangle$. Then the following holds:*

$$\mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I' \cap I''}, t\right) = \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I'}, t\right) + \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I''}, t\right) - \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I' + I''}, t\right).$$

Proof. Taking $J = I' + I''$ and $K = I''$ in the previous lemma, we have:

$$\begin{aligned}\mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/I'', t) &= \mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/(I' + I''), t) + \mathrm{Hilb}(I' + I''/I'', t)\\ &= \mathrm{Hilb}(\bar{F}\langle Y \cup Z, *\rangle/(I' + I''), t) + \mathrm{Hilb}(I'/(I' \cap I''), t).\end{aligned}$$

Now we complete the proof by using again the previous lemma with $J = I'$ and $K = I' \cap I''$:

$$\begin{aligned}\mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I' \cap I''}, t\right) &= \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I'}, t\right) + \mathrm{Hilb}\left(\frac{I'}{I' \cap I''}, t\right)\\ &= \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I'}, t\right) + \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I''}, t\right) - \mathrm{Hilb}\left(\frac{\bar{F}\langle Y \cup Z, *\rangle}{I' + I''}, t\right).\end{aligned}$$

□

Finally we have the key ingredients to prove the main result of this section.

Theorem 72. *Let $\bar{F}\langle Y \cup Z, *\rangle$ be the free $*$-algebra on the set of finite variables $Y = \{y_1^+, \ldots, y_p^+, y_1^-, \ldots, y_q^-\}$ and $Z = \{z_1^+, \ldots, z_r^+, z_1^-, \ldots, z_s^-\}$, where F is an algebraically closed field of characteristic zero. If I is a T_2^*-ideal of $\bar{F}\langle Y \cup Z, *\rangle$ containing at least an ordinary non-trivial identity, then the Hilbert series of the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/I$ is rational.*

Proof. Suppose that the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/I$ is non-rational. By the Specht's property for $*$-algebras (Theorem 59) there exists a T_2^*-ideal K of $\bar{F}\langle Y \cup Z, *\rangle$ containing an ordinary non-trivial identity and that it is maximal among T_2^*-ideals containing ordinary non-trivial identities and having non-rational Hilbert series $\bar{F}\langle Y \cup Z, *\rangle/K$. Indeed, if there is no such an ideal, then we get an infinite ascending chain of T_2^*-ideals containing an ordinary non-trivial identity that does not stabilize and this contradicts the fact that the union of the T_2^*-ideals is finitely generated.

The maximality of K implies that the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/K$ is T_2^*-equivalent to a single basic $*$-algebra A. Indeed, assuming the converse, by Corollary 35, we get that

$$\bar{F}\langle Y \cup Z, *\rangle/K \sim_{T_2^*} A_1 \oplus \cdots \oplus A_m,$$

where $A_1, \dots, A_m$ are basic $*$-algebras, $m \geq 2$ and $\mathrm{Id}^*(A_i) \nsubseteq \mathrm{Id}^*(A_j)$, $1 \leq i, j \leq m$ with $i \neq j$. Thus

$$\mathrm{Id}^*(\bar{F}\langle Y \cup Z, *\rangle / K) = \mathrm{Id}^*(A_1 \oplus \cdots \oplus A_m) = \bigcap_{i=1}^{m} \mathrm{Id}^*(A_i).$$

For every $i \in \{1, \dots, m\}$, clearly $\mathrm{Id}^*(\bar{F}\langle Y \cup Z, *\rangle / K) \subsetneq \mathrm{Id}^*(A_i)$. Let I_i be the evaluation on $\bar{F}\langle Y \cup Z, *\rangle$ of the T_2^*-ideal $\mathrm{Id}^*(A_i)$, $1 \leq i \leq m$. Then I_i properly contains K and their intersection is K. By the maximality of K, the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle / I_i$ is rational for every i and by Lemma 71 we obtain that the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle / K$ is rational, a contradiction. Hence, $m = 1$ and so $\bar{F}\langle Y \cup Z, *\rangle / K$ is T_2^*-equivalent to a single basic $*$-algebra A.

Let f be a Kemer $*$-polynomial of the basic $*$-algebra A (see Theorem 52) and let J be the T_2^*-ideal generated by f and K. Since f is not a $*$-identity of A, then it is not a $*$-identity of $\bar{F}\langle Y \cup Z, *\rangle / K$ and so $K \subsetneq J$. By the maximality of K, the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle / J$ is rational.

Our next goal is to show that the Hilbert series of J/K is a rational function.

Consider the decomposition $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ and let $\{\alpha_1^+, \dots, \alpha_k^+\}$, $\{\alpha_1^-, \dots, \alpha_l^-\}$, $\{\beta_1^+, \dots, \beta_m^+\}$, $\{\beta_1^-, \dots, \beta_n^-\}$ be F-bases of $A_0^+, A_0^-, A_1^+, A_1^-$, respectively. Let

$$\Lambda = \{\lambda_{ij}^+\}_{\substack{1 \leq i \leq p \\ 1 \leq j \leq k}} \cup \{\lambda_{ij}^-\}_{\substack{1 \leq i \leq q \\ 1 \leq j \leq l}} \cup \{\mu_{ij}^+\}_{\substack{1 \leq i \leq r \\ 1 \leq j \leq m}} \cup \{\mu_{ij}^-\}_{\substack{1 \leq i \leq s \\ 1 \leq i \leq n}}$$

be a set of commuting indeterminates which centralize with the elements of A. Now we consider the F-algebra $F\Lambda$. It is not difficult to see that the F-algebra $A \otimes_F F\Lambda$ is a $*$-algebra. The $\mathbb{Z}_2$-grading in A induces a $\mathbb{Z}_2$-grading in $A \otimes_F F\Lambda$:

$$A \otimes_F F\Lambda = (A_0 \otimes_F F\Lambda) \oplus (A_1 \otimes_F F\Lambda).$$

The superinvolution $\bar{*}$ in $A \otimes_F F\Lambda$ is given by $(a \otimes P)^{\bar{*}} = a^* \otimes P$, with $a \in A$, $P \in F\Lambda$ and $*$ the superinvolution defined on A.

Consider the map $\varphi \colon \bar{F}\langle Y \cup Z, *\rangle / K \to A \otimes_F F\Lambda$, induced by

$$y_i^+ \longmapsto \sum_{j=1}^{k} \alpha_j^+ \otimes \lambda_{ij}^+, \qquad y_i^- \longmapsto \sum_{j=1}^{l} \alpha_j^- \otimes \lambda_{ij}^-, \qquad z_i^+ \longmapsto \sum_{j=1}^{m} \beta_j^+ \otimes \mu_{ij}^+, \qquad z_i^- \longmapsto \sum_{j=1}^{n} \beta_j^- \otimes \mu_{ij}^-.$$

Clearly φ is a well-defined homomorphism of $*$-algebras. By definition, we have that φ is also injective. Hence $\mathcal{A} := \mathrm{Im}(\varphi)$ is isomorphic (as $*$-algebras) to $\bar{F}\langle Y \cup Z, *\rangle / K$. Thus, we can see $\bar{F}\langle Y \cup Z, *\rangle / K$ as a $*$-subalgebra of $A \otimes_F F\Lambda$.

Consider the following decompositions as $\mathbb{Z}_2$-graded algebras: $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $A = A_0 \oplus A_1$. Moreover, let $\bar{\mathcal{A}}_0$ and $\bar{A}_0$ be the semisimple parts of $\mathcal{A}_0$ and A_0, respectively. We can embed (embedding of $\mathbb{Z}_2$-graded algebras) $\bar{\mathcal{A}}_0$ into $\mathrm{End}_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda) \cong M_d(F\Lambda)$, where $d = \dim(\bar{A}_0)$, via the regular left $\bar{\mathcal{A}}_0$-action on $\bar{A}_0 \otimes_F F\Lambda$. Notice that each semisimple element $\bar{a} \in \bar{A}_0$ satisfies a Caley-Hamilton identity (characteristic polynomial of $\bar{a}$) of degree d.

By Remark 60 we get $\bar{F}\langle Y \cup Z, *\rangle / K$ is a PI-algebra. Hence Proposition 67 applies and we get that $\bar{F}\langle Y \cup Z, *\rangle / K$ has an essential Shirshov base. As a consequence, also $\mathcal{A}$ has an essential Shirshov base of elements of $\mathcal{A}_0$. Moreover, we may choose generators of $\mathcal{A}_0$ such that the corresponding essential Shirshov base is $\mathcal{B} = \bar{\mathcal{B}} \cup \mathcal{B}_J$ (disjoint union), where $\bar{\mathcal{B}} \subseteq \bar{\mathcal{A}}_0$ and $\mathcal{B}_J \subseteq J(\mathcal{A}_0)$ (the radical part of $\mathcal{A}_0$). Since $J(A) \otimes_F F\Lambda$ is nilpotent, the elements of $\mathcal{B}_J \subseteq J(\mathcal{A}_0) \subseteq J(A) \otimes_F F\Lambda$ are integrals over F.

In view of the embedding $\bar{\mathcal{A}}_0 \hookrightarrow \mathrm{End}_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda)$, each element of $\bar{\mathcal{B}}$ satisfies a characteristic polynomial of degree d with coefficients in $F\Lambda$. Let C be the F-subalgebra of $F\Lambda$ generated by these coefficients. Since $\mathcal{A}$ has unit, we may consider $\mathcal{B}$ having unit and therefore C has it too. Since the essential Shirshov base is finite, C is an affine commutative F-algebra and therefore a Noetherian F-algebra.

Consider the $*$-algebra $\mathcal{A}_C := C[\mathcal{A}]$. Notice that the elements of the essential Shirshov base of $\mathcal{A}$ are integrals over C because the elements of $\mathcal{B}_J$ are integrals over F and we may see F as the F-subspace spanned by the unit 1_C of C. On the other hand, given an element of $\bar{\mathcal{B}}$, by the Caley-Hamilton theorem, this satisfies its characteristic polynomial with coefficients in $F\Lambda$. But, by construction, these coefficients belong to C and so the elements of $\bar{\mathcal{B}}$ are integral over C. Thus, by Theorem 64, $\mathcal{A}_C$ is a finite module over C. By Proposition 65 we obtain that $\mathcal{A}_C$ has a rational Hilbert series.

We come back now to the study of the $*$-ideal J/K of the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle / K$. We denote by $\mathcal{J}$ the image through φ of J/K. By Lemma 68 and Remark 69, $\mathcal{J}$ is the $*$-ideal of $\mathcal{A}$ generated by all the evaluations on $\mathcal{A}$ of the Kemer $*$-polynomial f.

Now, we want to show that $\mathcal{J}$ is a C-submodule of $\mathcal{A}_C$, that is $\mathcal{J}$ is closed under the multiplication of the coefficients of the characteristic polynomials of the elements in $\mathcal{B}$. Given an element $b_0 \in \bar{\mathcal{B}}$ and its characteristic polynomial $\lambda^d + \sum_{i=1}^{d}(-1)^i\gamma_i\lambda^{d-i}$, it is sufficient to show that for the Kemer $*$-polynomial $f(X_d, Y)$, where X_d and Y are disjoint sets of variables and X_d has exactly d variables of degree zero, we have $\gamma_i f(\hat{X}_d, \hat{Y}) \in \mathcal{J}$, for every $i \in \{1, \ldots, d\}$, where $\hat{X}_d = \{\hat{x}_1, \ldots, \hat{x}_d\}$ and $\hat{Y}$ denote any evaluation by elements of $\mathcal{A}$. Since $d = \dim_F(\bar{A}_0) = \dim_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda)$ and $J(A) \otimes_F F\Lambda$ has the same nilpotency index of $J(A)$, we have that $A \otimes_F F\Lambda$ has the same Kemer index of A. Hence Remark 53 implies that the $\hat{x}_i$'s can only assume semisimple values in $\bar{A}_0 \subseteq \bar{A}_0 \otimes_F F\Lambda$, for $1 \leq i \leq d$. Denote these values by $a_1, \ldots, a_d$. Since f is alternating in the set of variables X_d, the value $f(a_1, \ldots, a_d, \hat{Y})$ is zero unless the elements $a_1, \ldots, a_d$ are linearly independent over $F\Lambda$. In this case, since $d = \dim_F(\bar{A}_0) = \dim_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda)$, the set $\{a_1, \ldots, a_d\}$ would be a linear basis of $\bar{A}_0 \otimes_F F\Lambda$ over $F\Lambda$. Finally, since we may see $b_0 \in \bar{\mathcal{B}}$ as an element of $\mathrm{End}_{F\Lambda}(\bar{A}_0 \otimes_F F\Lambda) \cong M_d(F\Lambda)$, we use Lemma 66 and conclude that

$$\gamma_i f(\hat{X}, \hat{Y}) = \gamma_i f(a_1, \ldots, a_d, \hat{Y}) = \sum_{\substack{k_1+\cdots+k_d=i \\ k_i\in\{0,1\}}} f((b_0)^{k_1}(a_1), \ldots, (b_0)^{k_d}(a_d), \hat{Y}) \in \mathcal{J}.$$

Since C is Noetherian, $\mathcal{J}$ is a finitely generated C-module and so, by Proposition 65, $\mathcal{J}$ has a rational Hilbert series. Since $\mathcal{A} = 1_C \cdot \mathcal{A} \subset \mathcal{A}_C$, we have that $\mathcal{J}$ is a common ideal of $\mathcal{A}$ and $\mathcal{A}_C$. We conclude that J/K has a rational Hilbert series.

So far we have proved that $\bar{F}\langle Y \cup Z, *\rangle/J$ and J/K have rational Hilbert series. Now, by applying Lemma 70, we get that the Hilbert series of $\bar{F}\langle Y \cup Z, *\rangle/K$ is rational, which is a contradiction. The contradiction arised from the assumption the Hilbert series of the relatively free $*$-algebra $\bar{F}\langle Y \cup Z, *\rangle/I$ is not a rational function. The proof is complete. □

We conclude this part of the paper concerning superalgebras with superinvolution highlighting that, through some results of this section, combined with the so-called Kemer's lemmas and using an approach similar to that of [2, 4, 38], it could be possible to obtain the proof of Theorem 11. Here we just cited such a result since in order to reach our purposes, it was only necessary to present Kemer's polynomials in a more exhaustive manner.

6. H-module algebras

In this section F will denote a fixed field of characteristic zero and H a Hopf algebra over F. We remand to the books [21, 49, 52, 58] for basic definitions, examples and further information about Hopf algebras. An algebra A is an *H-module algebra* if A is endowed with a left H-action $h \otimes a \mapsto ha$ or, equivalently, with a homomorphism $H \to \mathrm{End}_F(A)$ such that

1. $h(ab) = (h_{(1)}a)(h_{(2)}b)$,
2. $h(1_A) = \varepsilon(h)1_A$, for all $h \in H, a, b \in A$.

Here we use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$, where Δ and ε are the comultiplication and the counit in H, respectively.

An example of a finite dimensional semisimple Hopf algebra is the group algebra FG endowed with comultiplication $\Delta(g) = g \otimes g$, counit $\epsilon(g) = 1$ and antipodal map $S(g) = g^{-1}$ for every $g \in G$. It is well-known that, given any finite dimensional Hopf algebra H, then the dual structure H^* is also a Hopf algebra (see [21, Proposition 4.2.11]). Moreover, we recall that A is a G-graded algebra if and only if A is a (right) FG-comodule algebra (see [49, Example 1.6.7]) and, analogously, if and only if A is a (left) $(FG)^*$-module algebra (see [49, Lemma 1.6.4]). Clearly $(FG)^*$ is finite dimensional (because G is finite) and semisimple by the Maschke's theorem version for Hopf algebras (see [46, Proposition 3]). Then, if F is a field of characteristic zero, G is a finite group, we have that G-graded algebras provide an example of finite dimensional semisimple H-module algebras.

The first goal of this section is to introduce a free object in the class of H-module algebras. Let $F\langle X\rangle$ be the free algebra on the set of countable non-commutative variables $X = \{x_1, x_2, \ldots\}$ and consider the vector space $V = F\langle X\rangle \otimes_F H$. The free H-module algebra over X, denoted by $F^H\langle X\rangle$ is the tensor algebra over V. An element of $F^H\langle X\rangle$ is called *H-polynomial*. In what follows we shall use the notation:

$$x_{i_1}^{h_1} x_{i_2}^{h_2} \cdots x_{i_n}^{h_n} := (x_{i_1} \otimes h_1) \otimes (x_{i_2} \otimes h_2) \otimes \cdots \otimes (x_{i_n} \otimes h_n).$$

Now, let $\{b_1,\dots,b_m\}$ be a basis (as a vector space) of the Hopf algebra H. It follows that $F^H\langle X\rangle$ is isomorphic to the free algebra over F with free formal (non-commutative) generators x^{b_j}, $j\in\{1,\dots,m\}$, $x\in X$. Notice that $F^H\langle X\rangle$ has a structure of left H-module algebra by defining the next H-action:

$$h(x_{i_1}^{h_1}x_{i_2}^{h_2}\cdots x_{i_n}^{h_n})=x_{i_1}^{h_{(1)}h_1}x_{i_2}^{h_{(2)}h_2}\cdots x_{i_n}^{h_{(n)}h_n},$$

where $h_{(1)}\otimes h_{(2)}\otimes\cdots\otimes h_{(n)}$ is the image of $h\in H$ over the comultiplication Δ of H applied $(n-1)$ times. Thus $F^H\langle X\rangle$ is the free H-module algebra on X. This means that, for any H-module algebra W and for every function $\alpha\colon X\to W$, there exists a unique homomorphism of algebras and H-modules (we call this kind of homomorphisms simply H-homomorphisms) $\beta\colon F^H\langle X\rangle\to W$ extending α. In what follows, we shall identify X with the set $\{x^{1_H}|x\in X\}\subset F^H\langle X\rangle$.

Given any H-module algebra W, we say that an H-polynomial $f\in F^H\langle X\rangle$ is an *H-identity* of W if for every H-homomorphism $\varphi\colon F^H\langle X\rangle\to W$ the polynomial f is in the kernel of φ. In other words, $f(x_1,\dots,x_n)\in F^H\langle X\rangle$ is a H-identity of W if and only if $f(w_1,\dots,w_n)=0$, for all $w_1\dots,w_n\in W$. The set $\mathrm{Id}^H(W)$ of all identities satisfied by W is an ideal of $F^H\langle X\rangle$ and it is invariant under all H-endomorphisms of $F^H\langle X\rangle$. The ideals having such a property are called *T^H-ideals*. Moreover, all T^H-ideals are of this form: in fact, it is not difficult to see that, given a T^H-ideal I of $F^H\langle X\rangle$, then $\mathrm{Id}^H(F^H\langle X\rangle/I)=I$.

Two H-module algebras W_1 and W_2 are said to be *T^H-equivalent*, and we write $W_1\sim_{T^H}W_2$, if $\mathrm{Id}^H(W_1)=\mathrm{Id}^H(W_2)$.

Notice that the ordinary identities of any H-module algebra W (i.e. polynomials in the free algebra $F\langle X\rangle$) are H-identities of W (taking the identification $F\langle X\rangle\cong F\langle X\rangle\otimes_F 1_H\subset F^H\langle X\rangle$). Thus $\mathrm{Id}(W)\subset\mathrm{Id}^H(W)$. On the other hand, W does not necessarily have ordinary identities, even if it has H-identities. This is the case, for example, of the free non-commutative algebra W with H-action given by $1_Hw=w$ and $hw=0$, for all $w\in W$ and $h\in H$, $h\neq 1_H$.

Since the field F is of characteristic zero, every T^H-ideal is generated by multilinear H-polynomials, i.e. H-polynomials $f(x_1,\dots,x_n)\in F^H\langle X\rangle$ such that

$$f(x_1,\dots,x_{i-1},\alpha x_i+y,x_{i+1},\dots,x_n)=\alpha f(x_1,\dots,x_n)+f(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n),$$

for every $i\in\{1,\dots,n\}$ and $\alpha\in F$.

Now we recall the following results whose proof can be found in [38, Theorem 4.1].

Theorem 73. *Let W be an affine H-module algebra satisfying an ordinary non-trivial identity. Then there exists a finite dimensional H-module algebra A such that $\mathrm{Id}^H(A)\subset\mathrm{Id}^H(W)$.*

Remark 74. *Let I be a T^H-ideal of $F^H\langle X\rangle$ containing an ordinary non-trivial identity. Since $\mathrm{Id}^H(F^H\langle X\rangle/I)=I$, the relatively free H-module algebra $F^H\langle X\rangle/I$ contains an ordinary non-trivial identity.*

7. Specht's problem for H-module algebras

In this section we shall introduce some definitions and present several results concerning the theory of Specht in the setting of H-module algebras. We refer the reader to the paper [38] by Karasik for more details.

Let us start with the following definition.

Definition 75. *$f(x_1,\dots,x_n,Y)\in F^H\langle X\rangle$ be a multilinear H-polynomial, where Y is a set of variables disjoint from $x_1,\dots,x_n$. We say f is alternating in $\{x_1,\dots,x_n\}$ if there exists a multilinear H-polynomial $h(S_i,R_i):=h(x_1,\dots,x_n,Y)$ such that*

$$f(X)=\sum_{\sigma\in S_n}(-1)^\sigma h(x_{\sigma(1)},\dots,x_{\sigma(n)},Y).$$

Given an H-module algebra W, we say that W satisfies a Capelli identity of rank m if every H-polynomial $f(x_1,\dots,x_n,Y)$ alternating on $x_1,\dots,x_n$ is in $\mathrm{Id}^H(W)$. By Theorem 73, an affine H-module algebra satisfying an ordinary non-trivial identity satisfies a Capelli identity.

Definition 76. *Let W be an H-module algebra satisfying an ordinary non-trivial identity. The H-Kemer index of W is the ordered pair $(\beta(W),\gamma(W))\in\mathbb{N}\times\mathbb{N}$, where*

- $\beta(W)$ *is the maximal integer such that, for every* μ, *there exists a multilinear* H*-polynomial* $f = f(X_1, \ldots, X_\mu, Y) \notin \mathrm{Id}^H(W)$ *which is alternating with respect to the sets* $X_1, \ldots, X_\mu$, *which are all of cardinality* $\beta(W)$,
- $\gamma(W)$ *is the maximal integer such that, for every* μ, *there exists a multilinear* H*-polynomial* $g = g(X_1, \ldots, X_\mu, X'_1, \ldots, X'_{\gamma(W)}, Y) \notin \mathrm{Id}^H(W)$ *which is alternating with respect to the sets* $X_1, \ldots, X_\mu$, $X'_1, \ldots, X'_{\gamma(W)}$, *where* $X_1, \ldots, X_\mu$ *are of cardinality* $\beta(W)$ *and* $X'_1, \ldots, X'_{\gamma(W)}$ *are of cardinality* $\beta(W) + 1$.

The polynomials g *are called* H*-Kemer polynomials of rank* μ.

Let W be a finite dimensional H-module algebra and let $J = J(W)$ be the Jacobson radical of W. In [47] it was proved that J is H-invariant and so W/J is semisimple. By the Wedderburn-Malcev Theorem, W may be decomposed as $W = \bar{W} + J$ (decomposition as vector spaces), where $\bar{W}$ is a semisimple H-module subalgebra of W which is H-isomorphic to W/J. Let d_W be the dimension of $\bar{W}$ and let n_W be the nilpotency index of J. Denote by $\mathrm{Par}(W) = (d_W, n_W - 1)$ the *parameter* of W.

A finite dimensional H-module algebra A is called H*-basic* if there are no finite dimensional H-module algebras $B_1, \ldots, B_s$ such that $\mathrm{Par}(B_i) < \mathrm{Par}(A)$, $i \in \{1, \ldots, s\}$, and $A \sim_{T^H} B_1 \times \cdots \times B_s$. By induction on $\mathrm{Par}(W)$, every finite dimensional H-module algebra W is T^H equivalent to a finite product of H-basic algebras (see Remark 5.8 of [38]).

Kemer's Lemmas 1 and 2 for H-module algebras are given in [38, Lemmas 5.12 and 6.6]. They imply that, if A is H-basic, then $(d_A, n_A - 1) = (\beta(A), \gamma(A))$. It follows that A has an H-Kemer polynomial f having, say at least, n_A alternating sets of variables of cardinality d_A and a total of $n_A - 1$ alternating sets of variables of cardinality $d_A + 1$.

In particular, Kemer's Lemma 2 implies the following remark (see [38, Remark 6.8]).

Remark 77. *Any non-zero evaluation of* f *must consist only of semisimple evaluations in the sets of variables of cardinality* d_A.

Let W be an affine H-module algebra over a field F satisfying an ordinary non-trivial identity. The Representability Theorem for H-modules algebras (see [38]), says that there exists a field extension K of F and a finite dimensional H-module algebra A over K such that W is T^H-equivalent to A. As a consequence, W is T^H-equivalent to a finite product of H-basic algebras $A_1, \ldots, A_m$ over a field extension K of F. Notice that, since $\mathrm{Id}^H(A_1 \oplus \cdots \oplus A_m) = \cap_{i=1}^m \mathrm{Id}^H(A_i)$, we may assume that $\mathrm{Id}^H(A_i) \nsubseteq \mathrm{Id}^H(A_j)$, for every $1 \le i, j \le m$ with $i \neq j$. By passing to the algebraic closure of K, we may assume that the H-basic algebras A_i are finite dimensional over the same field F.

The following theorem is the *Specht property* for H-module algebras (see [38, Theorem 1.4]).

Theorem 78. *Let* W *be an affine* H*-module algebra satisfying an ordinary non-trivial identity. If* $I_1 \subset I_2 \subset \cdots$ *is an ascending chain of* T^H*-ideals of* W *containing an ordinary non-trivial identity, then the chain stabilizes.*

8. Rationality of the Hilbert series of relatively free H-module algebras

Let H be a Hopf algebra over F with basis $\{b_1, \ldots, b_m\}$. We denote by $F^H\langle X_r\rangle$ the free H-module algebra on the set of finite variables $X_r = \{x_1, \ldots, x_r\}$. Given a T^H-ideal I in $F^H\langle X_r\rangle$, then $F^H\langle X_r\rangle/I$ is the corresponding relatively free H-module algebra. Write Ω_n to denote the (finite) set of monomials of degree n on the variables $x_i^{b_j}$, $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, r\}$. If c_n is the dimension of the F-subspace of $F^H\langle X_r\rangle/I$ spanned by the monomials of Ω_n, then the Hilbert series of $F^H\langle X_r\rangle/I$ with respect to the generators $\{x_i^{b_j}\}_{ij}$ is defined by

$$\mathrm{Hilb}(F^H\langle X_r\rangle/I, t) = \sum_n c_n t^n.$$

Given any T^H-ideal I in $F^H\langle X_r\rangle$, it is convenient to view I as the evaluation on $F^H\langle X_r\rangle$ of a T^H-ideal $\mathcal{I}$ of the free H-module algebra $F^H\langle X\rangle$. As already mentioned in Section 6, every T^H-ideal is generated by multilinear H-polynomials. Unfortunately, passing from $\mathcal{I}$ to I (by evaluation) the multilinearity condition could no longer be true.

The main goal of this section is to show that, in case H is a semisimple Hopf F-algebra, then the Hilbert series of $F^H\langle X_r\rangle/I$ is a rational function. Let W be an H-module algebra and consider the T^H-ideal of the identities $\mathrm{Id}^H(W)$ satisfied by W. Since $\mathrm{char} F = 0$, we have $\mathrm{Id}^H(W) = \mathrm{Id}^H(W \otimes_F \bar{F})$, where $\bar{F}$ is the algebraic closure of F. This means that the ideal of identities of $W_{\bar{F}}$ over $\bar{F}$ is the span (over $\bar{F}$) of the T^H-ideal of identities of W over F. Therefore the Hilbert series remains the same when passing to the algebraic closure of F. From now on, we assume that $F = \bar{F}$.

We start by proving the following technical result.

Lemma 79. *Let $A \subseteq M_n(F)$ be an algebra which is a (left) H-module and let V be a d-dimensional subalgebra of $M_n(F)$ with an F-basis $a_1, \ldots, a_d$ of elements of A. Given an F-linear transformation $T\colon V \to V$, let $\lambda^d + \sum_{i=1}^{d}(-1)^i \gamma_i \lambda^{d-i}$ be its characteristic polynomial. Then for any H-polynomial $f(x_1, \ldots, x_d)$ which is alternating in the variables $x_1, \ldots, x_d$, the following equation holds:*

$$\gamma_i f(a_1, \ldots, a_d) = \sum_{\substack{k_1+\cdots+k_d=i \\ k_i \in \{0,1\}}} f(T^{k_1}(a_1), \ldots, T^{k_d}(a_d)).$$

Proof. We first show that the following equation holds:

$$\det(T) f(a_1, \ldots, a_d) = f(T(a_1), \ldots, T(a_d)).$$

Suppose that $T(a_j) = \sum_{i=1}^{d} c_{ij} a_i$, with $c_{ij} \in F$, $1 \leq i, j \leq d$. Since the H-action is linear, then $h \cdot (T(a_j)) = \sum_{i=1}^{d} c_{ij} h a_i$. Also, since $f(x_1, \ldots, x_d)$ is an alternating multilinear H-polynomial and T is an F-linear transformation, we get that

$$\begin{aligned} f(T(a_1), \ldots, T(a_d)) &= f\left(\sum_{i=1}^{d} c_{i1} a_i, \ldots, \sum_{i=1}^{d} c_{id} a_i\right) = \sum_{\sigma \in S_d} c_{\sigma(1),1} \cdots c_{\sigma(d),d} f(a_{\sigma(1)}, \ldots, a_{\sigma(d)}) \\ &= \sum_{\sigma \in S_d} (-1)^{\sigma} c_{\sigma(1),1} \cdots c_{\sigma(d),d} f(a_1, \ldots, a_d) = \det(T) f(a_1, \ldots, a_d). \end{aligned}$$

Here S_d is the symmetric group of order d.

Using the F-linear transformation $\lambda I_d - T$ in place of T, we get:

$$\det(\lambda I_d - T) f(a_1, \ldots, a_d) = f((\lambda I_d - T)(a_1), \ldots, (\lambda I_d - T)(a_d)).$$

Now we remark that

$$\begin{aligned} f((\lambda I_d - T)(a_1), \ldots, (\lambda I_d - T)(a_d)) &= f(\lambda a_1 - T(a_1), \ldots, \lambda a_d - T(a_d)) \\ &= \lambda^d f(a_1, \ldots, a_d) - \lambda^{d-1} \sum_{k_1+\cdots+k_d=1} f(T^{k_1}(a_1), \ldots, T^{k_d}(a_d)) \\ &\quad + \lambda^{d-2} \sum_{k_1+\cdots+k_d=2} f(T^{k_1}(a_1), \ldots, T^{k_d}(a_d)) + \cdots \\ &\quad (-1)^d \lambda^0 f(T(a_1), \ldots, T(a_d)), \end{aligned}$$

with $k_i \in \{0, 1\}$ for all $i \in \{1, \ldots, d\}$. On the other hand,

$$\det(\lambda I_d - T) = \lambda^d + \sum_{i=1}^{d} (-1)^i \gamma_i \lambda^{d-i},$$

the characteristic polynomial of T with coefficients $\gamma_i \in F$, $1 \leq i \leq d$. In conclusion we get

$$\gamma_i f(a_1, \ldots, a_d) = \sum_{\substack{k_1+\cdots+k_d=i \\ k_i \in \{0,1\}}} f(T^{k_1}(a_1), \ldots, T^{k_d}(a_d)).$$

□

Lemma 80. *Let S be a set of H-polynomials in $F^H\langle X\rangle$ and let I be the T^H-ideal generated by S. Given an H-module algebra W, consider $\mathcal{S}, \mathcal{I}$ to be the sets of all evaluations on W of the polynomials of S and I, respectively. Then $\mathcal{I} = \langle \mathcal{S} \rangle$ (the ideal generated by $\mathcal{S}$).*

Proof. The lemma can be proved following word by word the proof of Lemma 68. □

Remark 81. *Let K be a T^H-ideal of $F^H\langle X\rangle$ and let $f \in F^H\langle X\rangle$ be an H-polynomial such that $f \notin K$. Let J be the T^H-ideal generated by f and K. Taking $S = K \cup \{f\}$ and $W = F^H\langle X\rangle/K$ in the previous lemma, we have J/K is the ideal of $F^H\langle X\rangle/K$ generated by all evaluations on $F^H\langle X\rangle/K$ of the polynomial f.*

Let J be a T^H-ideal of $F^H\langle X_r\rangle$. In particular, J is an ideal of $F^H\langle X_r\rangle$ (as F-algebra). Then J becomes an H-module algebra with the operations of $F^H\langle X_r\rangle$ restricted to J.

The following lemmas can be proved by using the same arguments employed in the corresponding results of Section 5 (Lemmas 70 and 71).

Lemma 82. *Let K and J be T^H-ideals of $F^H\langle X_r\rangle$ such that $K \subset J$. Then the following holds:*

$$\mathrm{Hilb}(F^H\langle X_r\rangle/K, t) = \mathrm{Hilb}(F^H\langle X_r\rangle/J, t) + \mathrm{Hilb}(J/K, t).$$

Lemma 83. *Let I' and I'' be T^H-ideals of $F^H\langle X_r\rangle$. Then the following holds:*

$$\mathrm{Hilb}(F^H\langle X_r\rangle/(I' \cap I''), t)$$
$$= \mathrm{Hilb}(F^H\langle X_r\rangle/I', t) + \mathrm{Hilb}(F^H\langle X_r\rangle/I'', t) - \mathrm{Hilb}(F^H\langle X_r\rangle/(I' + I''), t).$$

Finally we are in a position to prove the main theorem of this section.

Theorem 84. *Let $F^H\langle X_r\rangle$ be the free H-module algebra on the set of variables $X_r = \{x_1, \cdots, x_r\}$, where H is a finite dimensional semisimple Hopf algebra and F is a field of characteristic zero. If I is a T^H-ideal of $F^H\langle X_r\rangle$ containing at least an ordinary non-trivial identity, then the Hilbert series of the relatively free H-module algebra $F^H\langle X_r\rangle/I$ is rational.*

Proof. The proof is very similar to the one given for the analogous result in the setting of $*$-algebras (Theorem 72). For this reason we will give here just a sketch of it.

Suppose that the Hilbert series of $F^H\langle X_r\rangle/I$ is non-rational. By the Specht's property for H-module algebras (Theorem 78) there exists a T^H-ideal K of $F^H\langle X_r\rangle$ containing an ordinary non-trivial identity and that it is maximal among T^H-ideals containing ordinary non-trivial identities and having non-rational Hilbert series $F^H\langle X_r\rangle/K$ is non-rational.

The maximality of K implies that the relatively free H-module algebra $F^H\langle X_r\rangle/K$ is T^H-equivalent to a single H-basic H-module algebra A. To this end we just need to use the Representability Theorem for H-module algebras and Lemma 83.

Now let f be a H-Kemer polynomial of the H-basic H-module algebra A and let J be the T^H-ideal generated by f and K. Since f is not an H-identity of A, then f is not an H-identity of $F^H\langle X_r\rangle/K$, and hence, $K \subsetneq J$. By the maximality of K, the Hilbert series of $F^H\langle X_r\rangle/J$ is rational.

In order to complete the proof we need to show that the Hilbert series of J/K is a rational function. In fact, once this is accomplished, we will have that $F^H\langle X_r\rangle/J$ and J/K have rational Hilbert series. Then by Lemma 82, the Hilbert series of $F^H\langle X_r\rangle/K$ is rational, which is a contradiction. The contradiction arises from having assumed that the Hilbert series of the relatively free H-module algebra $F^H\langle X_r\rangle/I$ is not a rational function.

From now on, our only goal is to prove that the Hilbert series of J/K is a rational function.

Suppose that $\{\alpha_1, \dots, \alpha_l\}$ is an F-basis of A and let $\Lambda = \{\lambda_{ij} \colon 1 \le i \le r, 1 \le j \le l\}$ be a set of commuting indeterminates centralizing with the elements of A. Consider the F-algebra $F\Lambda$ endowed with a formal H-action. It is not difficult to see that $A \otimes_F F\Lambda$ is an H-module algebra. Now, consider the H-homomorphism $\varphi \colon F^H\langle X_r\rangle/K \to A \otimes_F F\Lambda$, induced, for any $h \in H$, by

$$x_i^h \longmapsto \sum_{j=1}^{l} h(\alpha_j) \otimes \lambda_{ij}^h.$$

It is not difficult to see that φ is injective. Hence we get that $\mathcal{A} := \mathrm{Im}(\varphi)$ is H-isomorphic (isomorphic as H-module algebras) to $F^H\langle X_r\rangle/K$. Thus, we can see $F^H\langle X_r\rangle/K$ as a subalgebra of $A \otimes_F F\Lambda$.

Let $\bar{A}$ be the H-invariant semisimple part of A. We can embed (embedding of F-algebras) $\bar{A}$ into $\mathrm{End}_F(\bar{A}) \cong M_d(F)$, where $d = \dim(\bar{A})$, via the regular left $\bar{A}$-action on $\bar{A}$. This induce an embedding $\bar{A} \otimes_F F\Lambda$ into $\mathrm{End}_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$ via the regular action. Notice that each semisimple element $\bar{a} \in \bar{A}$ satisfies a Caley-Hamilton identity (characteristic polynomial of $\bar{a}$) of degree d.

Since A may be decomposed into the direct sum $\bar{A} \oplus J(A)$ where $J(A)$ is the Jacobson radical of A, we may decompose $\mathcal{A}$ into the direct sum $\bar{\mathcal{A}} \oplus \mathcal{A}_J$ where $\bar{\mathcal{A}} \subset \bar{A} \otimes_F F\Lambda \hookrightarrow \mathrm{End}_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$ and $\mathcal{A}_J \subset J(A) \otimes_F F\Lambda$. We shall call $\bar{\mathcal{A}}$ the semisimple part of $\mathcal{A}$ and $\mathcal{A}_J$ the radical part of $\mathcal{A}$.

Remark 74 implies that $F^H\langle X_r\rangle/K$ is a PI-algebra. By Theorem 63, $F^H\langle X_r\rangle/K$ has a Shirshov base, then $\mathcal{A}$ has a Shirshov base. Moreover, we may choose generators of $\mathcal{A}$ such that the corresponding Shirshov base is $\mathcal{B} = \bar{\mathcal{B}} \cup \mathcal{B}_J$ (disjoint union), where $\bar{\mathcal{B}} \subset \bar{\mathcal{A}}$ and $\mathcal{B}_J \subset \mathcal{A}_J$. In fact, if we choose generators $b_1, \ldots, b_s$ of $\mathcal{A}$ either from $\bar{A}$ or $\mathcal{A}_J$, a basic element $b_{i_1} b_{i_2} \cdots b_{i_t}$ belongs to $\bar{\mathcal{B}}$ if and only if $b_{i_j} \in \bar{A}$ for all $j \in \{1, \ldots, t\}$. Since $J(A) \otimes_F F\Lambda$ is nilpotent, the elements of $\mathcal{B}_J$ are integrals over F.

In view of the embedding $\bar{A} \hookrightarrow \mathrm{End}_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$, each element of $\bar{\mathcal{B}}$ satisfies a characteristic polynomial of degree d with coefficients in $F\Lambda$. Let C the F-subalgebra of $F\Lambda$ generated by these coefficients. Since $\mathcal{A}$ has unit, we may consider $\mathcal{B}$ having unit, and therefore C has unit. Since the Shirshov base is finite, C is an affine commutative F-algebra and therefore a Noetherian F-algebra.

Consider the H-module C-algebra $\mathcal{A}_C := C[\mathcal{A}]$. Notice that the elements of the Shirshov base of $\mathcal{A}$ are integrals over C because the elements of $\mathcal{B}_J$ are integrals over F and we may see F as the F-subspace spanned by the unit 1_C of C. On the other hand, given an element of $\bar{\mathcal{B}}$, by the Caley-Hamilton theorem this satisfies its characteristic polynomial with coefficients in $F\Lambda$. But, by construction, these coefficients belongs to C, then the elements of $\bar{\mathcal{B}}$ are integral over C. Thus, by Theorem 64 $\mathcal{A}_C$ is a finite module over C. Then $\mathcal{A}_C$ has a rational Hilbert series by Proposition 65.

We come back now to the study of the ideal J/K of the relatively free H-module algebra $F^H\langle X_r\rangle/K$. We denote by $\mathcal{J}$ the image by φ of J/K. By Lemma 80 and Remark 81, $\mathcal{J}$ is the ideal of $\mathcal{A}$ generated by all the evaluations on $\mathcal{A}$ of the H-Kemer polynomial f. We will show that $\mathcal{J}$ is a C-submodule of $\mathcal{A}_C$, that is, we show that $\mathcal{J}$ is closed under the multiplication of the coefficients of the characteristic polynomials of the elements in $\mathcal{B}$. So, given an element $b_0 \in \bar{\mathcal{B}}$ and $\lambda^d + \sum_{i=1}^d (-1)^i \gamma_i \lambda^{d-i}$ its characteristic polynomial, it is sufficient to show that for the H-Kemer polynomial $f(X_d, Y)$, where X_d and Y are sets of disjoint variables and X_d has d elements, we have $\gamma_i f(\hat{X}_d, \hat{Y}) \in \mathcal{J}$, where $\hat{X}_d = \{\hat{x}_1, \ldots, \hat{x}_d\}$ and $\bar{Y}$ denote an evaluation of elements of $\mathcal{A}$.

In view of the embedding $\mathcal{A} \subset A \otimes_F F\Lambda \subset (\bar{A} \otimes_F F\Lambda) \oplus (J(A) \otimes_F F\Lambda)$, an element $v \in \hat{X}_d \cup \hat{Y}$ can be written as $v = \bar{v} + v_J$ where $\bar{v} \in \bar{A} \otimes_F F\Lambda$ and $v_J \in J(A) \otimes_F F\Lambda$. Since $d = \dim_F(\bar{A}) = \dim_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$ and $J(A) \otimes_F F\Lambda$ has the same nilpotency index as $J(A)$, then $A \otimes_F F\Lambda$ has the same H-Kemer index as A. If we denote by a_i the semisimple part of $\hat{x}_i$ and by c_i the radical part of $\hat{x}_i$ for $1 \leq i \leq d$, Remark 77 implies $f(\hat{X}_d, \hat{Y}) = f(a_1, \ldots, a_d, \hat{Y})$. Since f is alternating in the set of variables X_d, the value $f(a_1, \ldots, a_d, \hat{Y})$ is zero unless the elements $a_1, \ldots, a_d$ are linearly independent over $F\Lambda$ and since $d = \dim_F(\bar{A}) = \dim_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$, the set $\{a_1, \ldots, a_d\}$ is a linear basis of $\bar{A} \otimes_F F\Lambda$ over $F\Lambda$.

Since we may see $b_0 \in \bar{\mathcal{B}}$ as an element of $\mathrm{End}_{F\Lambda}(\bar{A} \otimes_F F\Lambda)$, by Lemma 79 we get that

$$\gamma_i f(\hat{X}, \hat{Y}) = \gamma_i f(a_1, \ldots, a_d, \hat{Y}) = \sum_{\substack{k_1 + \cdots + k_d = i \\ k_i \in \{0,1\}}} f((b_0)^{k_1}(a_1), \ldots, (b_0)^{k_d}(a_d), \hat{Y}) \in \mathcal{J}.$$

Since C is Noetherian, $\mathcal{J}$ is a finitely generated C-module as well and again by Proposition 65, $\mathcal{J}$ has a rational Hilbert series. Since $\mathcal{A} = 1_C \cdot \mathcal{A} \subset \mathcal{A}_C$, we have that $\mathcal{J}$ is a common ideal of $\mathcal{A}$ and $\mathcal{A}_C$. We conclude that J/K has a rational Hilbert series. □

9. **Final remarks**

In the last section we have proved the rationality of the Hilbert series

$$\mathrm{Hilb}(F^H\langle X_r\rangle/(F^H\langle X_r\rangle \cap \mathrm{Id}^H(A)), t)$$

of the relatively free H-algebra of a certain H-module algebra A, where H is finite dimensional and semisimple. If we specialize $H = FG$, the group algebra generated by G over F, and we take G being finite (say $G = \{g_1, \ldots, g_s\}$) and abelian, then any finite dimensional H-module algebra A is a G-graded algebra too, and $\mathrm{Hilb}(F^H\langle X_r\rangle/(F^H\langle X_r\rangle \cap \mathrm{Id}^H(A)), t)$ takes a very strong PI "flavour" as we will see at the very end of this section.

First, keeping in mind our notation for G-graded algebras, it is worth recalling the following fact:

$$f(x_1, \ldots, x_n) \in \mathrm{Id}(A) \text{ if and only if } f(x_1^{g_1}, \ldots, x_1^{g_s}, \ldots, x_n^{g_1}, \ldots, x_n^{g_s}) \in \mathrm{Id}^G(A).$$

Then the knowledge of the behaviour of $\mathrm{Hilb}(F^H\langle X_r\rangle/(F^H\langle X_r\rangle \cap \mathrm{Id}^H(A)), t)$ gives good information about the behaviour of $\mathrm{Hilb}(F\langle X_r\rangle/(F\langle X_r\rangle \cap \mathrm{Id}(A)), t)$. Because of this, studying generalizations of the Hilbert series could provide us one more tool in the global understanding of the ordinary polynomial identities of a given algebra that is a very hard task.

Strictly related to the ideas besides the Hilbert series, we can see the growth of an algebra. In particular, we would like to spend some words toward the classical tool of the *Gelfand-Kirillov dimension* (sometimes GK dimension). Let A be an F-algebra generated by a finite set $\{a_1, \ldots, a_m\}$ and consider

$$V^n = \mathrm{span_F}\langle a_{i_1} \cdots a_{i_n} \mid i_j = 1, \ldots, m\rangle, \; n = 1, 2, \ldots.$$

Here we assume $V^0 = F$. The function of the non-negative argument n

$$g_V(n) = \dim_F(V^0 + V^1 + \cdots + V^n),\ n = 1, 2, \ldots,$$

is called the *growth function* of A (with respect to $V = V^1$). The Gelfand-Kirillov dimension of A is defined by

$$\mathrm{GKdim}(A) = \limsup_{n\to\infty}(\log_n g_V(n)) = \limsup_{n\to\infty} \frac{\log g_V(n)}{\log n}.$$

Notice that the definition of Gelfand-Kirillov dimension is independent of the generating space (see for example [42], Lemma 1.1). Hence we are allowed to remove the dependence on V from the symbol $g_V(n)$. Among the large amount of interesting properties that the GK dimension of an algebra carries inside, we would like to highlight the fact that the GK dimension of a finitely generated commutative algebra, for instance, coincides with its Krull dimension. For further details and results about GK dimension of algebras we refer to the books [42] by Krause and Lenagan and [48] by McConnell and Robson.

There are several results towards GK dimension of PI-algebras (see Section 10 of [42]) but in this section we want to discuss those ones regarding the GK dimension of the relatively free algebra of a PI-algebra. Let A be a PI-algebra and $r \geq 1$ an integer. We denote by $\mathrm{GKdim}_r(A)$ the GK dimension of $F\langle X_r\rangle/(F\langle X_r\rangle \cap \mathrm{Id}(A))$. In [10] the author studied several properties of $\mathrm{GKdim}_r(A)$. In particular, it can be proved $\mathrm{GKdim}_r(A)$ is defined by the complexity type of the algebra A or by a set of semidirect products of matrix algebras over the ring of polynomials from the variety generated by A. See also the paper [13] by Berele for explicit computations of the GK dimension of some remarkable PI-algebras or the surveys [23] by Drensky and [18] by Centrone.

In what follow, we shall introduce an H-module algebra version of the GK dimension of the relatively free H-module algebra. Let A be an finitely generated H-module algebra over F, where H is a finite dimensional Hopf algebra over F with F-basis $\{b_1, \ldots, b_m\}$. We shall denote with the symbol $F_k^H(A)$ the *relatively free H-module algebra of A in k variables*, that is,

$$F_k^H(A) := F^H\langle x_1, \ldots, x_k\rangle / (F^H\langle x_1, \ldots, x_k\rangle \cap \mathrm{Id}^H(A)).$$

Recall that $F^H\langle x_1, \ldots, x_k\rangle$ is isomorphic to the free algebra over F with free formal generators $x_i^{b_j}$, where $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, m\}$. Thus,

$$F_k^H(A) = \frac{F^H\left\langle x_1^{b_1}, \ldots, x_1^{b_m}, \ldots, x_k^{b_1}, \ldots, x_k^{b_m}\right\rangle}{F^H\left\langle x_1^{b_1}, \ldots, x_1^{b_m}, \ldots, x_k^{b_1}, \ldots, x_k^{b_m}\right\rangle \cap \mathrm{Id}^H(A)}.$$

Definition 85 (H-Gelfand-Kirillov dimension in k variables)**.** *Let H be a finite dimensional Hopf algebra over a field F with F-basis $\{b_1, \ldots, b_m\}$ and A a finitely generated H-module algebra over F. The H-Gelfand-Kirillov dimension of A in k variables is*

$$\mathrm{GKdim}_k^H(A) := \mathrm{GKdim}(F_k^H(A))$$

If the Hilbert series of a PI-algebra is rational, the following proposition tells us what are the possible behaviors of the growth of the algebra and how do its Hilbert series figures out.

Proposition 86. *Let $\mathrm{Hilb}(A, t)$ be the Hilbert series of an infinite dimensional algebra A, and assume that it is a rational function. Then either A has exponential growth, or $\mathrm{GKdim}(A) = d \in \mathbb{N}$ and*

$$\mathrm{Hilb}(A, t) = \frac{p(t)}{(1 - t^s)^d},$$

for some polynomial $p(t)$ with $p(1) \neq 0$.

By a result of Berele (see [14]), if A is an affine PI-algebra over an infinite field, then $\mathrm{GKdim}(A) < \infty$. Thus, the next result is a consequence of Theorem 84 and Proposition 86.

Theorem 87. *Let H be a semisimple finite dimensional semisimple Hopf algebra over field F of characteristic 0 and W be an affine H-module algebra over F satisfying an ordinary polynomial identity. Then the H-GK dimension of W in k variables is an integer d and*

$$\mathrm{Hilb}(W, t) = \frac{p(t)}{(1 - t^s)^d}$$

for some polynomial $p(t)$ with $p(1) \neq 0$.

Denote by P_n^H the space of all multilinear H-polynomials in $x_1, ..., x_n$, $n \in \mathbb{N}$, i.e.,

$$P_n^H := \langle x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \cdots x_{\sigma(n)}^{h_n} | h_i \in H, \sigma \in S_n \rangle \subseteq F^H\langle X \rangle.$$

The symmetric group S_n acts on the left on the space P_n^H by $\sigma(x_i^h) = x_{\sigma(i)}^h$ if $\sigma \in S_n$. Notice that the vector space $P_n^H \cap \mathrm{Id}^H(A)$ is stable under this S_n action, hence $P_n^H(A) := P_n^H/(P_n^H \cap \mathrm{Id}^H(A))$ is a left S_n-module. This leads us to consider the S_n-character of $P_n^H(A)$, namely $\chi_n^H(A)$, which is called *n-th cocharacter of polynomial H-identities* or the *n-th H-cocharacter* of A. By the classical theory of representations of the symmetric group (see for instance the book by Sagan [53]), the irreducible S_n-characters are in one-to-one correspondence with the partitions of the non-negative integer n (which carries a *Young Tableau*) because the ground field is of characteristic 0. In particular, if χ_λ denotes the irreducible S_n-character corresponding to the partition λ, then we are allowed to write

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m_\lambda^H \chi_\lambda,$$

where $m_\lambda^H \geq 0$ is the multiplicity of the irreducible character χ_λ in the decomposition of $\chi_n^H(A)$. Moreover the non-negative integer

$$c_n^H(A) := \dim_F(P_n^H(A))$$

is called the *n-th codimension of polynomial H-identities* or the *n-th H-codimension* of A. We shall also refer to the sequences $\{\chi_n^H(A)\}_{n\geq 0}$, $\{c_n^H(A)\}_{n\geq 0}$ as the *H-cocharacter sequence of* A and the *H-codimension sequence of* A, respectively.

It was proved independently by Berele (see [14]) and Drensky (see [22]) the following relation between the cocharacter sequence and Hilbert series of the relatively free algebra of a PI-algebra. Here we write the H-module algebra version that can be proved analogously.

Theorem 88. *If A is a PI-algebra and $\chi_n^H(A) = \sum_{\lambda\vdash n} m_\lambda^H \chi_\lambda$, then*

$$\mathrm{Hilb}(F^H\langle X_r\rangle/(F^H\langle X_r\rangle \cap \mathrm{Id}^H(A)), t) = \sum_{n\geq 0} \sum_{\lambda\vdash n} m_\lambda^H S_\lambda(t),$$

where $S_\lambda(t)$ is the Schur function of shape λ and content $\{t_1, \ldots, t_k\}$ and the summation in the Hilbert series runs on the partitions of height smaller than r.

Given an H-module algebra A, if the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n^H(A)}$$

exists, we shall call it *H PI-exponent* of A and we shall denote it by $\exp^H(A)$.

As we said above, if we specialize H with the dual algebra of the group algebra FG, where G is a finite abelian group, we get the notion of *G-graded identities, codimension, exponent*, etc. The existence of the exponent in the graded case when A is supposed to be associative and over a field of characteristic 0, has been studied by several authors as Giambruno and Zaicev in [30] when G is the trivial group, Benanti, Giambruno and Pipitone in [12] when $G = \mathbb{Z}_2$, by Aljadeff, Giambruno and La Mattina in [3] in case A is affine and G is abelian, by Giambruno and La Mattina (see [28]) if A is any G-graded algebra and G is abelian and in general by Aljadeff and Giambruno in [1].

In the general case of an H-algebra only partial results are known about the existence of such an exponent. If H is finite dimensional and semisimple acting on an associative algebra over a field of characteristic 0, then Karasik proved in [38] that the H-exponent exists and is an integer. It is easy to see Taft's algebras are not semisimple algebras. In [32] the author proved the existence of the exponent for finite dimensional algebras over an algebraically closed field of characteristic 0 that are simple under the action of a Taft algebra. We recall Taft's algebras are non-commutative, non-cocommutative and not semisimple Hopf algebras.

One of the authors of the present paper generalized the definition of $\mathrm{GKdim}_r(A)$ to graded algebras (see [17, 19]). Moreover, he found a deep relation between the PI-exponent of a PI-algebra and the Gelfand-Kirillov dimension. More precisely, given a finite dimensional G-graded algebra A, where G is a finite abelian group, and A satisfies an ordinary polynomial identity, then for any integer $k \geq 1$,

$$\mathrm{GKdim}_k^G(A) = \exp^G(A)k + \alpha, \tag{6}$$

where $\alpha \leq 0$ is an integer (In [17] the result is more general. If A is any G-graded algebra, where G is a finite abelian group, then there exists a finite dimensional $(G \times \mathbb{Z}_2)$-graded algebra R such that $Id^G(A) = Id^{G\times\mathbb{Z}_2}(\mathcal{G}(R))$, where $\mathcal{G}(R)$ denotes the Grassmann envelope of R. Then $\mathrm{GKdim}_k^G(A) = \exp^{(G,0)}(R)k + \alpha$, where α is as above and $\exp^{(G,0)}(R)$ denotes the contribution of the homogeneous component of degree 0 to the G-graded exponent of R).

We are almost done but we need to recall one more result before stating the crux of the matter. By the way, we suggest a recent proof that can be found in the paper [25].

Theorem 89. *If A is a finitely generated algebra so that its Hilbert series has the form*

$$\mathrm{Hilb}(A,t) = h(t)\prod_{i=1}^{s}\frac{1}{(1-t^{d_i})},$$

where $h(t)$ is a polynomial function, then its GK dimension equals the multiplicity of 1 as a pole of $\mathrm{Hilb}(A,t)$.

By Theorem 84, if A is an H-module algebra satisfying an ordinary polynomial identity, then the Hilbert series of the relatively free H-module algebra of A is a rational function. At light of Equation 6, if we prove a similar result for H-module algebras and because of Theorems 87 and Theorem 89, we could compute the H-exponent of A simply by looking at the multiplicity of 1 as a pole of the Hilbert series of the relatively free algebra of A. Hence, if we take $H = FG$, the G-graded exponent of A, here it comes, is the PI-flavour of the Hilbert series of a relatively free algebra.

Clearly, an analogous reasoning can be applied in the setting of $*$-algebras. If A is a $*$-algebra, we can define $\mathrm{GKdim}_k^*(A)$, the $*$-GK dimension of A in k variables. Moreover, as showed in Theorem 72, the Hilbert series of the relatively free $*$-algebra of A is a rational function. Here we want to highlight that the existence of the $*$-exponent of a $*$-algebra has been proved by one of the author in [34].

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Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via Edoardo Orabona, 4, 70125 Bari, Italy

IMECC, UNICAMP, Rua Sérgio Buarque de Holanda 651, 13083-859 Campinas, SP, Brazil
Email address: lucio.centrone@uniba.it, centrone@unicamp.br

IMECC, UNICAMP, Rua Sérgio Buarque de Holanda 651, 13083-859 Campinas, SP, Brazil
Email address: a227983@dac.unicamp.br

Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via R. Cozzi 55, 20126, Milano, Italy
Email address: antonio.ioppolo@unimib.it