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A BRÉZIS-NIRENBERG TYPE PROBLEM FOR A CLASS OF DEGENERATE ELLIPTIC PROBLEMS INVOLVING THE GRUSHIN OPERATOR

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Abstract. Motivated by the seminal paper due to Brézis-Nirenberg [12], we will establish the existence of solutions for the following class of degenerate elliptic equations with critical nonlinearity:

$$\begin{cases} -\Delta_\gamma v = \lambda |v|^{q-2} v + |v|^{2^*_\gamma - 2}\, v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_\gamma := \Delta_x + (1+\gamma)^2\, |x|^{2\gamma}\, \Delta_y$ is the Grushin operator, $z := (x, y) \in \mathbb{R}^N$, $N = m + n$, $m, n \geq 1$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda > 0$, $q \in [2, 2^*_\gamma)$ and $2^*_\gamma = \frac{2N_\gamma}{N_\gamma - 2}$ is the critical Sobolev exponent in this context, where $N_\gamma = m + (1+\gamma)n$ is the so-called homogeneous dimension attached to the Grushin operator Δ_γ. In order to prove our main results it was necessary to do a careful study involving the best constant $\mathcal{S}_\gamma(m, n)$ of the Sobolev embedding for the spaces associated with Δ_γ. In order to do that, we prove a version of the Lions' Concentration-Compactness Principle for the Grushin operator. We also provide existence results for a critical problem involving the Grushin operator on the whole space $\mathbb{R}^N$.

Contents

1. Introduction

In the last years many authors have considered the existence of solutions for some classes of elliptic problems with critical growth and the motivation to consider this type of problem comes from the seminal paper due to Brezis-Nirenberg [12], where the authors obtained the existence of positive solutions for the problem

$$\begin{cases} -\Delta v = \lambda |v|^{q-2} v + |v|^{2^* - 2}\, v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$ for $N \geq 3$ is a smooth bounded domain, $\lambda > 0$, $q \in [2, 2^*)$ and $2^* = \frac{2N}{N-2}$ for the following cases:
Case 1: $q = 2$ and $\lambda \in (0, \lambda_1(\Omega))$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ for $N \geq 4$ and λ belonging to a left neighborhood of $\lambda_1(\Omega)$ for $N = 3$;
Case 2: $q \in (2, 2^*)$, $\lambda > 0$ and $N \geq 4$;
Case 3: $q \in (2, 2^*)$ and $N = 3$: if $4 < q < 6$, existence of solutions is proved for any $\lambda > 0$; if $2 < q \leq 4$, existence is proved for λ sufficiently large. (see for this case [12], Example 2.3 and 2.4).

The main tool used in [12] is the variational method and a key point in the approach introduced in that paper are the properties of the best Sobolev constant S of the embedding $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \hookrightarrow L^{2^*}\left(\mathbb{R}^N\right)$. In particular, the functions $(\epsilon + |x|^2)^{-\frac{N-2}{2}}$, $\epsilon > 0$, play a natural role because they are the extremal functions for the Sobolev inequality in $\mathbb{R}^N$.

For the reader interested in this type of problems, we cite the papers [7], [13], [14], [15], [18], [36] and the references therein.

Motivated by the results in [12], in the present paper we intend to establish the existence of solutions for the following class of problems

$$\begin{cases} -\Delta_\gamma v = \lambda |v|^{q-2} v + |v|^{2^*_\gamma - 2}\, v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_γ is the so-called Grushin operator, defined as

$$\Delta_\gamma v = \Delta_x v + (1+\gamma)^2 \, |x|^{2\gamma} \, \Delta_y v, \tag{1.2}$$

where Δ_x and Δ_y are the Laplace operators in the variables x and y, respectively, $z := (x, y) \in \mathbb{R}^{m+n}$, $m, n \geq 1$ and $\gamma > 0$. In the considered problem (1.1), $\Omega \subset \mathbb{R}^{m+n}$ is a smooth bounded domain, $2^*_\gamma = \frac{2N_\gamma}{N_\gamma - 2}$ is the critical Sobolev exponent in this context, where $N_\gamma = m + (1+\gamma)n$ is the so-called *homogeneous dimension* attached to the operator Δ_γ, $q \in [2, 2^*_\gamma)$ and $\lambda \in \mathbb{R}$. Precisely, we search for weak solutions of (1.1) (see the definition (2.3) in Section 2), i.e. solutions in the Sobolev space $\overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\| := \left(\int_\Omega |\nabla_\gamma v|^2 \, dz \right)^{1/2},$$

where $\nabla_\gamma v = (\nabla_x v, (1+\gamma)|x|^\gamma \nabla_y v)$.

In order to prove our main results, it was necessary to do a careful study involving the best constant $\mathcal{S}_\gamma(m, n)$ of the Sobolev embedding $\mathcal{D}^{1,2}_\gamma\left(\mathbb{R}^N\right) \hookrightarrow L^{2^*_\gamma}\left(\mathbb{R}^N\right)$, where the space $\mathcal{D}^{1,2}_\gamma\left(\mathbb{R}^N\right)$ is defined in Sect. 2. Since the problem of the explicit knowledge of the Sobolev extremals for the Grushin operator has not been completely solved (see Monti [37] and the recent results in Dou et al [20]), following the approach in [33] we overcome this difficulty by means of a suitable a priori qualitative analysis of such extremals, which is, in fact, sufficient for our purpose. In particular, a preliminary crucial step will be to prove the existence of such minimizers for any $m, n \geq 1$, by means of a version of Lions' Concentration-Compactness Principle for the Grushin operator, which is a novelty for this class of problems, see Section 3 for more details. Then, we will use the knowledge of their asymptotic decay at infinity proved in [34] and recalled in Theorem 3.7.

Concerning the involved operator, we recall that in 1920, Tricomi [39] introduced a degenerate differential operator of the form

$$D := \frac{\partial^2}{\partial x^2} + x \frac{\partial^2}{\partial y^2}.$$

Later Grushin [25, 26] and Baouendi [9] generalized this definition to Δ_γ. Franchi and Lanconelli [23] obtained fundamental regularity and embedding results for a wide class of subelliptic operators including Δ_γ. This type of operator appears in the study of P.D.E. on manifolds, for example in Kogoj and Lanconelli [28], the authors showed that it appears in the study of a P.D.E. in the Heisenberg group, while in Bhakta and Sandeep [11] a strong relation with a P.D.E on the Hyperbolic space was pointed out. For recent results involving the Grushin operator, we refer to [17, 19, 20, 21, 22, 29, 31, 35, 37, 38, 40].

Our first main result concerns the existence of solutions for problem (1.1) with linear lower order term

$$\begin{cases} -\Delta_\gamma v = \lambda v + |v|^{2^*_\gamma - 2}\, v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain intersecting the degeneration set of the operator, i.e. such that $\Omega \cap \{x = 0\} \neq \emptyset$.

We emphasize that the condition $\Omega \cap \{x=0\} \neq \emptyset$ ensures that the considered problem with exponent $2^*_\gamma = \frac{2N_\gamma}{N_\gamma - 2}$ turns out to be "critical" in the usual sense of the Sobolev embeddings, since for these domains the Sobolev embedding $\overset{\circ}{H}{}^{1,2}_\gamma(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $p < 2^*_\gamma$, but fails to be compact for $p = 2^*_\gamma$, due to the action of the rescalings defined in (3.2). For domains that are far from the degeneration set, instead, the operator becomes uniformly elliptic and the underlying dimension $N = m + n$ takes over: in particular, for these domains the threshold of compactness is the usual exponent $\frac{2N}{N-2}$, which is higher than $2^*_\gamma = \frac{2N_\gamma}{N_\gamma - 2}$, and the problem of the existence of solutions for (1.3) can be provided by standard compactness arguments. So, our analysis will be devoted to domains intersecting the degeneration set $\{x = 0\}$. We will see that in the Brezis-Nirenberg problems for this class of domains the homogeneous dimension N_γ plays exactly the same role as the space dimension in the ordinary Laplacian case.

We also observe that the limit case when only the boundary of Ω touches the degeneration set $\{x = 0\}$ is more delicate from the geometric point of view and it will be addressed in a separate work.

Now, let $\lambda_1(\Omega)$ denote the first Dirichlet eigenvalue of $-\Delta_\gamma$ on $\overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$. We observe that, by the validity of the Poincaré inequality for the Grushin gradient and the compactness of the embedding of $\overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ into $L^2(\Omega)$, it follows by standard procedure in spectral theory that the spectrum of the Dirichlet $-\Delta_\gamma$ is discrete and that $\lambda_1(\Omega) > 0$ (see for more details [29]). We prove the following existence result:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $\Omega \cap \{x = 0\} \neq \emptyset$. Then, the following statements hold:*
i) If $N_\gamma \geq 4$, problem (1.3) has a nontrivial solution $v \in \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ for any $0 < \lambda < \lambda_1(\Omega)$;
ii) If $N_\gamma < 4$, there exists $\lambda_* > 0$ such that (1.3) has a nontrivial solution $v \in \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ for $\lambda \in (\lambda_*, \lambda_1(\Omega))$.

Moreover, in dimension $N_\gamma < 4$, we also prove that the problem does not admit nonnegative nontrivial solutions for small positive λ's, under some additional regularity assumptions up the boundary.

Theorem 1.2. *If $N_\gamma < 4$ and Ω is strictly δ_t-starshaped about a point of $\{x = 0\}$, there exists $\lambda_{**} > 0$ such that problem (1.3) has no nonnegative nontrivial solution $v \in C^2(\overline{\Omega})$ for $0 < \lambda < \lambda_{**}$.*

We observe that the above non-existence result of solutions for small positive λ's in dimension $N_\gamma < 4$ is obtained by means of a Pohozaev-type argument modelled on the Grushin geometry and it shows a phenomenon of so-called *critical dimensions* in this subelliptic context, analogous to the one observed for the ordinary Laplacian operator in dimension $N = 3$ (see [12]).

Our second main result involves the existence of solutions for the problem

$$\begin{cases} -\Delta_\gamma v = \lambda |v|^{q-2} v + |v|^{2^*_\gamma - 2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $2 < q < 2^*_\gamma$ and it can be stated as follows.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $\Omega \cap \{x = 0\} \neq \emptyset$ and let $q \in (2, 2^*_\gamma)$. Then, the following statements hold:*
a) If $N_\gamma \geq 4$, problem (1.4) has a nontrivial solution $v \in \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ for any $\lambda > 0$;
b) If $N_\gamma < 4$ and $\frac{4}{N_\gamma - 2} < q < 2^*_\gamma$, problem (1.4) has a nontrivial solution $v \in \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ for any $\lambda > 0$; if $2 < q \leq \frac{4}{N_\gamma - 2}$, there exists $\lambda_0 > 0$ such that problem (1.4) has a nontrivial solution for $\lambda > \lambda_0$.

In the last part of the paper, we consider a critical problem involving the Grushin operator on the whole space $\mathbb{R}^N$. Very recently, Alves and de Holanda [1] established the existence of nontrivial solutions of the following problem:

$$-\Delta_\gamma v + a(z) v = f(v) \quad \text{in} \quad \mathbb{R}^N, \tag{1.5}$$

where $\gamma > 0$, f has a subcritical growth, i.e., $|f(s)| = O(s^q)$, $1 < q < \frac{N_\gamma + 2}{N_\gamma - 2}$. More precisely, they have obtained critical points of the energy functional corresponding to (1.5) in the subspace $H^{1,2}_{\gamma, rad_{x,y}}(\mathbb{R}^N)$ when $a(z) = 1$ and in the subspace $H^a_{\gamma, rad_x}(\mathbb{R}^N)$ when $a(z) \geq 0$ is either periodic in the variable y or coercive in the variable y. They have made use of the *principle of symmetric criticality* (see, for instance, [41, Theor. 1.28]) to show that these critical points are solutions of (1.5). The existence of nontrivial solutions of (1.5) was open in the critical case, i.e., when $|f(s)| = O(s^{\frac{N_\gamma + 2}{N_\gamma - 2}})$. Here we complete this step.

Concerning problems for the Grushin operator in the whole space, we also quote the recent paper [2], where a problem of the type (1.5) is studied in $\mathbb{R}^2$ with nonlinearity combining exponential growth and critical Grushin exponent. Precisely, in [2], existence results are provided for the problem

$$-\Delta_\gamma v + a(z)v = f(z,v) \ \text{ in } \ \mathbb{R}^2,$$

where f is of the form $f(z,t) = h(|x|)|t|^{4/\gamma}t + (1-h(|x|))g(t)$, $\forall z = (x,y) \in \mathbb{R}^2$ and $t \in \mathbb{R}$, where the term $|t|^{4/\gamma}t$ has critical behavior, being $2^*_\gamma = 4/\gamma + 2$ for $m = n = 1$, $g(t)$ behaves like $e^{\alpha t^2}$, as $t \to \infty$, for some $\alpha > 0$, and h is a smooth function that modulates the behavior of the function f near the degeneracy region, that is, $0 \le h \le 1$, such that $h(t) = 1$ if $0 \le t \le \delta$ and $h(t) = 0$ if $t \ge 2\delta$, for some $\delta > 0$; moreover, a is assumed to be bounded from below by a positive constant, coercive in the variable x and periodic in the variable y.

We recall that there have been several works on the existence of solutions to elliptic equations in the whole $\mathbb{R}^N$ with nonlinearity having a critical growth. We refer to [3, 4, 16, 43] and the references therein, where the authors have established the existence of ground state solutions to nonlinear scalar field equations with critical growth and fractional scalar field equations under a general critical nonlinearity.

Motivated by the above research works, it is natural to ask the following question:

Q. Can we establish the existence of nontrivial solution to (1.5) with critical nonlinearity?

In the sequel, we answer this question. More precisely, our third main result establishes the existence of nontrivial solutions of (1.5) in the critical case under certain assumptions on a and f. In fact, we deal with the following class of problems:

$$-\Delta_\gamma v + a(z)v = \lambda\,|v|^{q-2}\,v + |v|^{2^*_\gamma - 2}\,v \ \text{ in } \mathbb{R}^N, \tag{1.6}$$

where $\lambda > 0$, $2 < q < 2^*_\gamma$ and a is a continuous real valued function satisfying the following assumptions:

(a1) $a(z) \ge a_0 > 0$ for all $z \in \mathbb{R}^N$.
(a2) **Radial symmetry in variable** x**:** $a(x,y) = a(x',y)$ for all $x, x' \in \mathbb{R}^m$ with $|x| = |x'|$ and for all $y \in \mathbb{R}^n$.
(a3) **Coercivity in variable** y **:** $a(x,y) \to +\infty$ as $|y| \to \infty$.

We have the following existence result of weak solutions to the above problem (1.6). For the definition of the involved Sobolev space $H^a_\gamma\left(\mathbb{R}^N\right)$, see Section 2.

Theorem 1.4. *Assume that a satisfies (a1)–(a3). Then,*
i) If $N_\gamma \ge 4$, *problem (1.6) has a nontrivial solution* $v \in H^a_\gamma\left(\mathbb{R}^N\right)$ *for each* $\lambda > 0$;
ii) If $N_\gamma < 4$ *and* $\frac{4}{N_\gamma - 2} < q < 2^*_\gamma$, *problem* (1.4) *has a nontrivial solution* $v \in H^a_\gamma\left(\mathbb{R}^N\right)$ *for any* $\lambda > 0$; *if* $2 < q \le \frac{4}{N_\gamma - 2}$, *there exists* $\lambda_* > 0$ *such that problem* (1.4) *has a nontrivial solution* $v \in H^a_\gamma\left(\mathbb{R}^N\right)$ *for* $\lambda > \lambda_*$.

A version of problem (1.6) has been considered for the Laplacian operator by Alves, Carrião and Miyagaki when a is periodic and asymptotic periodic, see [5]. Moreover, in Miyagaki [36], the author considered the case where the function a is coercive.

1.1. **Our approach.** As in the Laplacian case, one of the main difficulty to apply the variational method to this kind of problems is the fact that the embeddings $\mathring{H}^{1,2}_\gamma(\Omega) \hookrightarrow L^{2^*_\gamma}(\Omega)$ and $H^a_\gamma(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for any $q \ge 1$ are not compact. To overcome these difficulties, the techniques developed by Aubin [8], Brezis and Nirenberg [12] are useful in our analysis. For example, in connection with problems (1.4) and (1.6), a key step is to prove that the *mountain-pass* level c_λ associated with the energy functional and defined in (5.1) stands below a suitable compactness threshold, i.e.

$$c_\lambda < \frac{1}{N_\gamma}\mathcal{S}_\gamma^{\frac{N_\gamma}{2}}(m,n),$$

where $\mathcal{S}_\gamma(m,n)$ is the best constant for the Sobolev embedding $\mathcal{D}^{1,2}_\gamma\left(\mathbb{R}^N\right) \hookrightarrow L^{2^*_\gamma}\left(\mathbb{R}^N\right)$. To obtain the above inequality, a crucial role is played by the extremals of the Sobolev embedding, whose explicit knowledge is not yet complete in the Grushin case. In this regard, we recall that, for the case $\gamma = 1$ Beckner [10] proved that, for $N_1 = 3, 4$ such extremal functions have the form

$$u_{A,y_0}(x,y) = \left(\frac{A}{\left(|x|^2 + A\right)^2 + |y - y_0|^2}\right)^{\frac{N_1 - 2}{4}}, \tag{1.7}$$

where $A > 0$ and $y_0 \in \mathbb{R}^N$. In [37], existence and symmetry results were obtained for $m = 1$, $n \geq 1$, $\gamma > 0$. Recently the knowledge of the explicit form of such minimizers has been extended to a larger set of dimensions m, n in [20], by studying the best constant $S_{\gamma,x}(m,n)$ of the embedding $\mathcal{D}^{1,2}_{\gamma,radx}\left(\mathbb{R}^N\right) \hookrightarrow L^{2^*_\gamma}\left(\mathbb{R}^N\right)$ (where $\mathcal{D}^{1,2}_{\gamma,radx}\left(\mathbb{R}^N\right)$ is the space of functions in $\mathcal{D}^{1,2}_\gamma\left(\mathbb{R}^N\right)$ that are radially symmetric in the x variable, see Section 2) and its relation with $S_\gamma(m,n)$. However, a general result covering all the dimensions $m, n \geq 1$ and $\gamma \neq 1$ is not yet available.

To get around this difficulty we shall follow the approach in [33], developed to treat the case of sub-Laplacians on Carnot groups. We will perform the Brezis-Nirenberg asymptotic expansions on a family of abstract Sobolev concentrating functions, by exploiting only some qualitative properties of Sobolev extremals, namely their existence, which will be proved in Section 3 for all $m, n \geq 1$, and the knowledge of their exact rate of decay at infinity, proved in [34] and recalled in Theorem 3.7.

Concerning the sign of solutions, we point out that the approach explored in the present paper can be adapted to prove the existence of nonnegative solutions by replacing the functions $f(t) = |t|^{2^*_\gamma - 2}t$ and $g(t) = |t|^{q-2}t$ by $\tilde{f}(t) = t_+^{2^*_\gamma - 1}$ and $\tilde{g}(t) = t_+^{q-1}$, respectively, where $t_+ = \max\{t, 0\}$. Having this in mind, and using elliptic regularity and maximum principle away from the degeneration set, it is possible to prove the following:

(i) For problems (1.3) and (1.4), there exists a nonnegative solution u such that

$$u \in C^2(\Pi) \quad \text{and} \quad u(x) > 0, \quad \forall (x,y) \in \Pi$$

where $\Pi = \{(x,y) \in \Omega \,:\, x \neq 0\}$.

(ii) For problem (1.6), there exists a nonnegative solution u such that

$$u \in C^2(\Pi_*) \quad \text{and} \quad u(x) > 0, \quad \forall (x,y) \in \Pi_*$$

where $\Pi_* = \{(x,y) \in \mathbb{R}^N \,:\, x \neq 0\}$.

Finally, let us make some further comments about the case $q = 2$ with $\lambda = 0$ or $\lambda \geq \lambda_1(\Omega)$. As in the Laplacian case, the existence of nonnegative solutions when $\lambda = 0$ has a strong connection with the geometry of the domain, since we have a Pohozaev type identity for the Grushin operator that was proved in [28, Section 2], which enables us to prove a nonexistence result of nonnegative nontrivial solutions under suitable geometric assumptions on the domain. In order to state this result, we need to introduce some notations. To begin with, for each $t > 0$ let us denote by $\delta_t : \mathbb{R}^N \to \mathbb{R}^N$ the dilations naturally attached to the operator Δ_γ given by

$$\delta_t(x,y) = (tx, t^{1+\gamma}y), \quad \forall (x,y) \in \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n. \tag{1.8}$$

Recall that Δ_γ is homogeneous of degree two with respect to the dilations δ_t. Let us denote by $T : \mathbb{R}^N \to \mathbb{R}^N$ the infinitesimal generator of such dilations, i.e. the vector field defined by

$$T(x,y) = (x, (1+\gamma)y), \quad \forall (x,y) \in \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n.$$

Definition 1.5. We say that Ω is δ_t-starshaped (resp. *strictly* δ_t-starshaped) with respect to the origin if $0 \in \Omega$ and

$$\langle T(x,y), \nu \rangle \geq 0 \text{ (resp. } > 0), \quad \text{at every point } (x,y) \in \partial\Omega,$$

where ν is the outward normal to $\partial\Omega$.

Using the above notations, the result below is an immediate consequence of [28, Theorem 2.6].

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^N$ be a smooth connected bounded domain, δ_t-starshaped with respect to the origin. If $u \in C^2(\overline{\Omega})$ is a nonnegative solution of* (1.3) *with $\lambda = 0$, then $u = 0$.*

Among our results (see Theorem 1.2) we will prove that, for homogeneous dimensions $N_\gamma < 4$, a similar nonexistence result also holds in a right neighborhood of $\lambda = 0$, thus showing a phenomenon of so-called *critical dimensions* for the considered critical problem.

We don't want to address here the delicate issue concerning how the regularity assumptions in the non-existence theorems 1.2 and 1.6 can be weakened, we only quote in this regards the results in [29], where a Pohozaev-type identity for Grushin eigenfunctions is obtained under weaker regularity assumptions by means of a variational approach.

Finally, let us observe that, as in the Laplacian case, if we search for positive solutions to problem (1.1), we have to restrict our analysis to the range $0 < \lambda < \lambda_1(\Omega)$. This can be easily proved by using a nonnegative eigenfunction φ_1 associated with $\lambda_1(\Omega)$ as a test function in the equation. We recall this result below, omitting the proof for the sake of brevity.

Proposition 1.7. *If $\lambda \geq \lambda_1(\Omega)$ and $u \in \mathring{H}^{1,2}_\gamma(\Omega)$ is a nonnegative solution of* (1.3)*, then $u = 0$.*

1.2. **Organization of the article.** This paper is organized as follows. We recall important preliminary results and definitions and we introduce the appropriate functional setting in Section 2. In Section 3, we prove that the optimal constant for the Grushin Sobolev inequality is achieved. Then, we prove Theorems 1.1, 1.3 and 1.4 in Sections 4, 5 and 6, respectively.

1.3. **Notations.** In this paper, we use the following notations:

- For $q \in [1, +\infty)$, q' denotes the conjugate exponent of q, that is, $q' = \frac{q}{q-1}$.
- C and C_i denote any (possibly different) positive constants, whose values are not relevant.
- If $A \subset \mathbb{R}^N$ is a measurable set, we denote by $|A|$ its Lebesgue measure.

2. Preliminaries

In this section, let us recall some preliminaries on the Grushin operator and the related embedding results. For $\gamma > 0$, let $< \nabla_\gamma v, \nabla_\gamma u >:= \nabla_x v \cdot \nabla_x u + (1+\gamma)^2 |x|^{2\gamma} \nabla_y v \cdot \nabla_y u$ and $|\nabla_\gamma v|^2 := |\nabla_x v|^2 + (1+\gamma)^2 |x|^{2\gamma} |\nabla_y v|^2$. Let us define the space $\mathcal{D}^{1,2}_\gamma(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|v\|_{\mathcal{D}^{1,2}_\gamma(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla_\gamma v|^2 \, dz\right)^{\frac{1}{2}}.$$

In the sequel, $\mathcal{D}^{1,2}_{\gamma,rad_x}(\mathbb{R}^N)$ denotes the following subspace

$$\mathcal{D}^{1,2}_{\gamma,rad_x}(\mathbb{R}^N) := \left\{v \in \mathcal{D}^{1,2}_\gamma(\mathbb{R}^N) \mid v(x,y) = v(|x|, y) \text{ for all } (x,y) \in \mathbb{R}^{m+n}\right\}.$$

Another space we will use in the sequel is the following one:

$$H^{1,2}_\gamma(\mathbb{R}^N) := \left\{v \in L^2(\mathbb{R}^N) \mid \frac{\partial v}{\partial x_i}, |x|^\gamma \frac{\partial v}{\partial y_j} \in L^2(\mathbb{R}^N), i = 1, \ldots, n, j = 1, \ldots, m\right\}$$

endowed with the norm

$$\|v\|_{H^{1,2}_\gamma} := \left(\int_{\mathbb{R}^N} (|\nabla_\gamma v|^2 + v^2) dz\right)^{\frac{1}{2}}.$$

Associated with $H^{1,2}_\gamma(\mathbb{R}^N)$ we have the space below

$$H^{1,2}_{\gamma,rad_{x,y}}(\mathbb{R}^N) := \left\{v \in H^{1,2}_\gamma(\mathbb{R}^N) \mid v(x,y) = v(|x|, |y|) \text{ for all } (x,y) \in \mathbb{R}^{m+n}\right\}$$

and

$$H^{1,2}_{\gamma,rad_x}(\mathbb{R}^N) := \left\{v \in H^{1,2}_\gamma(\mathbb{R}^N) \mid v(x,y) = v(|x|, y) \text{ for all } (x,y) \in \mathbb{R}^{m+n}\right\}.$$

For each nonnegative continuous function $a(z) \geq 0$, let us define the following space

$$H^a_\gamma(\mathbb{R}^N) := \left\{v \in H^{1,2}_\gamma(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} a(z) |v|^2 \, dz < \infty\right\}$$

endowed with the norm

$$\|v\|_{H^a_\gamma} = \left(\int_{\mathbb{R}^N} (|\nabla_\gamma v|^2 + a(z)|v|^2) dz\right)^{\frac{1}{2}}.$$

Also, consider the subspace of functions which are radial in the variable x:

$$H^a_{\gamma,rad_x}(\mathbb{R}^N) = \left\{v \in H^a_\gamma(\mathbb{R}^N) \mid v(x,y) = v(|x|, y)\right\}.$$

For each smooth bounded domain $\Omega \subset \mathbb{R}^N$, we define the space

$$H^{1,2}_\gamma(\Omega) := \left\{v \in L^2(\Omega) \mid \frac{\partial v}{\partial x_i}, |x|^\gamma \frac{\partial v}{\partial y_j} \in L^2(\Omega), i = 1, \ldots, n, j = 1, \ldots, m\right\}$$

with the norm

$$\|v\|_{H^{1,2}_\gamma(\Omega)} = \left(\int_\Omega (|\nabla_\gamma v|^2 + |v|^2) dz\right)^{\frac{1}{2}}.$$

and we denote by $\overset{\circ}{H}{}^{1,2}_{\gamma}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\| = \left(\int_\Omega |\nabla_\gamma v|^2 \, dz\right)^{\frac{1}{2}}. \tag{2.1}$$

Through this paper, $\overset{\circ}{H}{}^{1,2}_{\gamma,rad_x}(\Omega)$ will denote the following subspace

$$\overset{\circ}{H}{}^{1,2}_{\gamma,rad_x}(\Omega) := \Big\{ v \in \overset{\circ}{H}{}^{1,2}_{\gamma}(\Omega) \mid v(x,y) = v(|x|\,,y) \text{ for all } (x,y) \in \Omega \Big\}.$$

Now, it is well known that $\mathcal{D}^{1,2}_\gamma(\mathbb{R}^N) \hookrightarrow L^{2^*_\gamma}(\mathbb{R}^N)$, where $2^*_\gamma = \frac{2N_\gamma}{N_\gamma - 2}$ and $N_\gamma = m + (1+\gamma)n$ (see e.g. [37], [32], [28]). In particular, the Sobolev embedding inequality

$$\left(\int_{\mathbb{R}^N} |v|^{2^*_\gamma} \, \mathrm{d}z\right)^{2/2^*_\gamma} \le C \int_{\mathbb{R}^N} |\nabla_\gamma v|^2 \, \mathrm{d}z, \quad \forall\, v \in C_0^\infty(\mathbb{R}^N), \tag{2.2}$$

where C is a positive constant depending on N_γ, can be derived as a consequence of an optimal embedding of $\mathcal{D}^{1,2}_\gamma(\mathbb{R}^N)$ into an ordinary anisotropic Sobolev space, proved by Franchi and Lanconelli in [23] (see for details [32, Section 5], where also Sobolev inequalities with remainder terms for the Grushin gradient were obtained). Moreover, we quote [28] for the Sobolev inequality for the general class of Δ_λ-operators; we also refer to [10], [37], [34], [20] for further related results about the qualitative properties of the optimizers in (2.2).

Moreover, it has been proved (see [28]) that, if Ω is a bounded domain, the embedding $\overset{\circ}{H}{}^{1,2}_{\gamma}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for each $q \in [1, 2^*_\gamma)$. We recall these known embedding results in the following propositions.

Proposition 2.1. *(1) The embedding* $H^{1,2}_\gamma\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ *is continuous for every* $q \in \left[2, 2^*_\gamma\right]$.

(2) If $\Omega \subset \mathbb{R}^N$ *is a smooth bounded domain, then the embedding* $\overset{\circ}{H}{}^{1,2}_{\gamma}(\Omega) \hookrightarrow L^q(\Omega)$ *is continuous for every* $q \in \left[1, 2^*_\gamma\right]$ *and compact for every* $q \in \left[1, 2^*_\gamma\right)$.

(3) Assume (a1) holds, then the embedding $H^a_\gamma\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ *is continuous for* $2 \le q \le 2^*_\gamma$.

Proposition 2.2. *(*[1, Lemma 2.5]*) Assume that* a *satisfies* $(a1)$ *and* $(a3)$. *Then the inclusion* $H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ *is compact for* $2 \le q < 2^*_\gamma$.

Definition 2.3. By a weak solution of (1.1), we mean a function $v \in \overset{\circ}{H}{}^{1,2}_{\gamma}(\Omega)$ satisfying

$$\int_\Omega \nabla_\gamma v \cdot \nabla_\gamma w \, dz = \int_\Omega \lambda v w \, dz + \int_\Omega |v|^{2^*_\gamma - 2} v w \, dz, \quad \forall\, w \in \overset{\circ}{H}{}^{1,2}_{\gamma}(\Omega). \tag{2.3}$$

Definition 2.4. By weak solution of (1.6), we mean a function $v \in H^a_\gamma\left(\mathbb{R}^N\right)$ satisfying

$$\int_{\mathbb{R}^N} \left(\nabla_\gamma v \cdot \nabla_\gamma w + a(z) v w\right) dz = \lambda \int_{\mathbb{R}^N} |v|^{q-2} v w \, dz + \int_{\mathbb{R}^N} |v|^{2^*_\gamma - 2} v w \, dz, \quad \forall\, w \in H^a_\gamma\left(\mathbb{R}^N\right).$$

Finally, let us recall the natural *gauge* associated to the Grushin operator: for $z = (x,y) \in \mathbb{R}^m \times \mathbb{R}^n$, let

$$d(z) = (|x|^{2(\gamma+1)} + |y|^2)^{\frac{1}{2(\gamma+1)}}. \tag{2.4}$$

The function $d(z)$ is homogeneous of degree one with respect to the anisotropic dilations δ_t defined in (1.8). Moreover, there exists a suitable constant $C > 0$ depending on γ and N_γ, such that $\Gamma(z) = \frac{C}{d(z)^{N_\gamma - 2}}$ is the fundamental solution of $-\Delta_\gamma$ with pole at the origin.

For $r > 0$, we shall denote by $B_r = B_r(0)$ the ball with center at 0 and radius r with respect to the homogeneous norm d, i.e., $B_r = \{z \in \mathbb{R}^N \mid d(z) < r\}$.

3. Optimal Constant and Sobolev extremals

Let $N_\gamma \ge 3$. The optimal constant in the Sobolev inequality (2.2) is defined as

$$\mathcal{S}_\gamma(m,n) := \inf_{v \in \mathcal{D}^{1,2}_\gamma(\mathbb{R}^N), \|v\|_{L^{2^*_\gamma}(\mathbb{R}^N)} = 1} \|\nabla_\gamma v\|^2_{L^2(\mathbb{R}^N)} > 0. \tag{3.1}$$

In this section, our goal is to prove that the infimum defined above is achieved, under no restrictions on $m, n \ge 1$. To accomplish our goal, we will generalize the results of [41, Section 1.9] for the Grushin operator and the corresponding Sobolev spaces.

Let $u \in \mathcal{D}_\gamma^{1,2}\left(\mathbb{R}^N\right)$ and $\rho > 0$. Define

$$u^{e,\rho}(z) := \rho^{\frac{N_\gamma - 2}{2}} u(\rho x, \rho^{1+\gamma} y + e), \tag{3.2}$$

where $z = (x, y) \in \mathbb{R}^{m+n}$ and $e \in \mathbb{R}^n$. It is easy to verify the following invariances

$$\|\nabla_\gamma u^{e,\rho}\|_{L^2(\mathbb{R}^N)} = \|\nabla_\gamma u\|_{L^2(\mathbb{R}^N)} \text{ and } \|u^{e,\rho}\|_{L^{2^*_\gamma}(\mathbb{R}^N)} = \|u\|_{L^{2^*_\gamma}(\mathbb{R}^N)}.$$

Analogously, it can be verified that the equation satisfied, up to a stretching constant, by the minimizers for $S_\gamma(m,n)$, that is $-\Delta_\gamma u = |u|^{2^*_\gamma - 2} u$ in $\mathbb{R}^N$, is invariant under the translations in the y variable and the rescaling defined in (3.2). Note that the same problem is not invariant under general translations in $\mathbb{R}^N$.

Let us recall some preliminary results and definitions from measure theory [41]. Let Ω be an open subset of $\mathbb{R}^N$ and define

$$K(\Omega) := \{v \in C(\Omega) : \operatorname{supp} v \text{ is a compact subset of } \Omega\}$$

and

$$B(\Omega) := \left\{ v \in C(\Omega) : \|v\|_{L^\infty(\Omega)} := \sup_{x \in \Omega} |v(x)| < \infty \right\}.$$

We denote by $C_0(\Omega)$ the closure of $K(\Omega)$ in $B(\Omega)$ with respect to the uniform norm. Following the approach in [42], we adopt the following definitions.

Definition 3.1. A finite measure on Ω is a continuous linear functional on $C_0(\Omega)$. The norm of the finite measure μ is defined by

$$\|\mu\| := \sup_{\substack{v \in C_0(\Omega) \\ \|v\|_{L^\infty(\Omega)} = 1}} |\langle \mu, v \rangle|.$$

We denote by

$\mathcal{M}(\Omega) -$ The space of finite measures on Ω.

$\mathcal{M}^+(\Omega) -$ The space of positive finite measures on Ω.

We say that a sequence $\mu_k \rightharpoonup \mu$ weakly in $\mathcal{M}(\Omega)$, whenever

$$\langle \mu_k, u \rangle \to \langle \mu, u \rangle, \quad \forall u \in C_0(\Omega).$$

Theorem 3.2. *(i) Every bounded sequence of finite measures on Ω contains a weakly convergent subsequence.*
(ii) If $\mu_k \rightharpoonup \mu$ in $\mathcal{M}(\Omega)$, then (μ_k) is bounded and

$$\|\mu\| \leq \liminf \|\mu_k\|.$$

(iii) If $\mu \in \mathcal{M}^+(\Omega)$, then $B(\Omega) \subset L^1(\Omega, \mu)$ and

$$\|\mu\| = \sup_{\substack{v \in B(\Omega) \\ \|v\|_{L^\infty(\Omega)} = 1}} |\langle \mu, v \rangle| = \langle \mu, 1 \rangle.$$

The proof of the above theorem can be found in [42], page 206.

Lemma 3.3. *(The Concentration-Compactness Principle) Let $\{v_k\} \subset \mathcal{D}_\gamma^{1,2}\left(\mathbb{R}^N\right)$ be a sequence such that*

$$\begin{aligned} v_k &\rightharpoonup v \text{ in } \mathcal{D}_\gamma^{1,2}\left(\mathbb{R}^N\right), \\ |\nabla_\gamma (v_k - v)|^2 &\rightharpoonup \mu \text{ in } \mathcal{M}\left(\mathbb{R}^N\right), \\ |v_k - v|^{2^*_\gamma} &\rightharpoonup \nu \text{ in } \mathcal{M}\left(\mathbb{R}^N\right), \\ v_k &\to v \text{ a.e. on } \mathbb{R}^N \end{aligned}$$

and define

$$\mu_\infty := \lim_{R \to \infty} \limsup_{k \to \infty} \int_{d(z) > R} |\nabla_\gamma v_k|^2 \, dz, \quad \nu_\infty := \lim_{R \to \infty} \limsup_{k \to \infty} \int_{d(z) > R} |v_k|^{2^*_\gamma} \, dz, \tag{3.3}$$

then it follows that

$$\|\nu\|^{\frac{2}{2^*_\gamma}} \leq \mathcal{S}_\gamma(m,n)^{-1}\|\mu\|, \tag{3.4}$$

$$\nu_\infty^{\frac{2}{2^*_\gamma}} \leq \mathcal{S}_\gamma(m,n)^{-1}\mu_\infty, \tag{3.5}$$

$$\limsup_{k\to\infty}\|\nabla_\gamma v_k\|_2^2 = \|\nabla_\gamma v\|_2^2 + \|\mu\| + \mu_\infty, \tag{3.6}$$

$$\limsup_{k\to\infty}\|v_k\|_{2^*_\gamma}^{2^*_\gamma} = \|v\|_{2^*_\gamma}^{2^*_\gamma} + \|\nu\| + \nu_\infty. \tag{3.7}$$

Moreover, if $v = 0$ *and* $\|\nu\|^{\frac{2}{2^*_\gamma}} = \mathcal{S}_\gamma(m,n)^{-1}\|\mu\|$, *then the measures* μ *and* ν *are concentrated at a single point.*

Proof. Case (i) Assume first $v = 0$. Let $\phi \in C_0^\infty(\mathbb{R}^N)$, then from the Sobolev embedding, we have

$$\mathcal{S}_\gamma(m,n)\left(\int_{\mathbb{R}^N}|\phi v_k|^{2^*_\gamma}\,dz\right)^{\frac{2}{2^*_\gamma}} \leq \int_{\mathbb{R}^N}|\nabla_\gamma(\phi v_k)|^2\,dz.$$

Now, using the Hölder inequality and the convergence $v_k \to 0$ in $L^2_{loc}(\mathbb{R}^N)$, we obtain

$$\mathcal{S}_\gamma(m,n)\left(\int_{\mathbb{R}^N}|\phi|^{2^*_\gamma}\,d\nu\right)^{\frac{2}{2^*_\gamma}} \leq \int_{\mathbb{R}^N}|\phi|^2\,d\mu. \tag{3.8}$$

Thus 3.4 follows.
For $R > 0$, let $\phi_R \in C^1(\mathbb{R}^N)$ be such that $0 \leq \phi_R \leq 1$ on $\mathbb{R}^N$, $\phi_R = 1$ in $B^c_{R+1} := \mathbb{R}^N \setminus B_{R+1}$ and $\phi_R = 0$ in B_R. Again, using Sobolev inequality, we obtain

$$\mathcal{S}_\gamma(m,n)\left(\int_{\mathbb{R}^N}|\phi_R v_k|^{2^*_\gamma}\,dz\right)^{\frac{2}{2^*_\gamma}} \leq \int_{\mathbb{R}^N}|\nabla_\gamma(\phi_R v_k)|^2\,dz.$$

Now, using Hölder's inequality and the convergence $v_k \to 0$ in $L^2_{loc}(\mathbb{R}^N)$, we obtain

$$\mathcal{S}_\gamma(m,n)\limsup_{k\to\infty}\left(\int_{\mathbb{R}^N}|\phi_R v_k|^{2^*_\gamma}\,dz\right)^{\frac{2}{2^*_\gamma}} \leq \limsup_{k\to\infty}\int_{\mathbb{R}^N}|\nabla_\gamma v_k|^2|\phi_R|^2\,dz. \tag{3.9}$$

It is easy to check that

$$\int_{B^c_{R+1}}|\nabla_\gamma v_k|^2\,dz \leq \int_{\mathbb{R}^N}|\nabla_\gamma v_k|^2\phi_R^2 dz \leq \int_{B^c_R}|\nabla_\gamma v_k|^2\,dz \tag{3.10}$$

and

$$\int_{B^c_{R+1}}|v_k|^{2^*_\gamma}\,dz \leq \int_{\mathbb{R}^N}|v_k|^{2^*_\gamma}\phi_R^{2^*_\gamma}dz \leq \int_{B^c_R}|v_k|^{2^*_\gamma}\,dz. \tag{3.11}$$

Now, (3.10) and (3.11) combine with (3.3) to give

$$\mu_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{B^c_R}|\nabla_\gamma v_k|^2\phi_R^2 dz, \quad \nu_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{B^c_R}|v_k|^{2^*_\gamma}\phi_R^{2^*_\gamma}dz.$$

Inequality (3.5) then follows from 3.9.
Assume, moreover, that $\|\nu\|^{\frac{2}{2^*_\gamma}} = \mathcal{S}_\gamma(m,n)^{-1}\|\mu\|$. From (3.8),

$$\mathcal{S}_\gamma^{\frac{1}{2}}(m,n)\left(\int_{\mathbb{R}^N}|\phi|^{2^*_\gamma}\,d\nu\right)^{\frac{1}{2^*_\gamma}} \leq \left(\int_{\mathbb{R}^N}|\phi|^2\,d\mu\right)^{\frac{1}{2}}.$$

Since $\frac{2}{N_\gamma} + \frac{2}{2^*_\gamma} = 1$, we use Hölder's inequality to obtain

$$\left(\int_{\mathbb{R}^N}|\phi|^2\,d\mu\right)^{\frac{1}{2}} \leq \|\mu\|^{\frac{1}{N_\gamma}}\left(\int_{\mathbb{R}^N}|\phi|^{2^*_\gamma}\,d\mu\right)^{\frac{1}{2^*_\gamma}}.$$

Therefore

$$\mathcal{S}_\gamma^{\frac{1}{2}}(m,n)\left(\int_{\mathbb{R}^N}|\phi|^{2^*_\gamma}\,d\nu\right)^{\frac{1}{2^*_\gamma}} \leq \|\mu\|^{\frac{1}{N_\gamma}}\left(\int_{\mathbb{R}^N}|\phi|^{2^*_\gamma}\,d\mu\right)^{\frac{1}{2^*_\gamma}},$$

leading to

$$\nu = \mathcal{S}_\gamma^{-\frac{2^*_\gamma}{2}}(m,n)\, \|\mu\|^{\frac{2}{N_\gamma - 2}}\, \mu.$$

It follows from (3.8), for each $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} |\phi|^{2^*_\gamma}\, d\nu\right)^{\frac{1}{2^*_\gamma}} \|\nu\|^{\frac{1}{N_\gamma}} \le \left(\int_{\mathbb{R}^N} |\phi|^2\, d\nu\right)^{\frac{1}{2}}.$$

Therefore, for each open set $\Omega \subset \mathbb{R}^N$,

$$\nu(\Omega)^{\frac{1}{2^*_\gamma}} \nu(\mathbb{R}^N)^{\frac{1}{N_\gamma}} \le \nu(\Omega)^{\frac{1}{2}},$$

showing that the measure ν is concentrated at a single point.
Case (ii) Consider, now, the general case. Define $w_k := v_k - v$. Then

$$w_k \rightharpoonup 0 \ \text{ in } \mathcal{D}^{1,2}_\gamma(\mathbb{R}^N),$$

$$|\nabla_\gamma v_k|^2 \rightharpoonup \mu + |\nabla_\gamma v|^2 \ \text{ in } \mathcal{M}(\mathbb{R}^N).$$

Now, using the Brézis-Lieb Lemma (see, e.g., Lemma 1.32 in [41]), we have for any $\phi \in K(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \phi\, |v|^{2^*_\gamma}\, dz = \lim_{k\to\infty}\left(\int_{\mathbb{R}^N} \phi\, |v_k|^{2^*_\gamma}\, dz - \int_{\mathbb{R}^N} \phi\, |w_k|^{2^*_\gamma}\, dz\right)$$

that is,

$$|v_k|^{2^*} \rightharpoonup \nu + |v|^{2^*_\gamma} \ \text{ in } \mathcal{M}\left(\mathbb{R}^N\right).$$

Hence, inequality (3.4) follows from the corresponding inequality for $\{w_k\}$. Since

$$\limsup_{k\to\infty} \int_{B_R^c} |\nabla_\gamma w_k|^2\, dz = \limsup_{k\to\infty} \int_{B_R^c} |\nabla_\gamma v_k|^2\, dz - \int_{B_R^c} |\nabla_\gamma v|^2\, dz,$$

we obtain

$$\lim_{k\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |\nabla w_k|^2 = \mu_\infty.$$

By the Brezis-Lieb lemma,

$$\int_{B_R^c} |v|^{2^*_\gamma}\, dz = \lim_{k\to\infty}\left(\int_{B_R^c} |v_k|^{2^*_\gamma}\, dz - \int_{B_R^c} |w_k|^{2^*_\gamma}\right) dz$$

and so

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |w_k|^{2^*_\gamma} = \nu_\infty.$$

Inequality (3.5) then follows from the corresponding inequality for $\{w_k\}$.

For every $R > 1$, we have

$$\begin{aligned}\limsup_{k\to\infty} \int_{\mathbb{R}^N} |\nabla_\gamma v_k|^2\, dz &= \limsup_{k\to\infty}\left(\int_{\mathbb{R}^N} \phi_R\, |\nabla_\gamma v_k|^2\, dz + \int_{\mathbb{R}^N} (1-\phi_R)\, |\nabla_\gamma v_k|^2\, dz\right)\\ &= \limsup_{k\to\infty} \int_{\mathbb{R}^N} \phi_R\, |\nabla_\gamma v_k|^2\, dz + \int_{\mathbb{R}^N} (1-\phi_R)\, d\mu + \int_{\mathbb{R}^N} (1-\phi_R)\, |\nabla_\gamma v|^2\, dz.\end{aligned}$$

When $R \to \infty$, we obtain, by Lebesgue theorem,

$$\limsup_{k\to\infty} \int_{\mathbb{R}^N} |\nabla_\gamma v_k|^2\, dz = \mu_\infty + \int_{\mathbb{R}^N} d\mu + \int_{\mathbb{R}^N} |\nabla_\gamma v|^2\, dz = \mu_\infty + \|\mu\| + \int_{\mathbb{R}^N} |\nabla_\gamma v|^2\, dz.$$

The proof of (3.7) is similar. □

Theorem 3.4. *Let $\{v_k\} \subset \mathcal{D}^{1,2}_\gamma(\mathbb{R}^N)$ be a minimizing sequence such that*

$$\|v_k\|_{2^*_\gamma} = 1, \ \|\nabla_\gamma v_k\|_2^2 \to \mathcal{S}_\gamma(m,n). \tag{3.12}$$

Then, there exists a sequence $\{e_k, \rho_k\} \subset \mathbb{R}^n \times (0,\infty)$ such that $\{v_k^{e_k,\rho_k}\}$ contains a convergent subsequence. In particular, there exists a minimizer for $\mathcal{S}_\gamma(m,n)$.

Proof. Up to a subsequence, we can assume that

$$v_k \rightharpoonup v \text{ in } \mathcal{D}^{1,2}_\gamma(\mathbb{R}^N),$$

so that

$$\|\nabla_\gamma v\|_2^2 \le \liminf_{k\to\infty} \|\nabla_\gamma v_k\|_2^2 = \mathcal{S}_\gamma(m,n).$$

Thus, v is a minimizer provided $\|v\|_{2^*_\gamma} = 1$. But we know only that $\|v\|_{2^*_\gamma} \le 1$. Therefore, we aim to show that $\|v\|_{L^{2^*_\gamma}} = 1$. Define a concentration function

$$Q_k(\rho) := \sup_{w:=(0,e)\in\mathbb{R}^N} \int_{B_\rho(w)} |v_k|^{2^*_\gamma}\, dz,$$

where $B_\rho(w) = \{z \in \mathbb{R}^N \,|\, d(z-w) < \rho\}$, d being the homogeneous norm defined in (2.4).

Note that for each k,

$$\lim_{\rho\to 0^+} Q_k(\rho) = 0, \quad \lim_{\rho\to\infty} Q_k(\rho) = 1.$$

Therefore, there exists $\rho_k > 0$ such that $Q_k(\rho_k) = \frac{1}{2}$. Moreover, since

$$\lim_{|e|\to\infty} \int_{B_{\rho_k}(w)} |v_k|^{2^*_\gamma}\, dz = 0,$$

there exists $e_k \in \mathbb{R}^n$ such that

$$\int_{B_{\rho_k}(w_k)} |v_k|^{2^*_\gamma}\, dz = Q_k(\rho_k) = \frac{1}{2},$$

where $w_k := (0,e_k) \in \mathbb{R}^N$. Let us define $u_k := v_k^{e_k,\rho_k}$. Hence

$$\|u_k\|_{2^*_\gamma} = 1, \;\|\nabla_\gamma u_k\|_2^2 \to \mathcal{S}_\gamma(m,n)$$

and

$$\frac{1}{2} = \int_{B_1} |u_k|^{2^*_\gamma}\, dz = \sup_{w=(0,e)\in\mathbb{R}^N} \int_{B_1(w)} |u_k|^{2^*_\gamma}\, dz. \tag{3.13}$$

Since $\{u_k\}$ is bounded sequence in $\mathcal{D}^{1,2}_\gamma(\mathbb{R}^N)$, up to a subsequence, we may assume that

$$\begin{aligned} u_k &\rightharpoonup u \text{ in } \mathcal{D}^{1,2}_\gamma\left(\mathbb{R}^N\right), \\ |\nabla_\gamma (u_k - u)|^2 &\rightharpoonup \mu \text{ in } \mathcal{M}\left(\mathbb{R}^N\right), \\ |u_k - u|^{2^*_\gamma} &\rightharpoonup \nu \text{ in } \mathcal{M}\left(\mathbb{R}^N\right), \\ u_k &\to u \text{ a.e. on } \mathbb{R}^N. \end{aligned}$$

Now, by an application of the previous Lemma 3.3,

$$\mathcal{S}_\gamma(m,n) = \lim_{k\to\infty} \int_{\mathbb{R}^N} |\nabla_\gamma u_k|^2\, dz = \mu_\infty + \|\mu\| + \int_{\mathbb{R}^N} |\nabla_\gamma u|^2\, dz, \tag{3.14}$$

$$1 = \lim_{k\to\infty} \|u_k\|_{2^*_\gamma}^{2^*_\gamma} = \|u\|_{2^*_\gamma}^{2^*_\gamma} + \|\nu\| + \nu_\infty, \tag{3.15}$$

where

$$\mu_\infty := \lim_{R\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |\nabla_\gamma u_k|^2\, dz, \quad \nu_\infty := \lim_{R\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |u_k|^{2^*_\gamma}\, dz.$$

Now, using (3.14), (3.4), (3.5) and Sobolev inequality, we get

$$\left(\left(\|u\|_{2^*_\gamma}^{2^*_\gamma}\right)^{\frac{2}{2^*_\gamma}} + \|\nu\|^{\frac{2}{2^*_\gamma}} + \nu_\infty^{\frac{2}{2^*_\gamma}}\right) \mathcal{S}_\gamma(m,n) \le \mathcal{S}_\gamma(m,n). \tag{3.16}$$

It follows from (3.15) and (3.16), taking into account that $2/2^*_\gamma < 1$, that $\|u\|_{2^*_\gamma}^{2^*_\gamma}$, $\|\nu\|$, ν_∞ are either equal to 0 or 1. By (3.13), $\nu_\infty \le \frac{1}{2}$, and hence $\nu_\infty = 0$. If $\|\nu\| = 1$ then $u = 0$. Then by the previous Lemma 3.3, the measure ν is concentrated at a single point z_0. Therefore, from (3.13) we deduce

$$\frac{1}{2} = \sup_{w=(0,e)\in\mathbb{R}^N} \int_{B_1(w)} |u_k|^{2^*_\gamma}\, dz \ge \int_{B_1(z_0)} |u_k|^{2^*_\gamma}\, dz \to \|\nu\| = 1,$$

a contradiction. Thus, $\|u\|_{2^*_\gamma} = 1$ and so

$$\|\nabla_\gamma u\|_2^2 = \mathcal{S}_\gamma(m,n) = \lim_{k\to\infty} \|\nabla_\gamma u_k\|_2^2 .$$

$\square$

A similar argument works to prove the following corollary:

Corollary 3.5. *Define*

$$\mathcal{S}_{\gamma,x}(m,n) := \inf_{v\in D^{1,2}_{\gamma,rad_x}(\mathbb{R}^N), \|v\|_{L^{2^*_\gamma}(\mathbb{R}^N)}=1} \|\nabla_\gamma v\|^2_{L^2(\mathbb{R}^N)} > 0. \tag{3.17}$$

Then there exists a minimizer for $\mathcal{S}_{\gamma,x}(m,n)$.

Concerning the relation between $S_\gamma(m,n)$ and $S_{\gamma,x}(m,n)$, in [20] the authors say that it is reasonable to expect that $S_\gamma(m,n) = S_{\gamma,x}(m,n)$, but they are able to prove it only when 2^*_γ is a positive integer (see [20, Prop. 1.13]). Their result about the explicit form of the minimizers for $\mathcal{S}_{\gamma,x}(m,n)$ is the following:

Theorem 3.6. *(Theorem 1.12 [20]) (1) Let* $\gamma = 1$. *Let* $m \neq 2$ *or* $m = 2, n \neq 1$. *Then, up to multiplicative constants, minimizers for* $\mathcal{S}_{1,x}(m,n)$ *are*

$$u_{A,y_0}(x,y) = \left(\frac{A}{\left(|x|^2 + A\right)^2 + |y-y_0|^2} \right)^{\frac{N_1-2}{4}}, \tag{3.18}$$

where $A > 0$ *and* $y_0 \in \mathbb{R}^n$.
(2) Let $\gamma > 0$, *then up to the multiple of some constant, minimizers for* $\mathcal{S}_{\gamma,x}(m,n)$ *are of the form*

$$u_{A,y_0}(x,y) = \left(\frac{A}{\left(|x|^{1+\gamma} + A\right)^2 + |y-y_0|^2} \right)^{\frac{N_\gamma-2}{2(\gamma+1)}} \psi\left(\left| \frac{(|x|^{1+\gamma} + A, y-y_0)}{\left(|x|^{1+\gamma} + A\right)^2 + |y-y_0|^2} - \left(\frac{1}{2A}, 0\right) \right| \right), \tag{3.19}$$

where $A > 0$, $y_0 \in \mathbb{R}^N$, $\psi > 0$ *is the unique solution of*

$$\begin{cases} \psi'' + \left(\frac{n}{r} - \frac{2\alpha r}{\frac{1}{4A^2} - r^2}\right)\psi' - \frac{\alpha(n+\alpha-1)}{\frac{1}{4A^2}-r^2}\psi = -CA^{\beta-\alpha}\left(\frac{1}{4A^2} - r^2\right)^{\beta-\alpha} \psi^{\frac{n+2\beta-\alpha+3}{n+\alpha-1}}, 0 < r < \frac{1}{2A}, \\ \psi\left(\frac{1}{2A}\right) = A^{\frac{n+\alpha-1}{2}}, \psi'(0) = 0, \lim_{r\to\left(\frac{1}{2A}\right)^-} \left(\frac{1}{4A^2} - r^2\right)^\alpha \psi'(r) = 0, \end{cases} \tag{3.20}$$

where α *and* β *satisfy*

$$\alpha > 0,\ \beta > -1,\ \frac{n-1}{n+1}\beta < \alpha < \beta + 2.$$

However, in our proofs, we will only use the following qualitative property of minimizers for $S_\gamma(m,n)$, namely their asymptotic decay at infinity, proved in [34].

Theorem 3.7. *(see [34]) If* $u \in \mathcal{D}^{1,2}_\gamma(\mathbb{R}^N)$ *is a minimizer for* $S_\gamma(m,n)$, *then*

$$u(z) \simeq d(z)^{2-N_\gamma} \quad \text{as } d(z) \to \infty.$$

where $d(z) = (|x|^{2(\gamma+1)} + |y|^2)^{\frac{1}{2(\gamma+1)}}$ *is the homogeneous norm defined in* (2.4).

Finally, if Ω is any open subset of $\mathbb{R}^N$, let us denote by $\mathcal{S}(\Omega)$ the best constant of the embedding $\mathring{H}^{1,2}_\gamma(\Omega) \hookrightarrow L^{2^*_\gamma}(\Omega)$, that is

$$\mathcal{S}(\Omega) := \inf_{v\in \mathring{H}^{1,2}_\gamma(\Omega), \|v\|_{L^{2^*_\gamma}(\Omega)}=1} \|\nabla_\gamma v\|^2_{L^2(\Omega)} > 0. \tag{3.21}$$

The following result holds.

Proposition 3.8. *Assume that* Ω *is a bounded domain with* $\Omega \cap \{x = 0\} \neq \emptyset$. *Then,* $\mathcal{S}(\Omega) = \mathcal{S}_\gamma(m,n)$.

Proof. By definition of $\mathcal{S}(\Omega)$ given in (3.21), we know that $\mathcal{S}_\gamma(m,n) \le \mathcal{S}(\Omega)$. Now, let $\{u_k\} \subset \mathcal{D}^{1,2}_\gamma(\mathbb{R}^N)$ be a minimizing sequence for $\mathcal{S}_\gamma(m,n)$, that is,

$$\|u_k\|_{2^*_\gamma} = 1, \quad \lim_{k\to\infty} \|\nabla_\gamma u_k\|_2 = \mathcal{S}_\gamma(m,n).$$

By density, we can assume $u_k \in C_0^\infty(\mathbb{R}^N)$. Since $\Omega \cap \{x=0\} \neq \emptyset$, without loss of generality, we can assume that $0 \in \Omega$. Thereby, there is a sequence $\rho_k > 0$ such that $u_k^{0,\rho_k} \in C_0^\infty(\Omega)$ for k sufficiently large. Hence,

$$\mathcal{S}(\Omega) \le \liminf_{k\to\infty} \left\|\nabla_\gamma u_k^{0,\rho_k}\right\|_2^2 = \liminf_{k\to\infty} \|\nabla_\gamma u_k\|_2^2 = \mathcal{S}_\gamma(m,n),$$

showing that $\mathcal{S}(\Omega) \le \mathcal{S}_\gamma(m,n)$, and so, $\mathcal{S}(\Omega) = \mathcal{S}_\gamma(m,n)$. □

4. Proof of Theorem 1.1

The main goal of this section is to prove Theorem 1.1 and we will look for critical points of the energy functional $J_\lambda : \mathring{H}^{1,2}_\gamma(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(v) := \frac{1}{2}\int_\Omega \left(|\nabla_\gamma v|^2 - \lambda |v|^2\right) dz - \frac{1}{2^*_\gamma}\int_\Omega |v|^{2^*_\gamma}\, dz. \tag{4.1}$$

Since the embedding $\mathring{H}^{1,2}_\gamma(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for $1 \le q \le 2^*_\gamma$, one can see that J_λ is well-defined and of class C^1 on the space $\mathring{H}^{1,2}_\gamma(\Omega)$.

In what follows, we denote by $\lambda_1(\Omega)$ the first homogeneous Dirichlet eigenvalue of $-\Delta_\gamma$ on $\mathring{H}^{1,2}_\gamma(\Omega)$, that is,

$$\lambda_1(\Omega) = \inf_{v \in \mathring{H}^{1,2}_\gamma(\Omega), \|v\|_{L^2(\Omega)}=1} \|\nabla_\gamma v\|^2_{L^2(\Omega)} > 0.$$

If $\lambda \in (0, \lambda_1(\Omega))$, a standard argument ensures that

$$\|\nabla_\gamma v\|^2_{L^2(\Omega)} - \lambda \|v\|^2_{L^2(\Omega)} \ge C \|\nabla_\gamma v\|^2_{L^2(\Omega)}, \forall v \in \mathring{H}^{1,2}_\gamma(\Omega),$$

for some $C > 0$. Hence, we can choose the following norm on the space $\mathring{H}^{1,2}_\gamma(\Omega)$:

$$\|v\|_\lambda := \left(\|\nabla_\gamma v\|^2_{L^2(\Omega)} - \lambda \|v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{4.2}$$

Lemma 4.1. *Assume* $\Omega \cap \{x=0\} \neq \emptyset$. *Then, any sequence* $\{v_k\} \subset \mathring{H}^{1,2}_\gamma(\Omega)$ *such that*

$$d := \sup_k J_\lambda(v_k) < c^* := \frac{\mathcal{S}_\gamma^{\frac{N_\gamma}{2}}(m,n)}{N_\gamma}$$

and

$$J'_\lambda(v_k) \to 0,$$

contains a convergent subsequence.

Proof. First we prove that the sequence $\{v_k\}$ is bounded in $\mathring{H}^{1,2}_\gamma(\Omega)$. For k big enough, we have

$$\begin{aligned} d + 1 + \|v_k\|_\lambda &\ge J_\lambda(v_k) - \frac{1}{2^*_\gamma} J_\lambda(v_k) v_k \\ &= \left(\frac{1}{2} - \frac{1}{2^*_\gamma}\right) \int_\Omega \left(|\nabla_\gamma v|^2 - \lambda v^2\right) dz \\ &= \left(\frac{1}{2} - \frac{1}{2^*_\gamma}\right) \|v_k\|^2_\lambda. \end{aligned}$$

Thus, we get $\{\|v_k\|_\lambda\}$ is bounded. Now, up to a subsequence, we can assume that

$$\begin{gathered} v_k \rightharpoonup v \text{ in } \mathring{H}^{1,2}_\gamma(\Omega), \\ v_k \to v \text{ in } L^2(\Omega), \\ v_k \to v \text{ a.e. in } \Omega. \end{gathered}$$

Since $\{v_k\}$ is bounded in $\mathring{H}^{1,2}_\gamma(\Omega)$, the continuous embedding $\mathring{H}^{1,2}_\gamma(\Omega) \hookrightarrow L^{2^*_\gamma}(\Omega)$ implies that $\left\{|v_k|^{2^*_\gamma-2}v_k\right\}$ is bounded in $L^{\frac{2^*_\gamma}{2^*_\gamma-1}}(\Omega)$. Thus, we can assume that

$$|v_k|^{2^*_\gamma-2}v_k \rightharpoonup |v|^{2^*_\gamma-2}v \text{ in } L^{\frac{2^*_\gamma}{2^*_\gamma-1}}(\Omega).$$

It follows that $J'_\lambda(v) = 0$, and so, $J'_\lambda(v)w = 0$ for all $w \in \mathring{H}^{1,2}_\gamma(\Omega)$. Consequently,

$$J_\lambda(v) = \frac{1}{2}\|v\|^2_\lambda - \frac{1}{2^*_\gamma}\int_\Omega |v|^{2^*_\gamma} = \left(\frac{1}{2} - \frac{1}{2^*_\gamma}\right)\int_\Omega |v|^{2^*_\gamma} = \frac{1}{N_\gamma}\int_\Omega |v|^{2^*_\gamma} \geq 0. \tag{4.3}$$

Let $w_k := v_k - v$. The Brezis-Lieb lemma gives us

$$\int_\Omega |v_k|^{2^*_\gamma} = \int_\Omega |v|^{2^*_\gamma} + \int_\Omega |w_k|^{2^*_\gamma} + o(1).$$

Assuming $J_\lambda(v_k) \to c \leq d$, one finds

$$J_\lambda(v) + \frac{\|w_k\|^2_\lambda}{2} - \int_\Omega |w_k|^{2^*_\gamma} \to 0. \tag{4.4}$$

Since $J'_\lambda(w_k)w_k \to 0$, one gets

$$\|w_k\|^2_\lambda - \int_\Omega |w_k|^{2^*_\gamma} \to \int_\Omega |v|^{2^*_\gamma} - \|v\|^2_\lambda = -J'_\lambda(v)v = 0.$$

Therefore, we may assume that

$$\|w_k\|^2_\lambda \to b$$

and

$$\int_\Omega |w_k|^{2^*_\gamma} \to b.$$

Since $w_k \to 0$ in $L^2(\Omega)$, it follows that $\|\nabla_\gamma w_k\|^2_{L^2(\Omega)} \to b$. By Sobolev inequality, one has

$$\|\nabla_\gamma w_k\|_{L^2(\Omega)} \geq \mathcal{S}_\gamma(m,n)\,\|w_k\|^2_{L^{2^*_\gamma}(\Omega)}$$

and so

$$b \geq \mathcal{S}_\gamma(m,n) b^{\frac{2}{2^*_\gamma}}.$$

Our aim is to show that $b = 0$, otherwise

$$b \geq \mathcal{S}_\gamma^{\frac{N_\gamma}{2}}(m,n),$$

and using (4.3) and (4.4), one gets

$$c^* = \frac{1}{N_\gamma}\mathcal{S}_\gamma^{\frac{N_\gamma}{2}}(m,n) \leq \left(\frac{1}{2} - \frac{1}{2^*_\gamma}\right) b \leq d < c^*,$$

that is a contradiction. □

Now, following the ideas in [33], we construct a family of Sobolev concentrating functions starting from a fixed Sobolev minimizer for $S_\gamma(m,n)$. Recall that Theorem 3.4 ensures the existence of minimizers for $S_\gamma(m,n)$ for all m, n. So, let $V > 0$ be a fixed minimizer for $S_\gamma(m,n)$. Then, up to a normalization constant, V solves the equation $-\Delta_\gamma V = V^{2^*_\gamma-1}$ in $\mathbb{R}^N$ and satisfies

$$\int_{\mathbb{R}^N} |\nabla_\gamma V|^2 dz = \int_{\mathbb{R}^N} |V|^{2^*_\gamma} dz = S_\gamma^{\frac{N_\gamma}{2}}(m,n). \tag{4.5}$$

For $A > 0$, define the family of rescaled functions

$$V_A(z) = A^{\frac{2-N_\gamma}{2}} V(\delta_{\frac{1}{A}} z).$$

Obviously, the functions V_A are also minimizers, they are solutions of the equation $-\Delta_\gamma V_A = V_A^{2^*_\gamma-1}$ in $\mathbb{R}^N$ and satisfy

$$\int_{\mathbb{R}^N} |\nabla_\gamma V_A|^2 dz = \int_{\mathbb{R}^N} |V_A|^{2^*_\gamma} dz = S_\gamma^{\frac{N_\gamma}{2}}(m,n), \quad \text{for all } A > 0. \tag{4.6}$$

Let $r > 0$ be such that $B_r = B_r(0) \subset \Omega$ (we can suppose $0 \in \Omega$, due to the translation invariance with respect to the y variable) and let $\phi \in C_0^\infty(B_r(0))$, $0 \leq \phi \leq 1$, $\phi = 1$ in $B_{r/2}(0)$. Define

$$v_A(z) := \phi(z)\, V_A(z). \tag{4.7}$$

The following asymptotic expansions hold:

Proposition 4.2. *The functions v_A satisfy the following estimates, as $A \to 0$*

$$\int_\Omega |\nabla_\gamma v_A|^2 dz \leq S_\gamma^{\frac{N_\gamma}{2}}(m,n) \,+\, O(A^{N_\gamma - 2}) \tag{4.8}$$

$$\int_\Omega v_A^{2^*_\gamma}\, dz = S_\gamma^{\frac{N_\gamma}{2}}(m,n) \,+\, O(A^{N_\gamma}) \tag{4.9}$$

$$\int_\Omega v_A^2\, dz \geq \begin{cases} C\,A^2 \,+\, O(A^{N_\gamma-2}) & \text{if } N_\gamma > 4 \\ C\,A^2 |\ln A| \,+\, O(A^2) & \text{if } N_\gamma = 4 \\ C\,A^{N_\gamma-2} + O(A^2) & \text{if } N_\gamma < 4. \end{cases} \tag{4.10}$$

Proof. The main ingredient of the proof is the asymptotic estimate on optimizers obtained in [34] and recalled in Theorem 3.7, i.e. there exist $C, r_0 > 0$ such that

$$C^{-1} d(z)^{2-N_\gamma} \leq V(z) \leq C d(z)^{2-N_\gamma}, \quad \text{for } d(z) > r_0.$$

The proof will therefore reduce to compute integrals of functions which only depend on the homogeneous norm d. We recall here the polar coordinates formula for d-radial functions: for every $0 \leq r_1 < r_2$ and for every measurable function $f : [r_1, r_2] \to \mathbb{R}$, we have

$$\int_{B_{r_2}(0) \setminus B_{r_1}(0)} f(d(z))\, dz \;=\; N_\gamma\, |B_1(0)| \int_{r_1}^{r_2} f(\rho)\, \rho^{N_\gamma - 1}\, d\rho,$$

if at least one of the two integrals exists. Let us begin to compute

$$\begin{aligned} \int_\Omega |\nabla_\gamma v_A|^2 dz &= \int_\Omega \phi^2 |\nabla_\gamma V_A|^2 dz + 2 \int_\Omega \phi V_A < \nabla_\gamma \phi, \nabla_\gamma V_A > dz + \int_\Omega |\nabla_\gamma \phi|^2 V_A^2 dz \\ &= \int_\Omega < \nabla_\gamma V_A, \nabla_\gamma (\phi^2 V_A) > \, dz + \int_\Omega |\nabla_\gamma \phi|^2 V_A^2\, dz \\ &= \int_\Omega \phi^2 V_A^{2^*_\gamma}\, dz + \int_\Omega |\nabla_\gamma \phi|^2 V_A^2\, dz \\ &= \int_{\mathbb{R}^N} V_A^{2^*_\gamma}\, dz + \int_\Omega |\nabla_\gamma \phi|^2 V_A^2\, dz + \alpha(\phi, A), \end{aligned} \tag{4.11}$$

where

$$\alpha(\phi, A) = -\int_{\Omega^C} {V_A}^{2^*_\gamma}\, dz + \int_\Omega (\phi^2 - 1)\, {V_A}^{2^*_\gamma}\, dz.$$

Let us evaluate the second integral in the right hand side of (4.11). Taking into account that $\phi = 1$ on the ball $B_{r/2}$ and $\phi = 0$ outside of B_r, we get

$$\begin{aligned} \int_\Omega |\nabla_\gamma \phi|^2 V_A^2\, dz \leq C \int_{B_r \setminus B_{r/2}} V_A^2\, dz &= C\, A^{2-N_\gamma} \int_{B_r \setminus B_{r/2}} V^2(\delta_{\frac{1}{A}} z)\, dz \\ &= CA^2 \int_{B_{r/A} \setminus B_{r/2A}} V^2\, dz \\ &\leq CA^2 \int_{r/2A < d(z) < r/A} \frac{1}{d(z)^{2N_\gamma - 4}}\, dz \\ &= CA^2 \int_{r/2A}^{r/A} \frac{1}{\rho^{N_\gamma - 3}}\, \mathrm{d}\rho \\ &= O(A^{N_\gamma - 2}), \end{aligned}$$

where we have used the estimate from above on V for $d(z)$ large.

Moreover, it is easily seen that $\alpha(\phi, A) = O(A^{N_\gamma})$. Indeed,

$$\begin{aligned}0 &\le \int_\Omega (1-\phi^2) V_A^{2^*_\gamma}\,dz \le \int_{d(z)>r/2} V_A^{2^*_\gamma}\,dz = \int_{d(z)>r/2A} V^{2^*_\gamma}\,dz \\ &\le C\int_{d(z)>r/2A} \frac{1}{d(z)^{2N_\gamma}}\,dz = C\int_{r/2A}^\infty \frac{1}{\rho^{N_\gamma+1}}\,\mathrm{d}\rho \\ &= O(A^{N_\gamma}),\end{aligned}$$

and an analogous estimate holds for the other term in $\alpha(\phi, A)$. So, taking into account (4.6), (4.8) is proved. Concerning $\|v_A\|_{2^*_\gamma}^{2^*_\gamma}$, we have

$$\begin{aligned}\int_\Omega v_A^{2^*_\gamma}\,dz &= \int_\Omega V_A^{2^*_\gamma}\,dz + \int_\Omega (\phi^{2^*_\gamma}-1)\,{V_A}^{2^*_\gamma}\,dz \\ &= \int_{\mathbb{R}^N} V_A^{2^*_\gamma}\,dz - \int_{\Omega^C} {V_A}^{2^*_\gamma}\,dz + \int_\Omega (\phi^{2^*_\gamma}-1)\,{V_A}^{2^*_\gamma}\,dz \\ &= S_\gamma^{\frac{N_\gamma}{2}}(m,n) + O(A^{N_\gamma}).\end{aligned}$$

Finally, we compute

$$\begin{aligned}\int_\Omega v_A^2\,dz &= \int_\Omega \phi^2 V_A^2 \ge \int_{B_{r/2}} V_A^2(z)\,dz = A^2\int_{B_{r/2A}} V^2(z)\,\mathrm{d}z \\ &= A^2\left(\int_{B_{r_0}} V^2(z)\,dz + \int_{B_{r/2A}\setminus B_{r_0}} V^2(z)\,dz\right) \\ &\ge CA^2\left(1 + \int_{r_0}^{r/2A} \frac{1}{\rho^{N_\gamma-3}}\,\mathrm{d}\rho\right) \\ &= \begin{cases} C\,A^2 + O(A^{N_\gamma-2}) & \text{if } N_\gamma > 4 \\ C\,A^2|\ln A| + O(A^2) & \text{if } N_\gamma = 4 \\ C\,A^{N_\gamma-2} + O(A^2) & \text{if } N_\gamma < 4 \end{cases}\end{aligned}$$

where we have used the estimate from below on V for $d(z)$ large.

The proof is therefore complete. □

Proof of Theorem 1.1 *Proof of part i)* First we claim that, for $N_\gamma \ge 4$, there exists some nonnegative function $u \in \mathring{H}_\gamma^{1,2}(\Omega) \setminus \{0\}$ such that

$$Q_\lambda(u) := \frac{\|u\|_\lambda^2}{\|u\|_{L^{2^*_\gamma}(\Omega)}^2} < S_\gamma(m,n). \tag{4.12}$$

Indeed, by Proposition 4.2, we get, for $N_\gamma > 4$

$$Q_\lambda(v_A) \le S_\gamma(m,n) - \lambda C A^2 + O(A^{N_\gamma-2}) < S_\gamma(m,n)$$

for $A > 0$ sufficiently small.

Analogously, for $N_\gamma = 4$ and $A > 0$ sufficiently small, we have

$$Q_\lambda(v_A) \le S_\gamma(m,n) - \lambda C A^2|\ln A| + O(A^2) < S_\gamma(m,n).$$

Hence, (4.12) holds. Now, define

$$g(t) := J_\lambda(tu) = \frac{t^2}{2}\,\|u\|_\lambda^2 - \frac{t^{2^*_\gamma}}{2^*_\gamma}\int_\Omega u^{2^*_\gamma},\ t \ge 0.$$

We obtain

$$
\begin{aligned}
0 < \max_{t\geq 0} g(tu) &= \frac{1}{N_\gamma}\left(\frac{\|u\|_\lambda^2}{\|u\|^2_{L^{2^*_\gamma}(\Omega)}}\right)^{\frac{N_\gamma}{2}}\\
&< \frac{S_\gamma^{\frac{N_\gamma}{2}}(m,n)}{N_\gamma} = c^*.
\end{aligned}
$$

Since

$$
\begin{aligned}
J_\lambda(v) &\geq \frac{\|v\|_\lambda^2}{2} - \frac{1}{2^*_\gamma}\|v\|^{2^*_\gamma}_{L^{2^*_\gamma}(\Omega)}\\
&\geq \frac{\|v\|_\lambda^2}{2} - \frac{1}{2^*_\gamma S_\gamma^{\frac{2^*_\gamma}{2}}(m,n)}\|\nabla_\gamma v\|^{2^*_\gamma}_{L^2(\Omega)},
\end{aligned}
$$

there exists $r > 0$ such that

$$
b := \inf_{\|u\|_\lambda = r} J_\lambda(u) > 0 = J_\lambda(0).
$$

Also, one can see that there exists $t_0 > 0$ such that $\|t_0 u\|_\lambda > r$ and $J_\lambda(t_0 u) < 0$. Therefore, we get

$$
\max_{t\in[0,1]} J_\lambda(tt_0 u) < c^*.
$$

Now, using the preceding Lemma 4.1 and the Mountain-Pass theorem [6], we see that J_λ has a critical point v in $\overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ and corresponding critical value $c \in [b, c^*)$. Hence, we get a solution of (1.3).

Proof of part ii) For $N_\gamma < 4$, let us consider an eigenfunction φ_1 associated with $\lambda_1(\Omega)$. Then $Q_{\lambda_1}(\varphi_1) = 0$; by continuity, there exists $\lambda_* < \lambda_1$ such that $Q_\lambda(\varphi_1) < \mathcal{S}_\gamma(m,n)$ for $\lambda_* < \lambda < \lambda_1$. □

Finally, we complete the study of problem (1.3) in low dimensions $N_\gamma < 4$ by proving the non-existence result for small positive $\lambda's$ stated in Theorem 1.2. We need the following preliminary result.

Lemma 4.3. *Let $f \in L^p(\Omega)$, $p \geq 1$ and let v be a solution to problem $-\Delta_\gamma v = f$ in $\overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$. Then, v satisfies the following estimate*

$$
\|v\|_{L^r(\Omega)} \leq C\|f\|_{L^p(\Omega)},
$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, and $q < \frac{N_\gamma}{N_\gamma - 2}$.

Proof. To prove the above estimate, we shall use the representation of Lax-Milgram solutions to problem $-\Delta_\gamma v = f$ in terms of approximate Green's functions (see [27] for the Laplacian case). As proved in [32], the following representation formula holds

$$
v_\rho(\xi) := \fint_{B_\rho(\xi)} v = \int_\Omega G^\rho(\xi, z) f(z)\, dz,
$$

where $G^\rho_\xi = G^\rho(\xi, \cdot)$ is the ρ-approximate Green's function of Ω with pole at ξ. Now, adapting the proof of Young's theorem on convolution in $\mathbb{R}^N$, if $f \in L^p$, $p \geq 1$, and r is such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, write

$$
G^\rho(\xi, z)|f(z)| = \left[G^\rho(\xi,z)^q |f(z)|^p\right]^{1/r} |f(z)|^{p(1-1/q)} G^\rho(\xi,z)^{q(1-1/p)}.
$$

Then, from generalized Hölder's inequality

$$
|v_\rho(\xi)| \leq \left(\int_\Omega G^\rho(\xi,z)^q |f(z)|^p\, dz\right)^{1/r} \left(\int_\Omega G^\rho(\xi,z)^q\, dz\right)^{1/p'} \left(\int_\Omega |f(z)|^p\, dz\right)^{1/q'}
$$

and integrating the latter inequality

$$
\left(\int_\Omega |v_\rho(\xi)|^r\, d\xi\right)^{1/r} \leq \sup_{\xi\in\Omega} \|G^\rho(\xi,\cdot)\|_q \left(\int_\Omega |f(z)|^p\, dz\right)^{1/p}.
$$

Now, we take into account that, as proved in [32],

$$\sup_{\xi\in\Omega}\|G^\rho(\xi,\cdot)\|_q \le C, \quad q < \frac{N_\gamma}{N_\gamma - 2},$$

uniformly with respect to ρ. Hence, since $v_\rho \to v$ a.e. in Ω, we conclude by dominated convergence. □

Proof of Theorem 1.2 We adapt the original idea in [12, Theorem 1.2]. By assumption, Ω is a smooth bounded open set of $\mathbb{R}^{m+n}$, with $N_\gamma = m + (\gamma+1)n < 4$, strictly δ_t-starshaped with respect to a point of the axis $x = 0$, which we shall suppose to be 0 for simplicity and v is a nonnegative solution $u \in C^2(\overline{\Omega})$ of problem (1.3) in Ω.

In what follows, it will be convenient to write Δ_γ as a divergence form operator on $\mathbb{R}^N$. To this aim, consider the $N \times N$ matrix

$$A_\gamma = \left(\begin{array}{c|c} I_{\mathbb{R}^m} & 0 \\ \hline 0 & |x|^{2\gamma} I_{\mathbb{R}^n} \end{array}\right) \tag{4.13}$$

Then

$$\Delta_\gamma = \mathrm{div}(A_\gamma\nabla) \quad \text{and} \quad < A_\gamma\nabla v, \nabla v >= |\nabla_\gamma v|^2,$$

where div and ∇ are the usual Euclidean divergence and gradient taken with respect to the variable $z \in \mathbb{R}^N$.

Let us recall the Pohozaev identity in the Grushin context (see [28]): if $v \in C^2(\overline{\Omega})$ is a solution of the equation $-\Delta_\gamma v = f(v)$ in Ω, $v = 0$ on $\partial\Omega$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then v satisfies the identity

$$N_\gamma \int_\Omega F(v)\,dz - \left(\frac{N_\gamma - 2}{2}\right)\int_\Omega v f(v)\,dz = \frac{1}{2}\int_{\partial\Omega} < A_\gamma\nabla v, \nabla v >< T, \nu > d\sigma$$

where $F(v) = \int_0^v f(t)dt$, T is the infinitesimal generator of the dilations δ_t and ν denotes the outward unit normal to $\partial\Omega$. The above identity, in the particular case $f(v) = v^{2^*_\gamma - 1} + \lambda v$, reads

$$\lambda\int_\Omega v^2 dz = \frac{1}{2}\int_{\partial\Omega} < A_\gamma\nabla v, \nabla v >< T, \nu > d\sigma. \tag{4.14}$$

Then, by (4.14), we get

$$\begin{aligned} \lambda\int_\Omega v^2 dz &\ge a\int_{\partial\Omega} < A_\gamma\nabla v, \nabla v > d\sigma \\ &\ge b\int_{\partial\Omega} (< A_\gamma\nabla v, \nu >)^2 d\sigma \ge c\left(\int_{\partial\Omega} < A_\gamma\nabla v, \nu > d\sigma\right)^2 \\ &= c\left(\int_\Omega \mathrm{div}(A_\gamma\nabla v)\,dz\right)^2 = c\left(\int_\Omega \Delta_\gamma v\,dz\right)^2 \\ &= c\left(\int_\Omega |\Delta_\gamma v|\,dz\right)^2 \\ &\ge d\int_\Omega v^2\,dz, \end{aligned} \tag{4.15}$$

where a, b, c, d are positive constants. In the above calculation, we have used first the strict starshapedness of Ω. Then, we have used that, since $\nabla v = -\nu|\nabla v|$ on $\partial\Omega$, it holds $(< A_\gamma\nabla v, \nu >)^2 =< A_\gamma\nabla v, \nabla v >< A_\gamma\nu, \nu >$ on $\partial\Omega$, and $< A_\gamma\nu, \nu >\le C$, being Ω bounded. Finally, in the last inequality, we have concluded by using the estimate $\|v\|_2 \le C\|\Delta_\gamma v\|_1$, which holds exactly for $N_\gamma < 4$ by Lemma 4.3.

Hence, from (4.15) it follows that, if problem (1.3) admits a nontrivial nonnegative solution for $N_\gamma < 4$, then it must be $\lambda \ge d > 0$. □

5. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. To this aim, we will find a critical point of the following energy functional corresponding to (1.4):

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla_\gamma v|^2\,dz - \int_\Omega \left(\frac{\lambda}{q}|v|^q + \frac{1}{2^*_\gamma}|v|^{2^*_\gamma}\right)dz \quad \text{for } v \in \mathring{H}^{1,2}_\gamma(\Omega), \quad 2 < q < 2^*_\gamma.$$

As in the previous section, J_λ is well-defined and of class C^1. We point out that in this section we are using on $\overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ the norm $\| \ \|$ given in (2.1).

In the next lemma, we will show that the functional J_λ verifies the Mountain-Pass geometry.

Lemma 5.1. *The functional J_λ satisfies:*

(i) There exists positive constants δ and R such that $J_\lambda(v) \geq \delta$ for all $\|v\| = R$;

(ii) There exists $v_1 \in \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ with $\|v_1\| > R$ and $J_\lambda(v_1) < 0$.

Proof. (i) The continuous embedding $\overset{\circ}{H}{}^{1,2}_\gamma(\Omega) \hookrightarrow L^q(\Omega)$ for $2 < q \leq 2^*_\gamma$ implies that

$$\begin{aligned} J_\lambda(v) &= \frac{1}{2}\int_\Omega |\nabla_\gamma v|^2\, dz - \frac{\lambda}{q}\int_\Omega |v|^q\, dz - \frac{1}{2^*_\gamma}\int_\Omega |v|^{2^*_\gamma}\, dz \\ &\geq \frac{1}{2}\|v\|^2 - \frac{\lambda C_1}{q}\|v\|^q - \frac{C_2}{2^*_\gamma}\|v\|^{2^*_\gamma}. \end{aligned}$$

Now, if $\|v\| = R$ is small enough, it follows that

$$J_\lambda(v) \geq \frac{1}{2}\|v\|^2 - \frac{\lambda C_1}{q}\|v\|^q - \frac{C_2}{2^*_\gamma}\|v\|^{2^*_\gamma} := \delta > 0.$$

(ii) Note that for any fixed nonzero $\psi \in C_0^\infty(\Omega)$, it holds that

$$\begin{aligned} J_\lambda(t\psi) &= \frac{t^2}{2}\int_\Omega |\nabla_\gamma \psi|^2\, dz - \frac{t^q\lambda}{q}\int_\Omega |\psi|^q\, dz - \frac{t^{2^*_\gamma}}{2^*_\gamma}\int_\Omega |\psi|^{2^*_\gamma}\, dz \\ &\leq \frac{t^2}{2}\int_\Omega |\nabla_\gamma \psi|^2\, dz - t^q\int_\Omega \left(\frac{\lambda}{q}|\psi|^q + \frac{1}{2^*_\gamma}|\psi|^{2^*_\gamma}\right) dz \to -\infty \text{ as } t \to \infty. \end{aligned}$$

This proves (ii). ☐

Lemma 5.2. *Let $\{u_k\} \subset \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ be a sequence such that*

$$J_\lambda(u_k) \to c \in \left(0, \frac{S_\gamma^{\frac{N_\gamma}{2}}(m,n)}{N_\gamma}\right)$$

and

$$J'_\lambda(u_k) \to 0, \quad in \ \left(\overset{\circ}{H}{}^{1,2}_\gamma(\Omega)\right)^{-1}.$$

Then, there exists $u \in \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$ such that $u_k \rightharpoonup u$, up to a subsequence, $J'_\lambda(u) = 0$ and u is a nontrivial solution to problem (1.4)*.*

Proof. The proof is analogous to the proof for the case $q = 2$, see Lemma 4.1. ☐

Because of Lemma 5.1, the *mountain-pass* level defined by

$$c_\lambda := \inf_{\alpha \in \mathcal{A}} \max_{t \in [0,1]} J_\lambda(\alpha(t)), \tag{5.1}$$

where $\mathcal{A} = \left\{\alpha \in C\left([0,1], \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)\right) \mid \alpha(0) = 0, J_\lambda(\alpha(1)) < 0\right\}$ is well-defined. By Theorem 2.2 in [12], J_λ admits a PS sequence at level c_λ; moreover, such a sequence may be chosen in the cone of nonnegative functions since $J_\lambda(|u|) \leq J_\lambda(u)$, for all $u \in \overset{\circ}{H}{}^{1,2}_\gamma(\Omega)$.

Lemma 5.3. *Fix $v \in \overset{\circ}{H}{}^{1,2}_\gamma(\Omega) \setminus \{0\}$. Then there exists a unique t_v such that*

$$J_\lambda(t_v v) = \max_{t>0} J_\lambda(tv).$$

Proof. Since by (i) of Lemma 5.1, there exists $t_0 > 0$ such that $J_\lambda(t_0 v) > 0$. Also, it is easy to see that $J_\lambda(tv) \to -\infty$ as $t \to \infty$ (see, proof of Lemma 5.1 part (ii)). Therefore, there exists $t_v > 0$ such that

$$J_\lambda(t_v v) = \max_{t>0} J_\lambda(tv).$$

To see the uniqueness, suppose there are $0 < t_1 < t_2$ such that

$$\max_{t>0} J_\lambda(tv) = J_\lambda(t_1 v) = J_\lambda(t_2 v).$$

Note that

$$J'_\lambda(t_1 v) = 0 = J'_\lambda(t_2 v).$$

Therefore,

$$\begin{aligned}\|v\|^2 &= t_2^{q-2}\lambda \int_\Omega |v|^q\,dz + t_2^{2^*_\gamma}\int_\Omega |v|^{2^*_\gamma}\,dz\\ &> t_1^{q-2}\lambda \int_\Omega |v|^q\,dz + t_1^{2^*_\gamma}\int_\Omega |v|^{2^*_\gamma}\,dz\\ &= \|v\|^2,\end{aligned}$$

a contradiction. ☐

Lemma 5.4. *For* $\max\{2, \frac{N_\gamma}{N_\gamma-2}, \frac{4}{N_\gamma-2}\} < q < 2^*_\gamma$*, the number* c_λ *given by (5.1) satisfies*

$$c_\lambda < \frac{1}{N_\gamma}\mathcal{S}_\gamma^{\frac{N_\gamma}{2}}(m,n), \quad \forall\lambda > 0. \tag{5.2}$$

Proof. Let us consider

$$w_A := \frac{v_A}{\|v_A\|_{L^{2^*_\gamma}}}, \tag{5.3}$$

where the functions v_A are defined as in (4.7). From Lemma 5.3, there exists a unique $t_A > 0$ such that

$$J_\lambda(t_A w_A) = \max_{t>0} J_\lambda(t w_A).$$

Since $J_\lambda(t w_A) \to -\infty$ as $t \to \infty$, there exists $R > 0$ such that $J_\lambda(R w_A) < 0$. Now, defining $v_1 = R w_A$, and using Lemma 5.1, we can write

$$c_\lambda = \inf_{\alpha\in\mathcal{A}} \max_{t\in[0,1]} J_\lambda(\alpha(t)) \le \max_{t>0} J_\lambda(t w_A).$$

So, in order to prove the lemma, it is sufficient to show that

$$J_\lambda(t_A w_A) < \frac{1}{N_\gamma}\mathcal{S}_\gamma^{\frac{N_\gamma}{2}}(m,n).$$

Since $\|w_A\|_{L^{2^*_\gamma}} = 1$, we have for $t > 0$

$$\begin{aligned}J_\lambda(t w_A) &= \frac{t^2}{2}\int_\Omega |\nabla_\gamma w_A|^2\,dz - \frac{t^q\lambda}{q}\int_\Omega |w_A|^q\,dz - \frac{t^{2^*_\gamma}}{2^*_\gamma}\int_\Omega |w_A|^{2^*_\gamma}\,dz\\ &= \left(\frac{t^2}{2}\int_\Omega |\nabla_\gamma w_A|^2\,dz - \frac{t^{2^*_\gamma}}{2^*_\gamma}\right) - \frac{t^q\lambda}{q}\int_\Omega |w_A|^q\,dz.\end{aligned}$$

Define

$$h(s) := \frac{s^2}{2}\int_\Omega |\nabla_\gamma w_A|^2\,dz - \frac{s^{2^*_\gamma}}{2^*_\gamma}.$$

It is easy to check that h achieves its maximum at point $s_A := \left(\int_\Omega |\nabla_\gamma w_A|^2\,dz\right)^{\frac{1}{2^*_\gamma-2}}$ and it is increasing on $[0, s_A]$. Moreover, from

$$0 = J'_\lambda(t_A) = t_A\left(\int_\Omega |\nabla_\gamma w_A|^2\,dz - t_A^{2^*_\gamma-2} - \lambda t_A^{q-2}\int_\Omega |w_A|^q dz\right),$$

we have

$$\int_\Omega |\nabla_\gamma w_A|^2\,dz = t_A^{2^*_\gamma-2} - \lambda t_A^{q-2}\int_\Omega |w_A|^q dz \ge t_A^{2^*_\gamma-2}.$$

Hence

$$t_A \le \left(\int_\Omega |\nabla_\gamma w_A|^2\right)^{\frac{1}{2^*_\gamma-2}} = s_A.$$

Therefore, also taking into account that, by Proposition 4.2, it holds:

$$\int_\Omega |\nabla_\gamma w_A|^2\,dz = \mathcal{S}_\gamma(m,n) + O(A^{N_\gamma-2}),$$

we get that

$$\begin{align}
J_\lambda(t_A w_A) &= h(t_A) - \frac{\lambda}{q} t_A^q \int_\Omega |w_A|^q\, dz \tag{5.4}\\
&\le h(s_A) - \frac{\lambda}{q} t_A^q \int_\Omega |w_A|^q\, dz \tag{5.5}\\
&= \frac{1}{N_\gamma}\left(\int_\Omega |\nabla_\gamma w_A|^2\, dz\right)^{\frac{N_\gamma}{2}} - \frac{\lambda}{q} t_A^q \int_\Omega |w_A|^q\, dz \tag{5.6}\\
&\le \frac{1}{N_\gamma}\mathcal{S}_\gamma^{\frac{N_\gamma}{2}}(m,n) + O(A^{N_\gamma-2}) - \lambda C_A \int_\Omega |w_A|^q dz, \tag{5.7}
\end{align}$$

where $C_A = \frac{t_A^q}{q}$. We observe that there is a positive constant C_0 such that $C_A \ge C_0 > 0, \forall A > 0$. Otherwise, we could find a sequence $A_k \to 0$, as $k \to \infty$ such that $t_{A_k} \to 0$ as $k \to \infty$, since $C_A \ge 0$. Now, up to a subsequence, we have $t_{A_k} w_{A_k} \to 0$, as $k \to \infty$. Therefore,

$$0 < c_\lambda \le \max_{t\ge 0} J_\lambda(t w_{A_k}) = J_\lambda(t_{A_k} w_{A_k}) = J_\lambda(0) = 0,$$

which is a contradiction. Next, we claim that

$$\lim_{A\to 0} \frac{\lambda C_A}{A^{N_\gamma-2}} \int_\Omega |w_A|^q dz = \infty. \tag{5.8}$$

Reasoning exactly as in Proposition 4.2, we have the following estimates, as $A \to 0$:

$$\int_\Omega |w_A|^q\, dz \ge \begin{cases} O(A^{N_\gamma - \frac{N_\gamma-2}{2}q}) & \text{if } \frac{N_\gamma}{N_\gamma-2} < q < 2_\gamma^*, \\ O(A^{N_\gamma - \frac{N_\gamma-2}{2}q}|\ln A|) & \text{if } q = \frac{N_\gamma}{N_\gamma-2}, \\ O(A^{\frac{q(N_\gamma-2)}{2}}) & \text{if } 1 \le q < \frac{N_\gamma}{N_\gamma-2}. \end{cases} \tag{5.9}$$

So, if $\max\{2, \frac{N_\gamma}{N_\gamma-2}, \frac{4}{N_\gamma-2}\} < q < 2_\gamma^*$, from the first line in (5.9) and being

$$N_\gamma - \frac{N_\gamma-2}{2} q < N_\gamma - 2,$$

we get (5.8). Hence, from (5.4) and (5.8), we have

$$J_\lambda(t_A w_A) < \frac{\mathcal{S}_\gamma^{\frac{N_\gamma}{2}}(m,n)}{N_\gamma}, \text{ for } A > 0 \text{ sufficiently small,}$$

for any $\lambda > 0$. This proves our aim. ☐

Proof of Theorem 1.3: a) Let $N_\gamma \ge 4$. By Lemma 5.4, we get that condition (5.2) holds for any $2 < q < 2_\gamma^*$. Therefore, we get a nontrivial solution to problem 1.4 for any $\lambda > 0$, for the whole range of q.
b) Let $N_\gamma < 4$. In this case, condition (5.2) in Lemma 5.4 holds for $\frac{4}{N_\gamma-2} < q < 2_\gamma^*$, hence for this range of $q's$ the existence of a nontrivial solution to problem (1.4) is ensured for any $\lambda > 0$.

For $2 < q \le \frac{4}{N_\gamma-2}$, we use a different argument. Observe that, for any fixed nonnegative function $v \in \overset{\circ}{H}{}_\gamma^{1,2}(\Omega)\backslash\{0\}$, it holds

$$M_\lambda := \max_{t\ge 0}\left\{\frac{t^2}{2}\int_\Omega |\nabla_\gamma v|^2\, dz - \frac{\lambda t^q}{q}\int_\Omega |v|^q\, dz\right\} = \frac{1}{\lambda^{\frac{2}{q-2}}}\left(\frac{1}{2} - \frac{1}{q}\right)\left(\frac{\|v\|^2}{\|v\|_{L^q(\Omega)}^q}\right)^{\frac{q}{q-2}} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Thereby, by definition of c_λ,

$$0 < c_\lambda \le \max_{t\ge 0} J_\lambda(tv) \le M_\lambda, \quad \forall \lambda > 0.$$

Hence, there exists $\lambda_0 > 0$ such that

$$c_\lambda < \frac{1}{N_\gamma}\mathcal{S}_{\gamma,x}^{\frac{N_\gamma}{2}}(m,n), \quad \forall \lambda \ge \lambda_0.$$

So, for this range of q's, the existence of a solution is ensured for λ sufficiently large.

6. Proof of Theorem 1.4

In this section, we turn to problem (1.6) on the whole space $\mathbb{R}^N$ and we give the proof of Theorem 1.4. Our goal is to obtain a ground state solution of (1.6). To this aim, we will find a critical point of the energy functional corresponding to (1.6):

$$J_a(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla_\gamma v|^2 + a(z)v^2\right)dz - \int_{\mathbb{R}^N}\left(\frac{\lambda}{q}|v|^q + \frac{1}{2^*_\gamma}|v|^{2^*_\gamma}\right)dz,\ v \in H^a_\gamma\left(\mathbb{R}^N\right).$$

Proposition 2.1 implies that the functional J_a is well-defined and of class C^1. In order to overcome the loss of compactness involving the space $H^a_\gamma\left(\mathbb{R}^N\right)$, we restrict the functional J_a to the space $H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right)$ and find a critical point there. Then, we use the *principle of symmetric criticality* to obtain a solution in the space $H^a_\gamma\left(\mathbb{R}^N\right)$.

In the next lemma, we show that the functional J_a verifies the Mountain-Pass geometry.

Lemma 6.1. *Suppose that* $(a1)$ *holds. Then the functional* J_a *satisfies:*

- *(i) There exist positive constants* δ *and* R *such that* $J_a(v) \geq \delta$ *for all* $\|v\|_{H^a_\gamma} = R$;
- *(ii) There exists* $v_1 \in H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right)$ *with* $\|v_1\|_{H^a_\gamma} > R$ *and* $J_a(v_1) < 0$.

Proof. (i) By the continuous embedding $H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right) \hookrightarrow L^q(\mathbb{R}^N)$ for $2 < q \leq 2^*_\gamma$ proved in [1] and recalled in Prop. 2.1, we get that

$$\begin{aligned} J_a(v) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla_\gamma v|^2 + a(z)v^2 dz\right)dz - \frac{\lambda}{q}\int_{\mathbb{R}^N}|v|^q\,dz - \frac{1}{2^*_\gamma}\int_{\mathbb{R}^N}|v|^{2^*_\gamma}\,dz \\ &\geq \frac{1}{2}\|v\|^2_{H^a_\gamma} - \frac{\lambda C_1}{q}\|v\|^q_{H^a_\gamma} - \frac{C_2}{2^*_\gamma}\|v\|^{2^*_\gamma}_{H^a_\gamma}. \end{aligned}$$

Now, if $\|v\|_{H^a_\gamma} = R$ is small enough, it follows that

$$J_a(v) \geq \frac{1}{2}\|v\|^2_{H^a_\gamma} - \frac{\lambda C_1}{q}\|v\|^q_{H^a_\gamma} - \frac{C_2}{2^*_\gamma}\|v\|^{2^*_\gamma}_{H^a_\gamma} := \delta > 0.$$

(ii) For any fixed nonzero $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} J_a(t\psi) &= \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla_\gamma\psi|^2\,dz + \frac{t^2}{2}\int_{\mathbb{R}^N}a(z)|\psi|^2\,dz - \frac{t^q\lambda}{q}\int_{\mathbb{R}^N}|\psi|^q\,dz - \frac{t^{2^*_\gamma}}{2^*_\gamma}\int_{\mathbb{R}^N}|\psi|^{2^*_\gamma}\,dz \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla_\gamma\psi|^2\,dz + \frac{t^2}{2}\int_{\mathbb{R}^N}a(z)|\psi|^2\,dz - t^q\int_{\mathbb{R}^N}\left(\frac{\lambda}{q}|\psi|^q + \frac{1}{2^*_\gamma}|\psi|^{2^*_\gamma}\right)dz \to -\infty,\ \text{as } t \to \infty. \end{aligned}$$

This proves (ii). ☐

Because of Lemma 6.1, the *mountain-pass* level defined by

$$c_a := \inf_{\alpha\in\mathcal{A}}\max_{t\in[0,1]} J_a(\alpha(t)), \tag{6.1}$$

where $\mathcal{A} = \left\{\alpha \in C\left([0,1], H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right)\right) \mid \alpha(0) = 0, J_a(\alpha(1)) < 0\right\}$ is well-defined.

The next lemma follows as in Section 5 and we omit its proof.

Lemma 6.2. *Assume that (a1) holds and fix* $v \neq 0$ *in* $H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right)$. *Then there exists unique* t_v *such that*

$$J_a(t_v v) = \max_{t>0} J_a(tv).$$

Our next lemma is crucial in our approach. In its proof, we will use a family of concentrating Sobolev functions constructed by means of a fixed minimizer for the constant $S_{\gamma,x}(m,n)$ defined in (3.17). We recall that the existence of such minimizers was proved in Corollary 3.5.

So, let $\tilde{V} > 0$ be a fixed minimizer for $S_{\gamma,x}(m,n)$ and consider, for $A > 0$, the family of rescaled functions

$$\tilde{V}_A(z) = A^{\frac{2-N_\gamma}{2}}\tilde{V}(\delta_{\frac{1}{A}}z).$$

Obviously, the functions $\tilde{V}_A$ are also minimizers and they are solutions, up to a normalization, of the equation $-\Delta_\gamma \tilde{V}_A = \tilde{V}_A^{2^*_\gamma - 1}$ in $\mathbb{R}^N$. Moreover, they satisfy

$$\int_{\mathbb{R}^N} |\nabla_\gamma \tilde{V}_A|^2 dz = \int_{\mathbb{R}^N} |\tilde{V}_A|^{2^*_\gamma} dz = S_{\gamma,x}^{N_\gamma/2}(m,n), \quad \text{for all } A > 0. \tag{6.2}$$

Let $\phi \in C_0^\infty(B_r(0))$, $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_{r/2}(0)$ and define

$$\tilde{v}_A(z) := \phi(z)\, \tilde{V}_A(z). \tag{6.3}$$

Moreover, define

$$\tilde{w}_A = \frac{\tilde{v}_A}{\|\tilde{v}_A\|_{2^*_\gamma}}.$$

Lemma 6.3. *Assume that (a1) holds. The number c_a given by (6.1) satisfies*

$$c_a < \frac{1}{N_\gamma} \mathcal{S}_{\gamma,x}^{\frac{N_\gamma}{2}}(m,n) \tag{6.4}$$

for any $\lambda > 0$ if $N_\gamma \ge 4$ or $N_\gamma < 4$ and $\frac{4}{N_\gamma - 2} < q < 2^*_\gamma$; for $\lambda > 0$ sufficiently large if $2 < q \le \frac{4}{N_\gamma - 2}$.

Proof. From Lemma 6.2, we have a unique $t_A > 0$ such that

$$J_a(t_A \tilde{w}_A) = \max_{t>0} J_a(t\tilde{w}_A),$$

where $\tilde{w}_A$ was given in (5.3). Since, $J_a(tw_A) \to -\infty$ as $t \to \infty$, there exists $R > 0$ such that $J_a(R\tilde{w}_A) < 0$. Now, defining $v_1 = R\tilde{w}_A$, and using Lemma 6.1, we can write

$$c_a = \inf_{\alpha \in \mathcal{A}} \max_{t \in [0,1]} J_a(\alpha(t)) \le \max_{t>0} J_a(t\tilde{w}_A).$$

So, in order to prove the lemma, it is sufficient to show that

$$J_a(t_A \tilde{w}_A) < \frac{1}{N_\gamma} \mathcal{S}_{\gamma,x}^{\frac{N_\gamma}{2}}(m,n).$$

Let $r > 0$ be fixed and will be chosen later. Since $\|\tilde{w}_A\|_{2^*_\gamma} = 1$, we have for $t > 0$

$$\begin{aligned} J_a(t\tilde{w}_A) &= \frac{t^2}{2}\int_{\mathbb{R}^N} \nabla_\gamma \tilde{w}_A^2 dz + \frac{t^2}{2}\int_{\mathbb{R}^N} a(z)\tilde{w}_A^2 dz - \frac{t^q \lambda}{q}\int_{\mathbb{R}^N} \tilde{w}_A^q dz - \frac{t^{2^*_\gamma}}{2^*_\gamma}\int_{\mathbb{R}^N} \tilde{w}_A^{2^*_\gamma} dz \\ &= \left(\frac{t^2}{2}\int_{\mathbb{R}^N} \nabla_\gamma \tilde{w}_A^2 dz - \frac{t^{2^*_\gamma}}{2^*_\gamma} \right) + \frac{t^2}{2}\int_{\mathbb{R}^N} a(z)\tilde{w}_A^2 dz - \frac{t^q \lambda}{q}\int_{\mathbb{R}^N} \tilde{w}_A^q dz. \end{aligned}$$

Define

$$h(s) := \frac{s^2}{2}\left(\int_{\mathbb{R}^N} |\nabla_\gamma \tilde{w}_A|^2 dz + \int_{B_r} a(z)\tilde{w}_A^2 dz \right) - \frac{s^{2^*_\gamma}}{2^*_\gamma}.$$

It is easy to check that h achieves maximum at point $s_A := \left(\int_{\mathbb{R}^N} |\nabla_\gamma \tilde{w}_A|^2 dz + \int_{B_r} a(z)\tilde{w}_A^2 dz \right)^{\frac{1}{2^*_\gamma - 2}}$ and it is increasing on $[0, s_A]$. Moreover, it is easily seen that

$$t_A \le \left(\int_{\mathbb{R}^N} |\nabla_\gamma \tilde{w}_A|^2 + \int_{B_r} a(z)\tilde{w}_A^2 dz \right)^{\frac{1}{2^*_\gamma - 2}} = s_A.$$

Hence, taking into account that

$$\int_{\mathbb{R}^N} |\nabla_\gamma \tilde{w}_A|^2 dz = \mathcal{S}_{\gamma,x}(m,n) + O(A^{N_\gamma - 2}),$$

we get that

$$J_a(t_A \tilde{w}_A) \le \frac{1}{N_\gamma}\left(\mathcal{S}_{\gamma,x}(m,n) + O(A^{N_\gamma - 2}) + \int_{B_r} a(z)\tilde{w}_A^2 dz \right)^{\frac{N_\gamma}{2}} - \lambda C_A \int_{B_r} \tilde{w}_A^q dz, \tag{6.5}$$

where $C_A = \frac{t_A^q}{q}$. We can assume that there is a positive constant C_0 such that $C_A \geq C_0 > 0, \forall A > 0$. Otherwise, we could find a sequence $A_k \to 0$, as $k \to \infty$ such that $t_{A_k} \to 0$ as $k \to \infty$, since $C_A \geq 0$. Now, up to a subsequence, we have $t_{A_k}\tilde{w}_{A_k} \to 0$, as $k \to \infty$. Therefore

$$0 < c_a \leq \max_{t \geq 0} J_a(t\tilde{w}_{A_k}) = J_a(t_{A_k}\tilde{w}_{A_k}) = J_a(0) = 0,$$

which is a contradiction. Now, by using the following inequality: If $\alpha \geq 0$, then

$$(a+b)^{\alpha} \leq a^{\alpha} + \alpha(a+b)^{\alpha+1}b,$$

for all positive constants a and b, we conclude that

$$J_A(t_A\tilde{w}_A) \leq \frac{\mathcal{S}_{\gamma,x}^{\frac{N_\gamma}{2}}(m,n)}{N_\gamma} + O(A^{N_\gamma-2}) + \int_{B_r} \left(Ca(z)\tilde{w}_A^2 - \lambda C_A \tilde{w}_A^q\right) dz,$$

for some constant $C > 0$. Next, we claim that, for $N_\gamma \geq 4$ or $N_\gamma < 4$ and $q > \frac{4}{N_\gamma - 2}$ it holds

$$\lim_{A \to 0} \frac{1}{A^{N_\gamma-2}} \int_{B_r} \left(Ca(z)\tilde{w}_A^2 - \lambda C_A \tilde{w}_A^q\right) dz = -\infty. \tag{6.6}$$

Assuming that (6.6) has been proved, we get

$$J_A(t_A\tilde{w}_A) < \frac{\mathcal{S}_{\gamma,x}^{\frac{N_\gamma}{2}}(m,n)}{N_\gamma}, \text{ for } A > 0 \text{ sufficiently small.} \tag{6.7}$$

This proves our statement. Now, we prove our claim (6.6). It suffices to show that

$$\lim_{A \to 0} \frac{1}{A^{N_\gamma-2}} \int_{B_r} \left(Ca(z)\tilde{V}_A^2 - \lambda C_A \tilde{V}_A^q\right) dz = -\infty \tag{6.8}$$

Case 1. $N_\gamma > 4$.
Reasoning as in Prop. 4.2, we get that

$$\int_{B_r} \tilde{V}_A^2(z)\,dz = CA^2 \int_{B_{r/A}} \tilde{V}^2(z)\,\mathrm{d}z \leq CA^2 + O(A^{N_\gamma-2}).$$

Moreover, as in (5.9), we get, for $N_\gamma > 4$

$$\int_{B_r} \tilde{V}_A^q\,dz \geq O(A^{N_\gamma - \frac{N_\gamma-2}{2}q}),$$

for any $2 < q < 2_\gamma^*$, since in this case $\frac{N_\gamma}{N_\gamma-2} < 2$. Hence, (6.6) follows, noting that $N_\gamma - \frac{N_\gamma-2}{2}q < N_\gamma - 2$.

Case 2. $N_\gamma = 4$.
We have

$$J_A(t_A\tilde{w}_A) \leq \frac{\mathcal{S}_{\gamma,x}^{\frac{N_\gamma}{2}}(m,n)}{N_\gamma} + O(A^2) + CA^2|\ln A| - \lambda O(A^{4-q}),$$

hence (6.7) holds as before, being $q > 2$.

Case 3. $N_\gamma < 4$.
In this case, we get

$$J_A(t_A\tilde{w}_A) \leq \frac{\mathcal{S}_{\gamma,x}^{\frac{N_\gamma}{2}}(m,n)}{N_\gamma} + O(A^{N_\gamma-2}) - \lambda O(A^{\frac{N_\gamma-2}{2}q}),$$

hence, if $\frac{4}{N_\gamma-2} < q < 2_\gamma^*$, (6.7) holds.

If, instead, $2 < q \leq \frac{4}{N_\gamma-2}$, by using a different argument analogous to the one used in the proof of Theorem 2.1, part b), we can prove that (6.4) holds for λ sufficiently large. We omit the details.

□

Lemma 6.4. *Assume that (a1) and (a3) hold. If* (6.4) *holds, there exists a bounded sequence* $\{v_k\}$ *in* $H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right)$ *such that*

$$J_a(v_k) \to c_a \tag{6.9}$$

and

$$\|J_a'(v_k)\|_{\left(H^a_{\gamma,rad_x}\right)^*} \to 0. \tag{6.10}$$

Moreover, the weak limit v of v_k is nonzero and satisfies

$$J_a'(v)u = 0 \textit{ for all } u \in H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right).$$

Proof. The proof is analogous to the proof of Lemma 4.1 and Lemma 5.2. Assumption (a3) plays a role in proving this lemma as we need a compact embedding $H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right)$ for $2 \leq q < 2^*_\gamma$, see Proposition 2.2. ☐

Now, we are in a position to prove Theorem 1.4. The previous lemma shows that the weak limit v of the sequence $\{v_k\}$ is a critical point of J_a in the subspace $H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right)$. In fact, we will show that v is a critical point of J_a in the space $H^a_\gamma\left(\mathbb{R}^N\right)$ with $J_a(v) = c_a$.

Proof of Theorem 1.4: By the previous Lemma 6.4, we have shown that

$$J_a'(v)u = 0 \text{ for all } u \in H^a_{\gamma,rad_x}\left(\mathbb{R}^N\right).$$

Now, by (a2), we can employ the *principle of symmetric criticality* (see, for instance, [41, Theor. 1.28]) to conclude that

$$J_a'(v)u = 0 \text{ for all } u \in H^a_\gamma\left(\mathbb{R}^N\right).$$

We refer to [1], where the authors use the *principle of symmetric criticality* to show the same. Thus, v is a critical point of J_a in the space $H^a_\gamma\left(\mathbb{R}^N\right)$. Hence, we obtain a weak solution to (1.6). Next, we claim that $J_a(v) = c_a$. Note that $J_a(v) \geq c_a$. Now, using Fatou's lemma, we get

$$\begin{aligned}
c_a &= \liminf_{k\to\infty}\left(J_a(v_k) - \frac{1}{2}J_a'(v_k)v_k\right)\\
&= \liminf_{k\to\infty} \frac{\lambda(q-2)}{2q}\int_{\mathbb{R}^N}|v_k|^q\,dz + \frac{1}{N_\gamma}\int_{\mathbb{R}^N}|v_k|^{2^*_\gamma}\,dz\\
&\geq \frac{\lambda(q-2)}{2q}\int_{\mathbb{R}^N}|v|^q\,dz + \frac{1}{N_\gamma}\int_{\mathbb{R}^N}|v|^{2^*_\gamma}\,dz\\
&= J_a(v) - \frac{1}{2}J_a'(v)v\\
&= J_a(v),
\end{aligned}$$

showing that $J_a(v) = c_a$. This completes the proof. ☐

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REFERENCES

[1] C. O. Alves, A. R. F. de Holanda, *A Berestycki-Lions type result for a class of degenerate elliptic problems involving the Grushin operator,* Proc. R. Soc. Edinb., 128 (2022), DOI:10.1017/prm.2022.43.

[2] C. O. Alves, A. R. F. de Holanda, *Existence of the solution for a class of the semilinear degenerate elliptic equation involving the Grushin operator in $\mathbb{R}^2$: The interaction between Grushin's critical exponent and exponential growth*, Math. Nachr. (2023), DOI: 10.1002/mana.202300171

[3] C. O. Alves, M. Montenegro and M. A. Souto, *Existence of a ground state solution for a nonlinear scalar field equation with critical growth*, Calc. Var. PDEs 43 (2012), 537-554.

[4] C. O. Alves, G. M. Figueiredo and G. Siciliano, *Ground state solutions for fractional scalar field equations under a general critical nonlinearity*, Comm. Pure Appl. Anal. 18 (2019), 2199-2215.

[5] C. O. Alves, P. C. Carrião, O. H. Miyagaki, *Nonlinear perturbations of a periodic elliptic problem with critical growth*, J. Math. Anal. Appl. 260 (2001), no. 1, 133-146.

[6] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Functional Anal, 1973, 14, 349-381.

[7] A Ambrosetti, M. Struwe, *A note on the problem $-\Delta u = \lambda u + |u|^{2^*-2}u$,* Manuscripta Math., 1986, 54, 373-379.

[8] T. Aubin, *Equations différentielles non linéaires et probléme de Yamabe concernant la courbure scalaire,* J. Math. Pures et Appl. 55, 1976, 269-293.

[9] M. S. Baouendi, *Sur une Classe d'Operateurs Elliptiques Degenerés,* Bull. Soc. Math. France 95 (1967), 4587.

[10] W. Beckner, *On the Grushin operator and hyperbolic symmetry*, Proc. Amer. Math. Soc. 129 (2001), no. 4, 1233-1246.

[11] M. Bhakta and K. Sandeep, *Poincaré-Sobolev equations in the hyperbolic space,* Calc. Var. and PDEs (2012) 44:247-269 DOI 10.1007/s00526-011-0433-8.

[12] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical exponents,* Comm. Pure Appl. Math. 36 (1983) 437-477.

[13] A. Capozzi,D. Fortunato and G. Palmieri, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent,* Ann. Inst. H. Poincaré Analyse non Lineaire, 1985, 2(6), 463-470.

[14] G. Cerami, D. Fortunato and M. Struwe, *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents,* Ann Inst. H. Poincaré Analyse non Lineaire, 1984, l(5), 341-350.

[15] G. Cerami, S. Solimini, and M. Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents,* J. Funct. Analysis, 1986, 69, 289-306.

[16] X. J. Chang and Z. Q. Wang, *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity,* Nonlinearity 26 (2013), 479494.

[17] N. T. Chung. *On a class of semilinear elliptic systems involving Grushin type operator,* Comm. Korean Math. Soc. 29 (2014), 37-50.

[18] E.N. Dancer, *A note on an equation with critical exponent,* Bull. London Math. Soc., 1988, 20, 600-602.

[19] L. D'Ambrosio, *Hardy inequalities related to Grushin type operators,* Proc. Amer. Math. Soc. 132 (2004), no. 3, 725-734.

[20] J. Dou, L. Sun, L. Wang, M. Zhu, *Divergent operator with degeneracy and related sharp inequalities,* J. Functional Anal. 282 (2022), no. 2, Paper No. 109294, 85 pp.

[21] A. T. Duong and N. T. Nguyen, *Liouville type theorems for elliptic equations involving Grushin operator and advection,* Electron. J. Differ. Equ. 2017 (2017), 111.

[22] A. T. Duong and Q. H. Phan, *Liouville type theorem for nonlinear elliptic system involving Grushin operator,* J. Math. Anal. Appl. 454 (2017), 785-801.

[23] B. Franchi, E. Lanconelli, *An embedding theorem for Sobolev Spaces related to non-smooth vector fields and Harnack inequality*, Comm. Partial Differential Equations 9 (1984) 1237-1264.

[24] N. Garofalo and E. Lanconelli, *Existence and nonexistence results for semilinear differential equations on the Heisenberg group,* Indiana Univ. Math. J. 41 (1992) 71-98.

[25] V. V. Grushin, *On a class of hypoelliptic operators,* 1970 Math. USSR Sb. 12 458.

[26] V. V. Grushin, *On a class of hypoelliptic pseudodifferential operators degenerate on a submanifold,* 1971 Math. USSR Sb. 13 155.

[27] M. Grüter, K. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. 37 (1982), 303-342.

[28] A. E. Kogoj and E. Lanconelli, *On semilinear Δ_λ-Laplace equation,* Nonlinear Anal. 75 (2012), 4637-4649.

[29] P.D. Lamberti, P. Luzzini, P. Musolino, *Shape perturbation of Grushin eigenvalues,* J. Geom. Anal., 31 (11) (2021), pp. 10679-10717.

[30] F. Lascialfari, D. Pardo, *Compact embedding of a degenerate Sobolev space and existence of entire solutions to a semilinear equation for a Grushin-type operator,* Rend. Sem. Mat. Univ. Padova 107 (2002), 139152.

[31] M. Liu, Z. Tang and C. Wang, *Infinitely many solutions for a critical Grushin-type problem via local Pohozaev identities*, Ann. Mat. Pura Appl. 199 (2019), 1737-1762.

[32] A. Loiudice, *Sobolev inequalities with remainder terms for sublaplacians and other subelliptic operators*, NoDEA Nonlinear Differential Equations Appl. 13 (2006), no. 1, 119-136.

[33] A. Loiudice, *Semilinear subelliptic problems with critical growth on Carnot groups*, Manuscripta Math. 124 (2007), 247-259.

[34] A. Loiudice, *Asymptotic behaviour of solutions for a class of degenerate elliptic critical problems*, Nonlinear Anal. 70, no. 8 (2009), 2986-2991.

[35] A. Loiudice, *Asymptotic estimates and nonexistence results for critical problems with Hardy term involving Grushin-type operators*, Ann. Mat. Pura Appl. 198 (2019), 1909-1930.

[36] O. H. Miyagaki, *On a class of semilinear elliptic problems in $\mathbb{R}^N$ with critical growth,* Nonlinear Anal. 29 (1997), no. 7, 773-781.

[37] R. Monti, Sobolev inequalities for weighted gradients, Commun. Partial Differ. Equ. 31(2006), 1479-1504.

[38] R. Monti and D. Morbidelli, *Kelvin transform for Grushin operators and critical semilinear equations,* Duke Math. J. 131 (2006), 167-202.

[39] F. G. Tricomi, *Sulle Equazioni Lineari alle Derivate Parziali di 2^o Ordine di Tipo Misto*, Mem Lincei 14 (1923), 133-247.

[40] C. Wang, Q. Wang, J. Yang, *On the Grushin critical problem with a cylindrical symmetry,* Adv. Differ. Equ. 20, 77-116 (2015).

[41] M. Willem, *Minimax Theorems,* Birkhäuser, Boston 123, 1996.

[42] M. Willem, *Analyse harmonique réelle*, Hermann, Paris, 1995.

[43] X. Zhang, B. Zhang, M. Xiang, *Ground states for fractional Schrödinger equations involving a critical nonlinearity*, Adv. Nonlinear Anal. 5 (2016), no. 3, 293-314.

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