# DIFFERENTIAL POLYNOMIAL IDENTITIES OF UPPER TRIANGULAR MATRICES OF SIZE THREE

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Abstract. We determine the differential polynomial identities of $3 \times 3$ upper triangular matrices over a base field of characteristic zero, under the action of its full Lie algebra of derivations. We compute the exact differential codimension sequence of the multilinear ones and describe their $S_n$-structure by means of an explicit decomposition of the $S_n$-cocharacter of their proper part.

## 1. Introduction

The word *derivation* brings to mind the word *Analysis* almost instantly. By the way, topological considerations are in order just to build and employ some specific derivations, but the very essence of the definition is of purely algebraic nature: a linear map satisfying the Leibniz rule. This was recognized by J. F. Ritt, who was an analyst at heart, and fathered a new branch in algebra named *Differential Algebra* after his work [Ritt] in 1950. Few years later I. Kaplansky wrote a book on the subject [Ka], and 22 years after Ritt's death, E. R. Kolchin provided a unified exposition in his book [Kol].

The study of algebras with derivations received new impetus by the works of Kharchenko ([Khar1], [Khar2]; see also his book [KharB]): he brought up the notion of *differential identity*, which provided a systematic and uniform approach to a number of problems on derivations, so avoiding the considerably clever but *ad hoc* computations often needed earlier.

Loosely speaking, differential polynomial identities are a direct generalization of polynomial identities: they are the identical relations holding in a structure (ring, algebra) endowed with some derivation action, and classically they have their place among the so-called Generalized Polynomial Identities (see [B&Ma&Mi] for a thorough exposition and a rich list of references about this subject).

In present days, however, the several generalizations of the notion of polynomial identities of an algebra (superidentities and graded identities, $*$-identities, differential identities) may be dealt with in a unifying setup: starting from a sparkling intuition of Berele in his influential paper [Be] (more precisely, the Remark at page 878), all of them are encompassed by the notion of $H$-polynomial identities, where $H$ denotes a Hopf algebra acting on the prescribed algebra (actually, a generalized Hopf-algebra action in case of involutions). Thus, at least for finite dimensional algebras on a field of characteristic zero, these different kind of identities share

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a common and sound ground, and admit the same basic investigation tools ($H$-multilinear identities) and quantitative description (the $H$-codimension sequence). In particular Amitsur's conjecture, solved by Giambruno and Zaicev in the ordinary case (see [Gi&Za] for a complete overview) holds within this general framework, as proved in [Go], [Go&Ko] under some particular condition and, more specifically, it holds for the differential codimensions of finite dimensional associative algebras with an action of an arbitrary Lie algebra by derivation, as proved in [Go1], Theorem 3. All these works give a common reason to the several results in the several different situations.

Through the decades, many results have been achieved about ordinary, super, graded, $*$- identities of relevant algebras (the matrix algebras $M_n(F)$, $UT_n(F)$, the Grassmann algebra $E$ of an infinite-dimensional vector space, and related algebras), but the general problem of describing the concrete polynomial identities of an algebra proved a goal too hard to achieve, and remains still unsolved for many of them. This is the case, for instance, for the ordinary polynomial identities of $M_3(F)$.

Even worse is the situation about the differential polynomial identities of algebras: so far, to the best of my knowledge, the only known results are on the Grassmann algebra $E$ of an infinite-dimensional vector spaces under the action of a finite dimensional abelian Lie algebra [Ri], and the algebras $UT_m(F)$ under the action of the non-abelian two-dimensional Lie algebra [Gi&Ri], [DV&N1]. A partial reason for this is, roughly speaking, that while in general super- and graded-polynomial identities of an algebra tend to decrease in complexity with respect to the ordinary one, or keep the same level ($*$-identities), the differential ones follow the inverse trend and tend to increase in complexity. Actually, in the few known cases mentioned above, the ordinary polynomial identities are needed in order to describe the differential ones, thus turning a hard problem into a harder one.

In the present paper we are going to face the description of the differential polynomial identities of upper triangular matrices of size three under the derivation action of its full Lie algebra of derivations. In fact, the case $UT_2(F)$, investigated in [Gi&Ri], is far too small to give but a pale picture of the general one; on the other hand, the general algebra $UT_m(F)$ has been investigated [DV&N1] just in case the derivation action is provided by a two-dimensional subalgebra of the full derivation algebra of $UT_m(F)$, and far too small as well. The algebra $UT_3(F)$ is instead small enough to keep the concrete computations at a reasonable level, but big enough to display the full impact of derivations on identical relations. Since in general it is unclear how algebra derivations and identical relations interact, and which new (differential) identities arise, any new result and concrete description in this direction is of interest. Incidentally, the investigation of the case $m = 3$ may hopefully provide some hints and clues to handle the general case $UT_m(F)$.

The paper develops around a small core consisting of basic notions and tools concerning Lie algebras and PI-Theory, with the main role played by the so-called multilinear polynomials proper with respect to a distinguished set of indeterminates and their properties, but prominently carries on along concrete and direct arguments of combinatorial nature, providing in the end a finite generating set for the differential polynomial identities of the algebra, their exact rate of growth (the differential codimension sequence) and a description of their structure (the proper differential cocharacter sequence).

## 2. Notation and basic results

Throughout this paper, let $F$ denote a fixed ground field of characteristic zero. By the word *algebra* we mean an associative, unitary algebra over $F$. By the word *derivation on* $A$ we mean any $F$-linear transformation $d : A \to A$ satisfying the *Leibniz rule* $(ab)^d = a^d b + ab^d$ for all $a, b \in A$ (we adopt the exponential notation for derivations; consequently, derivations will compose from left to right). The set of all derivations on $A$ is denoted $Der(A)$ and is a Lie algebra sitting inside $End_F(A)$. If $L$ is a Lie algebra, we say that $L$ *acts on* $A$ *by derivation* (for short: $A$ is an $L$-*algebra*) provided that a Lie-homomorphism $\varphi : L \to Der(A)$ has been assigned; then $A$ turns into a right module of $U(L)$, the universal enveloping algebra of $L$. Those algebra homomorphisms between $L$-algebras preserving the $L$-action (that is the $U(L)$-module structure) are called $L$-*homomorphisms.*

It is possible to define a free object in the class of $L$-algebras: let $X$ denote a countable set of free indeterminates, and let us consider the tensor algebra, denoted $F\langle X|L\rangle$, of the vector space $FX \otimes_F U(L)$. This algebra is endowed with a natural $L$-action by derivation, hence it is an $L$-algebra; moreover, any set-theoretic map from $X$ to any $L$-algebra $A$ extends uniquely to an $L$-homomorphism $F\langle X \mid L\rangle \to A$, that is $F\langle X \mid L\rangle$ is free on $X$ in the class of $L$-algebras.

This abstract, *basis free* definition is perhaps the purest way of presenting the free $L$-algebra; by the way, in order to make computations, we need to represent elements of $L$, *i.e.* to fix an $F$-basis in $L$. This necessity will involve the properties of $U(L)$ and definitely a new presentation of the free $L$-algebra.

So, let $\mathscr{L}$ denote any fixed, linearly ordered $F$-basis of $L$. Then the Poincaré-Birkhoff-Witt Theorem provides both an embedding of $L$ into $U(L)$ and a canonical $F$-basis of $U(L)$, namely constituted by the *semistandard words* $w = b_1 b_2 \dots b_n \in U(L)$, for all $n \in \mathbb{N}$, on the alphabet $\mathscr{L}$, that is by all words fulfilling $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$ where $b_1, \dots, b_n \in \mathscr{L}$. Let the simple tensors (*diads*) $x \otimes w$, for $x \in X$ and $w$ in the basis of $U(L)$, be denoted by $x^w$ and call them *letters.* When $w = 1 \in U(L)$ we identify the letter $x^1 = x \otimes 1$ with the indeterminate $x \in X$ and call $x$ an *ordinary* letter, otherwise we call $x^w$ a *truly differential letter.* The set of letters will be denoted by $X^L$, and after the mentioned identification it holds $X \subseteq X^L$. Then $F\langle X \mid L\rangle$ is isomorphic to the free associative algebra $F\langle X^L\rangle$ freely generated by $X^L$; also, the free algebra $F\langle X\rangle$ generated by $X$ is a subalgebra of $F\langle X^L\rangle$. Since elements of $F\langle X^L\rangle$ can be viewed as polynomials in the (noncommutative) letters $x^w$, they are called *differential* polynomials, or $L$-polynomials.

If $A$ is an assigned $L$-algebra, the set of $L$-polynomials lying in the kernel of all $L$-homomorphisms $F\langle X^L\rangle \to A$ is an ideal, denoted $T_L(A)$, stable under all $L$-endomorphisms of $F\langle X^L\rangle$. The elements of $T_L(A)$ are called the $L$-*differential identities of* $A$, and $T_L(A)$ is called the $T_L$-ideal of differential identities of $A$. Notice that if $Id(A)$ denotes the ideal of ordinary polynomial identities of $A$, then $Id(A) \subseteq T_L(A)$ is a genuine inclusion; actually, if $L \to Der(A)$ is the zero homomorphism, then $T_L(A) = Id(A)$. Hence, from this point of view, ordinary polynomial identities are just the (non trivial) differential polynomial identities under a trivial derivation action of $L$ on $A$, and the study of differential polynomial identities embodies the study of the ordinary polynomial identities of an algebra.

If $\mathscr{G}$ is a set of differential polynomials, the least $T_L$-ideal of $F\langle X^L\rangle$ containing $\mathscr{G}$ is called the $T_L$-ideal *generated by* $\mathscr{G}$. A main objective in studying the differential polynomial identities of an algebra is to find a generating set for $T_L(A)$ (and possibly a *minimal* generating set). Within our settings, the differential proper multilinear polynomials provide an extremely powerful tool toward this goal, and we are going to carefully define them. For all $n \in \mathbb{N}$, let $P_n^L$ be the vector subspace of $F\langle X^L\rangle$ spanned by all monomials of length $n$ and involving exactly all the indeterminates $x_1, \dots, x_n$, that is $P_0^L = F$ and

$$P_n^L := \operatorname{span}_F\langle x_{\sigma(1)}^{w_1} x_{\sigma(2)}^{w_2} \cdots x_{\sigma(n)}^{w_n} \mid \sigma \in S_n,\ w_i \text{ basis elements of } U(L)\rangle.$$

Standard Vandermonde arguments yield that $T_L(A)$ is generated, as a $T_L$-ideal, by the slices $P_n^L \cap T_L(A)$. One should notice that, since $U(L)$ is infinite dimensional, for all $n \geqslant 1$ the space $P_n^L$ is infinite dimensional, too, unlike the ordinary case. However, it is easy to prove that in case $A$ is finite dimensional the factor space $P_n^L/(T_L(A) \cap P_n^L)$ is in fact finite dimensional, hence it is possible to define the sequence $c_n^L(A) := \dim_F P_n^L/(P_n^L \cap T_L(A))$, the *L-codimension sequence of* $A$, measuring how many differential multilinear polynomials of degree $n$ are not differential identities of $A$ and, indirectly, provides a measure on how big $T_L(A)$ is. Again, this extends the usual ordinary codimension sequence, brought up by Regev [Re] as a valuable numerical invariant.

It has to be mentioned that these constructions are available in a more general framework, namely those of algebras under an Hopf-algebra action (or a generalized Hopf-algebra action in order to include algebras with involution): for differential identities, the acting Hopf-algebra is $U(L)$ [Go&Ko]. Also, the sequence $c_n^L(A)$ gives rise to another numerical invariant, $\text{PI-exp}^L(A) := \lim_n \sqrt[n]{c_n^L(A)}$, the so-called *differential PI-exponent of* $A$, when the limit does exist; sufficient conditions for the existence have been provided by Gordienko ([Go], Theorem 2) in the general framework of algebras under a Hopf-algebra action, and in the specific case of $L$-algebras the existence of the differential exponent was proved in [Go1], Theorem 3.

After recalling differential multilinear polynomials, let us briefly recall what differential proper polynomials are: for any $z_1, z_2 \in X^L$ define the commutators *of length* 2 by $[z_1, z_2] := z_1z_2 - z_2z_1$ and, for $k \geqslant 3$, recursively define the commutators *of length* $k$ by $[z_1, z_2, \dots, z_k] := [[z_1, z_2, \dots, z_{k-1}], z_k]$. The unitary subalgebra of $F\langle X^L\rangle$ generated by the commutators of any length will be denoted $B^L$, and we will call its elements *proper L-polynomials*. The relation between $F\langle X^L\rangle$ and $B^L$ can be clearly described through Lie algebras: let $\mathfrak{L}$ be the free Lie algebra generated by $X^L$ and let $\mathfrak{L}' = [\mathfrak{L}, \mathfrak{L}]$ be its derived ideal. Then $\mathfrak{L}'$ is also a free Lie algebra, and $\mathfrak{L}$ is spanned by $X^L$ modulo $\mathfrak{L}'$. By Witt's Theorem, $F\langle X^L\rangle$ is the universal enveloping algebra of $\mathfrak{L}$, while $B^L$ is the universal enveloping algebra of $\mathfrak{L}'$. Furthermore, let us fix a linear order in $X^L$ such that the ordinary letters precede the differential ones. Then the *semistandard commutators* $[z_1, z_2, \dots, z_k]$ (that is such that $z_1 > z_2 \leqslant z_3 \leqslant \cdots \leqslant z_k$) form a basis for $\mathfrak{L}'$ (see [Ba], Corollary of Proposition 8, (ii), p. 55), and can be completed to a basis for $\mathfrak{L}$ by adding the elements of $X^L$. The linear ordering on $X^L$ can be extended to a total order on this basis such that elements of $X^L$ precede any commutator. Then the semistandard polynomials $wb$, where $w$ is a semistandard monomial on $X^L$ and $b$ is a semistandard sequence of the $\mathfrak{L}'$-basis, constitute a basis for $F\langle X^L\rangle$ by the Poincarè-Birkhoff-Witt Theorem.

We actually need just polynomials which are *proper with respect to ordinary letters*, that is elements of the subalgebra $B_X^L$ of $F\langle X^L\rangle$ generated by commutators and truly differential letters. Thus we may consider the truly differential letters as commutators of length 1, similarly to what has been made in [DV&N], and consider *normal* semistandard commutators (*nssc's* for short) of length $\geqslant 1$: a commutator $[z_1, z_2, \ldots, z_k]$ is normal if at most a single truly differential letter occurs in it, and in this case it is $z_1$. By Proposition 7 in [DV&N], the products of nssc's together with 1 constitute an $F$-basis $\mathscr{B}$ for $B_X^L$. Partitioning $\mathscr{B}$ with respect to the number of the factors of its elements, we get $\mathscr{B} = \biguplus_{k\geqslant 0} \mathscr{B}_k$, where $\mathscr{B}_0 = \{1_F\}$, $\mathscr{B}_1$ includes all the single nssc's (comprising the single truly differential letters), and for a generic $k \geqslant 2$ it is $\mathscr{B}_k = \{c_1 c_2 \ldots, c_k \mid c_i \text{ nssc}\}$.

The meaning of *multilinear proper $L$-polynomials* should now be clear: they are simply the elements of the vector spaces $\Gamma_n^L := P_n^L \cap B_X^L$, for $n \geqslant 1$ (it is customary to define $\Gamma_0^L := F$). An $F$-basis for $\Gamma_n^L$ is provided by $\mathscr{P}_n^L := \Gamma_n^L \cap \mathscr{B}$; setting $\mathscr{P}_{n,k} := \Gamma_n^L \cap \mathscr{B}_k$ for all $k \geqslant 0$, it holds $\mathscr{P}_{0,0} = \{1_F\}$ and, for $n \geqslant 1$, $\mathscr{P}_n^L = \biguplus_{1\leqslant k\leqslant n} \mathscr{P}_{n,k}$.

The very reason why we are interested in proper multilinear $L$-polynomials is that $T_L(A)$ is generated, as a $T_L$-ideal, by the slices $\Gamma_n^L \cap T_L(A)$ for $n \geqslant 1$, as proved in [DV&N1], Theorem 1. The sequence $\gamma_n^L(A) := \dim_F \Gamma_n^L/(\Gamma_n^L \cap T_L(A))$, for $n \geqslant 0$, is called the *proper $L$-codimension sequence of $A$*, and is related to the $L$-codimension sequence by the simple relation $c_n^L(A) = \sum_{k=0}^n \binom{n}{k} \gamma_k^L(A)$.

A more refined description of the differential identities of $A$ moves its steps from the fact that the usual renaming action of $S_n$ on the letters of $X^L$, namely $\sigma \cdot x_i^d := x_{\sigma(i)}^d$, turns $P_n^L$ into a left $S_n$-module. The same happens to $\Gamma_n^L$ and $T_L(A)$, hence the factor spaces $P_n^L(A) := P_n^L/(P_n^L \cap T_L(A))$ and $\Gamma_n^L(A) := \Gamma_n^L/(\Gamma_n^L \cap T_L(A))$ are left $S_n$-modules, too. As for codimensions, in case $A$ is finite dimensional the $S_n$-character $\chi_n^L(A)$ of $P_n^L(A)$ is well defined and provides the structure of the multilinear differential identities of degree $n$ satisfied by $A$ by complete reducibility. By the way, all information on $\chi_n^L(A)$ are encoded in the simpler $S_n$-characters $\xi_n^L(A)$ of $\Gamma_n^L(A)$. In the very essence, these considerations follow from a renowned paper of Drensky, although in a different language ([Dr], Section 2); in terms of $S_n$-characters, the $S_n$-characters $\chi_n^L(A)$ are *Young-derived* from the proper ones (see [Re1]) . Therefore one just needs to know the decomposition of $\xi_n^L(A)$ into irreducible $S_n$-characters in order to know not only how many $L$-polynomial identities of $A$ are in $P_n^L$, but also their $S_n$-structure.

In the end of this preliminary section, let us agree on a light simplification of the notation: since we are dealing with differential polynomials, identities, modules, etc, we simply suppress the word *differential* and talk of multilinear polynomials, cocharacters and so on. Also, we agree in keeping the $L$-prefix/suffix only when necessary (for instance, in distinguishing $X^L$ from $X$). Hence, for instance $\Gamma_n^L$ will be denoted $\Gamma_n$. The full notation will be used just in the main Definitions and Theorems, in order to provide precise statements for quick reading and referencing.

## 3. The algebra $L = Der(UT_3(F))$

If $A$ is any $F$-algebra and $a \in A$, the *inner* derivation *induced by $a$* is the $F$-linear endomorphism $[\cdot, a]$ of $A$, that is the map sending $x \in A \to [x, a] := xa - ax$. From now on, let us fix $A := UT_3(F)$, the $F$-algebra of upper triangular matrices of size 3, and let $e_{ij}$ be the $(i, j)$-unit matrix, whose only nonzero entry is $1_F$ in position $(i, j)$.

**Definition 3.1.** Let $\varepsilon_1, \varepsilon_2$ denote the inner derivations induced by $-e_{11}$ and $e_{33}$ respectively; then, let $\eta_1, \eta_2$ denote the inner derivations induced by $-e_{12}, e_{23}$ respectively. Finally, denote by $\delta$ the inner derivation induced by $e_{13}$.

**Lemma 3.2.** *The set $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2, \delta$ constitutes an $F$-basis of the Lie algebra $Der(A)$.*

*Proof.* Any derivation of $A$ is inner by [C&PM]. Hence the map $\varphi : A \to Der(A)$, sending $a \in A$ into the inner derivation induced by $a$, is a surjective Lie homomorhism whose kernel is $Z(A) = F\mathbf{1}_3$. Hence the factor algebra $A/Z(A)$ has a representative basis constituted by the unit matrices $e_{11}, e_{12}, e_{13}, e_{23}, e_{33}$. Switching the vectors $e_{11}$ and $e_{12}$ into their opposites provides a basis, as well, and its $\varphi$-image is a basis for $\varphi(A)$. Therefore, $\varphi$ induces a Lie-isomorphism between $A/Z(A)$ and $Der(A)$, and the set $\{\varepsilon_1, \varepsilon_2, \eta_1, \eta_2, \delta\}$ is a basis of $Der(A)$. □

Thus we get a concrete representation of the Lie algebra $L := Der(A)$. The nonzero (Lie) products in $L$ are just

$$[\eta_i, \varepsilon_i] = \eta_i, \quad [\delta, \varepsilon_i] = \delta, \quad [\eta_1, \eta_2] = -\delta \qquad (\text{ for } i = 1, 2),$$

and they encode the structure of $L$: $L$ is a 5-dimensional solvable, non nilpotent, Lie algebra with trivial center, whose derived ideal $L' = [L, L]$ is the 3-dimensional Heisenberg algebra with basis $\eta_1, \eta_2, \delta$.

The derivation action of $L$ on $A$ gives rise to an action of the universal enveloping algebra $U(L)$ on $A$, thus turning $A$ into a (right) $U(L)$-module. Let $K$ denote the kernel of this action; so, in fact, $A$ is a $U(L)/K$-right module. So far, these facts are purely theoretic: in order to do computations, we need to fix an ordered basis of $L$. Let us choose $\varepsilon_1 < \varepsilon_2 < \eta_1 < \eta_2 < \delta$: as a matter of fact, this choice makes computations easier. Then by the PBW Theorem the semistandard words $\varepsilon_1^{e_1}\varepsilon_2^{e_2}\eta_1^{e_3}\eta_2^{e_4}\delta^{e_5}$, for all nonnegative integers $e_1, \dots, e_5$, constitute a basis for $U(L)$.

**Lemma 3.3.** *Let $\varepsilon_1 < \varepsilon_2 < \eta_1 < \eta_2 < \delta$ be the linear order assigned to the fixed basis of $L$, and let $\varphi' : U(L) \to End_F(A)$ be the unique algebra homomorphism factoring the inclusion $L \hookrightarrow End_F(A)$. Then*

(1) $\ker \varphi' = K$ *is the twosided ideal of $U(L)$ generated by*

$$\{\varepsilon_i^2 - \varepsilon_i, \eta_i^2, \varepsilon_i\eta_i, \varepsilon_i\delta, \eta_i\delta, \delta^2 \mid i \in \{1, 2\}\};$$

(2) $\mathscr{U} := \{1, \varepsilon_1, \varepsilon_2, \eta_1, \eta_2, \delta, \varepsilon_1\varepsilon_2, \varepsilon_1\eta_2, \varepsilon_2\eta_1, \eta_1\eta_2\}$ *is an $F$-basis of $U(L)/K$.*

*Proof.* Let $I$ denote the twosided ideal of $U(L)$ generated by the selected relations. The fact that $I \subseteq \ker \varphi' = K$ is easily verified, so we have to prove the reverse inclusion. First, notice that since $\eta_i^2, \delta^2$ are among the generators of $I$ and $\varepsilon_i^2 \equiv \varepsilon_i \pmod I$, any semistandard word involving a generator with exponent $> 1$ is congruent $\pmod I$ to a standard word (that is, with all exponents $\leqslant 1$), if not congruent to 0. Then, let $w$ be a standard word of length 3: if $w = \varepsilon_1 x_2 x_3$ then $x_2 \in \{\varepsilon_2, \eta_2\}$, otherwise $w \in I$. Assume $x_2 = \varepsilon_2$. Then in any case $w \in I$: if $x_3 \in \{\eta_2, \delta\}$ because $\varepsilon_2\eta_2, \varepsilon_2\delta \in I$, and if $x_3 = \eta_1$ because $\varepsilon_1$ and $\varepsilon_2$ commute, thus $\varepsilon_1\varepsilon_2\eta_1 = \varepsilon_2\varepsilon_1\eta_1$ and $\varepsilon_1\eta_1 \in I$. Hence we have to check $w = \varepsilon_1\eta_2 x_3$. But then $x_3 = \delta$ is the only choice, so $w \in I$ as well. Similar arguments apply to all monomials of length 3 (the only non trivial case is to check that $\varepsilon_2\eta_1\eta_2 \in I$: actually $\varepsilon_2\eta_1 = \eta_1\varepsilon_2$ since $[\varepsilon_2, \eta_1] = 0$), so the only elements of the PBW-basis of $U(L)$ not necessarily in $I$ must be of length $\leqslant 2$. A part of them is still in the set generating $I$, and the remaining ones are those of $\mathscr{U}$. It is easily checked that they are linearly independent modulo $K$, hence $I = K$ and the cosets of the elements in $\mathscr{U}$ form a basis for $U(L)/K$. □

Thus we get a handful of differential identities, arising from the generators of $K$:

**Corollary 3.4.** *The polynomials*

$$x^{\varepsilon_i^2} - x^{\varepsilon_i}, \quad x^{\varepsilon_i\eta_i}, \quad x^{\varepsilon_i\delta}, \quad x^{\eta_i^2}, \quad x^{\eta_i\delta}, \quad x^{\delta^2}$$

*(for $x \in X$ and $i = 1, 2$) are differential polynomial identities of $A$. We call them the* structural differential identities *of $A$.*

**Remark 3.5.** A whole lot of differential polynomial identities follows from the structural ones: in fact, if $u \in K$ then $x^u$ is a differential PI, since the linear operator $\varphi'(u)$ is zero. They comprise, for instance, all the differential letters $x^w$ where the basis element $w$ has length $\geqslant 3$, or length 2 but lies among the generators of $K$, such as $x^{\delta^2}, x^{\varepsilon_1\eta_1}$ and so on. Hence the only differential letters we have to deal with in describing the differential polynomial identities of $A$ are those surviving, that is $x^w$ for $w \in \mathscr{U}$.

We remark that, according to the chosen settings, other identities of kind $x^p$, for $p$ a polynomial in $\varepsilon_i, \eta_i, \delta$, cannot be regarded as vanishing differential letters, but also follow from the structural identities. For instance, it is true that $x^{\delta\eta_1}$ is a differential polynomial identity, but it has NOT to be regarded as a vanishing differential letter, because the word $\delta\eta_1$ is not semistandard. Actually, it still depends on the structural identities, because $\delta\eta_1 = \eta_1\delta$ in $U(L)$, and $\eta_1\delta \in K$. Similar arguments apply to the identity $x^{\eta_1\varepsilon_1} - x^{\varepsilon_1\eta_1} - x^{\eta_1}$, which also follows from the structural identities since $\eta_1\varepsilon_1 \equiv \eta_1 \pmod K$.

We are not going deeper into these (and other) distinctions, but it seems worth to point them out, once and for all.

In the rest of the paper, we shall write $x^d$ meaning that $1 \neq d \in \mathscr{U}$, that is $x^d$ is a truly differential letter, unless different explicit specifications. Also, $\mathscr{U}^* := \mathscr{U} \setminus \{1\}$.

**Lemma 3.6.** *$x^{\varepsilon_2}y^{\varepsilon_1} \in T_L(A)$. For all $(d_1, d_2) \notin \{\varepsilon_1, \eta_1\} \times \{\varepsilon_2, \eta_2\}$, the monomial $x^{d_1}y^{d_2}$ follows from $x^{\varepsilon_2}y^{\varepsilon_1}$ together with the structural identities. In particular, $x^{d_1}y^{d_2}$ is an identity of $A$.*

*Proof.* It holds $A^{\varepsilon_1} = \operatorname{span}_F\langle e_{12}, e_{13}\rangle$, $A^{\varepsilon_2} = \operatorname{span}_F\langle e_{13}, e_{23}\rangle$. It follows at once that $x^{\varepsilon_2}x^{\varepsilon_1} \in T_L(A)$. The proof of the second part of the statement consists of a rather lengthy (76 cases) sequence of direct verifications, and I shall omit it here. Instead, I provide a few instances. Let us work modulo the $T_L$-ideal generated by the structural identities together with $x^{\varepsilon_2}y^{\varepsilon_1}$:

(1) $x^{\varepsilon_1}y^{\varepsilon_1}$: just compute

$$0 = (xy)^{\varepsilon_1^2} - (xy)^{\varepsilon_1} = (x^{\varepsilon_1^2} - x^{\varepsilon_1})y + x(y^{\varepsilon_1^2} - y^{\varepsilon_1}) + 2x^{\varepsilon_1}y^{\varepsilon_1},$$

hence $x^{\varepsilon_1}y^{\varepsilon_1}$ is a pure consequence of the structural identity $x^{\varepsilon_1^2} - x^{\varepsilon_1}$.

(2) $x^{\varepsilon_1}y^{\eta_1}$: at first compute

$$0 = (xy)^{\varepsilon_1\eta_1} = x^{\varepsilon_1\eta_1}y + xy^{\varepsilon_1\eta_1} + x^{\varepsilon_1}y^{\eta_1} + x^{\eta_1}y^{\varepsilon_1} = x^{\varepsilon_1}y^{\eta_1} + x^{\eta_1}y^{\varepsilon_1}.$$

Now, substitute $y$ by $y^{\varepsilon_1}$ getting

$$x^{\varepsilon_1}y^{\varepsilon_1\eta_1} + x^{\eta_1}y^{\varepsilon_1^2} = x^{\eta_1}y^{\varepsilon_1},$$

hence $x^{\eta_1}y^{\varepsilon_1} = 0$, so $x^{\varepsilon_1}y^{\eta_1} = 0$ as well. Also in this case, $x^{\varepsilon_1}y^{\eta_1}$ is a pure consequence of the structural identities.

(3) $x^{\varepsilon_2}y^{\eta_1}$: from $[\eta_1,\varepsilon_1]=\eta_1$ it follows
$$x^{\varepsilon_2}y^{\eta_1}=x^{\varepsilon_2}y^{[\eta_1,\varepsilon_1]}=x^{\varepsilon_2}(y^{\eta_1\varepsilon_1}-y^{\varepsilon_1\eta_1})=x^{\varepsilon_2}(y^{\eta_1})^{\varepsilon_1}-x^{\varepsilon_2}y^{\varepsilon_1\eta_1},$$
a consequence of $x^{\varepsilon_2}y^{\varepsilon_1}$ and $y^{\varepsilon_1\eta_1}$.

□

For short, just the monomials $x^{\varepsilon_1}y^{\varepsilon_2}$, $x^{\varepsilon_1}y^{\eta_2}$, $x^{\eta_1}y^{\varepsilon_2}$ and $x^{\eta_1}y^{\eta_2}$ are not in $T_L(A)$.

The following argument, on consequences of an $L$-polynomial and depending upon properties of the derivations, will be freely used without further mention in the rest of the paper.

**Lemma 3.7.** *Let $u=x^{d_1}y^{d_2}\in T_L(A)$ and let $w$ be any monomial in the letters of $X^L$. Then the monomial $x^{d_1}wy^{d_2}$ is a consequence of the structural identities together with the identity $x^{\varepsilon_2}y^{\varepsilon_1}$, so in particular it is in $T_L(A)$. The same holds for $c_1c_2$, where $c_i$ is any commutator of length $\geqslant 1$ involving $x^{d_i}$.*

*Proof.* By induction on $n=len(w)$, the length of $w$. The case $n=0$ has been dealt with in Lemma 3.6, so assume the statement is true for any word $w$ of length $n\geqslant 0$, let $z$ be any letter in $X^L$ and consider $x^{d_1}wzy^{d_2}$. The statement follows easily when $d_1$ or $d_2$ belongs to $\{\varepsilon_i,\eta_i,\delta\mid i\in\{1,2\}\}$: namely, assume this holds for $d_2$. Then
$$x^{d_1}wzy^{d_2}=x^{d_1}w(zy^{d_2})=x^{d_1}w(zy)^{d_2}-x^{d_1}w(z^{d_2}y),$$
because $d_2$ is a derivation. Each of the two summands is a consequence of $u$ by inductive assumption, hence the same holds for $x^{d_1}wzy^{d_2}$. Since $u$ is a consequence of the structural identities and $x^{\varepsilon_2}y^{\varepsilon_1}$, the statement follows.

So assume neither $d_1$ nor $d_2$ is a derivation, that is $d_1,d_2\in\{\varepsilon_1\varepsilon_2,\varepsilon_1\eta_2,\varepsilon_2\eta_1,\eta_1\eta_2\}$; say $d_2=b_1b_2$. Then
$$x^{d_1}wzy^{b_1b_2}=x^{d_1}w(zy^{b_1b_2})=x^{d_1}w(zy)^{b_1b_2}-x^{d_1}w(z^{b_1}y^{b_2})-x^{d_1}w(z^{b_2}y^{b_1}),$$
Since $b_2>b_1$, the last summand is a consequence of $z^{b_2}y^{b_1}\in T_L(A)$ (hence a consequence of the structural identities and $x^{\varepsilon_2}y^{\varepsilon_1}$), by Lemma 3.6; since $d_1\notin\{\varepsilon_1,\eta_1\}$ it follows that $x^{d_1}z^{b_1}\in T_L(A)$, and by induction hypothesis the second summand is a consequence of the selected identities, as well. The same holds for the consequence $x^{d_1}w(zy)^{b_1b_2}$ of $x^{d_1}y^{d_2}$.

Finally, since any commutator $[x^d,z_1,\dots,z_m]$ is a sum of monomials involving $x^d$, the second part of the statement is true as well. □

Therefore, if $w$ is a monomial in $F\langle X^L\rangle$ non vanishing on $A$, then at most two truly differential letters may occur in it and, when it happens, the monomial has to be $w=w_1x_1^{d_1}w_2x_2^{d_2}w_3$, where $w_1,w_2,w_3$ are (possibly empty) words in ordinary letters only, $d_1\in\{\varepsilon_1,\eta_1\}$ and $d_2\in\{\varepsilon_2,\eta_2\}$.

Beside $x^{\varepsilon_2}y^{\varepsilon_1}$, there is another fundamental identity:

**Lemma 3.8.** $[x,y]^{\varepsilon_1\varepsilon_2}-[x,y]^{\varepsilon_1}-[x,y]^{\varepsilon_2}+[x,y]\in T_L(A)$.

Apart from the structural identities, both this last identity and $x^{\varepsilon_2}y^{\varepsilon_1}$ have a deep impact on the differential identities of $A$, and in fact adding to them to the generator of the ordinary identities of $A$ we get a candidate set of generators of $T_L(A)$:

**Definition 3.9.** Let $I$ denote the $T_L$-ideal generated by the following differential polynomials:

(1) $x^{\varepsilon_i^2} - x^{\varepsilon_i}$, $x^{\varepsilon_i\eta_i}$, $x^{\varepsilon_i\delta}$, $x^{\eta_i^2}$, $x^{\eta_i\delta}$, $x^{\delta^2}$ (for $i \in \{1,2\}$);
(2) $x^{\varepsilon_2}y^{\varepsilon_1}$;
(3) $[x,y]^{\varepsilon_1\varepsilon_2} - [x,y]^{\varepsilon_1} - [x,y]^{\varepsilon_2} + [x,y]$;
(4) $[x_1,y_1][x_2,y_2][x_3,y_3]$.

Notice that all the generators of $I$ are proper and multilinear. In the following sections our aim will be to show that, in fact, $I = T_L(A)$. Since a number of useful and nontrivial differential identities follows from those generating $I$, I am listing them here:

**Proposition 3.10.** *Let us set $c = [x,y]$, $c_i = [x_i, y_i]$, and assume $d, d_i \in \mathscr{U}^*$. The following polynomials are in $I$:*
(1) $x_1^{d_1}c$, $cx_2^{d_2}$ *for all* $d_i \notin \{\varepsilon_i, \eta_i\}$*;*
(2) $c^\delta$, $c^{\eta_1\eta_2}$, $c^{\varepsilon_1\eta_2} - c^{\eta_2}$, $c^{\varepsilon_2\eta_1} - c^{\eta_1}$*;*
(3) $c_1c_2^{\eta_2}$, $x_1^dc_2^{\eta_2}$, $c_1^{\eta_1}c_2$, $c_1^{\eta_1}x_2^d$ *for all* $d \in \mathscr{U}^*$*;*
(4) $c_1(c_2^{\varepsilon_2} - c_2)$, $x_1^d(c_2^{\varepsilon_2} - c_2)$, $(c_1^{\varepsilon_1} - c_1)c_2$, $(c_1^{\varepsilon_1} - c_1)x_2^d$ *for all* $d \in \mathscr{U}^*$*;*
(5) $x^{\varepsilon_1}c_2c_3$, $c_1c_2x^{\varepsilon_2}$, $x_1^{\varepsilon_1}cx_2^{\varepsilon_2}$.

*Proof.*
(1) $x_1^{d_1}c \equiv x_1^{d_1}(c^{\varepsilon_1} + c^{\varepsilon_2} - c^{\varepsilon_1\varepsilon_2}) \pmod I$. Since $d_1 \notin \{1, \varepsilon_1, \eta_1\}$, each summand is in $I$. Instead, note that $x^{\varepsilon_1}c$, $x^{\eta_1}c$ do not belong to $I$ (nor to $T_L(A)$, too). The same arguments apply to $cx_2^{d_2}$.
(2) Again, $c^\delta \equiv (c^{\varepsilon_1} + c^{\varepsilon_2} - c^{\varepsilon_1\varepsilon_2})^\delta = c^{\varepsilon_1\delta} + c^{\varepsilon_2\delta} - c^{\varepsilon_1\varepsilon_2\delta} \pmod I$, and each summand is a consequence of the structural identities. The same arguments apply to $c^{\eta_1\eta_2}$. $c^{\varepsilon_1\eta_2} - c^{\eta_2} = (c^{\varepsilon_1} - c)^{\eta_2} \equiv (c^{\varepsilon_1\varepsilon_2} - c^{\varepsilon_2})^{\eta_2} \equiv 0 \pmod I$ by the structural identities, and all the same $c^{\varepsilon_2\eta_1} - c^{\eta_1} \in I$.
(3) Since $c^{\varepsilon_1\eta_2} \equiv c^{\eta_2} \pmod I$ by (2), it holds

$$c_1c_2^{\eta_2} \equiv (c_1^{\varepsilon_1} + c_1^{\varepsilon_2} - c_1^{\varepsilon_1\varepsilon_2})c_2^{\varepsilon_1\eta_2} \pmod I,$$

and applying distributivity any summand is a consequence of a monomial identity, hence in $I$. Moreover $x_1^dc_2^{\eta_2} \in I$ for all $d \in \mathscr{U}^*$: it holds

$$x^dc^{\eta_2} \equiv x^dc^{\varepsilon_1\eta_2} \pmod I$$

by (2), so it is a consequence of the monomial identity $x_1^dy^{\varepsilon_1\eta_2}$. The same arguments apply to show $c_1^{\eta_1}c, c^{\eta_1}x^d \in I$.
(4) $c_1(c_2^{\varepsilon_2} - c_2) \equiv (c_1^{\varepsilon_1} + c_1^{\varepsilon_2} - c_1^{\varepsilon_1\varepsilon_2})(c_2^{\varepsilon_1\varepsilon_2} - c_2^{\varepsilon_1}) \pmod I$, hence again applying distributivity each summand is a consequence of a monomial identity. Moreover, $x_1^d(c_2^{\varepsilon_2} - c_2) \equiv x_1^d(c_2^{\varepsilon_1\varepsilon_2} - c_2^{\varepsilon_1})$ is a consequence of the monomial identities $x^{d_1}y^{d_2}$ for $d_2 \notin \{\varepsilon_2, \eta_2\}$. The same arguments apply to $(c_1^{\varepsilon_1} - c_1)c_2$ and $(c_1^{\varepsilon_1} - c_1)x_2^d$.
(5) $x^{\varepsilon_1}c_2c_3 \equiv x^{\varepsilon_1}c_2^{\varepsilon_1}c_3$ by (4), and $x^{\varepsilon_1}c_2^{\varepsilon_1}$ follows from the monomial identities, hence it is in $I$. Similar arguments apply to the remaining polynomials.

□

**Remark 3.11.** Apart from $c_1c_2^{\varepsilon_2}, c_1^{\varepsilon_1}c_2$, any other product $c_1c_2^{d_2}$ or $c_1^{d_1}c_2$ is a consequence of the identities listed so far, and apart from $x_1^{\varepsilon_1}c_2^{\varepsilon_2}, x_1^{\eta_1}c_2^{\varepsilon_2}$, $c_1^{\varepsilon_1}x_2^{\varepsilon_2}, c_1^{\varepsilon_1}x_2^{\eta_2}$, all products $x_1^{d_1}c_2^d$, $c_1^d x_2^{d_2}$ are in $I$ as well. Finally, the following congruences do hold modulo $I$:

$$c_1^{\varepsilon_1}c_2^{\varepsilon_2} \equiv c_1^{\varepsilon_1}c_2 \equiv c_1c_2^{\varepsilon_2} \equiv c_1c_2, \qquad c_1^{d_1}c_2^{d_2} \equiv 0 \quad (\text{for all } (d_1,d_2) \neq (\varepsilon_1,\varepsilon_2)),$$

as well as

$$(c_1^{\varepsilon_1\eta_2} - c_1^{\eta_2})c_2,\ c_1(c_2^{\varepsilon_2\eta_1} - c_2^{\eta_1}) \in I.$$

## 4. Normal forms of differential multilinear proper polynomials

Let $\Gamma$ denote the full space of proper multilinear $L$-polynomials. Its standard basis $\mathscr{P}$ consists of products of normal standard commutators of any length, and can be partitioned according to the number $k \geqslant 1$ of the factors (commutators) and the differential letters $x^d$, for $d \in \mathscr{U}$, occurring (as first or only letter) in the factors. We denote their set by $\mathscr{P}_k^{d_1|\dots|d_l}$, where $l \leqslant k$ and $1 < d_1 < \cdots < d_l$ (omitting multiple occurrences of the same $d \in \mathscr{U}$, as well of $1 \in \mathscr{U}$). We reserve the writing $\mathscr{P}_k^1$ to mean that just ordinary letters are involved, and $\mathscr{P}_k^{\mathbf{d}}$ if we are not really interested on which $d_1 < \cdots < d_l$ do occur. When necessary, we explicitly write $\mathscr{P}_{n,k}^{d_1|\dots|d_l}$ meaning that $\mathscr{P}_{n,k}^{d_1|\dots|d_l} \subseteq P_n^L \cap \mathscr{B}_k$.

For instance, $[x_1^\delta, x_2][x_3^{\varepsilon_1\varepsilon_2}, x_5, x_7][x_6, x_4, x_8]x_9^\delta$ is an element of $\mathscr{P}_{9,4}^{\delta|\varepsilon_1\varepsilon_2} \subsetneq \mathscr{P}_4^{\delta|\varepsilon_1\varepsilon_2}$. According with this notation,

**Notation.** Denote $\Gamma_{n,k}^{d_1|\dots|d_l} := \operatorname{span}_F\langle \mathscr{P}_{n,k}^{d_1|\dots|d_l}\rangle$.

Although this notation is quite natural, it may cause misunderstandings. Let us work out an example to see what may go wrong:

**Example 4.1.** The polynomial $c := [x_1^{\varepsilon_1}, x_2, x_4, x_3]$ is a proper multilinear polynomial, written as a single commutator, and involving just one $\varepsilon_1$-letter, and yet $c \notin \Gamma_{4,1}^{\varepsilon_1}$. Indeed, it is not a nsc. In fact, since

$$[x_1^{\varepsilon_1}, x_2, x_4, x_3] = [x_1^{\varepsilon_1}, x_2, x_3, x_4] + [x_1^{\varepsilon_1}, x_2][x_4, x_3] - [x_4, x_3][x_1^{\varepsilon_1}, x_2],$$

we get $c \in \Gamma_{4,1}^{\varepsilon_1} + \Gamma_{4,2}^{\varepsilon_1}$. These kind of situations may cause confusion, especially when working modulo $I$.

Of course, almost all the spaces $\Gamma_{n,k}^{\mathbf{d}}$ are actually contained in $I \subseteq T_L(A)$, and just a finite set is spared: if $k \geqslant 3$, then $\Gamma_k^{\mathbf{d}} \subseteq I$, since $c_1c_2c_3 \in I$ for any nsc's $c_1, c_2, c_3$. So our interest will be in $\Gamma_k^{\mathbf{d}}$ for $k = 1, 2$. We are going to describe a spanning set $\mathscr{S}$ for the factor space $\Gamma/(\Gamma \cap I)$. If $f \equiv b \pmod{\Gamma \cap I}$, for some $b \in F\mathscr{S}$, we will say for short that $b$ is a *normal form* for $f$. Let us start by describing $\Gamma^\delta$:

**Lemma 4.2.** *$\Gamma^\delta$ is spanned, modulo $I$, by the nsc's $[x_n^\delta, x_1, \dots, x_{n-1}]$, for all $n \geqslant 1$.*

*Proof.* Let $u \in \mathscr{P}_k^\delta$, and assume $u \notin I$. Since $x^\delta y^d$ and $y^d x^\delta$ are among the basic monomial identities of $A$, exactly one $\delta$-letter occurs in $u$. Moreover, since $x^\delta c, cx^\delta$ are among the generators of $I$, $u$ consists of a single commutator. Say $u = [x^\delta, y_1, \dots, y_{n-1}]$, with $y_1 < \cdots < y_{n-1}$ and $\{x, y_1, \dots, y_{n-1}\} = \{x_1, \dots, x_n\}$. The statement of the Lemma is trivially true if $n = 1$; if $n \geqslant 2$ and $x = x_n$ we are done, so assume $x \neq x_n$ and $n \geqslant 2$. Then $u = [x^\delta, y_1, \dots, y_{n-2}, x_n]$. The single special case $n = 2$ can be easily dealt with: since $0 \equiv [x_1, x_2]^\delta = [x_1^\delta, x_2] + [x_1, x_2^\delta]$ (mod $I$), one has $u = [x_1^\delta, x_2] \equiv [x_2^\delta, x_1]$ (mod $I$).

So, assume $n > 2$. Then $u = \Big[[x^\delta, y_1, \dots, y_{n-3}], y_{n-2}, x_n\Big] =: [c, y_{n-2}, x_n]$ and, by Jacobi law, $[c, y_{n-2}, x_n] = [c, x_n, y_{n-2}] + [c, [y_{n-2}, x_n]] \equiv [c, x_n, y_{n-2}]$ since $x^\delta[y_{n-2}, x_n]$ and $[y_{n-2}, x_n]x^\delta$ are among the (images of) generators of $I$. Iteratively, we get $u \equiv [x^\delta, x_n, y_1, \dots, y_{n-2}]$.

Now, as in case $n = 2$, it holds $[x^\delta, x_n] \equiv [x_n^\delta, x]$ modulo $I$, so $u \equiv [x_n^\delta, x, y_1, \dots, y_{n-2}]$ and we may rearrange $x, y_1, \dots, y_{n-2}$ in the correct order and get $u \equiv [x_n^\delta, x_1, \dots, x_{n-1}]$. □

This simple normal form for proper polynomials involving a $\delta$-letter is no longer available for other differential letters, even for polynomials involving $\eta_1\eta_2$: for instance, if $\hat{x}_i$ denotes that $x_i$ is missing, it is false that $[x_i^{\eta_1\eta_2}, x_1, \dots, \hat{x}_i, \dots, x_n] \equiv [x_n^{\eta_1\eta_2}, x_1, \dots, x_{n-1}] \pmod I$ when $i \neq n$, even if $x^d c, cx^d, c^d \in I$, both for $d = \delta$ and $d = \eta_1\eta_2$.

To see why this happens, and how to bypass this difficulty and efficiently determine the suitable normal forms of proper multilinear $L$-polynomials, let us check the following facts:

- if $x^d c, cx^d \in I$ hold, the same arguments used for $\delta$ allow us to straighten $[x^d, y_{\sigma(1)}, \dots, y_{\sigma(n)}] \equiv [x^d, y_1, \dots, y_n] \pmod I$ for all $n \geqslant 1$ and $\sigma \in S_n$. So it remains valid through $d \in \{\varepsilon_1\varepsilon_2, \varepsilon_1\eta_2, \varepsilon_2\eta_1, \eta_1\eta_2\}$;
- the relation $[x^d, y] \equiv [y^d, x] \pmod I$ is, on the contrary, false when $d \neq \delta$. More precisely, when $d \in \{\varepsilon_1\varepsilon_2, \varepsilon_1\eta_2, \varepsilon_2\eta_1, \eta_1\eta_2\}$, the relation holds just up to extra summands, in the form of proper multilinear $L$-polynomials involving differential $\{1, \varepsilon_i, \eta_i\}$-letters (simple differential letters).

When $d \in \{\varepsilon_1\varepsilon_2, \varepsilon_1\eta_2, \varepsilon_2\eta_1, \eta_1\eta_2\}$, it is the central role of the relations $[x^d, y] \equiv [y^d, x] \pmod{I + V}$ for a suitable $V \subseteq \bigoplus_{(d_1,d_2)} \Gamma^{d_1|d_2}$ (for $d_i \in \{1, \epsilon_1, \epsilon_2, \eta_1, \eta_2\}$), which must be checked. This is done in

**Proposition 4.3.** *Let us work modulo $I$. The following relations hold:*

(1) $[x^{\eta_1\eta_2}, y] = [y^{\eta_1\eta_2}, x] + x^{\eta_1}y^{\eta_2} - y^{\eta_1}x^{\eta_2}$. *Hence*
$$[x^{\eta_1\eta_2}, y] \equiv [y^{\eta_1\eta_2}, x] \pmod{\Gamma^{\eta_1|\eta_2}};$$
(2) $[x^{\varepsilon_1\eta_2}, y] = [y^{\varepsilon_1\eta_2}, x] - x^{\varepsilon_1}y^{\eta_2} + y^{\varepsilon_1}x^{\eta_2} + [x^{\eta_2}, y] - [y^{\eta_2}, x]$. *Hence*
$$[x^{\varepsilon_1\eta_2}, y] \equiv [y^{\varepsilon_1\eta_2}, x] \pmod{\Gamma^{\varepsilon_1|\eta_2} + \Gamma^{\eta_2}};$$
(3) $[x^{\varepsilon_2\eta_1}, y] = [y^{\varepsilon_2\eta_1}, x] - x^{\eta_1}y^{\varepsilon_2} + y^{\eta_1}x^{\varepsilon_2} + [x^{\eta_1}, y] - [y^{\eta_1}, x]$. *Hence*
$$[x^{\varepsilon_2\eta_1}, y] \equiv [y^{\varepsilon_2\eta_1}, x] \pmod{\Gamma^{\varepsilon_2|\eta_1} + \Gamma^{\eta_1}};$$
(4) $[x^{\varepsilon_1\varepsilon_2}, y] = [y^{\varepsilon_1\varepsilon_2}, x] - x^{\varepsilon_1}y^{\varepsilon_2} + y^{\varepsilon_1}x^{\varepsilon_2} + [x^{\varepsilon_1}, y] + [x^{\varepsilon_2}, y] - [y^{\varepsilon_1}, x] - [y^{\varepsilon_2}, x] + [y, x]$. *Hence*
$$[x^{\varepsilon_1\varepsilon_2}, y] \equiv [y^{\varepsilon_1\varepsilon_2}, x] \pmod{\Gamma^{\varepsilon_1|\varepsilon_2} + \Gamma^{\varepsilon_1} + \Gamma^{\varepsilon_2} + \Gamma^1};$$

*Proof.* Similarly to the case $d = \delta$, already settled, simply compute the double derivation $[x, y]^d$ – the difference with $\delta$ is in fact that $d = d_1d_2$ is an element of the universal enveloping algebra, not a derivation – then apply the proper differential identity in $I$:

(1) compute
$$[x, y]^{\eta_1\eta_2} = [x^{\eta_1\eta_2}, y] + [x, y^{\eta_1\eta_2}] + [x^{\eta_1}, y^{\eta_2}] + [x^{\eta_2}, y^{\eta_1}],$$
then recall $x^{\eta_2}y^{\eta_1}, y^{\eta_2}x^{\eta_1} \in I$ to get the result;

(2) compute $[x,y]^{\varepsilon_1\eta_2}$, but recall $[x,y]^{\varepsilon_1\eta_2} - [x,y]^{\eta_2} \in I$;
(3) same computation, but using $c^{\varepsilon_2\eta_1} - c^{\eta_1} \in I$;
(4) use $c^{\varepsilon_1\varepsilon_2} \equiv c^{\varepsilon_1} + c^{\varepsilon_2} - c \pmod I$.

□

Thus, for the "composite" elements $d \in \{\varepsilon_1\varepsilon_2, \varepsilon_1\eta_2, \varepsilon_2\eta_1, \eta_1\eta_2\}$, any $f \in \Gamma_n^d \setminus I$ must consist of a single commutator $[x_n^d, x_1, \ldots, x_{n-1}]$, just as in case $d = \delta$, plus a sum of polynomials involving ordinary letters and "simple" differential letters $x^{\varepsilon_i}, x^{\eta_i}$.

**Notation.** Let $x \in X$ and $Y \subseteq X$. Write $x > Y$ if $x > y$ for all $y \in Y$. Moreover, if $Y = \{y_1, y_2, \ldots, y_m\}$ and $y_1 < y_2 < \cdots < y_m$, denote

$$[x, Y] := [x, y_1, y_2, \ldots, y_m].$$

In case $Y = \varnothing$, $[x^d, Y] := x^d$ has to be a differential letter (in order to get a nsc).

According to this notation, we may write

**Definition 4.4.** Let $\mathscr{S}^\delta := \{[x^\delta, Y] \mid X \ni x > Y \subseteq X\} \cap P^L$.

Then, working modulo $I$, Lemma 4.2 can be restated as $\Gamma^\delta = \mathrm{span}_F\langle \mathscr{S}^\delta \rangle$. We record the following

**Corollary 4.5.** *Let us work* $\bmod I$, *and define*

- $\mathscr{S}^{\eta_1\eta_2} := \{[x^{\eta_1\eta_2}, Y] \mid X \ni x > Y \subseteq X\} \cap P^L$,
- $\mathscr{S}^{\varepsilon_1\eta_2} := \{[x^{\varepsilon_1\eta_2}, Y] \mid X \ni x > Y \subseteq X\} \cap P^L$,
- $\mathscr{S}^{\varepsilon_2\eta_1} := \{[x^{\varepsilon_2\eta_1}, Y] \mid X \ni x > Y \subseteq X\} \cap P^L$.
- $\mathscr{S}^{\varepsilon_1\varepsilon_2} := \{[x^{\varepsilon_1\varepsilon_2}, Y] \mid X \ni x > Y \subseteq X\} \cap P^L$.

*Then:*

- $\Gamma^{\eta_1\eta_2} = span_F\langle \mathscr{S}^{\eta_1\eta_2} \rangle \pmod{\Gamma^{\eta_1|\eta_2}}$;
- $\Gamma^{\varepsilon_1\eta_2} = span_F\langle \mathscr{S}^{\varepsilon_1\eta_2} \rangle \pmod{\Gamma^{\varepsilon_1|\eta_2} + \Gamma^{\eta_2}}$;
- $\Gamma^{\varepsilon_2\eta_1} = span_F\langle \mathscr{S}^{\varepsilon_2\eta_1} \rangle \pmod{\Gamma^{\varepsilon_2|\eta_1} + \Gamma^{\eta_1}}$;
- $\Gamma^{\varepsilon_1\varepsilon_2} = span_F\langle \mathscr{S}^{\varepsilon_1\varepsilon_2} \rangle \pmod{\Gamma^{\varepsilon_1} + \Gamma^{\varepsilon_2} + \Gamma^1}$;

*Proof.* Let $d \in \{\eta_1\eta_2,\ \varepsilon_1\eta_2,\ \varepsilon_2\eta_1,\ \varepsilon_1\varepsilon_2\}$. The whole $\Gamma_2^d$ lies inside $I$, because the identities $x^d c, cx^d$. Hence, $\Gamma^d$ is spanned modulo $I$ by the single commutators $u = [x_i^d, y_1, \ldots, y_{n-1}]$, for $n \geqslant 1$ and $x_i \in \{x_1, \ldots, x_n\}$, and the statement is true if $n = 1$. If $n \geqslant 2$ and $i = n$ then $u \in \mathscr{S}^d$, so assume $i < n$. Since $x_i^d c, cx_i^d \in I$, it holds $u \equiv [x_i^d, x_n, x_1, \ldots] \pmod I$. Now, apply the previous Proposition to get the desired result. We just check the single case $d = \varepsilon_2\eta_1$. From the previous Lemma, we get

$$[x_i^{\varepsilon_2\eta_1}, x_n] = [x_n^{\varepsilon_2\eta_1}, x_i] + p_{\varepsilon_2|\eta_1} + p_{\eta_1},$$

where $p_{\varepsilon_2|\eta_1} = -x_i^{\eta_1} x_n^{\varepsilon_2} + x_n^{\eta_1} x_i^{\varepsilon_2}$ and $p_{\eta_1} = [x_i^{\eta_1}, x_n] - [x_n^{\eta_1}, x_i]$. Then

$$[x_i^d, x_n, x_1, \ldots] = [x_n^d, x_i, x_1, \ldots] + [p_{\varepsilon_2|\eta-1}, x_1, \ldots] + [p_{\eta_1}, x_1, \ldots].$$

Notice that both $x_i^{\eta_1} x_n^{\varepsilon_2}$ and $x_n^{\eta_1} x_i^{\varepsilon_2}$ are proper multilinear polynomials, precisely inside $\Gamma_2^{\varepsilon_2|\eta_1}$. Hence $[p_{\varepsilon_2|\eta_1}, x_1, \ldots]$ is actually in $\Gamma_2^{\varepsilon_2|\eta_1}$. Even more easily, $[p_{\eta_1}, x_1, \ldots] \in \Gamma^{\eta_1}$. Then, $[x_n^d, x_i, x_1, \ldots] \equiv [x_n^d, x_1, \ldots, x_{n-1}] \pmod I$ and the result follows. □

Working $\pmod I$, the spaces $\Gamma^{\varepsilon_1|\varepsilon_2}, \Gamma^{\varepsilon_1|\eta_2}, \Gamma^{\varepsilon_2|\eta_1}$ and $\Gamma^{\eta_1|\eta_2}$ are necessarily spanned by products of two commutators, the first one involving a $\varepsilon_1$- or a $\eta_1$-letter, and the latter a $\varepsilon_2$- or a $\eta_2$-letter. They also have normal forms:

**Lemma 4.6.** *Let us work* $\mod I$*, and define*

- $\mathscr{S}^{\varepsilon_1|\varepsilon_2} := \{[z_1^{\varepsilon_1}, Y_1][z_2^{\varepsilon_2}, Y_2] \mid X \ni z_i > Y_i \subseteq X,\ i \in \{1,2\}\} \cap P^L$;
- $\mathscr{S}^{\varepsilon_1|\eta_2} := \{[z_1^{\varepsilon_1}, Y_1][z_2^{\eta_2}, Y_2] \mid X \ni z_i > Y_i \subseteq X,\ i \in \{1,2\}\} \cap P^L$;
- $\mathscr{S}^{\varepsilon_2|\eta_1} := \{[z_1^{\eta_1}, Y_1][z_2^{\varepsilon_2}, Y_2] \mid X \ni z_i > Y_i \subseteq X,\ i \in \{1,2\}\} \cap P^L$;
- $\mathscr{S}^{\eta_1|\eta_2} := \{[z_1^{\eta_1}, Y_1][z_2^{\eta_2}, Y_2] \mid X \ni z_i > Y_i \subseteq X,\ i \in \{1,2\}\} \cap P^L$.

*Then:*

(1) $\Gamma^{\varepsilon_1|\varepsilon_2} = span_F\langle \mathscr{S}^{\varepsilon_1|\varepsilon_2}\rangle \pmod{\Gamma_2^{\varepsilon_1} + \Gamma_2^{\varepsilon_2} + \Gamma_2^1}$;
(2) $\Gamma^{\varepsilon_1|\eta_2} = span_F\langle \mathscr{S}^{\varepsilon_1|\eta_2}\rangle \pmod{\Gamma_2^{\eta_2}}$;
(3) $\Gamma^{\varepsilon_2|\eta_1} = span_F\langle \mathscr{S}^{\varepsilon_2|\eta_1}\rangle \pmod{\Gamma_2^{\eta_1}}$;
(4) $\Gamma^{\eta_1|\eta_2} = span_F\langle \mathscr{S}^{\eta_1|\eta_2}\rangle$.

*Proof.* The trickiest case is about $\Gamma^{\varepsilon_1|\varepsilon_2}$, so I shall deal with it in full details.

The generic (non zero $\pmod I$) basis element involving a $\varepsilon_1$- and a $\varepsilon_2$-letter is a product $c_1c_2$ of snc's, namely $c_1 = [\overline{x}_1^{\varepsilon_1}, y_{1,1}, \dots, y_{1,h}]$ and $c_2 = [\overline{x}_2^{\varepsilon_2}, y_{2,1}, \dots, y_{2,k}]$ where the $\overline{x}_i$'s, $y_{i,j}$'s are pairwise distinct elements of $X$, $y_{1,1} < y_{1,2} < \cdots < y_{1,h}$, and $y_{2,1} < y_{2,2} < \cdots < y_{2,k}$. Hence $z_1 = \max\{\overline{x}_1, y_{1,h}\}$ and $z_2 = \max\{\overline{x}_2, y_{2,k}\}$.

Let us assume $z_1 \neq \overline{x}_1$ (hence $z_1 = y_{1,h} > \overline{x}_1$ and $h \geqslant 1$): in case $z_1 = \overline{x}_1$ the first commutator has the desired form, and we may switch our considerations to $c_2$. Also, it is not that relevant to assume $k > 0$: we will employ the identity $x^{\varepsilon_1}cy^{\varepsilon_2}$.

As a first step, it yields $c_1c_2 \equiv [\overline{x}_1^{\varepsilon_1}, z_1, Y_1']c_2 \pmod I$, where $Y_1'$ is the sequence $y_{1,1}, \dots, y_{1,h-1}$: indeed, for any $a, x, y, z \in X$ and $Y \subseteq X$ it holds

$$[x^{\varepsilon_1}, Y, a, z]y^{\varepsilon_2} = [x^{\varepsilon_1}, Y, z, a]y^{\varepsilon_2} + \Big[[x^{\varepsilon_1}, Y], [a, z]\Big]y^{\varepsilon_2}.$$

The second summand follows from $x^{\varepsilon_1}cy^{\varepsilon_2}$ (since $[x^{\varepsilon_1}, Y]$ is a sum of monomials involving $x^{\varepsilon_1}$), and hence it is in $I$. Thus we iterate the process to get $c_1c_2 \equiv [\overline{x}_1^{\varepsilon_1}, z_1, Y_1']c_2 \pmod I$. Now recall the identity $([\overline{x}_1, z_1]^{\varepsilon_1} - [\overline{x}_1, z_1])y^{\varepsilon_2} \in I$, providing $[\overline{x}_1^{\varepsilon_1}, z_1]c_2 \equiv [z_1^{\varepsilon_1}, \overline{x}_1]c_2 + [z_1, \overline{x}_1]c_2 \pmod I$. Notice that the second summand is in $\Gamma_2^{\varepsilon_2}$, hence if $h = 1$ we are already done. If $h > 1$, instead, it holds $c_1c_2 \equiv [z_1^{\varepsilon_1}, \overline{x}_1, Y_1']c_2 \pmod{I + \Gamma_2^{\varepsilon_2}}$, but we cannot simply insert $Y_1'$ inside $[z_1^{\varepsilon_1}, \overline{x}_1]c_2 + [z_1, \overline{x}_1]c_2$: a bit more care is required. In fact, from $([\overline{x}_1, z_1]^{\varepsilon_1} - [\overline{x}_1, z_1])y^{\varepsilon_2} \in I$ it follows $([\overline{x}_1, z_1, Y_1']^{\varepsilon_1} - [\overline{x}_1, z_1, Y_1'])y^{\varepsilon_2} \in I$, since $[\overline{x}_1, z_1, Y_1']$ is a specialization of $[\overline{x}_1, z_1]$. Since $\varepsilon_1$ is a derivation, it holds

$$[\overline{x}_1, z_1, Y_1']^{\varepsilon_1} = [\overline{x}_1^{\varepsilon_1}, z_1, Y_1'] + [\overline{x}_1, z_1^{\varepsilon_1}, Y_1'] + \sum_{1 \leqslant j < h} [\overline{x}, z_1, y_{1,1}, \dots, y_{1,j}^{\varepsilon_1}, \dots, y_{1,h-1}].$$

For all $1 \leqslant j < h$, the summand $[\overline{x}, z_1, y_{1,1}, \dots, y_{1,j}^{\varepsilon_1}, \dots, y_{1,h-1}]$ is a linear combination of products, each one involving the 2-commutator $[\overline{x}_1, z_1]$ and an $\varepsilon_1$-letter (namely $y_{1,j}^{\varepsilon_1}$), thus $[\overline{x}, z_1, y_{1,1}, \dots, y_{1,j}^{\varepsilon_1}, \dots, y_{1,h-1}]c_2 \equiv 0 \pmod I$ because of the identities $x^{\varepsilon_1}c_1c_2$ and $cx^{\varepsilon_1}$. Therefore

$$[\overline{x}_1, z_1, Y_1']^{\varepsilon_1}c_2 \equiv [\overline{x}_1^{\varepsilon_1}, z_1, Y_1']c_2 + [\overline{x}_1, z_1^{\varepsilon_1}, Y_1']c_2 \pmod I.$$

Then $([\overline{x}_1, z_1, Y_1']^{\varepsilon_1} - [\overline{x}_1, z_1, Y_1'])y^{\varepsilon_2} \equiv 0 \pmod I$ is equivalent to

$$[\overline{x}_1^{\varepsilon_1}, z_1, Y_1']y^{\varepsilon_2} \equiv [z_1^{\varepsilon_1}, \overline{x}_1, Y_1']y^{\varepsilon_2} + [z_1, \overline{x}_1, Y_1']y^{\varepsilon_2},$$

with the second summand lying in $\Gamma_2^{\varepsilon_2}$. Rearranging the sequence $(\overline{x}_1, Y_1')$ modulo $I$ as in the first step one finally gets $c_1c_2 \equiv [z_1^{\varepsilon_1}, Y_1]c_2 \pmod{I + \Gamma_2^{\varepsilon_2}}$. Now we can repeat the procedure to $c_2$ and get the final normal form modulo $I + (\Gamma_2^{\varepsilon_1} + \Gamma_2^{\varepsilon_2} + \Gamma_2^1)$.

Similar arguments provide the proofs of the remaining statements. □

For the remaining spaces, if $d \in \{1, \varepsilon_1, \varepsilon_2, \eta_1, \eta_2\}$ simply select $\mathscr{S}_1^d = \mathscr{P}_1^d$ (the standard basis of $\Gamma_1^d$), as well as $\mathscr{S}_2^1 = \mathscr{P}_2^1$ (no vector of the standard bases is left off): then of course $\mathscr{S}_1^d$ spans $\Gamma_1^d$ modulo $I$, too, as well as $\mathscr{S}_2^1$ spans $\Gamma_2^1 \pmod I$. We turn to the spaces $\Gamma_2^d$ for $d \in \{\varepsilon_1, \varepsilon_2, \eta_1, \eta_2\}$:

**Lemma 4.7.** *Let us work* $\mod I$*, and define*

- $\mathscr{S}_2^{\varepsilon_1} := \{[x^{\varepsilon_1}, Y_1][y, Y_2] \mid X \ni x > Y_1 \subseteq X,\, Y_2 \neq \varnothing,\, y > \min(Y_2)\} \cap P^L$;
- $\mathscr{S}_2^{\eta_1} := \{[x^{\eta_1}, Y_1][y, Y_2] \mid X \ni x > Y_1 \subseteq X,\, Y_2 \neq \varnothing,\, y > \min(Y_2)\} \cap P^L$;
- $\mathscr{S}_2^{\varepsilon_2} := \{[y, Y_1][x^{\varepsilon_2}, Y_2] \mid X \ni x > Y_2 \subseteq X,\, Y_1 \neq \varnothing,\, y > \min(Y_1)\} \cap P^L$;
- $\mathscr{S}_2^{\eta_2} := \{[y, Y_1][x^{\eta_2}, Y_2] \mid X \ni x > Y_2 \subseteq X,\, Y_1 \neq \varnothing,\, y > \min(Y_1)\} \cap P^L$.

*Then:*

1. $\Gamma_2^{\varepsilon_1} = span_F\langle \mathscr{S}_2^{\varepsilon_1} \rangle \pmod{\Gamma_2^1}$;
2. $\Gamma_2^{\eta_1} = span_F\langle \mathscr{S}_2^{\eta_1} \rangle$;
3. $\Gamma_2^{\varepsilon_2} = span_F\langle \mathscr{S}_2^{\varepsilon_2} \rangle \pmod{\Gamma_2^1}$;
4. $\Gamma_2^{\eta_2} = span_F\langle \mathscr{S}_2^{\eta_2} \rangle$.

*Proof.* Let $u = c_1c_2 \in \mathscr{P}_2^{\varepsilon_1}$ be an element of the standard basis of $\Gamma_2^{\varepsilon_1}$, not lying in $I$. Then $c_1 = [x^{\varepsilon_1}, y_1, \dots, y_n]$ and $c_2$ is an ordinary standard commutator (hence of length at least two), $c_2 = [y, y_{2,1}, \dots, y_{2,k}]$, hence $y > y_{2,1}$ and $c_2$ is already of the desired form. Let us consider $c_1$: if $x > y_n$ then $c_1c_2 \in \mathscr{S}_2^{\varepsilon_1}$; so, assume on the contrary $x < y_n$. Since $x^{\varepsilon_1}cc_2 \in I$, we may rearrange the sequence in $c_1$ getting $u \equiv [x^{\varepsilon_1}, y_n, y_1, \dots, y_{n-1}]c_2 \pmod I$. Now, recall the identity $([x, y_n]^{\varepsilon_1} - [x, y_n])c_2 \in I$, getting $[x^{\varepsilon_1}, y_n]c_2 \equiv [y_n^{\varepsilon_1}, x]c_2 + [x, y_n]c_2 \pmod I$, and hence $u \equiv [y_n^{\varepsilon_1}, x, y_1, \dots, y_{n-1}]c_2 + [x, y_n, y_1, \dots, y_{n-1}]c_2 \pmod I$ by the same arguments of the proof of the preceding Lemma, where $c_2$ is now replacing the former $y^{\varepsilon_2}$. We may straighten the sequence $x, y_1, \dots, y_{n-1}$ in the first summand into its natural order modulo $I$, while the second summand belongs to $\Gamma_2^1$. Hence $u \equiv [y_n^{\varepsilon_1}, Y_1']c_2 \pmod{I + \Gamma_2^1}$ and $y_n > Y_1'$, getting the first statement of the Lemma.

Similar arguments can be used to prove the other statements. □

**Example 4.8.** Let us work out an example, in order to view how the sets $\mathscr{S}_i^d$ look like. Since there are several of them, let us stick to the case where the degree of the polynomials is 5, and just ordinary, or $\varepsilon_1$ or $\varepsilon_1|\varepsilon_2$ letters are involved. In order to simplify notation, let us identify the indeterminates with their names, so $x_1^{\varepsilon_1}$ is written $1^{\varepsilon_1}$:

- $\mathscr{S}_{5,1}^1$: it is formed by single commutators in the ordinary letters $1, 2, 3, 4, 5$, and having in second position the least one. There are four of them, namely $[2, 1, 3, 4, 5]$, $[3, 1, 2, 4, 5]$, $[4, 1, 2, 3, 5]$ and $[5, 1, 2, 3, 4]$;
- $\mathscr{S}_{5,2}^1$: its elements are products of two commutators, each of length at least two, hence $[\cdot, \cdot][\cdot, \cdot, \cdot]$ or simmetrically $[\cdot, \cdot, \cdot][\cdot, \cdot]$. After choosing the letters occurring in the 3-commutator, their minimum must occur in the second position of the 3-commutator. The other commutator has therefore a fixed writing. For instance, after choosing $2, 4, 5$, we get the elements $[4, 2, 5][3, 1]$, $[5, 2, 4][3, 1]$ and symmetrically $[3, 1][4, 2, 5]$ and $[3, 1][5, 2, 4]$. Since there are $\binom{5}{3} = 10$ ways to choose three elements, there are 40 vectors in $\mathscr{S}_{5,2}^1$.
- $\mathscr{S}_{5,2}^{\varepsilon_1|\varepsilon_2}$: its vectors are products of commutators $c_1c_2$ where $len(c_i) \geqslant 1$, $c_i$ involving one $\varepsilon_i$-letter. For a choosen $l_1$, there are $\binom{5}{l_1}$ ways to select the letters occurring in $c_1$, and the greatest among them will be the $\varepsilon_1$-one. The remaining $5 - l_1$ will occur in $c_2$, and the greatest among them will

be the $\varepsilon_2$-letter. For instance, to $l_1 = 3$ and letters $2, 4, 5$, corresponds just one vector, namely $[5^{\varepsilon_1}, 2, 4][3^{\varepsilon_2}, 1]$. So there are 30 vectors in $\mathscr{S}_2^{\varepsilon_1|\varepsilon_2}$, selected among the 160 elements of $\mathscr{P}_2^{\varepsilon_1|\varepsilon_2}$. For instance, $1^{\varepsilon_1}[2^{\varepsilon_2}, 3, 4, 5]$, $1^{\varepsilon_1}[3^{\varepsilon_2}, 2, 4, 5]$, $1^{\varepsilon_1}[4^{\varepsilon_2}, 2, 3, 5]$, $1^{\varepsilon_1}[5^{\varepsilon_2}, 2, 3, 4]$ all belong to $\mathscr{P}_{5,2}^{\varepsilon_1|\varepsilon_2}$, but just $1^{\varepsilon_1}[5^{\varepsilon_2}, 2, 3, 4]$ is in $\mathscr{S}_{5,2}^{\varepsilon_1|\varepsilon_2}$.

- $\mathscr{S}_{5,1}^{\varepsilon_1}$: it is formed by single nsc's, where any of the 5 names may occur in first position as $\varepsilon_1$-letter. Hence there are five of them, namely $[1^{\varepsilon_1}, 2, 3, 4, 5]$, $\ldots$, $[5^{\varepsilon_1}, 1, 2, 3, 4]$. Actually, $\mathscr{S}_{5,1}^{\varepsilon_1} = \mathscr{P}_{5,1}^{\varepsilon_1}$.
- $\mathscr{S}_{5,2}^{\varepsilon_1}$: its vectors are products of a nsc $c_1$ involving an $\varepsilon_1$-letter, and an ordinary standard commutator $c_2$ of length $\geqslant 2$. Once the indeterminates of $c_1$ have been selected, the greatest one will be $\varepsilon_1$-derived and occupy the first position in $c_1$. For instance, for $c_1$ involving $3, 5$ only, the vectors $[5^{\varepsilon_1}, 3][2, 1, 4]$, $[5^{\varepsilon_1}, 3][4, 1, 2]$ are in $\mathscr{S}_{5,2}^{\varepsilon_1}$; if $c_1$ involves $2, 3, 5$, instead, then we get just the vector $[5^{\varepsilon_1}, 2, 3][4, 1]$, while the vectors $[2^{\varepsilon_1}, 3, 5][4, 1]$, $[3^{\varepsilon_1}, 2, 5][4, 1]$ are in $\mathscr{P}_{5,2}^{\varepsilon_1}$ but not in $\mathscr{S}_{5,2}^{\varepsilon_1}$. Actually, just 45 among the 85 elements of $\mathscr{P}_{5,2}^{\varepsilon_1}$ have been selected to form $\mathscr{S}_{5,2}^{\varepsilon_1}$.

**Definition 4.9.** Let us define $\mathbf{D} := \mathscr{U} \cup \{\varepsilon_1|\varepsilon_2,\ \varepsilon_1|\eta_2,\ \varepsilon_2|\eta_1,\ \eta_1|\eta_2\}$, and set

$$\mathscr{S} := \bigcup_{\mathbf{d}\in\mathbf{D}} \mathscr{S}^{\mathbf{d}},$$

where $\mathscr{S}^{\mathbf{d}}$ are the sets defined from Definition 4.4 to Lemma 4.7.

Of course, we are contracting, for instance, $\mathscr{S}^{\delta} := \mathscr{S}_1^{\delta}$, since $\mathscr{S}_2^{\delta} = \varnothing$ and, at the opposite, $\mathscr{S}^{\varepsilon_1|\varepsilon_2} := \mathscr{S}_2^{\varepsilon_1|\varepsilon_2}$, since $\mathscr{S}_1^{\varepsilon_1|\varepsilon_2} = \varnothing$. Each set can be further partitioned upon the degree, that is on the total number of names involved in its elements. For instance, $\mathscr{S}^{\delta} = \mathscr{S}_{1,1}^{\delta} \cup \mathscr{S}_{2,1}^{\delta} \cup \ldots$. Then we have the main result of this section:

**Theorem 4.10.** *The factor space $\Gamma^L/(\Gamma^L \cap I)$ is spanned by $\mathscr{S}$ modulo $I$.*

*Proof.* Each vector of the standard basis $\mathscr{P}^L$ of $\Gamma^L$ has a normal form modulo $I$, which is an $F$-linear combination of elements of $\mathscr{S}$ modulo $I$, by all the previous results. Hence the statement follows. □

## 5. Differential identities of $UT_3(F)$

So far, we selected a set $\mathscr{S} \subseteq \Gamma$, which can be partitioned into subsets $\mathscr{S}^{\mathbf{d}}$ for $\mathbf{d} \in \mathbf{D}$. Each of them can be further decomposed taking into account the degrees and the number of nsc's occurring in its elements, that is the sets $\mathscr{S}_{n,k}^{\mathbf{d}}$ for $1 \leqslant n \in \mathbb{N}$ and $k \in \{1, 2\}$. Some of them are actually empty, depending on $n$ and $\mathbf{d}$: for instance, if $n = 1$ then $\mathscr{S}_{1,2}^{\mathbf{d}} = \varnothing$ for all $\mathbf{d} \in \mathbf{D}$ and, if $n \geqslant 2$ then $\mathscr{S}_{n,2}^{d} = \varnothing$ for all $d \in \{\delta, \varepsilon_1\varepsilon_2, \varepsilon_1\eta_2, \eta_1\varepsilon_2, \eta_1\eta_2\}$; also, $\mathscr{S}_{n,1}^{\mathbf{d}} = \varnothing$ for all $\mathbf{d} \in \{\varepsilon_1|\varepsilon_2,\ \varepsilon_1|\eta_2,\ \varepsilon_2|\eta_1,\ \eta_1|\eta_2\}$. However, if $n \geqslant 4$, we get exactly 19 non-empty disjoint sets $\mathscr{S}_{n,k}^{\mathbf{d}}$, each one participating in spanning $\Gamma_n$ modulo $I$. Since $I \subseteq T := T_L(A)$, their union spans $\Gamma_n$ modulo $T$, as well. If we are able to prove that for all $n \geqslant 1$ the elements in $\mathscr{S}_n := \bigcup_{\mathbf{d},k} \mathscr{S}_{n,k}^{\mathbf{d}}$ are linearly independent modulo $T$, then $\Gamma_n \cap I = \Gamma_n \cap T$ for all $n \geqslant 1$, and hence $I = T$.

It is clearly unreasonable to set up and solve a linear system, so another strategy is needed. We are going to set up a sort of elimination algorithm:

(1) Initialize $S := \mathscr{S}_n = \bigcup_{k,\mathbf{d}} \mathscr{S}_{n,k}^{\mathbf{d}}$;

(2) pick an element $b \in S$ and set $S_b := S \setminus \{b\}$;
(3) exhibit an $L$-homomorphism $\varphi : F\langle X^L \rangle \to A$ such that $\varphi(b) \notin \mathrm{span}_F\langle \varphi(S_b)\rangle$ (so, in particular, $\varphi(b) \neq 0$);
(4) replace $S$ by $S_b$ (that is: delete $b$ from $S$);
(5) if $S \neq \varnothing$ go to step 2.

If this elimination algorithm reaches $S = \varnothing$, then $\mathscr{S}$ is indeed linearly independent modulo $T$. Since $F\langle X^L\rangle$ is free on $X$, in order to choose the suitable $\varphi$ we only have to specialize the indeterminates $x \in X$ to elements of $A$, and the more elements of $S$ are zero in the resultant evaluation, the better for us. Our first task will be to delete from $S := \mathscr{S}_n$ the more elements of $\mathscr{S}_{n,1}$ we can.

As it will be evident in the sequel, the difficulties arise not only in finding a suitable homomorphism once an element $b \in S$ has been selected, but also in finding a suitable sequence of choices for the elements of $S$: they simply cannot be chosen randomly.

**Proposition 5.1.** *For all $n \geqslant 1$ it holds*

$$\Gamma_n(A) = \bigoplus_{d \in \mathscr{U}} \Gamma^d_{n,1} \oplus \mathit{span}_F\langle \mathscr{S}_{n,2}\rangle \pmod{T_L(A)},$$

*and $\mathscr{S}^d_{n,1}$ is a basis for $\Gamma^d_{n,1} \pmod{\Gamma^d_{n,1} \cap T_L(A)}$, for all $d \in \mathscr{U}$.*

*Proof.* For convenience of the reader, we list the polynomials we are interested in:

| $d$ | Vectors |
|---|---|
| $1$ | $[x_2, Y_2] := [x_2, x_1, Y_2], [x_3, Y_3], \ldots, [x_n, Y_n]$ |
| $\varepsilon_1$ | $[x_1^{\varepsilon_1}, Y_1], [x_2^{\varepsilon_1}, Y_2], \ldots, [x_n^{\varepsilon_1}, Y_n]$ |
| $\varepsilon_2$ | $[x_1^{\varepsilon_2}, Y_1], [x_2^{\varepsilon_2}, Y_2], \ldots, [x_n^{\varepsilon_2}, Y_n]$ |
| $\eta_1$ | $[x_1^{\eta_1}, Y_1], [x_2^{\eta_1}, Y_2], \ldots, [x_n^{\eta_1}, Y_n]$ |
| $\eta_2$ | $[x_1^{\eta_2}, Y_1], [x_2^{\eta_2}, Y_2], \ldots, [x_n^{\eta_2}, Y_n]$ |
| $\delta$ | $[x_n^{\delta}, x_1, \ldots, x_{n-1}]$ |
| $\varepsilon_1\varepsilon_2$ | $[x_n^{\varepsilon_1\varepsilon_2}, x_1, \ldots, x_{n-1}]$ |
| $\varepsilon_1\eta_2$ | $[x_n^{\varepsilon_1\eta_2}, x_1, \ldots, x_{n-1}]$ |
| $\eta_1\varepsilon_2$ | $[x_n^{\eta_1\varepsilon_2}, x_1, \ldots, x_{n-1}]$ |
| $\eta_1\eta_2$ | $[x_n^{\eta_1\eta_2}, x_1, \ldots, x_{n-1}]$ |

where, of course, $\mathscr{S}^1_{1,1} = \varnothing$.

(1) Let $\varphi : X \to e_{33}$ denote the constant function sending $\varphi(x) = e_{33}$ for all $x \in X$. Then any element of $\mathscr{S}^{\mathbf{d}}_{n,2}$ has zero evaluation in $\varphi$. The same holds for almost all elements of $\mathscr{S}^{\mathbf{d}}_{n,1}$: indeed, any ordinary commutator is zero-valued, and if $d \neq \delta, \eta_2$ then $e_{33} \in \mathit{ker}(d)$. By the way, $0 \neq [e_{33}^{\delta}, e_{33}, \ldots, e_{33}] = [-e_{13}, e_{33}, \ldots, e_{33}] \in \mathrm{span}_F\langle e_{13}\rangle$, while $[e_{33}^{\eta_2}, e_{33}, \ldots, e_{33}] = [-e_{23}, e_{33}, \ldots, e_{33}] \in \mathrm{span}_F\langle e_{23}\rangle$. We cannot delete the $[x_i^{\eta_2}, Y_i]$'s from $S$, since there are $n$ of them, but we may safely remove from $S$ the vector $[x_n^{\delta}, Y_n]$.
(2) Let $\varphi : X \to e_{11}$ be the substitution sending any $x \in X$ to $e_{11}$. As before, any ordinary commutator vanishes under $\varphi$, as well as any element of $\mathscr{S}^{\mathbf{d}}_{n,2}$. Since $e_{11} \in \ker d$ for all $d \neq \eta_1\eta_2, \eta_1$, just $[x_n^{\eta_1\eta_2}, Y]$ and $[x_i^{\eta_1}, Y_i]$ may be nonzero valued. In fact, $0 \neq [e_{11}^{\eta_1\eta_2}, e_{11}, \ldots, e_{11}] = [-e_{13}, e_{11}, \ldots, e_{11}] \in \mathrm{span}_F\langle e_{13}\rangle$, while $0 \neq [e_{11}^{\eta_1}, e_{11}, \ldots, e_{11}] = [-e_{12}, e_{11}, \ldots, e_{11}] \in \mathrm{span}_F\langle e_{12}\rangle$. Again, we cannot delete from $S$ the $[x_i^{\eta_1}, Y_i]$'s, but we can safely remove $[x_n^{\eta_1\eta_2}, x_1, \ldots, x_{n-1}]$.

(3) Let $i < n$, and let $\varphi_i : (x_i, Y_i) \to (e_{23}, e_{33})$ denote the map sending $x_i \to e_{23}$ and $Y_i \to e_{33}$. Then $0 \neq \varphi_i([x_i^{\eta_2}, Y_i]) \in \text{span}_F\langle e_{13}\rangle$. The other non-vanishing elements are $[x_i, Y_i]$ (if $i > 1$) and $[x_i^{\varepsilon_2}, Y_i]$, taking values in $\text{span}_F\langle e_{23}\rangle$. Notice that for $i = 1$ all $[x_j, Y_j]$, for $j \in \{2, \ldots, n\}$, are still non vanishing but with values in $\text{span}_F\langle e_{23}\rangle$. So we may delete the elements $[x_1^{\eta_2}, Y_2], \ldots, [x_{n-1}^{\eta_2}, Y_{n-1}]$ from $S$. We instead cannot delete $[x_n^{\eta_2}, Y_n]$: it holds $0 \neq \varphi_n([x_n^{\varepsilon_2\eta_1}, Y_n]) \in \text{span}_F\langle e_{13}\rangle$, too.

(4) Similarly to the previous case, let $\varphi_i : (x_i, Y_i) \to (e_{12}, e_{11})$, for $i < n$. Then just $[x_i^{\eta_2}, Y_i]$, $[x_i^{\varepsilon_1}, Y_i]$ and elements from $\mathscr{S}^1_{n,1}$ (the single $[x_i, Y_i]$ if $i > 1$, the whole set if $i = 1$) do not vanish under $\varphi_i$, but the former gets its value in $\text{span}_F\langle e_{13}\rangle$, the other ones in $\text{span}_F\langle e_{12}\rangle$. Just as before, this does not work if $i = n$, because $[x_n^{\varepsilon_1\eta_2}, Y_n]$ too gets a nonzero value in $\text{span}_F\langle e_{13}\rangle$. Hence we delete from $S$ just the elements $[x_1^{\eta_2}, Y_1], \ldots, [x_{n-1}^{\eta_2}, Y_{n-1}]$, but still keep $[x_n^{\eta_2}, Y_n]$.

(5) Let us employ once again $\varphi : X \to e_{11}$ on the remaining vectors. Now just $[x_n^{\eta_1}, Y_n]$ does not vanish, so we can remove it from $S$. After that, let us use the constant assignment $\varphi : X \to e_{33}$, so to delete the only non vanishing vector $[x_n^{\eta_2}, Y_n]$.

(6) Repeat the same trick, this time assigning $(x_n, Y_n) \to (e_{12}, e_{11})$: just $[x_n^{\varepsilon_1\eta_2}, Y_n]$ takes a non zero value in $\text{span}_F\langle e_{13}\rangle$ and can be deleted from $S$; then the assignment $(x_n, Y_n) \to (e_{23}, e_{33})$ allows us to delete $[x_n^{\eta_1\varepsilon_2}, Y_n]$, because it is the only one taking a non zero value in $\text{span}_F\langle e_{13}\rangle$.

(7) So far, $S = \mathscr{S}^1_{n,1} \cup \mathscr{S}^{\varepsilon_1}_{n,1} \cup \mathscr{S}^{\varepsilon_2}_{n,1} \cup \mathscr{S}^{\varepsilon_1\varepsilon_2}_{n,1} \cup \mathscr{S}_{n,2}$. It is a bit tricky to delete the elements $[x_i^{\varepsilon_2}, Y_i]$ , $i < n$: choose at first $1 < i < n$ and assume that $\sum_{s\in S} \alpha_s s \in T$ for some scalars $\alpha_s \in F$. Then let $\varphi$ be the map sending $x_i \to e_{12}$ and $Y_i \to e_{11}$, once again. It holds $\varphi([x_i, Y_i]) = (-1)^{n+1}e_{12} = \varphi([x_i^{\varepsilon_1}, Y_i])$, while $\varphi(s) = 0$ for all other $s \in S$. Hence the relation $\alpha_{[x_i,Y_i]} + \alpha_{[x_i^{\varepsilon_1},Y_i]} = 0$ among their coefficients must hold.

Now consider $\psi$, sending $x_i \to e_{13}$ and $Y_i \to e_{11}$. We have first of all to check carefully that $\psi(\mathscr{S}_{n,2}) = 0$: so assume that $u = c_1c_2 \in \mathscr{S}_{n,2}$. Notice that if $x_i$ occurs in a commutator $c$, either derived or not, then $\psi(c) \in \text{span}_F\langle e_{13}\rangle$ (possibly, $\psi(c) = 0$). Since the letter $x_i$ must occur either in $c_1$ or in $c_2$, one among $\psi(c_1), \psi(c_2)$ is a multiple of $e_{13}$, and the other one is in $J(A)$; it follows that $\psi(u) = \psi(c_1)\psi(c_2) = 0$, because $J(A)$ annihilates $e_{13}$.

Among the elements of $\mathscr{S}_{n,1} \cap S$, just $[x_i, Y_i]$, $[x_i^{\varepsilon_1}, Y_i]$ and $[x_i^{\varepsilon_2}, Y_i]$ are not zero under $\psi$. More precisely, all of them take value $(-1)^{n+1}e_{13}$. So, among their coefficients it holds $\alpha_{[x_i,Y_i]} + \alpha_{[x_i^{\varepsilon_1},Y_i]} + \alpha_{[x_i^{\varepsilon_2},Y_i]} = 0$. Comparing the two relations, we get $\alpha_{[x_i^{\varepsilon_2},Y_i]} = 0$, so we can delete $[x_i^{\varepsilon_2}, Y_i]$ from $S$.

The case $i = 1$ can be dealt with in the same spirit, taking into account that under the assignment $\varphi : (x_1, Y_1) \to (e_{12}, e_{11})$ it holds $\varphi([x_j, Y_j]) = (-1)^n e_{12}$ for all $j > 1$, and a more complex relation among coefficients appears, namely

$$\sum_{j=2}^{n} \alpha_{[x_j,Y_j]} - \alpha_{[x_1^{\varepsilon_1},Y_1]} = 0.$$

Similarly, assigning $\psi : (x_1, Y_1) \to (e_{13}, e_{11})$ it holds $\psi([x_j, Y_j]) = (-1)^n e_{13}$ for all $j > 1$, so

$$\sum_{j=2}^{n} \alpha_{[x_j,Y_j]} - \alpha_{[x_1^{\varepsilon_1},Y_1]} - \alpha_{[x_1^{\varepsilon_2},Y_1]} = 0.$$

In the end, however, still $\alpha_{[x_1^{\varepsilon_2},Y_1]} = 0$, and we may delete $[x_1^{\varepsilon_2}, Y_1]$ from $S$ as well. Instead, we have to keep $[x_n^{\varepsilon_2}, Y_n]$.

(8) For any $1 < i < n$ the substitution $\varphi : (x_i, Y_i) \to (e_{23}, e_{33})$ allows us to delete the ordinary commutator $[x_i, Y_i]$, since just $[x_i, Y_i]$ does not vanish among the elements of $S$ (after we deleted the $[x_i^{\varepsilon_2}, Y_i]$). After deleting $[x_2, Y_2], \ldots, [x_{n-1}, Y_{n-1}]$, the substitution $(x_1, Y_1) \to (e_{23}, e_{33})$ allows us to delete $[x_n, Y_n]$ as well (since just $[x_n, Y_n]$ does not vanish), hence deleting the last ordinary commutator.

(9) After deleting from $S$ all ordinary single commutators, the substitution $(x_i, Y_i) \to (e_{12}, e_{11})$ works in deleting each $[x_i^{\varepsilon_1}, Y_i]$ for all $i \in \{1, \ldots, n\}$, since any $S \ni s \neq [x_i^{\varepsilon_1}, Y_i]$ is sent to zero.

(10) Now $S = \{[x_n^{\varepsilon_2}, Y_n], [x_n^{\varepsilon_1\varepsilon_2}, Y_n]\} \cup \mathscr{S}_{n,2}$, and we may employ the substitutions $\varphi : (x_n, Y_n) \to (e_{23}, e_{33})$ to cancel $[x_n^{\varepsilon_2}, Y_n]$, then $\psi : (x_n, Y_n) \to (e_{13}, e_{33})$ to delete $[x_n^{\varepsilon_1\varepsilon_2}, Y_n]$.

□

Hence it holds $\Gamma_n(A) = \bigoplus_{\mathbf{d}} \Gamma_{n,1}^{\mathbf{d}}(A) \oplus (\mathrm{span}_F\langle \mathscr{S}_{n,2}\rangle + T)/T$ and $\mathscr{S}_{n,1}$ is part of a basis of $\Gamma_n(A)$; we still have to apply the algorithm to $S = \mathscr{S}_{n,2}$ in order to reach $S = \varnothing$. Needless to say, it will be much harder. Therefore, instead of stating a short sentence with a long proof, let us separate the several steps in a sequence of statements, each with a proof of reasonable length.

**Notation.** Let $c = [z^d, y_1, \ldots, y_k]$ a normal standard commutator, for $d \in \mathscr{U}$ (including $d = 1$). The set $\mathscr{X}(c) := \{z, y_1, \ldots, y_k\} \subseteq X$ is the *content of* $c$, and define $\|c\| := z^d$ (the *leading letter of* $c$). We say that $c$ is a *leading commutator* if $z = \max \mathscr{X}(c)$. Moreover, let $len(c) := |\mathscr{X}(c)|$ (the *length* of $c$).

Notice that, according to this terminology, for any element $c_1c_2 \in \mathscr{S}^{\varepsilon_1|\varepsilon_2}$ both $c_1$ and $c_2$ are leading commutators; if instead $c_1c_2 \in \mathscr{S}^{\varepsilon_1}$ then $c_1$ is a leading commutator, while $c_2$ may be not. If $c_1c_2 \in \mathscr{S}^1$, both $c_1$ and $c_2$ may be non-leading commutators: for instance, $[x_2, x_1, x_3][x_5, x_4, x_6]$. If $c = [z^d, y_1, \ldots, y_h]$ is non-leading, this amounts to say that $z < y_h = \max \mathscr{X}(c)$. On the contrary, $c = [z, Y]$ is a leading commutator if and only if $z = \max \mathscr{X}(c)$, that is $z > Y$. Of course, any differential letter is a leading commutator.

A key point in simplifying our considerations is the following easy

**Remark 5.2.** Let $J = J(A)$ be the Jacobson radical of $A$, let $b = e_{ij}$ be a unit matrix and let $c$ be a commutator involving $b$. If $i = 1$ then $c \in \mathrm{span}_F\langle e_{12}, e_{13}\rangle$ (possibly $c = 0$), and $Jc = 0$. Similarly, if $j = 3$ then $c \in \mathrm{span}_F\langle e_{13}, e_{23}\rangle$ and $cJ = 0$.

**Lemma 5.3.** *We may delete from $S$ all the vectors $[z_1, Y_1][z_2, Y_2]$ such that $z_1 < \max Y_1$ and $z_2 < \max Y_2$.*

Equivalently, we may delete all ordinary elements of $S$ which are a product of two non-leading commutators.

*Proof.* Let us pick any $v = [z_1, Y_1][z_2, Y_2] =: [z_1, y_{1,1}, \dots, y_{1,h}][z_2, y_{2,1}, \dots, y_{2,k}]$ where neither $[z_1, Y_1]$ nor $[z_2, Y_2]$ is a leading commutator; this amounts to say that $z_1 < y_{1,h}$ and $z_2 < y_{2,k}$. Consider the substitution

$$\varphi : \begin{array}{cccc} z_1 & Y_1 & z_2 & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ e_{12} & e_{11} & e_{23} & e_{33} \end{array} .$$

Clearly, $\varphi(v) \neq 0$, and we aim to prove that for any $v \neq u \in S$ it holds $\varphi(u) = 0$.

So, let $u = c_1c_2 \in S$ and assume that $\varphi(u) \neq 0$. We claim that $z_2, y_{2,1}, \dots, y_{2,k} \in \mathscr{X}(c_2)$. In fact, if $y_{2,j}$ does not occur in $c_2$ then $y_{2,j}$ occurs in $c_1$ as an ordinary letter, since $e_{33}^{\varepsilon_1} = 0 = e_{33}^{\eta_1}$. By the Remark 5.2, it follows that $\varphi(c_1) \in \mathrm{span}_F\langle e_{13}, e_{23}\rangle$ and (since $\varphi(c_2) \in J$) we get $\varphi(u) = 0$, a contradiction. If $z_2$ occurs in $c_1$ as an ordinary letter, then $\varphi(c_1)$ involves $e_{23}$ and $\varphi(u) = 0$, as well. So it remains to assume that $z_2$ occurs in $c_1$ as an $\varepsilon_1$- or $\eta_1$-letter. Since $e_{23}^{\varepsilon_1} = 0$, the only possibility is that $z_2^{\eta_1}$ occurs in $c_1$, but then $\varphi(c_1)$ involves $e_{13}$ and $\varphi(u) = 0$ once again. Similar arguments apply to prove that $z_1, y_{1,1}, \dots, y_{1,h} \in \mathscr{X}(c_1)$. Hence, $c_1$ and $c_2$ involve exactly the same indeterminates as $[z_1, Y_1]$ and $[z_2, Y_2]$, respectively.

If $c_2$ is ordinary, then $\|c_2\| = y_{2,j}$ for some $j \in \{2, \dots, k\}$ forces $\varphi(c_2) = [e_{33}, e_{33}, \dots] = 0$, so either $\|c_2\| = z_2$ or $c_2$ involves a differential letter, that is $\|c_2\| \in \{y_{2,k}^{\varepsilon_2}, y_{2,k}^{\eta_2}\}$ (recall $z_2 < y_{2,k}$: this is precisely the point in working with non leading commutators). The case $\|c_2\| = z_2^{\varepsilon_2}$ is not possible because $e_{33}^{\varepsilon_2} = 0$, hence let us suppose $\|c_2\| = e_{33}^{\eta_2}$. In this case $\varphi(c_2) = [-e_{23}, e_{33}, \dots, e_{23}, \dots] = 0$ so $\varphi(c_2) = 0$ too. Therefore, $\|c_2\| = z_2$ and this forces $c_2 = [z_2, Y_2]$. Since $c_1$ is non leading as well, similar considerations apply and therefore $u = v$ is the only non vanishing element of $S$ under $\varphi$. Now we may delete $v$ from $S$ and choose another element, until all products of two non leading ordinary commutators have been cancelled from $S$. □

**Lemma 5.4.** *We can delete any* $[z_1, Y_1][z_2, Y_2]$ *such that* $z_1 > Y_1$ *and* $z_2 < \max Y_2$.

*Proof.* Let $v = [z_1, y_{1,1}, \dots, y_{1,h}][z_2, y_{2,1}, \dots, y_{2,k}] =: [z_1, y_{1,1}, Y_1][z_2, Y_2]$ an ordinary element of $S$, with $z_1 > y_{1,h}$ and $z_2 < y_{2,k}$, and consider the substitution

$$\varphi : \begin{array}{ccccc} z_1 & y_{1,1} & Y_1 & z_2 & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_{11} & e_{12} & e_{11} & e_{23} & e_{33} \end{array} .$$

Clearly, $\varphi$ is tailored to make $\varphi(v) \neq 0$, and we are going to prove that $v$ is the only non vanishing element of $S$. So, assume that $u = c_1c_2 \in S$ and $\varphi(u) \neq 0$. As in the previous Lemma, this forces $\mathscr{X}(c_1) = \mathscr{X}([z_1, y_{1,1}, Y_1])$ and $\mathscr{X}(c_2) = \mathscr{X}([z_2, Y_2])$. If $c_2$ involves a differential letter, that must be $y_{2,k}$ because $[z_2, Y_2]$ is not a leading commutator, hence $\|c_2\| \in \{y_{2,k}^{\varepsilon_2}, y_{2,k}^{\eta_2}\}$. However, $\varphi(y_{2,k}) = e_{33}$ and $e_{33}^{\varepsilon_2} = 0$, so $\|c_2\| = y_{2,k}^{\eta_2}$, but then $\varphi(c_2) = [-e_{23}, e_{33}, \dots, e_{23}, \dots] = 0$ too. Hence $c_2$ must be ordinary. If $\|c_2\| \neq z_2$ then $\varphi(c_2) = [e_{33}, e_{33}, \dots, e_{23}, \dots] = 0$, so $\|c_2\| = z_2$ and hence $c_2 = [z_2, Y_2]$. Now, since we already deleted all products of non-leading commutators, $c_1$ must be a leading commutator, hence $\|c_1\| \in \{z_1, z_1^{\varepsilon_1}, z_1^{\eta_1}\}$. By the way, $\varphi(z_1) = e_{11}$, and $e_{11}^{\varepsilon_1} = 0$, so $\|c_1\| = z_1^{\eta_1}$. Hence $\varphi(c_1) = [-e_{12}, e_{12}, \dots] = 0$, and we conclude that $\|c_1\| = z_1$, so $c_1 = [z_1, y_{1,1}, Y_1]$, that is $u = v$. We may then delete $v$ from $S$ and iteratively remove from $S$ all ordinary elements which are a product of a leading commutator and a non-leading one. □

The following statement is a perfect analogous of the previous one, and we omit its proof:

**Lemma 5.5.** *We can delete from $S$ all products $[z_1, Y_1][z_2, Y_2]$ with $z_1 < \max Y_1$ and $z_2 > Y_2$.*

After that, the only ordinary elements still in $S$ are products of leading commutators. We use this fact to delete the elements $c_1c_2 \in \mathscr{S}^{\varepsilon_1}_{n,2}$ for $c_2$ non leading, as well as those in $\mathscr{S}^{\varepsilon_2}_{n,2}$ with $c_1$ non leading.

**Lemma 5.6.** *We can delete from $S$ the vectors $[z_1^{\varepsilon_1}, Y_1][z_2, Y_2]$ such that $z_2 < \max Y_2$, as well as the vectors $[z_1, Y_1][z_2^{\varepsilon_2}, Y_2]$ such that $z_1 < \max Y_1$.*

*Proof.* Pick an element $v = [z_1^{\varepsilon_1}, Y_1][z_2, Y_2] \in S$ such that $z_2 < \max Y_2$, and consider the substitution

$$\varphi : \begin{array}{cccc} z_1 & Y_1 & z_2 & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ e_{12} & e_{11} & e_{23} & e_{33} \end{array}$$

once again, where it may happen $Y_1 = \varnothing$ (that is $v = z_1^{\varepsilon_1}[z_2, Y_2]$), so that actually no indeterminate is sent to $e_{11}$. Assume that $u = c_1c_2 \in S$, with $\varphi(u) \neq 0$. The same considerations of the previous results apply here as well, hence $\varphi(u) \neq 0$ forces $c_2 = [z_2, Y_2]$, because $[z_2, Y_2]$ is not a leading commutator. Now checking $c_1$, the statement is true if $len(c_1) = 1$: it must be $c_1 = z_1^{\varepsilon_1}$. If $len(c_1) > 1$, it is true that $\varphi([z_1, Y_1][z_2, Y_2]) = \varphi(v)$, but $[z_1, Y_1]$ is a leading commutator, $[z_2, Y_2]$ not, and we already deleted $[z_1, Y_1][z_2, Y_2]$ from our list. Hence $c_1 = [z_1^{\varepsilon_1}, Y_1]$, thus $u = v$.

The proof of the second statement is completely analogous. □

After the products $c_1c_2 \in \mathscr{S}^{\varepsilon_1}_{n,2}$ for non leading ordinary commutators $c_2$ have been removed from $S$, it is easy to delete the remaining elements of $\mathscr{S}^{\varepsilon_1}_{n,2}$, namely those $[z_1^{\varepsilon_1}, Y_1][z_2, Y_2]$ with $z_2 > Y_2$. Symmetrically, the same can be done for elements of $\mathscr{S}^{\varepsilon_2}_{n,2}$ still in $S$:

**Lemma 5.7.** *We can delete from $S$ the vectors $[z_1^{\varepsilon_1}, Y_1][z_2, Y_2]$, $[z_1, Y_1][z_2, Y_2]$ and $[z_1, Y_1][z_2^{\varepsilon_2}, Y_2]$ with $z_1 > Y_1$ and $z_2 > Y_2$.*

*Proof.* Let $v = [z_1^{\varepsilon_1}, Y_1][z_2, y_{2,1}, Y_2]$ with of course $z_1 > Y_1$ (recall that $Y_1$ is possibly empty) and $z_2 > Y_2 > y_{2,1}$. We repeat the trick employed in the proof of Lemma 5.4, but we have to distinguish the cases $|Y_1| = 0$ and $|Y_1| > 0$. At first assume $|Y_1| = 0$, so that the first commutator of $v$ is actually $z_1^{\varepsilon_1}$. The substitution

$$\varphi : \begin{array}{cccc} z_1 & z_2 & y_{2,1} & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ e_{12} & e_{33} & e_{23} & e_{33} \end{array}$$

will do the job. Infact $\varphi(v) \neq 0$, and if $u = c_1c_2 \in S$ is such that $\varphi(u) \neq 0$ then it follows $\mathscr{X}(c_1) = z_1$, $\mathscr{X}(c_2) = \mathscr{X}([z_2, y_{2,1}, Y_2])$. Since $c_2$ must be a leading commutator, by the preceding cancellations, it holds $\|c_2\| \in \{z_2, z_2^{\varepsilon_2}, z_2^{\eta_2}\}$. If $\|c_2\| = z_2^{\varepsilon_2}$ then $\varphi(c_2) = 0$ because $e_{33}^{\varepsilon_2} = 0$, and if $\|c_2\| = z_2^{\eta_2}$ then $\varphi(c_2) = [-e_{23}, e_{23}, \dots] = 0$. Hence $\|c_2\| = z_2$, so $c_2 = [z_2, y_{2,1}, Y_2]$ and $u = v$. Similar considerations allow us to deal with the case $[z_1, Y_1]z_2^{\varepsilon_2}$.

Now assume $|Y_1| > 0$, and let $v = [z_1, y_{1,1}, Y_1][z_2, y_{2,1}, Y_2]$ with $z_1 > Y_1 > y_{1,1}$ and $z_2 > Y_2 > y_{2,1}$. We have to modify slightly $\varphi$ in order to work: define

$$\varphi : \begin{array}{cccccc} z_1 & y_{1,1} & Y_1 & z_2 & y_{2,1} & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_{11} & e_{12} & e_{11} & e_{33} & e_{23} & e_{33} \end{array}$$

(a sort of double switch). Now if $u = c_1c_2 \in S$ is such that $\varphi(u) \neq 0$ we know that the factors $c_1$ and $c_2$ of $u$ must have the same contents as those of $v$, hence their leading letters will completely determine $u$. In principle, $\|c_1\| \in \{z_1, z_1^{\varepsilon_1}, z_1^{\eta_1}\}$, but $e_{11}^{\varepsilon_1} = 0$ and if $\|c_1\| = z_1^{\eta_1}$ then $\varphi(c_1) = [-e_{12}, e_{12}, \dots] = 0$. Therefore $\|c_1\| = z_1$ and $c_1 = [z_1, y_{1,1}, Y_1]$. The same reasons provide $c_2 = [z_2, y_{2,1}, Y_2]$, hence just $u = v$ does not vanish under $\varphi$, and we may delete all vectors $[z_1, Y_1][z_2, Y_2]$ from $S$.

Finally, let us consider $v = [z_1^{\varepsilon_1}, Y_1][z_2, Y_2]$ for $|Y_1| > 0$: now that the interfering vector $[z_1, Y_1][z_2, Y_2]$ has been removed from $S$, the earlier version of $\varphi$, namely

$$\varphi : \begin{array}{ccccc} z_1 & Y_1 & z_2 & y_{2,1} & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_{12} & e_{11} & e_{33} & e_{23} & e_{33} \end{array},$$

does not vanish on $v$ only.

After deleting the elements $[z_1^{\varepsilon_1}, Y_1][z_2, Y_2]$ as well as $[z_1, Y_1][z_2, Y_2]$, it is now the turn of the elements $[z_1, y_{1,1}, Y_1][z_2^{\varepsilon_2}, Y_2]$ for $|Y_2| > 0$, and the substitution

$$\varphi : \begin{array}{ccccc} z_1 & y_{1,1} & Y_1 & z_2 & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_{11} & e_{12} & e_{11} & e_{23} & e_{33} \end{array}$$

will work perfectly: $\|c_1\| = z_1$ is forced if $\varphi(c_1c_2) \neq 0$, so $c_1 = [z_1, y_{1,1}, Y_1]$. Since $\mathscr{X}(c_2) = \mathscr{X}([z_2, Y_2])$ and $[z_1, y_{1,1}, Y_1][z_2, Y_2]$ has just been deleted from $S$, it follows $\|c_2\| = z_2^{\varepsilon_2}$, hence just $v$ does not vanish under $\varphi$ and can be removed from $S$ too. $\square$

Now that all elements in $\mathscr{S}^1_{n,2} \cup \mathscr{S}^{\varepsilon_1}_{n,2} \cup \mathscr{S}^{\varepsilon_2}_{n,2}$ have been removed from $S$, with little effort we may delete from $S$ those of $\mathscr{S}^{\varepsilon_1|\varepsilon_2}_{n,2}$, too.

**Lemma 5.8.** *We may delete from $S$ all vectors $[z_1^{\varepsilon_1}, Y_1][z_2^{\varepsilon_2}, Y_2]$.*

*Proof.* Of course, the substitution $(z_1, Y_1, z_2, Y_2) \to (e_{12}, e_{11}, e_{23}, e_{33})$ will not vanish on $[z_1^{\varepsilon_1}, Y_1][z_2^{\varepsilon_2}, Y_2]$ only (of course, if $|Y_i| = 0$ then simply omit the corresponding assignment). $\square$

The vectors still in $S$ are those involving $\eta_i$-letters, that is those in $\mathscr{S}^{\varepsilon_1|\eta_2} \cup \mathscr{S}^{\varepsilon_2|\eta_1} \cup \mathscr{S}^{\eta_1|\eta_2}$ as well as those in $\mathscr{S}^{\eta_1} \cup \mathscr{S}^{\eta_2}$.

**Lemma 5.9.** *We can delete from $S$ all vectors $[z_1^{\eta_1}, Y_1][z_2, Y_2]$.*

*Proof.* Choose, among the vectors $[z_1^{\eta_1}, Y_1][z_2, Y_2]$ still in $S$, one whose second commutator is of maximal length and not a leading one, if any. In the beginning, this means choosing $z_1^{\eta_1}[z_2, Y_2]$ with $z_2 < \max Y_2$. Then set the following substitution

$$\varphi : \begin{array}{ccc} z_1 & z_2 & Y_2 \\ \downarrow & \downarrow & \downarrow \\ e_{22} & e_{23} & e_{33} \end{array}$$

Since $e_{11}^{\eta_1} = e_{12}$, we get $\varphi(v) \neq 0$. Now let $u = c_1c_2 \in S$ such that $\varphi(u) \neq 0$. Still we can check that $z_2$ and all elements of $Y_2$ belong to $\mathscr{X}(c_2)$, as a consequence of Remark 5.2, so this forces $u = z_1^{\eta_1} c_2$, and $\|c_2\|$ will determine completely $c_2$. If $\|c_2\|$ is a derived letter, then $\|c_2\| \in \{y^{\varepsilon_2}, y^{\eta_2}\}$ where $y := \max Y_2 > z_2$, but then $\varphi(y) = e_{33}$, and $e_{33}^{\varepsilon_2} = 0$. All the same $\|c_2\| = y^{\eta_1}$ yields $\varphi(c_2) = 0$: indeed $e_{33}^{\eta_2} = -e_{23}$, so $\varphi(c_2) = [-e_{23}, e_{33}, \dots, e_{23}, \dots] = 0$. Therefore $c_2$ must be an ordinary commutator, and if $\|c_2\| \neq z_2$ then $\varphi(c_2) = [e_{33}, e_{33}, \dots] = 0$. So if $\varphi(u) \neq 0$ then $u = v$ is the only possibility. We then delete $v$ from $S$ and choose a new vector whose second commutator is an ordinary, non leading one of maximal length, build a new substitution tailored on it and finally delete it, and so on until just leading commutators are available as a second factor of maximal length.

We now have to change the substitution: let $v = z_1^{\eta_1}[z_2, y_{2,1}, Y_2] \in S$ (this time $z_2 > Y_2 > y_{2,1}$), and define

$$\psi : \begin{array}{cccc} z_1 & z_2 & y_{2,1} & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ e_{22} & e_{33} & e_{23} & e_{33} \end{array}$$

(apply the "switch"). By Remark 5.2, if $u = c_1 c_2 \in S$ and $\varphi(u) \neq 0$ then $\mathscr{X}(c_2) = \{z_2, y_{2,1}, \dots, y_{2k}\}$, and $\|c_2\| \in \{z_2, z_2^{\varepsilon_2}, z_2^{\eta_2}\}$. Since however $z_2 = e_{33}$, once again $\|c_2\|$ cannot be a derived letter or otherwise $\varphi(c_2) = 0$, hence $c_2$ is an ordinary leading commutator, that is $u = v$. We then proceed in deleting all the remaining elements of $\mathscr{S}_{n,2}^{\eta_1}$ in which the second commutator has length $n - 1$.

This selection algorithm applies until all elements $[z_1^{\eta_2}, Y_1][z_2, Y_2]$ have been deleted: select in $S$ an element with $[z_2, Y_2]$ of maximal length and not leading. If this can be done, consider the substitution

$$\varphi : \begin{array}{cccc} z_1 & Y_1 & z_2 & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ e_{22} & e_{11} & e_{23} & e_{33} \end{array}$$

and check which elements $u = c_1 c_2 \in S$ do not vanish under $\varphi$. By Remark 5.2, $z_2$ and all $y_{2,j}$ must belong to $\mathscr{X}(c_2)$. If a letter $y_{1,j}$ occurs in $c_2$, it cannot be derived because $e_{11}^{\varepsilon_2} = 0 = e_{11}^{\eta_2}$, but if it occurs as ordinary letter then $\varphi(c_2) \in \mathrm{span}_F\langle e_{12}, e_{13}\rangle$ and hence $\varphi(u) = 0$. If $z_1$ occurs in $c_2$ as a derived letter, then $\varphi(c_2) = 0$, because $e_{22}^{\varepsilon_2} = 0$ and $e_{22}^{\eta_2} = e_{23} = \varphi(z_2)$, so in case $z_1$ has to occur as an ordinary letter. Then, since $len(c_2) > len([z_2, Y_2])$, $c_2$ has to involve a differential letter, precisely $y_{2,k} = \max Y_2 > z_2$. This forces $\varphi(c_2) = 0$, since $\varphi(y_{2,k}) = e_{33}$.

Summarizing, the maximality of $len([z_2, X_2])$ forces $\mathscr{X}(c_2) = \mathscr{X}([z_2, Y_2])$ and $\mathscr{X}(c_1) = \mathscr{X}([z_1, Y_1])$. Now we may use the same arguments as before to conclude that $u = v$, then safely delete $v$.

When no non leading commutators $[z_2, Y_2]$ of maximal length are available, pick one leading commutator of maximal length and use "the switch"

$$\psi : \begin{array}{ccccc} z_1 & Y_1 & z_2 & y_{2,1} & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_{22} & e_{11} & e_{33} & e_{23} & e_{33} \end{array} .$$

Again, the maximality and the usual considerations work in order to prove that just the one vector used to define $\psi$ does not vanish. Delete it, then step to the next one until all vectors $[z_1^{\eta_1}, Y_1][z_2, Y_2]$ have been deleted. □

**Remark 5.10.** The different strategy in cancelling the vectors of $\mathscr{S}_{n,2}^{\eta_1}$ is a necessity: on one hand, we cannot avoid to involve $e_{22}$ into the evaluations, in order to pursuing the cancellations; on the other hand, the derivation $\eta_1$, combined with $e_{22}$, has a more elusive nature than $\varepsilon_1$. Essentially, a *migration of letters* from one commutator to another may happen, because Remark 5.2 guarantees just $\mathscr{X}(c_2) \supseteq \mathscr{X}([z_2, Y_2])$ when $e_{22}$ and $\eta_i$ are involved.

For instance, let $v = [x_7^{\eta_1}, x_1, x_2, x_3][x_5, x_4, x_6]$. The second commutator is not a leading one (this was our best help when working with $\varepsilon_1$), and build

$$\varphi : \begin{array}{cccc|ccc} x_7 & x_1 & x_2 & x_3 & x_5 & x_4 & x_6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_{22} & e_{11} & e_{11} & e_{11} & e_{23} & e_{33} & e_{33} \end{array} .$$

Then it really happens that $\varphi(c_1c_2) \neq 0$, but $\{x_4, x_5, x_6\} \subsetneq \mathscr{X}(c_2)$: take for instance $u = [x_3^{\eta_1}, x_1, x_2][x_7, x_4, x_5, x_6]$ (*migration* of $x_7$ to the right). This is the very reason to modify the approach, taking into account a further detail: the length of the right commutator.

Moreover, the use of the "switch" is necessary too, in order to prevent $\varphi([z_2, Y_2]) = \varphi([z_2^{\varepsilon_2}, Y_2])$ when $[z_2, Y_2]$ is a leading commutator.

Once the $[z_1^{\eta_1}, Y_2][z_2, Y_2]$ have been deleted from $S$, the same strategy and, more or less, the same substitutions work in deleting almost all the remaining ones:

**Lemma 5.11.** *We can delete all the vectors* $[z_1^{\eta_1}, Y_1][z_2^{\varepsilon_2}, Y_2]$*, as well as the vectors* $[z_1, Y_1][z_2^{\eta_2}, Y_2]$ *and* $[z_1^{\varepsilon_1}, Y_1][z_2^{\eta_2}, Y_2]$.

*Proof.* Without giving all the details, we continue in picking right commutators of maximal lengths, and apply the following selections and cancellations:

(1) just $[z_1^{\eta_1}, Y_1][z_2^{\varepsilon_2}, Y_2]$ does not vanish under the substitution

$$\begin{array}{cccc} z_1 & Y_1 & z_2 & Y_2 \\ e_{22} & e_{11} & e_{23} & e_{33} \end{array} ;$$

and may be removed from $S$; repeat the step until no vector in $\mathscr{S}_{n,2}^{\varepsilon_2|\eta_1}$ is left; then go to the next step.

(2) If $z_1 < \max Y_1$, just $[z_1, Y_1][z_2^{\eta_2}, Y_2]$ does not vanish under the substitution

$$\begin{array}{cccc} z_1 & Y_1 & z_2 & Y_2 \\ e_{12} & e_{11} & e_{22} & e_{33} \end{array} ;$$

and may be removed from $S$. Repeat the step until just vectors $[z_1, Y_1][z_2^{\eta_2}, Y_2]$, with $len([z_2^{\eta_2}, Y_2]) =: l$ maximal and $z_1 > Y_1$ are available in $\mathscr{S}_{n,2}^{\eta_2}$. Then go to the next step.

(3) If no non-leading left commutator is available, then just $[z_1, y_{1,1}, Y_1][z_2^{\eta_2}, Y_2]$ ($z_1 > Y_1$, $[z_2^{\eta_2}, Y_2]$ of maximal length $l$) does not vanish under the switch

$$\begin{array}{ccccc} z_1 & y_{1,1} & Y_1 & z_2 & Y_2 \\ e_{11} & e_{12} & e_{11} & e_{22} & e_{33} \end{array} ;$$

delete it and repeat the step until no vector has a right commutator of length $l$. If $l > 1$, decree the length to $l-1$ and go to step 2. If $l = 1$, go to step 4.

(4) After deleting all vectors in $\mathscr{S}^{\eta_2}$, just $[z_1^{\varepsilon_1}, Y_1][z_2^{\eta_2}, Y_2]$ does not vanish under

$$\begin{array}{cccc} z_1 & Y_1 & z_2 & Y_2 \\ e_{12} & e_{11} & e_{22} & e_{33} \end{array} ;$$

delete it and repeat the step until no vector in $\mathscr{S}_{n,2}^{\varepsilon_1|\eta_2}$ is left.

□

Almost all of elements of $\mathscr{S}$ have now been deleted: just those in $\mathscr{S}_{n,2}^{\eta_1|\eta_2}$ have been spared. However, not for long:

**Lemma 5.12.** *All elements* $[z_1^{\eta_1}, Y_1][z_2^{\eta_2}, Y_2]$ *can be deleted from* $S$.

*Proof.* Pick an element $v := [z_1^{\eta_1}, Y_1][z_2^{\eta_2}, Y_2]$ with $[z_2^{\eta_2}, Y_2]$ of maximal length, and define

$$\varphi : \begin{array}{cccc} z_1 & Y_1 & z_2 & Y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ e_{11} & e_{11} & e_{33} & e_{33} \end{array} .$$

Then just $v$ does not vanish under $\varphi$, and can be deleted from $S$. Repeat the process until $S$ is empty. □

We can therefore state the main result of the paper:

**Theorem 5.13.** *$T_L(A)$ is generated by the set of polynomials listed in Definition 3.9, and the cosets of $\mathscr{S}_{n,1} \cup \mathscr{S}_{n,2}$ constitute an $F$-basis of $\Gamma_n^L(A)$.*

As an immediate consequence of this description, we record

**Proposition 5.14.** *The proper differential codimension sequence of $A$ is $\gamma_0^L(A) = 1$ and, for $n \geqslant 1$,*

$$\gamma_n^L(A) = (n^2 + 3n + 4)2^{n-2} + 3n + 2.$$

*Proof.* Elementary combinatorial arguments show that for $n \geqslant 1$ the dimensions of the spaces $\Gamma_n^{\mathbf{d}}(A)$ can be summarized in the following table:

| $\mathbf{d}$ | $\gamma_{n,1}^{\mathbf{d}}(A)$ | $\gamma_{n,2}^{\mathbf{d}}(A)$ |
|---|---|---|
| $1$ | $n-1$ | $(n-1)(n-4)2^{n-2} + 2(n-1)$ |
| $\varepsilon_1$ | $n$ | $(n-2)(2^{n-1} - 1)$ |
| $\varepsilon_2$ | $n$ | $(n-2)(2^{n-1} - 1)$ |
| $\eta_1$ | $n$ | $(n-2)(2^{n-1} - 1)$ |
| $\eta_2$ | $n$ | $(n-2)(2^{n-1} - 1)$ |
| $\delta$ | $1$ | $0$ |
| $\varepsilon_1\varepsilon_2$ | $1$ | $0$ |
| $\varepsilon_1\eta_2$ | $1$ | $0$ |
| $\varepsilon_2\eta_1$ | $1$ | $0$ |
| $\eta_1\eta_2$ | $1$ | $0$ |
| $\varepsilon_1\|\varepsilon_2$ | $0$ | $2^n - 2$ |
| $\varepsilon_1\|\eta_2$ | $0$ | $2^n - 2$ |
| $\eta_1\|\varepsilon_2$ | $0$ | $2^n - 2$ |
| $\eta_1\|\eta_2$ | $0$ | $2^n - 2$ |

and $\gamma_n^L(A)$ is just the sum of all those dimensions. □

Once this is done, the differential codimension sequence follows at once:

**Theorem 5.15.** *For all $n \geqslant 1$ it holds*

$$c_n^L(A) = (n^2 + 5n + 9)3^{n-2} + (3n + 4)2^{n-1} - 2.$$

*Proof.* Differential codimensions and proper differential codimensions are related by the formula

$$c_n^L(A) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k^L(A) = 1 + \sum_{k=1}^{n} \binom{n}{k} \left( (k^2 + 3k + 4)2^{k-2} + 3k + 2 \right).$$

Combining the binomial expansion of $(x+y)^n$ with the differential operator $E = x\frac{\partial}{\partial x}$, it is easy to compute the partial sums

$$\sum_{k=1}^{n}\binom{n}{k}k^2 2^{k-2} = \frac{n(2n+1)}{18}3^n \qquad \sum_{k=1}^{n}\binom{n}{k}3k2^{k-2} = \frac{n}{2}3^n$$

$$\sum_{k=1}^{n}\binom{n}{k}2^k = 3^n - 1 \qquad \sum_{k=1}^{n}\binom{n}{k}3k = 3n2^{n-1} \qquad \sum_{k=1}^{n}\binom{n}{k}2 = 2^{n+1} - 2$$

and thus getting $c_n^L(A)$. □

As an easy consequence, we record the following

**Corollary 5.16.** *The differential PI-exponent PI-exp$^L(A)$ exists, and equals the ordinary PI-exponent. More precisely,*

$$\textit{PI-exp}^L(A) = \lim_n \sqrt[n]{c_n^L(A)} = 3.$$

**Remark 5.17.** As already mentioned in the introduction of this paper, it has been proved by Gordienko that the differential PI-exponent of a finite dimensional associative algebra exists ([Go1], Theorem 3), and Gordienko and Kotchetov showed that it equals the ordinary PI-exponent, provided that $L$ is a finite dimensional *semisimple* Lie algebra ([Go&Ko], Theorem 15). In our settings, $L = Der(A)$ is a *solvable*, thus not semisimple, Lie algebra, but the conclusions are the same.

Actually, the common reason is that all derivations of $A$ are inner, as already remarked by the same Authors immediately after Theorem 15.

## 6. Differential cocharacters

The symmetric group $S_n$ acts on $\Gamma_n$, turning it into a left $FS_n$-module; since $I = T_L(A)$ is also $S_n$-invariant, the vector space $\Gamma_n \cap I$ is a left $FS_n$-module, too, and one can consider the factor module $\Gamma_n(A) := \Gamma_n/(\Gamma_n \cap I)$. Let us denote $\xi_n(A)$ its $S_n$-character. Our aim is to decompose $\xi_n(A)$ into its irreducible $S_n$-characters, that is to determine the multiplicities $m_\lambda \geqslant 0$ of the irreducible $S_n$-characters $\chi_\lambda$ of $S_n$, for each $\lambda \vdash n$, so that $\xi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$. In order to avoid round-parentheses proliferation, a partition $\lambda \vdash_k n$ with parts $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ will be denoted by $\lambda =: [\![\lambda_1, \dots, \lambda_k]\!]$. Once and for all, we record the trivial cases

**Lemma 6.1.** $\Gamma_0(A) = F$ *and* $\Gamma_1(A) = span_F\langle x_1^d \mid d \in \mathscr{U}^*\rangle$, *hence* $\xi_0(A) = 1$ *and* $\xi_1(A) = 9\chi_{[\![1]\!]}$.

So in the rest of the section we assume $n \geqslant 2$. In order to simplify the notation, we are going to commit an abuse, and employ the same symbol $\lambda$ to denote the partition $\lambda \vdash n$, the $S_n$-character $\chi_\lambda$ as well as any irreducible $S_n$-module whose $S_n$-character is $\chi_\lambda$, when no confusion may arise.

A finer description of the elements of $\mathscr{S}_n^{\mathbf{d}}$ is needed, in order to keep the exposition as clear as possible.

**Notation.** Let $w$ be any element of the basis $\mathscr{S}_n$ of $\Gamma_n^L(A)$. If $w$ consists of a product of two nsc's, the length and the involved letters of each single commutators are of relevance for us. So assume $w = c_1c_2$, and $c_1 = [y_1^{d_1}, y_2, \dots, y_p]$, $c_2 = [z_1^{d_2}, z_2, \dots, z_q]$ for some $p, q \geqslant 1$, $d_i \in \mathscr{U}$ and $\{y_1, \dots, y_p, z_1, \dots, z_q\} = \{x_1, \dots, x_n\}$.

- The pair $l = l(w) := (p, q)$ is a weak 2-compositon of $n$ with limitations $p, q \geqslant 1$. In symbols, we will write $l(w) \vDash_2 n$, and call $l(w)$ the *structure of* $w$;
- as already agreed, $\mathscr{X}_1 = \mathscr{X}(c_1) = \{y_1, \dots, y_p\}$ is the set of indeterminates occurring in $c_1$, and similarly $\mathscr{X}_2 = \mathscr{X}(c_2) = \{z_1, \dots, z_q\}$ the set of those occurring in $c_2$. Recall that $\|c_1\| = y_1^{d_1}$ is a letter, not an indeterminate, unless $d_1 = 1$. We set $\mathscr{X}(w) := (\mathscr{X}_1, \mathscr{X}_2)$, a 2-partition of $\{x_1, \dots, x_n\}$;
- $[n] := \{1, \dots, n\}$; if $\mathscr{A} \subseteq [n]$, the subgroup of $S_n$ made of all permutations on $\mathscr{A}$ will be denoted $Sym(\mathscr{A})$;
- $\mathscr{N}_1 = \mathscr{N}(c_1) := \{j \in [n] \mid x_j \in \mathscr{X}_1\}$, the set of the names of the indeterminates occurring in $c_1$, and similarly define $\mathscr{N}_2 = \mathscr{N}(c_2)$. Then $\mathscr{N}(w) := (\mathscr{N}_1, \mathscr{N}_2)$ is a 2-partition of $[n]$;
- since $Sym(\mathscr{N}_1)$, $Sym(\mathscr{N}_2)$ are disjoint subgroups of $S_n$, they mutally centralize and their (internal) direct product $H_w$ is a subgroup of $S_n$, isomorphic to the (external) direct product $S_p \times S_q$.

Why is this necessary? Since the cosets $\{w + I \mid w \in \mathscr{S}_n\}$ form an $F$-basis for $\Gamma_n(A)$, and $\mathscr{S}_n$ is partitioned into the subsets $\mathscr{S}^{\mathbf{d}}_{n,k}$ for $\mathbf{d} \in \mathbf{D}$ and $k = 1, 2$, we get the decomposition $\Gamma_n(A) = \bigoplus_{\mathbf{d},k} \Gamma^{\mathbf{d}}_{n,k}(A)$. It would be nice if all the subspaces $\Gamma^{\mathbf{d}}_{n,k}(A)$ were $S_n$-submodules of $\Gamma_n(A)$. Sometimes it happens, but in general it fails:

**Example 6.2.** Let $w := [x_2^{\varepsilon_1}, x_1][x_4, x_3] \in \mathscr{S}^{\varepsilon_1}_{4,1}$, and let $\sigma := (1\,2)$. Then $w + I \in \Gamma^{\varepsilon_1}_{4,2}(A)$, but $\sigma(w + I) = [x_1^{\varepsilon_1}, x_2][x_4, x_3] + I$ does not belong to $\Gamma^{\varepsilon_1}_{4,2}(A)$. Actually,

$$[x_1^{\varepsilon_1}, x_2][x_4, x_3] + I = ([x_2^{\varepsilon_1}, x_1][x_4, x_3] - [x_2, x_1][x_4, x_3]) + I \in \Gamma^{\varepsilon_1}_{4,2}(A) \oplus \Gamma^{1}_{4,2}(A).$$

In fact, we must recall that by definition $\Gamma^{\mathbf{d}}_{n.k} = \operatorname{span}_F\langle \mathscr{P}^{\mathbf{d}}_{n,k}\rangle$, so it is not an $S_n$-invariant subspace of $\Gamma_n$ (recall the warning just before Example 4.1). Hence, some care is required in getting the decomposition of $\xi_n(A)$. Let us start by investigating the $S_n$-submodules of $\Gamma_n(A)$: the character of the mean one is actually recovered for free:

**Proposition 6.3.** *For all $n \geqslant 2$ the vector space $\Gamma^1_n(A)$ is an $S_n$-submodule of $\Gamma_n(A)$ and $\xi^1_n(A) = \sum_{\lambda \vdash n} m_\lambda \lambda$, where*

- $m_\lambda = 1$ *if* $\lambda = [\![n-1, 1]\!]$*;*
- $m_\lambda = n - 3$ *if* $\lambda = [\![n-2, 1, 1]\!]$*;*
- $m_\lambda = b + 1$ *if* $\lambda = [\![2 + a + b, 2 + a]\!]$*,* $\lambda = [\![2 + a + b, 2 + a, 2]\!]$ *or* $\lambda = [\![1 + a + b, 1 + a, 1, 1]\!]$*, for all* $a, b \geqslant 0$*;*
- $m_\lambda = 2(b + 1)$ *if* $\lambda = [\![1 + a + b, 1 + a, 1]\!]$*, if* $a \geqslant 1$ *and* $b \geqslant 0$*;*
- $m_\lambda = 0$ *for any other* $\lambda \vdash n$*.*

*Proof.* Since $\Gamma^1_n(A) = \Gamma_n(UT_3(F))$, the $n$-th ordinary proper multilinear space of $UT_3(F)$, by [Dr&Ka] for all $n \geqslant 2$ it holds

$$\xi^1_n(A) = [\![n-1, 1]\!] + \sum_{\substack{(p_1,p_2) \vDash_2 n \\ p_1, p_2 \geqslant 2}} \left([\![p_1 - 1, 1]\!] \otimes [\![p_2 - 1, 1]\!]\right)^{S_n},$$

where $(p_1, p_2)$ runs among all the weak compositions of $n$ in two parts subject to the limitations $p_1, p_2 \geqslant 2$, and $[\![\alpha \otimes \beta]\!]^{S_n}$ denotes the $S_n$-character induced from $S_{p_1} \times S_{p_2}$ to $S_n$. Then the Littlewood-Richardson rule provides the listed multiplicities of the irreducible $S_n$-characters. □

Almost for free one also gets

**Proposition 6.4.** *For all $n \geqslant 2$ the vector space $\Gamma_n^\delta(A)$ is an $S_n$ submodule of $\Gamma_n(A)$, and it holds $\xi_n^\delta(A) = [\![n]\!]$.*

*Proof.* $\Gamma_n^\delta(A)$ is spanned by the only vector $[x_n^\delta, x_1, \ldots, x_{n-1}] + I$, which is fixed by $S_n$. ☐

There are three more submodules:

**Proposition 6.5.** *For all $n \geqslant 2$ the vector space $\Gamma_n^{\eta_1|\eta_2}(A)$ is an $S_n$-sumodule of $\Gamma_n(A)$, and it holds $\xi_n^{\eta_1|\eta_2}(A) = \sum_{\lambda\vdash n} m_\lambda \lambda$, where*

- *$m_\lambda = n-1$ if $\lambda = n$;*
- *$m_\lambda = b+1$ if $\lambda = [\![1+a+b, 1+a]\!] \vdash n$;*
- *$m_\lambda = 0$ for all other $\lambda \vdash n$.*

*Proof.* Let $w \in \mathscr{S}_{n,2}^{\eta_1|\eta_2}$, of structure $l(w) = (l_1, l_2) \vDash_2 n$ (with $l_1, l_2 \geqslant 1$) and $\mathscr{X}(w) = (\mathscr{X}_1, \mathscr{X}_2)$. If $\sigma \in S_n$, the polynomial $\sigma w$ is a product of commutators still of structure $l(w)$, and contents $\sigma\mathscr{X} = (\sigma\mathscr{X}_1, \sigma\mathscr{X}_2)$. By Lemma 4.6 (4) its normal form is an element of $\mathscr{S}_{n,2}^{\eta_1,\eta_2}$ still of structure $l(w)$ and contents $\sigma\mathscr{X}$. In particular, the vector space $\Gamma_{n,2}^{\eta_1|\eta_2}(A)$ is $S_n$-invariant and, as such, an $S_n$-submodule of $\Gamma_n(A)$.

In particular, any element of $H_w$ acts trivially on $w$, so $FH_w w$ is a one-dimensional $H_w$-module, $FH_w w \cong_{H_w} [\![p_1]\!] \otimes [\![p_2]\!]$. Inducing up to $S_n$, we get a submodule $W(l)$ of $\Gamma_{n,2}^{\eta_1|\eta_2}(A)$ of dimension $\binom{n}{l_1}$. Since there are exactly $\binom{n}{l_1}$ elements of $\mathscr{S}_{n,2}^{\eta_1|\eta_2}$ of structure $l$, we get the (internal) $S_n$-decomposition

$$\Gamma_n^{\eta_1|\eta_2}(A) = \Gamma_{n,2}^{\eta_1|\eta_2}(A) = \bigoplus_{\substack{(l_1,l_2)\vDash_2 n\\ l_1,l_2\geqslant 1}} W(l)$$

with character

$$\xi_n^{\eta_1|\eta_2}(A) = \sum_{\substack{(l_1,l_2)\vDash_2 n\\ l_1,l_2\geqslant 1}} \left([\![l_1]\!] \otimes [\![l_2]\!]\right)^{S_n}.$$

Now the Young-Pieri rule provides the stated decomposition. ☐

**Proposition 6.6.** *The vector spaces $\Gamma_{n,2}^{\eta_1}(A)$, $\Gamma_{n,2}^{\eta_2}(A)$ are $S_n$-submodules of $\Gamma_n(A)$, with character $\xi_{n,2}^{\eta_1}(A) = \xi_{n,2}^{\eta_2}(A) = \sum_{\lambda\vdash n} m_\lambda\lambda$ where*

- *$m_\lambda = n-2$ if $\lambda = [\![n-1, 1]\!]$;*
- *$m_\lambda = b+1$ if $\lambda = [\![2+a+b, 2+a]\!]$ and $a, b \geqslant 0$;*
- *$m_\lambda = b+1$ if $\lambda = [\![1+a+b, 1+a, 1]\!]$ and $a, b \geqslant 0$;*
- *$m_\lambda = 0$ for all other $\lambda \vdash n$.*

*Proof.* The proof is similar for both vector spaces, so let us deal with $\Gamma_{n,2}^{\eta_1}(A)$. If $w \in \mathscr{S}_{n,2}^{\eta_1}$, let $l = l(w)$ and $\mathscr{X}(w) = (\mathscr{X}_1, \mathscr{X}_2)$ be its structure and content, with $l_1 \geqslant 1$ and $l_2 \leqslant 2$. Any $\sigma \in S_n$ may vary the contents of $w$, not the structure, hence by Lemma 4.7, (2) and (4), the normal form of $\sigma w$ is still in $\mathscr{S}_{n,2}^{\eta_1}$. Hence $\Gamma_{n,2}^{\eta_1}(A) \leqslant_{S_n} \Gamma_n(A)$.

In particular, $FH_w w$ is an $H_w$-module isomorphic to $[\![l_1]\!] \otimes [\![l_2 - 1, 1]\!]$, and induces an $S_n$-submodule $W(l)$ of $\Gamma_{n,2}^{\eta_1}(A)$ of dimension $[S_n : H_w](l_2 - 1) = \binom{n}{l_1}(l_2 - 1)$, which is the number of vectors in $\mathscr{S}^{\eta_1}_{n,2}$ with structure $l$. This means that $W(l)$ is the vector space spanned by all such vectors, and one gets

$$\Gamma^{\eta_1}_{n,2}(A) = \bigoplus_{\substack{(l_1,l_2)\vDash_2 n\\ l_1\geqslant 1\\ l_2\geqslant 2}} \left(\llbracket l_1\rrbracket \otimes \llbracket l_2-1,1\rrbracket\right)^{S_n},$$

whose character is

$$\xi^{\eta_1}_{n,2}(A) = \sum_{\substack{(l_1,l_2)\vDash_2 n\\ l_1\geqslant 1\\ l_2\geqslant 2}} \left(\llbracket l_1\rrbracket \otimes \llbracket l_2-1,1\rrbracket\right)^{S_n}.$$

The Young-Pieri rule once again gives the stated multiplicities $m_\lambda$ of its irreducible components. □

$\Gamma^{\varepsilon_1}_{n,2}(A)$ (and the same for $\varepsilon_2$) is not an $S_n$-submodule of $\Gamma_n(A)$, as already remarked. By the way, let us factor out the submodule $N := \Gamma^1_n(A) \oplus \Gamma^{\eta_1}_{n,2}(A) \oplus \Gamma^{\eta_2}_{n,2}(A) \leqslant_{S_n} \Gamma_n(A)$ and consider the factor module $\Gamma_n/N$. Here $\Gamma^{\varepsilon_1}_{n,2}(A)$ turns into and $S_n$-submodule, as well as several other vector spaces $\Gamma^{\mathbf{d}}_{n,k}(A)$.

**Proposition 6.7.** *Let $\pi_N : \Gamma_n(A) \to \Gamma_n(A)/N$ the canonical homomorphism. Then the $\pi$-images of $\Gamma^{\varepsilon_i}_{n,2}(A), \Gamma^{\varepsilon_1|\eta_2}_{n,2}(A)$, $\Gamma^{\varepsilon_2|\eta_1}_{n,2}(A)$ are $S_n$-submodules of $\Gamma_n(A)/K$, and*

- *$\xi^{\eta_1}_{n,2}(A)$ equals the character of $\pi_N(\Gamma^{\varepsilon_i}_{n,2}(A))$;*
- *$\xi^{\eta_1|\eta_2}_{n,2}(A)$ equals the character of $\pi_N(\Gamma^{\varepsilon_1|\eta_2}_{n,2}(A))$ and of $\pi_N(\Gamma^{\varepsilon_2|\eta_1}_{n,2}(A))$.*

*Proof.* Let $w \in \mathscr{S}^{\varepsilon_1}_{n,2}$ and let $\sigma \in S_n$: $\sigma w$ is then a product of two commutators, the first one involving a single $\varepsilon_1$-letter in its first position. It may be not an element of $\mathscr{S}^{\varepsilon_1}_{n,2}$ still, but its normal form is an element of $\mathscr{S}^{\varepsilon_1}_{n,2}$ up to an element in $\Gamma^1_{n,2}(A)$, by Lemma 4.7, (1). Hence, working modulo $N$, $\Gamma^{\varepsilon_1}_{n,2}(A)$ becomes an $S_n$-module. Moreover, since $FH_w \cong_{FH_w} \llbracket l_1(w)\rrbracket \otimes \llbracket l_2(w)-1,1\rrbracket$, it induces an $S_n$-submodule of dimension $\binom{n}{l_1(w)}(l_2(w)-1)$, which is exactly the cardinality of $\mathscr{S}^{\varepsilon_1}_{n,2}$, and so it equals the image of $\Gamma^{\varepsilon_1}_{n,2}(A)$. Hence the character of $\pi_n(\Gamma^{\varepsilon_1}_{n,2}(A))$ is

$$\sum_{\substack{(l_1,l_2)\vDash_2 n\\ l_1\geqslant 1\\ l_2\geqslant 2}} \left(\llbracket l_1\rrbracket \otimes \llbracket l_2-1,1\rrbracket\right)^{S_n} = \xi^{\eta_1}_{n,2}(A).$$

Similar arguments, employing 4.6, points (2) and (3), provide the second equality. □

The arguments employed in the preceding proof yield that the vector space

$$N \oplus \Gamma^{\varepsilon_1}_{n,2}(A) \oplus \Gamma^{\varepsilon_2}_{n,2}(A) \oplus \Gamma^{\varepsilon_1|\eta_2}_{n,2}(A) \oplus \Gamma^{\varepsilon_2|\eta_1}_{n,2}(A) \oplus \Gamma^{\eta_1|\eta_2}_{n,2}(A)$$

is in fact an $S_n$-submodule of $\Gamma_n(A)$, and by complete reducibility we also got its $S_n$-character. We can now repeat the trick: replace the former $N$ with this module and factor it out from $\Gamma_n(A)$, getting

**Proposition 6.8.** *Let $\pi_N : \Gamma_n(A) \to \Gamma_n(A)/N$ be the canonical $S_n$-epimorphism. Then $\pi_N(\Gamma^{\varepsilon_1|\varepsilon_2}_{n,2}(A))$ is an $S_n$-submodule of $\Gamma_n(A)/N$, and its character equals $\xi^{\eta_1|\eta_2}_{n,2}(A)$.*

*Proof.* By Lemma 4.6, (1), $\Gamma_{n,2}^{\varepsilon_1|\varepsilon_2}(A)$ is spanned by $\mathscr{S}_{n,2}^{\varepsilon_1|\varepsilon_2}$ modulo $N$, hence its $\pi_N$-image is a submodule of $\Gamma_n(A)/N$. In particular, the $N$-coset of any $w \in \mathscr{S}_{n,2}^{\varepsilon_1|\varepsilon_2}$ is fixed by $H_w$, hence $FH_w w \cong_{H_w} [\![l_1(w)]\!] \otimes [\![l_2(w)]\!]$, so its character is

$$\sum_{\substack{(l_1,l_2)\models_2 n \\ l_1,l_2\geqslant 1}} \big([\![l_1]\!] \otimes [\![l_2]\!]\big)^{S_n} = \xi_{n,2}^{\eta_1|\eta_2}(A).$$

□

Now that $\Gamma_{n,2}(A)$ has been ruled out, let us turn our attention to $\Gamma_{n,1}(A)$; it is definitely not a submodule of $\Gamma_n(A)$, but again we can study it factoring out what is already known:

**Proposition 6.9.** *Let* $N := \Gamma_{n,1}^{1}(A) \oplus \bigoplus_{\mathbf{d}} \Gamma_{n,2}^{\mathbf{d}}(A)$. *Then the character of the factor module* $\Gamma_n(A)/N$ *is* $\xi_{n,1}(A) = 8[\![n]\!] + 4[\![n-1,1]\!]$.

*Proof.* The vector $[x_n^{\varepsilon_1\varepsilon_2}, x_1, \dots, x_{n-1}] + N$ is fixed by the $S_n$-action, hence it provides an irreducible one-dimensional component $[\![n]\!]$, and the same holds for $\varepsilon_1\eta_2$, $\varepsilon_2\eta_1$ and $\eta_1\eta_2$.

Then, consider $w = [x_1^{\varepsilon_1}, x_2, \dots, x_n] + N$. The group $H := S_1 \times Sym(2, \dots, n)$ acts trivially on it, so $FHw = [\![1]\!] \otimes [\![n-1]\!]$. Inducing it to $S_n$, we get

$$\big([\![1]\!] \otimes [\![n-1]\!]\big)^{S_n} = [\![n]\!] \oplus [\![n-1,1]\!],$$

of dimension $n$, which is exactly the number of elements of $\mathscr{S}_{n,1}^{\varepsilon_1}$. Therefore it is precisely the vector space $(\Gamma_{n,1}^{\varepsilon_1}(A) + N)/N$. The same arguments apply to $\varepsilon_2, \eta_1$ and $\eta_2$, and the statement follows. □

Now, by complete reducibility, we can summarize the results obtained so far:

**Theorem 6.10.** *Let* $\xi_n^L(A) = \sum_{\lambda\vdash n} m_\lambda \lambda$ *be the* $S_n$*-character of* $\Gamma_n^L(A)$. *Then*

- $\xi_2^L(A) = 13[\![2]\!] + 9[\![1,1]\!]$;
- $\xi_3^L(A) = 17[\![3]\!] + 17[\![2,1]\!] + 4[\![1,1,1]\!]$

*and, for* $n \geqslant 4$ *and for any* $\lambda \vdash n$, *it holds*

- $m_\lambda = 4n + 5$ *if* $\lambda = [\![n]\!]$;
- $m_\lambda = 12n - 15$ *if* $\lambda = [\![n-1,1]\!]$;
- $m_\lambda = 9(b+1)$ *if* $\lambda = [\![2+a+b, 2+a]\!]$ *for* $a, b \geqslant 0$;
- $m_\lambda = 5n - 11$ *if* $\lambda = [\![n-2,1,1]\!]$;
- $m_\lambda = 6(b+1)$ *if* $\lambda = [\![2+a+b, 2+a, 1]\!]$ *for* $a, b \geqslant 0$;
- $m_\lambda = b+1$ *if* $\lambda = [\![2+a+b, 2+a, 2]\!]$ *for* $a, b \geqslant 0$;
- $m_\lambda = b+1$ *if* $\lambda = [\![1+a+b, 1+a, 1, 1]\!]$ *for* $a, b \geqslant 0$;
- $m_\lambda = 0$ *for all other* $\lambda$.

*Proof.* The cases $n = 2$ and $n = 3$ have to be investigated directly, but are easily checked. Then, by complete reducibility, the multiplicities for $n \geqslant 4$ follow from all preceding partial results. □

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