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Varieties of Null-Filiform Leibniz Algebras Under the Action of Hopf Algebras

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Abstract
Let L be an n-dimensional null-filiform Leibniz algebra over a field K. We consider a finite dimensional cocommutative Hopf algebra or a Taft algebra H and we describe the H-actions on L. Moreover we provide the set of H-identities and the description of the S_n-module structure of the relatively free algebra of L.

Keywords Polynomial identities · Leibniz Algebras · H-module algebras are the correct captured words

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1 Introduction

The study of polynomial identities of non-associative algebras plays a crucial role in the investigation of genetic populations. In 1939 Etherington (see [25]) introduced the so called *genetic algebras* that are algebras arising from the multiplication table given by the composition of the gametic or zygotic (and many other) types of some population. Genetic algebras are, in general, non-associative algebras and, depending on some special identities they satisfy, they give rise to peculiar "genetic facts". For instance, see the Bernstein algebras that are baric algebras satisfying the identity $(x^2)^2 = \omega(x)(x^2)x^2$. The Bernstein algebras describe the equilibrium state of a certain population in future generations.

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A complete understanding of the equilibrium states of any population is strictly related to the study of the varieties generated by Bernstein algebras. In this paper, following this line of research, we would like to study the varieties generated by some special Leibniz algebras that are non-associative algebras carrying on the defining identities of Lie algebras but the anticommutativity. We recall the important role played by Lie algebras in genetics. The genetic code is usually represented mathematically as a Lie algebra and new such structures are studied by several groups of biomathematicians (see for instance [47]). In particular, here we study varieties of null-filiform Leibniz algebras under the action of a big class of Hopf algebras. Although we still have not a literature containing the application in genetics of the action of any Hopf algebra, we strongly believe this should be a necessary step in founding a bridge between mathematics and genetics. Nevertheless we know that abelian gradings on Lie algebras (that is a very particular example of an action of a Hopf algebra on a Lie algebra) and the knowledge of their representations, identities, etc. can be useful in the description of degeneracies of a genetic code (see [26] and [32]).

It is worth saying Leibniz algebras were introduced by Bloh in [10] and Loday in [40], and they have many applications either in pure and applied mathematics or in physics. Because of this, many known results of the theory of Lie algebras as well as combinatorial group theory have been spread to Leibniz algebras during the last two decades, (see, for instance, [8, 13, 16, 21, 38] and [50]). It is well known that a Leibniz algebra decomposes into a semidirect sum of the solvable radical and a semisimple Lie algebra [8]. Therefore, restricting the study of solvable Leibniz algebras to nilpotent ones, it is sufficient studying nilpotent Leibniz algebras with certain conditions such as the conditions on the index of nilpotency, various types of gradings and characteristic sequence (see, for example, [2, 15] and [37]). Regarding the index of nilpotency, we recall that in the theory of Lie algebras a finite dimensional *filiform* Lie algebra over field K is a nilpotent Lie algebra L whose nil index is maximal and equal to $dim(L) - 1$. The notion of p-filiform Lie algebras makes sense for $p \geq 1$ (see [11]), while for $p = 0$ it is not possible to define that, since a Lie algebra has at least two generators. In the case of Leibniz algebras the class of null-filiform Leibniz algebras and their properties was originally introduced in [5]. The maximal index of nilpotency of n-dimensional Leibniz algebra is equal to $n + 1$, whereas, the maximal index of nilpotency of an n-dimensional Lie algebra is n.

In the classical and purely mathematical literature towards varieties of algebras (as well as polynomial identities of algebras), one of the main problems is the so-called *Specht problem* which concerns the existence of a finite basis of identities for any subvariety of a given variety. It is known that every variety of associative algebras over a field of characteristic 0 satisfies the Specht property [36]. In the case of Lie algebras this problem is still open in general even in the case the ground field is of characteristic 0. Partial cases were studied and solved by Iltyakov in [33] including the case of finite dimensional Lie algebras. The study of polynomial identities for Leibniz algebras was started by Drensky and Piacentini Cattaneo in [24]. Drensky et al. described the free metabelian Leibniz algebras and provided a complete list of left-nilpotent of class 2 varieties of Leibniz algebras. Abanina and Mishchenko in [1] studied the variety of Leibniz algebras determined by the identity $x(y(zt)) \equiv 0$ and proved many properties similar to the variety of abelian-by-nilpotent Lie algebras of class 2. Several years later Mishchenko and Valenti in [43] studied the variety of Leibniz algebras defined by identity $y_1(y_2y_3)(y_4y_5) \equiv 0$ and provided a description of multilinear identities in the language of Young diagrams through representation theory of symmetric groups. In [49] Vais and Zelmanov proved that any finitely generated Jordan algebra in characteristic 0 satisfies the Specht property by showing that it has the same identities of a finite dimensional generalized Jordan pair. Unfortunately, we do not know yet whether the answer is positive nor negative in case of infinitely generated Jordan algebras. Some special cases of the Specht property for graded Jordan algebras can be found in [19].

The following pages will contain a full description of the variety of a finite dimensional *null-filiform* Leibniz algebras under the action of a finite dimensional pointed cocommutative Hopf algebra first and under the action of a Taft algebra too. This means we will furnish information such as the description of the action, the polynomial identities, the S_n-structure of the relatively free algebras (cocharacters, codimensions, Hilbert series). As a consequence we get an H-variety generated by a null-filiform Leibniz algebra always satisfies the Specht property if H is a Taft's algebra whereas it satisfies the Specht property only if the underlying group of H is finite in the case H is pointed cocommutative.

2 Preliminaries

Although almost all of the arguments below can be performed for any algebra (associative, Lie, Jordan, etc.) every algebra in the sequel will be a Leibniz algebra. We recall a *Leibniz algebra* is a vector space L over a field K with bilinear product $(-, -)$ which satisfies the Leibniz identity

$$(x(yz)) = ((xy)z) - ((xz)y). \tag{1}$$

Indeed we have the notion of Leibniz subalgebras, ideals, homomorphisms, gradings. On this purpose, we would like to construct a free set in the class of graded Leibniz algebras. Hence let G be a group and let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets. Set $X = \bigcup_{g\in G} X^g$ and denote by $\mathcal{L}\langle X|G\rangle$ the free Leibniz algebra freely generated by the set X. An indeterminate (or variable) $x \in X$ is said to be of *homogeneous G-degree g*, written $\deg(x) = g$, if $x \in X^g$. We always write x^g if $x \in X^g$. The homogeneous G-degree of a monomial $m = x_{i_1}x_{i_2}\cdots x_{i_k}$ is defined to be $\deg(m) = \deg(x_{i_1})\cdot\deg(x_{i_2})\cdots\deg(x_{i_k})$. For every $g \in G$ we denote by $\mathcal{L}\langle X|G\rangle^g$ the subspace of $\mathcal{L}\langle X|G\rangle$ spanned by all monomials having homogeneous G-degree g. Notice that $\mathcal{L}\langle X|G\rangle^g\mathcal{L}\langle X|G\rangle^{g'} \subseteq \mathcal{L}\langle X|G\rangle^{gg'}$ for all $g, g' \in G$. Thus

$$\mathcal{L}\langle X|G\rangle = \bigoplus_{g\in G}\mathcal{L}\langle X|G\rangle^g$$

and $\mathcal{L}\langle X|G\rangle$ is a G-graded algebra. We refer to the elements of $\mathcal{L}\langle X|G\rangle$ as *G-graded polynomials* or just *graded polynomials*. An ideal I of $\mathcal{L}\langle X|G\rangle$ is said to be a *T_G-ideal* (or *graded T-ideal*) if it is invariant under all G-graded endomorphisms $\varphi : \mathcal{L}\langle X|G\rangle \to \mathcal{L}\langle X|G\rangle$ such that $\varphi(\mathcal{L}\langle X|G\rangle^g) \subseteq \mathcal{L}\langle X|G\rangle^g$ for all $g \in G$. If A is a G-graded Leibniz algebra, a G-graded polynomial $f(x_1, \ldots, x_n)$ is said to be a *graded polynomial identity* of A if $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in \bigcup_{g\in G} A^g$ such that $a_k \in A^{\deg(x_k)}$, $k = 1, \ldots, n$. We denote by $Id^G(A)$ the ideal of all graded polynomial identities of A. It is a T_G-ideal of $\mathcal{L}\langle X|G\rangle$ in the sense that it is invariant under all graded homomorphism of $\mathcal{L}\langle X|G\rangle$. We shall call *substitution* with elements of A any graded homomorphism $\mathcal{L}\langle X|G\rangle \to A$ and we sometimes use the notation $\overline{x} = a \in A$ in order to denote explicitly such evaluation of the variable x.

Given a subset $S \subseteq \mathcal{L}\langle X|G\rangle$ one can talk about the least T_G-ideal of $\mathcal{L}\langle X|G\rangle$ containing the set S. Such T_G-ideal will be denoted by $\langle S\rangle^{T_G}$ and will be called the *T_G-ideal generated by S*. We say that elements of $\langle S\rangle^{T_G}$ are *consequences* of elements of S, or simply that they follow from S. If $Id^G(A) = \langle S\rangle^{T_G}$, we say that S is a *basis* for the graded polynomial identities of A.

Let K be a field of characteristic 0, and let H be a Hopf algebra over K with comultiplication $\Delta : H \to H \otimes H$. Here we use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

Definition 2.1 A K-algebra A is called an *H-module algebra* or *an algebra with an H-action*, if A is a left H-module with action

$$h \otimes a = h \cdot a,$$

for all $h \in H, a \in A$ such that

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b).$$

We shall construct a free object inside the class of H-module algebras too. Let H be a Hopf algebra with unit 1 and let us consider a countable set of indeterminates $X := \{x_1, x_2, \ldots\}$; we set $x_j := x_j^1$. We choose a linear basis $(\gamma_\beta)_{\beta\in\Lambda}$ in H and we denote by $\mathcal{L}\langle X|H\rangle$ the free Leibniz algebra over K generated by the free formal generators $x_i^{\gamma_\beta}$ lying in the set $X^H = \{x_i^{\gamma_\beta} | i \in \mathbb{N}, \beta \in \Lambda, i \in \mathbb{N}\}$. If $h \in H$ let $x_i^h := \sum_{\beta\in\Lambda} \alpha_\beta x_i^{\gamma_\beta}$ for $h = \sum_{\beta\in\Lambda} \alpha_\beta \gamma_\beta, \alpha_\beta \in F$, where only a finite number of α_β's is non-zero. We refer to the elements of $\mathcal{L}\langle X|H\rangle$ as *H-polynomials*. Note that here we do not consider any H-action on $\mathcal{L}\langle X|H\rangle$.

Let A be a Leibniz algebra with an H-module algebra structure. Any map $\psi : X \to A$ has a unique homomorphic extension $\overline{\psi} : F\langle X|H\rangle \to A$ such that $\overline{\psi}(x_i^h) = h\psi(x_i)$ for all $i \in \mathbb{N}$ and $h \in H$. An H-polynomial $f \in \mathcal{L}\langle X|H\rangle$ is an *H-identity* of A if $\overline{\psi}(f) = 0$ for all maps $\psi : X \to A$ which are called *substitutions*. In other words, $f(x_1, x_2, \ldots, x_n)$ is an H-identity of A if and only if

$$\overline{\psi}(f) = f(\overline{x}_1, \ldots, \overline{x}_n) = f(a_1, a_2, \ldots, a_n) = 0$$

for any $a_i \in A$, where $\overline{x}_i := \psi(x_i)$. In this case we write $f \equiv 0$. The set $\mathrm{Id}^H(A)$ of all H-identities of A is an ideal of $\mathcal{L}\langle X|H\rangle$ which is invariant under all endomorphisms of $\mathcal{L}\langle X|H\rangle$, i.e., it is a T^H-ideal. Note that our definition of $\mathcal{L}\langle X|H\rangle$ depends on the choice of the basis $(\gamma_\beta)_{\beta\in\Lambda}$ in H. However such algebras can be identified in a natural way and $Id^H(A)$ turns out to be the same.

It is worth noticing that if we consider the trivial Hopf algebra $H = F$, then we are simply studying *ordinary polynomial identities* and we shall omit any index or super-index to refer to its H-identities or related stuffs whereas, if $H = KG$ and G is abelian, we are studying *G-graded polynomial identities*. For further lectures about polynomial identities of associative algebras we strongly recommend the books [23] by Drensky and [29] by Giambruno and Zaicev. We also address the reader to the book [6] by Bahturin that is more focused on identities of Lie algebras.

Denote by P_n^H the space of all multilinear H-polynomials in x_1, ..., x_n, $n \in \mathbb{N}$, i.e.,

$$P_n^H := \langle x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \cdots x_{\sigma(n)}^{h_n} | h_i \in H, \sigma \in S_n\rangle \subset \mathcal{L}\langle X|H\rangle.$$

The symmetric group S_n acts on the left on the space P_n^H by $\sigma(x_i^h) = x_{\sigma(i)}^h$ if $\sigma \in S_n$. Notice that the vector space $P_n^H \cap \mathrm{Id}^H(A)$ is stable under this S_n action, hence $P_n^H(A) := P_n^H/(P_n^H \cap \mathrm{Id}^H(A))$ is a left S_n-module. This leads us to consider the S_n-character of $P_n^H(A)$, namely $\chi_n^H(A)$, which is called *n-th cocharacter of polynomial H-identities* or the *n-th H-cocharacter* of A. By the classical theory of representations of the symmetric group (see for instance the book by Sagan [46]), the irreducible S_n-characters are in one-to-one correspondence with the partitions of the non-negative integer n (which carries a

Young Tableau) because the ground field is of characteristic 0. In particular, if χ_λ denotes the irreducible S_n-character corresponding to the partition λ, then we are allowed to write

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m_\lambda^H \chi_\lambda,$$

where $m_\lambda^H \geq 0$ is the multiplicity of the irreducible character χ_λ in the decomposition of $\chi_n^H(A)$. Moreover the non-negative integer

$$c_n^H(A) := \dim_F(P_n^H(A))$$

is called the *n-th codimension of polynomial H-identities* or the *n-th H-codimension* of A. We shall also refer to the sequences $\{\chi_n^H(A)\}_{n\geq 0}$, $\{c_n^H(A)\}_{n\geq 0}$ as the *H-cocharacter sequence of A* and the *H-codimension sequence of A* respectively.

Given an H-module algebra A, if the limit

$$\lim_n \sqrt[n]{c_n^H(A)}$$

exists we shall call it *H PI-exponent* of A and we shall denote it by $\exp^H(A)$.

As we said above, if we specialize H with the dual algebra of the group algebra FG, where G is a finite abelian group, we get the notion of *G-graded identities, codimension, exponent*, etc. The existence of the exponent in the graded case when A is supposed to be associative and over a field of characteristic 0, has been studied by several authors as Giambruno and Zaicev in [28] when G is the trivial group, Benanti, Giambruno and Pipitone in [9] when $G = \mathbb{Z}_2$, by Aljadeff, Giambruno and La Mattina in [4] in the case A is affine and G is abelian, by Giambruno and La Mattina (see [27]) if A is any G-graded algebra and G is abelian and in general by Aljadeff and Giambruno in [3]. In the general case of an H-algebra only partial results are known about the existence of such exponent. If H is finite dimensional and semisimple acting on an associative algebra over a field of characteristic 0, then Karasik proved in [35] the H-exponent exists and is an integer. It is easy to see Taft's algebras are not semisimple algebras. Also the result by Gordienko [30] is another good step in this direction. In fact, he proved the existence of the exponent for finite dimensional algebras over an algebraically closed field of characteristic 0 that are simple under the action of a Taft algebra. We recall Taft's algebras are non-commutative, non-cocommutative and not semisimmple Hopf algebras. In [45] the authors constructed the first example of a non-associative unital algebras whose PI-exponent does not exist whereas in [31] the author proved the existence of the exponent for finite dimensional Lie and associative algebras graded by any group. The first example of an infinite dimensional Lie algebra with a non-integer ordinary PI-exponent was constructed by Mishchenko and Zaicev in [44].

Finally, given an H-module algebra A, we would also give a new definition of *H-Hilbert series* of the relatively free algebra of A. Let us set the framework. We take H being finite dimensional and let us choose a linear basis $\mathcal{B} = \{h_1, \ldots, h_d\}$ of H. We choose a natural number $k \geq 1$ and the set of H-variables

$$X_k^H = \{x_1^{h_1}, \ldots, x_k^{h_1}, x_1^{h_2}, \ldots, x_k^{h_2}, \ldots, x_1^{h_d}, \ldots, x_k^{h_d}\}.$$

Denoting by $Hilb(B, t)$ the Hilbert series of an algebra B in the variable t, we have the next definition which should be considered as a generalization to the Hopf algebra case of the graded Hilbert series of an algebra as appeared for the first time in [18]. See also the paper [17] for an interesting relation with the graded exponent.

Definition 2.2 Let A be an H-module algebra and $k \geq 1$ a natural number. We define the H-Hilbert series of A in k-variables as

$$Hilb_k^H(A,t) := Hilb(\mathcal{L}\langle X_k^H\rangle/(\mathcal{L}\langle X_k^H\rangle \cap Id^H(A)), t).$$

Let us show briefly a bridge between H-identities and graded identities that would be helpful for our purposes in this paper. The proof of Proposition 3.3.6 of [29] gives us a well known and nice duality between G-gradings and G-actions of finite dimensional algebras provided that G is a finite abelian group. However one can define G-polynomials as KG-polynomials, where the group algebra FG is endowed with its canonical Hopf algebra structure. Notice also in the proof we mentioned above the authors take an opportune basis of KG as a vector space (corresponding to "projections") so that the KG-polynomials correspond to G-graded polynomials, where the G-grading is constructed adequately. In few words, given a finite abelian group G and a finite-dimensional algebra A with a G-action we obtain a G-grading on A and viceversa; furthermore, the G-polynomial identities and the G-graded identities coincide, that is

$$Id^{KG}(A) = Id^{gr}(A).$$

Due to the results in [7] we can associate to every group grading a certain *signature*. We recall the definition of a signature. We say that a vector space $\mathcal{A}$ is an Ω-algebra and Ω is a *signature* of A, where $\Omega = \bigcup_{n\geq 0}\Omega_n$, if each $\omega_n \in \Omega_n$ defines an n-linear map $\omega_n : \mathcal{A}\times\cdots\times\mathcal{A} \to \mathcal{A}$. For instance, our definition of algebra is a Ω-algebra, where $|\Omega_2| = 1$, and $\Omega_n = \emptyset$, for $n \neq 2$. We can construct the free Ω-algebra, so we can talk about Ω-polynomials identities (see, for instance, [34, Chapter 2]). Let $\mathcal{A}$ be a G-graded algebra, where G is finite and define $\pi_g : \mathcal{A} \to \mathcal{A}$ as the projection with respect to the decomposition $\mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$. Hence we can consider the signature $\Omega_G = \Omega_1 \cup \Omega_2$, where $|\Omega_2| = 1$, and $\Omega_1 = \{\pi_g \mid g \in G\}$. In [7], the authors prove that the elements $\pi_g(x)$ in the relatively free Ω_G-algebra correspond to graded variables of degree g. Thus

$$Id_{\Omega_G}(\mathcal{A}) = Id^{gr}(\mathcal{A}).$$

3 General Overview on Free Leibniz Algebras and Null-Filiform Leibniz Algebras

We shall denote by $\mathcal{L}\langle X\rangle$ the *free Leibniz algebra* freely generated by X over K. Then X is called set of free generators of $\mathcal{L}\langle X\rangle$. The next proposition shows that the Leibniz identity allows us to reduce every polynomial to a linear combination of left-normed monomials.

Proposition 3.1 [39, 41, 42] *Every multilinear polynomial in free Leibniz algebra $\mathcal{L}\langle X\rangle$ can be written as a linear combination of left-normed monomials.*

Proof Let us denote by $\deg(m)$ the length (usual degree) of a monomial m. The proof will be performed by induction on the degree. Let $x_1, x_2, x_3 \in X$, then the Leibniz identity gives

$$x_{i_1}(x_{i_2}x_{i_3}) = (x_{i_1}x_{i_2})x_{i_3} - (x_{i_1}x_{i_3})x_{i_2}.$$

We have four types of monomials of length 4 as listed below:

(a) $(x_{i_1}(x_{i_2}x_{i_3}))x_{i_4} = ((x_{i_1}x_{i_2})x_{i_3})x_{i_4} - ((x_{i_1}x_{i_3})x_{i_2})x_{i_4}$,
(b) $(x_{i_1}x_{i_2})(x_{i_3}x_{i_4}) = ((x_{i_1}x_{i_2})x_{i_3})x_{i_4} - ((x_{i_1}x_{i_2})x_{i_4})x_{i_3}$,

(c) $(x_{i_1}((x_{i_2}x_{i_3})x_{i_4})) = (x_{i_1}(x_{i_2}x_{i_3}))x_{i_4} - (x_{i_1}x_{i_4})(x_{i_2}x_{i_3})$,
(d) $x_{i_1}(x_{i_2}(x_{i_3}x_{i_4})) = x_{i_1}((x_{i_2}x_{i_3})x_{i_4}) - x_{i_1}((x_{i_2}x_{i_4})x_{i_3})$,

The monomials (a) and (b) are already in the required form. We note that in the second case $(x_{i_1}x_{i_2})$ is considered as a single variable. Moreover, cases (c) and (d) are immediate consequence of the cases (a) and (c), respectively. Suppose the assertion true for monomials of length less than $n - 1 \geq 4$ and let us prove it for monomials of degree n. Let w be a monomial so that $\deg(w) = n$. Then w has one of the following forms:

$$w = (u)x_{i_n}, \quad w = x_{i_0}(u), \quad \text{or } w = (u)(v),$$

where $\deg(u) = n - 1$ and $\deg(u) + \deg(v) = n$. If $w = (u)x_{i_n}$ we are done because w is left-normed. We consider now the case $w = x_{i_0}(u)$. We write

$$w = x_{i_0}(((\ldots((x_{i_1}x_{i_2})x_{i_3})\ldots)x_{i_{n-2}})x_{i_{n-1}}),$$

and we denote $y := (\cdots((x_{i_1}x_{i_2})x_{i_3})\ldots)x_{i_{n-2}}$, then we get

$$x_{i_0}(yx_{i_{n-1}}) = (x_{i_0}y)x_{i_{n-1}} - (x_{i_0}x_{i_{n-1}})y,$$

by the Leibniz identity. We consider now the summand $(x_{i_0}y)x_{i_{n-1}}$ and we notice that it can be written as a linear combination of left-normed monomials because of induction hypothesis because $\deg(y) = n - 2$. Now we consider $(x_{i_0}x_{i_{n-1}})y$ and, as above, we set $z := (\cdots((x_{i_1}x_{i_2})x_{i_3})\cdots)x_{i_{n-2}}$ and we get, always by the Leibniz identity,

$$(x_{i_0}x_{i_{n-1}})(y) = ((x_{i_0}x_{i_{n-1}})(z))x_{i_{n-2}} - ((x_{i_0}x_{i_{n-1}})x_{i_{n-2}})(z).$$

Note that $\deg(z) = \deg(y) - 1 = \deg(u) - 2$, then, as above, the first summand is a linear combination of left-normed monomials by induction hypothesis whereas the second one can be further decomposed as above.

Suppose now $w = (u)(v)$, then we have

$$w = ((\ldots((x_{i_1}x_{i_2})x_{i_3})\ldots)x_{i_k})((\ldots((x_{j_1}x_{j_2})x_{j_3})\ldots)x_{j_r}).$$

Let us set $a := ((\ldots((x_{j_1}x_{j_2})x_{j_3})\ldots)x_{j_{r-1}})$, then we obtain

$$w = ((u)(a))x_{j_r} - ((u)x_{j_r})(a),$$

which completes the proof. □

For a given Leibniz algebra L, the sequence of two-sided ideals defined as

$$L^1 = L, \; L^{k+1} = (L^k L), \; k \geq 1,$$

is said to be the lower central series of L.

Definition 3.2 A Leibniz algebra L is said to be *nilpotent*, if there exists $n \in \mathbb{N}$ such that $L^n = 0$. The minimal number n such that $L^n = 0$ is said to be the *nilpotency index* of L.

Definition 3.3 An n-dimensional Leibniz algebra L is called *null-filiform* if $dimL^i = n + 1 - i, 1 \leq i \leq n + 1$.

Obviously, nilpotent null-filiform Leibniz algebras have maximal nilpotency index. In [5] the authors achieved the following characterization of null-filiform Leibniz algebras.

Theorem 3.4 *An arbitrary n-dimensional null-filiform Leibniz algebra L is isomorphic to the algebra NF_n whose multiplication table is given by the following rules*

$$(e_i e_1) = e_{i+1}, \; 1 \leq i \leq n - 1,$$

where $\{e_1, \dots, e_n\}$ *is a suitable linear basis of* L.

At light of Theorem 3.4 from now on we shall work only on NF_n. The next is an easy but useful consequence of Theorem 3.4.

Lemma 3.5 *For every* $i \geq 2$ *we have* $e_i e_j = 0$ *for every* i *and for every* $j \geq 2$.

Proof We proceed by induction on j. If $j = 2$, then by the Leibniz rule we have $e_i e_2 = e_i(e_1 e_1) = (e_i e_1)e_1 - (e_i e_1)e_1 = 0$. Suppose the lemma true for $j - 1 \geq 2$ and let us prove it for j. Hence $e_i e_j = e_i(e_{j-1}e_1) = (e_i e_{j-1})e_1 - (e_i e_1)e_{j-1}$ but $e_i e_{j-1} = 0$ by induction whereas $(e_i e_1)e_{j-1} = e_{i+1}e_{j-1} = 0$ by induction again and the result follows. □

The *support* of a G-grading is the set $Supp(G) = \{g \in G \mid A^g \neq 0\}$. It has been proved in [12] that any G-grading on a null-filiform Leibniz algebra, where G is an abelian group, is cyclic. In particular, the next result holds.

Theorem 3.6 *Let* L *be a null-filiform Leibniz algebra of dimension* n. *Then, up to equivalence, all gradings over* L *are the following:*

(1) *The trivial grading given by* $L = \langle e_1, \dots, e_n \rangle$;
(2) *The* $\mathbb{Z}$*-grading given by* $L = \langle e_1 \rangle_1 \oplus \langle e_2 \rangle_2 \oplus \cdots \oplus \langle e_n \rangle_n$;
(3) *For any* $2 \leq i \leq n - 1$, *the* $\mathbb{Z}_i$*-grading given by* $L = L_{\bar{0}} \oplus L_{\bar{1}} \oplus \cdots \oplus L_{\overline{i-1}}$, *where homogeneous subspaces are the followings:*

$$
\begin{aligned}
L_{\bar{0}} &= \langle e_i, e_{2i}, \dots, e_{mi} \rangle \\
L_{\bar{1}} &= \langle e_1, e_{i+1}, \dots, e_{mi+1} \rangle \\
&\dots \\
L_{\overline{p+1}} &= \langle e_{p+1}, e_{i+p+1}, \dots, e_{(m-1)i+p+1} \rangle \\
L_{\overline{i-1}} &= \langle e_{i-1}, e_{2i-1}, \dots, e_{mi-1} \rangle,
\end{aligned}
$$

where $n = mi + p$ *with* $0 \leq p \leq i - 1$ *and* $p, m \in \mathbb{N}$.

4 Actions of Pointed Cocommutative Hopf Algebras on Null-Filiform Leibniz Algebras

In this section we shall describe the actions of finite dimensional pointed cocommutative Hopf algebras on null-filiform Leibniz algebras NF_n. We shall also give a complete description of identities and the multilinear and homogeneous structure of the relatively free algebra of NF_n.

We consider the following structure generated by the action of a Hopf algebra H on any algebra (associative, Lie, Leibniz, etc.).

Definition 4.1 Let A be an H-module algebra over a field K. Then the *smash product* algebra $H\#A$ is defined as follows: as a vector space $H\#A = H \otimes A$ and we write $h\#a$ instead of $h \otimes a$ while the multiplication is given by

$$(h\#a)(k\#b) = \sum h_2 k \# a(h_1 \cdot b),$$

for all $a, b \in A$, $h, k \in H$.

It is easy to see $A \cong 1\#A$ and $H \cong H\#1$. We also recall for a given Hopf algebra (H, Δ, ϵ) we define the set of *group-like* elements as

$$G(H) := \{h \in H | \Delta(h) = h \otimes h\},$$

while we define the set of *primitive elements* as

$$P(H) := \{h \in H | \Delta(h) = h \otimes 1 + 1 \otimes h\}.$$

Notice that if $\mathfrak{g}$ is a Lie algebra and $U(\mathfrak{g})$ is its universal enveloping algebra, then $P(U(\mathfrak{g})) = \mathfrak{g}$.

Moreover a Hopf algebra is said to be *pointed* if every simple subcoalgebra has dimension 1 whereas it is said to be *connected* if the sum of its simple subcoalgebras has dimension 1. We have a nice description of pointed cocommutative Hopf algebras attributed to Cartier and Gabriel in [22] and to Konstant in [48].

Theorem 4.2 *Let H be a Hopf algebra with $G = G(H)$, then if H is pointed cocommutative we get $FG\#H_1 \cong H$ via $x\#h \mapsto xh$ for any $x \in G$, $h \in H$, where H_1 is a suitable sub Hopf algebra of H containing the unit element.*

In the connected case we have the next result due to the independent works by Cartier [14] and Kostant which remained unpublished.

Theorem 4.3 *Let H be a cocommutative connected Hopf algebra over a field K of characteristic 0. Then $H \cong U(\mathfrak{g})$ for $\mathfrak{g} = P(H)$.*

Keeping in mind these last two classical results, we assume K is an algebraically closed field of characteristic zero and $H = KG\#H_1$, where $G = G(H)$ is a finite abelian group, H_1 is a KG-module algebra via $g \cdot h = ghg^{-1}$ for $g \in G$, $h \in H_1$, and $H_1 = U(\mathfrak{g})$, where $\mathfrak{g} = P(H_1)$ (the set of primitive elements).

We first note that $G \cdot \mathfrak{g} \subseteq \mathfrak{g}$. Thus $\mathfrak{g}$ is a G-graded algebra and, since G is abelian, this G-grading on $\mathfrak{g}$ induces a G-grading on $U(\mathfrak{g})$.

Now let A be a finite dimensional H-module algebra. Then A is a G-graded algebra because, as remarked before, KG can be identified as a subalgebra of H. Moreover, the G-grading on A induces naturally a G-grading on $\mathrm{End}_K(A)$ and then a G-grading on $\mathrm{Der}(A)$, the set of all derivations of A. Also, we get $\mathfrak{g}$ acts as a set of derivations on A. This means we have a Lie homomorphism

$$\iota : \mathfrak{g} \to \mathrm{Der}(A)$$

which is a graded homomorphism too. Hence, we have a graded homomorphism $U(\mathfrak{g}) \to \mathrm{End}_K(A)$.

Conversely, a G-grading on A and on $\mathfrak{g}$ and a graded Lie homomorphism $\mathfrak{g} \to \mathrm{Der}(A)$ defines a structure of H-module algebra on A. This is the content of the next result.

Theorem 4.4 *Let K be an algebraically closed field of characteristic zero. Let $H = FG\#U(\mathfrak{g})$, then a structure of H-module algebra on a finite-dimensional algebra A is uniquely determined by*

(1) *a G-grading on A,*
(2) *a G-grading on $\mathfrak{g}$,*
(3) *a graded Lie homomorphism $\mathfrak{g} \to \mathrm{Der}(A)$.*

Looking back at the previous result, we can also consider *graded differential polynomials*, that is, graded polynomials under the action of the graded Lie algebra $\mathfrak{g}$. In this case, using Theorem 4.4, the H-identities coincide with the differential graded polynomial identities. More precisely, we get the next result.

Proposition 4.5 *Let $\mathfrak{g}$ be a graded Lie algebra and let A be a finite-dimensional associative $\mathfrak{g}$-module algebra. Then*

$$Id^H(A) = Id^{\mathrm{gr},U(\mathfrak{g})}(A).$$

Indeed a graded Lie algebra $\mathfrak{g}$ acts on A as a *graded derivation*, i.e., if $d \in \mathfrak{g}^g$ is homogeneous in the grading and $a \in A^h$, then $d(a) \in A^{g\cdot h}$.

In the sequel, we describe the graded derivations of NF_n. Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be a basis of NF_n, then we have

$$d(e_k) = d(e_{k-1}e_1) = d(e_{k-1})e_1 + e_{k-1}d(e_1),$$

which means d is completely determined by $d(e_1)$. If $d(e_1) = \sum_{i=1}^{n} \alpha_i e_i$, then we can write $d(e_1)$ as

$$(\alpha_1 id_{NF_n} + \alpha_2 R_{e_1} + \alpha_3 R_{e_1}^2 + \cdots + \alpha_n R_{e_1}^{n-1})(e_1),$$

where R_{e_1} is the right multiplication by e_1. Of course, for every $i = 1, \ldots, n-1$, we have $R_{e_1}^i = R_{e_i}$. This means a derivation d is such that

$$d = \alpha_1 id_{NF_n} + \alpha_2 R_{e_1} + \alpha_3 R_{e_1}^2 + \cdots + \alpha_n R_{e_1}^{n-1}.$$

Because sum of derivations is again a derivation, we get any derivation d has the following presentation:

$$d = \alpha_2 R_{e_1} + \alpha_3 R_{e_1}^2 + \cdots + \alpha_n R_{e_1}^{n-1}$$

as linear combination of derivations. Of course the $R_{e_1}^i$'s are linearly independent. This means the Lie algebra $Der(NF_n)$ of derivations of NF_n is one generated, abelian and is a vector space of dimension $n-1$ and, of course, any Lie subalgebra of $Der(NF_n)$ is a finite dimensional vector space of dimension less than or equal to $n-1$.

The next result describes explicitly the action of a derivation on the basis elements of NF_n taking into account the remarks above.

Proposition 4.6 *Let $d \in \mathfrak{g}^g$ and $e_1 \in NF_n^h$. Let us set*

$$d(e_1) = \sum_{\substack{i=2 \\ \deg(e_i) = g\cdot h}}^{n} \alpha_i e_i.$$

Then we get:

$$d(e_k) = \sum_{\substack{i=2 \\ \deg(e_{i+k-1}) = g\cdot \deg(e_k)}}^{n+1-k} \alpha_i e_{i+k-1}$$

for every $k = 1, \ldots, n-1$;

$$d(e_n) = 0.$$

Proof In what follows, for the sake of simplicity, we shall omit the homogenous degree of any basis element. We have

$$d(e_2) = d(e_1e_1) = d(e_1)e_1 + e_1d(e_1)$$
$$= \left(\sum_{i=2}^{n} \alpha_i e_i\right) e_1 + e_1 \left(\sum_{i=2}^{n} \alpha_i e_i\right) = \sum_{i=2}^{n-1} \alpha_i e_{i+1}.$$

Suppose the assertion true for smaller values than k and prove it for k. Then

$$d(e_k) = d(e_{k-1}e_1) = d(e_{k-1})e_1 + e_{k-1}d(e_1)$$
$$= \left(\sum_{i=2}^{n+2-k} \alpha_i e_{i+k-2}\right) e_1 = \sum_{i=2}^{n+1-k} \alpha_i e_{i+k-1}.$$

and we are done. □

5 *H*-Identities of Null Filiform Leibniz Algebras Under the Action of a Pointed Cocommutative Hopf Algebra

Given any Lie algebra $\mathfrak{g}$ of dimension l and a group $G = \{g_1, g_2, \ldots\}$ with neutral element 1_G, by Proposition 4.5, in order to study the $H = U(\mathfrak{g})\#G$-graded polynomial identities of NF_n, we have to study the differential graded polynomial identities of NF_n. On this purpose, let $\mathcal{B} = \{\delta_1, \ldots, \delta_l\}$ be a G-homogeneous basis of $\mathfrak{g}$. The elements of $\mathcal{B}$ act on NF_n as a set of G-graded derivations. Indeed $\mathcal{B} = \cup_{i=1}^{|G|} \mathcal{B}_i$, where $\mathcal{B}_i$ is the linear basis of the homogeneous component $\mathfrak{g}^{g_i}$ of $\mathfrak{g}$.

Let us set

$$Supp^{Der}(NF_n) = \{(g, \delta_g) | x^{g,\delta_g} \text{is not an } H\text{-identity}\},$$

and

$$Supp^{H}(NF_n) := Supp^{Der}(NF_n) \cup Supp(NF_n),$$

then the next shows up a set of H-identities of NF_n.

Lemma 5.1 *Let NF_n be a G-graded null-filiform Leibniz algebra, where G is any group. Let $Supp(NF_n) = S$, then the following polynomials are H-identities for the null-filiform Leibniz algebra NF_n:*

$$x^g,\ g \notin Supp(NF_n),\ x^{g,\delta_g},\ (g, \delta_g) \notin Supp^{Der}(NF_n),$$
$$x_1^{\delta_i,g_1} x_2^{\delta_j,g_2},\ x_1^{\delta_i,g_1} x_2^{g},\ g \neq h,\ x_1^{g} x_2^{\delta_j,g_2},\ x_1^{g_1} x_2^{g_2},\ g_2 \neq h$$
$$(x_1^{\delta_k,g_1} x_2^{h})x_3^{h} - (x_1^{\delta_k,g_1} x_3^{h})x_2^{h},$$
$$(x_1^{g} x_2^{h})x_3^{h} - (x_1^{g} x_2^{h})x_3^{h},$$
$$x_1^{h_1} \cdots x_{n+1}^{h_{n+1}},\ h_i \in H.$$

Proof We will argue only for the identities of type $(x_1^{\delta_k,g_1} x_2^{h})x_3^{h} - (x_1^{\delta_k,g_1} x_3^{h})x_2^{h}$ because the remaining cases can be handled similarly. Observe the only non-trivial substitution of both $(x_1^{\delta_k,g_1} x_2^{\delta_i,g_2})x_3^{\delta_j,g_3}$ and $(x_1^{\delta_k,g_1} x_2^{\delta_j,g_3})x_3^{\delta_i,g_2}$ is $(ee_1)e_1$, where e is a non-zero element of $\delta_k(NF_n^{g_1})$ and we are done. □

Let now I be the T_H-ideal generated by the polynomial identities of Lemma 5.1.

Theorem 5.2 *Let NF_n be a null-filiform Leibniz algebra with H-action, where the H-action is defined as above. Then*

$$Id^H(NF_n) = I.$$

Moreover, we have

$$c_m^H(NF_n) = |Supp^H(NF_n)|m \ if \ m \le n \ and \ 0 \ otherwise.$$

Then, consequently, $exp^H(NF_n) = 0$. We also have

$$Hilb_k^H(NF_n, t) = 1 + |Supp^H(NF_n)| \sum_{m=1}^{n} \binom{k+m-2}{m-1} t^m.$$

Proof We shall work in the free Leibniz algebra $\mathcal{L}\langle X|H\rangle$ modulo the H-ideal I. Let $f = f(x_1^{g_1}, \dots, x_s^{g_s})$ be a multilinear H-identity of NF_n, where each variable $x_i^{\delta,g}$ appearing in f is such that $g \in Supp(NF_n)$. Moreover, let $e_1 \in NF_n^h$. Due to the identities $x_1^{\delta_i,g_1} x_2^{\delta_j,g_2}$, $x_1^{g_1} x_2^{g_2}$ and the nilpotency identity, f can be written as $f = \sum \alpha_i f_i$, where

$$f_i = (\cdots((x_{i_1}^{\delta_1,g_1} x_{i_2}^h)x_{i_3}^h)\cdots)x_{i_s}^h.$$

where $s \le n$ or $f = \sum \alpha_i g_i$, where

$$g_i = (\cdots((x_{i_1}^{g_1} x_{i_2}^h)x_{i_3}^h)\cdots)x_{i_s}^h.$$

where $s \le n$. By the remaining identities of Lemma 5.1 we get the indexes are strictly increasing. Using the substitution ϕ such that $x_{i_1}^{\delta_1,g_1} \mapsto e \ x_{i_j}^{\delta_j,g_j} \mapsto e_1$, if $j \ge 2$, where $e \ne 0$, we get $0 = \phi(f) = \alpha(\cdots((e)e_1)\cdots)e_1 = \alpha e_s$. Hence $\alpha = 0$ and we are done. We also get

$$c_m^H(NF_n) = |Supp^H(NF_n)|m, \text{ if } m \le n \text{ and } 0 \text{ otherwise}$$

and consequently

$$\exp^H(NF_n) = 0.$$

Working with multihomogeneous polynomials in k-H- variables inside $F_k^H(NF_n)$ we have

$$f_m := (\cdots((x_{i_1}^{\delta_1,g_1} x_{i_2}^h)x_{i_3}^h)\cdots)x_{i_s}^h,$$

where $j \ge 2$, are the only non-trivial monomials of degree $m \le n$. This means

$$Hilb_k^H(NF_n, t) = 1 + |Supp^H(NF_n)| \sum_{m=1}^{n} \binom{k+m-2}{m-1} t^m$$

and the growth function

$$g(m, NF_n) = 1 + |Supp^H(NF_n)| \sum_{i=1}^{m} \binom{k+i-2}{i-1},$$

if $m \le n$. □

At light of the last result, because of the nilpotency identity, given a pointed cocommutative Hopf algebra $H = FG\#U(\mathfrak{g})$,

an H-variety generated by a null-filiform Leibniz algebra

satisfies the Specht property if and only if G is finite.

In what follows we shall describe the H-cocharacters of NF_n.

Lemma 5.3 *Let δ_i, $i \in \{1, \ldots, m\}$ be a set of linearly independent graded derivations of NF_n of a certain fixed degree. Let $e \in NF_n$ be a homogeneous element such that $\delta_j(e) \neq 0$ for some $j \in \{1, \ldots, m\}$. If there are $\alpha_1, \ldots, \alpha_m \in K$ such that $\sum_{i=1}^{m} \alpha_i \delta_i(e) = 0$, then some of the α_i must be 0.*

Proof Let $i \in \{1, \ldots, m\}$, then $\delta_i = \sum_{k=1}^{l} a_{i,k} R_{s_k}$, where $a_{i,k} \in K$ and $s_k \in \{1, \ldots, n-1\}$. Because the δ_i's are linearly independent, then the matrix $A = [a_{i,k}]$ has maximal rank and we choose a square submatrix $A' = [a_{t,r}]$ of A having maximal rank. Let us denote by A_i the rows of A and by A'_t those of A'. If $\delta_j(e) \neq 0$, then $X' = (X_1, \ldots, X_l)$, where $X_i = \delta_i(e)$, is a non-trivial solution of the homogeneous system $A''X = 0$ because $\sum_{i=1}^{m} \alpha_i \delta_i(e) = 0$, where $A''_i = \alpha_i A_i$ for every $i \in \{1, \ldots, m\}$. This means A'' has not maximal rank and $rank(A'') < rank(A)$. Suppose further for every $i \in \{1, \ldots, m\}$ we have $\alpha_i \neq 0$. Notice that A''' such that $A'''_t = \alpha_t A't$ is a square submatrix of A'' such that $\det(A''') = \alpha \det(A')$, where $\alpha \in K$ is the product of some of the α_i and is of course non-zero. Hence $\det(A''') \neq 0$ that means $rank(A''') \geq rank(A'') = rank(A)$ that is an absurd. Then some of the α_i must be zero and we are done. □

Theorem 5.4 *Let $m \in \mathbb{N}$, then*

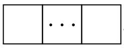
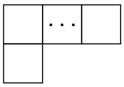
$$\chi_m^H(NF_n) = |Supp^H(NF_n)|\chi_{(m)} + |Supp^H(NF_n)|\chi_{(m-1,1)}$$

if $m \leq n$ and

$$\chi_m^H(NF_n) = \emptyset$$

otherwise.

Proof Consider the Young tableaux

$\chi_{(m)}^H :=$ , $\chi_{(m-1,1)}^H :=$ 

both corresponding (up to specialize g with h) the set of H-polynomials:

1. $(\cdots((x^{g,\delta}x^h)x^h)\cdots)x^h, \ \delta \in \mathcal{B}$,

2. $(\cdots((x^g x^h)x^h)\cdots)x^h$.

The families of polynomials **1.** and **2.** are linearly independent. We need to prove the polynomials of the family **1.** are linearly independent because the other case can be treated similarly. If they are not linearly independent there exist $\alpha_\delta \in K$ not all 0 such that

$$\sum_{\delta \in \mathcal{B}} \alpha_\delta (\cdots((x^{g,\delta}x^h)x^h)\cdots)x^h = 0.$$

Let e be a non-zero homogeneous element of NF_n^g and consider the substitution ϕ sending $x^g \mapsto e$ and $x^h \mapsto e_1$. Then the previous relation says

$$(\cdots((e^{\sum_{\delta \in \mathcal{B}} \alpha_\delta \delta} e_1)e_1)\cdots)e_1 = 0$$

that is $\sum_{\delta \in \mathcal{B}} \alpha_\delta \delta(e) = 0$. Without loss of generality, we may suppose all of the δ's of a certain fixed degree. Moreover, notice that $(\delta, g) \in Supp^{Der}(NF_n)$. Now we can apply Lemma 5.3 several times in order to get every α_δ is 0 and the proof follows. Hence $m_\lambda^H \geq |Supp^H(NF_n)|$ if $\lambda = (m)$ or $\lambda = (m-1, 1)$. On the other side we have

$$m_{(m)}\chi_{(m)}^H(1) + m_{(m-1,1)}\chi_{(m-1,1)}^H(1) = c_m^H(NF_n) = |Supp^H(NF_n)|m.$$

Because of the previous relation, if both $m_{(m)}$ and $m_{(m-1,1)}$ are strictly greater than $|Supp^H(NF_n)|$ we get a contradiction, then either $m_{(m)}$ or $m_{(m-1,1)}$ is equal to $|Supp^H(NF_n)|$ but this forces $m_{(m)}$ and $m_{(m-1,1)}$ being equal to $|Supp^H(NF_n)|$ and we are done. □

As a consequence of Theorem 5.3 and Theorem 5.4, we have the following description of the G-graded identities and cocharacters of NF_n, where G is any group.

Theorem 5.5 *Let G be a finite group and let NF_n be a G-graded n-dimensional null-filiform Leibniz algebra. Then*

$$Id^G(NF_n) = \langle \{x^g \mid g \notin Supp(NF_n)\} \cup \{x_1^{g_1} x_2^{g_2} \mid e_1 \notin NF_n^{g_2}\} \rangle^{T_G}.$$

Moreover, we have

$$c_m^G(NF_n) = |Supp(NF_n)|m \; if \; m \leq n \; and \; 0 \; otherwise.$$

Then, consequently, $exp^G(NF_n) = 0$. We also have

$$Hilb_k^G(NF_n, t) = 1 + |Supp(NF_n)| \sum_{m=1}^{n} \binom{k+m-2}{m-1} t^m.$$

Theorem 5.6 *Let G be a finite group and $m \in \mathbb{N}$, then*

$$\chi_m^G(NF_n) = |Supp^G(NF_n)|\chi_{(m)} + |Supp^G(NF_n)|\chi_{(m-1,1)}$$

if $m \leq n$ and

$$\chi_m^G(NF_n) = \emptyset$$

otherwise.

6 The Case of Taft's Hopf Algebras

In this section we shell exploit the actions of a Taft Hopf algebra on the null-filiform Leibniz algebra NF_n and we shall show up the set of its identities and the description of its relatively free algebra.

First we shall give a small account on Taft's algebras (see also [20]). Let K be a field containing a primitive p-th root of the unit γ for some positive integer p. Let $(H_p, \Delta, \epsilon, S)$ be the Hopf algebra so that

$$H_p = F\langle c, x \mid c^p = 1, x^p = 0, xc = \gamma cx \rangle$$

as an algebra with comultiplication Δ such that

$$\Delta(c) = c \otimes c, \;\; \Delta(x) = x \otimes 1 + c \otimes x$$

and counit ϵ defined by

$$\epsilon(\mathrm{c}) = 1, \;\; \epsilon(\mathrm{x}) = 0.$$

Moreover the antipode S is such that

$$S(c) = c^{-1}, \;\; S(x) = -c^{-1}x.$$

Thus, H_p is an p^2-dimensional algebra which is neither commutative nor cocommutative. This algebra is known as the *Taft's Hopf algebra of order p*. A particular case of a Taft's algebra occurs when $p = 2$ and the latter algebra is known as the Sweedler's Hopf algebra.

From now on let A be a finite dimensional (associative, Lie, Jordan, etc.) algebra over a field K. Notice that the element c acts as a homomorphism of algebras on A. Moreover, since $c^p = 1$, we obtain that c acts as an automorphism of A of order p. Using the same idea, x acts as a c-derivation (also known as a *skew-derivation*), that is, it satisfies

$$x(ab) = x(a)b + c(a)x(b), \quad \forall a, b \in A.$$

Moreover, the actions of x and c are related by the following relation $xc = \gamma cx$. On the other hand, the choice of an automorphism α of A of order p and an α-derivation d satisfying $d^p = 0$, and $d\alpha = \gamma\alpha d$, defines an H_p-action on A. In fact it is sufficient to consider the F-algebra $F\langle\alpha, d\rangle$ which turns out to be a Hopf algebra isomorphic to H_p. Hence we get the next result.

Proposition 6.1 *Let A be a finite dimensional algebra over a field K containing a p-th primitive root of unit γ, assume that its characteristic does not divide p, then an action of H_p on A is completely determined by a choice of:*

(1) *an automorphism α of A of order p,*
(2) *an α-derivation d of A such that $d^p = 0$, and $\alpha d = \gamma d\alpha$.*

Equivalently, the structure of H_p-module algebra on A is uniquely determined by a choice of:

(1) *a $\mathbb{Z}_p$-grading $A = \bigoplus_{i\in\mathbb{Z}_p} A_i$,*
(2) *an α-derivation d (where α defines the $\mathbb{Z}_p$-grading above) such that $d(A_i) \subseteq A_{i-1}$, and $d^p = 0$.*

Note that if we consider a linear basis $\mathcal{B}_1$ of the subalgebra $\langle c\rangle$ of H_p generated by c, then $\{x^i\beta \mid \beta \in \mathcal{B}_1, i = 0, 1, \ldots, p-1\}$ is a basis of H_p. Let A be an H_p-module algebra. The proof of [29, Proposition 3.3.6] gives us a basis $\{\chi_1, \ldots, \chi_p\}$ of $\langle c\rangle$ such that each χ_i corresponds to a projection of a certain $\mathbb{Z}_p$-grading on A. So $\mathcal{B} = \{x^j\chi_i\}$ is a basis of H_p and $\Omega = \Omega_1 \cup \Omega_2$ is a signature, where $\Omega_1 = \mathcal{B}$, and $|\Omega_2| = 1$. Let $D_p = F\langle x\rangle = span_F\langle\{1, x, x^2, \ldots, x^{p-1}\}\rangle$. By [7] again we have the variables $x^i(\chi_j(x))$ correspond to graded variables under the action of x^j. In few words the H_p-polynomials correspond to $\mathbb{Z}_p$-graded polynomials with the action of D_p and the polynomial identities coincide. Hence we can establish the following.

Proposition 6.2 *Let A be a finite-dimensional associative H_p-module algebra. Consider the corresponding $\mathbb{Z}_p$-grading and the skew-derivation d and let $D_p = F\langle x\rangle$. We consider the G-graded polynomials with the action of d, then*

$$Id^{H_p}(A) = Id^{\mathrm{gr}, D_p}(A).$$

At light of point (2) of Proposition 6.1, of the previous proposition and by the description of the sole $\mathbb{Z}_p$-grading of NF_n given in Theorem 3.6, if d is a skew-derivation of NF_n such that $d(e_1) = \sum_{i=1}^n \alpha_i e_i$, then for every $j = 2, \ldots, n$ we get $d(e_j) = \sum_{i=1}^n \beta_i e_i$, where

$\beta_1 = 0$. This means the monomial $x_1^h x_2^d$ is an H_p-identity for every $h \in H_p$. Let J be the H_p-ideal generated by the following sets of H_p-polynomials:

$$x^g,\ g \notin Supp(NF_n),\ x^{g,d},\ (g,d) \notin Supp^{Der}(NF_n),$$
$$x_1^{d,g_1} x_2^{d,g_2},\ x_1^{d,g_1} x_2^{g},\ g \neq h,\ x_1^{g} x_2^{d,g_2},\ x_1^{g_1} x_2^{g_2},\ g_2 \neq h$$
$$(x_1^{d,g_1} x_2^{h}) x_3^{h} - (x_1^{d,g_1} x_3^{h}) x_2^{h},$$
$$(x_1^{g} x_2^{h}) x_3^{h} - (x_1^{g} x_2^{h}) x_3^{h},$$
$$x_1^{h_1} \cdots x_{n+1}^{h_{n+1}},\ h_i \in H_p.$$

Indeed the previous polynomials turn out to be H_p-identities of NF_n. Arguing verbatim as in Theorems 5.2 and 5.4 we get the next results.

Theorem 6.3 *Let NF_n be a null-filiform Leibniz algebra with H_p-action. Then*

$$Id_{H_p}(NF_n) = J.$$

Moreover, we have

$$c_m^{H_p}(NF_n) = |Supp^{H_p}(NF_n)| m \ if \ m \leq n \ and \ 0 \ otherwise.$$

Then, consequently, $exp^{H_p}(NF_n) = 0$. We also have

$$Hilb_k^{H_p}(NF_n, t) = 1 + |Supp^{H_p}(NF_n)| \sum_{m=1}^{n} \binom{k+m-2}{m-1} t^m.$$

We also furnish the description of the sequence of H_p-cocharacters of NF_n.

Theorem 6.4 *Let $m \in \mathbb{N}$, then*

$$\chi_m^{H_p}(NF_n) = |Supp^{H_p}(NF_n)| \chi_{(m)} + |Supp^{H_p}(NF_n)| \chi_{(m-1,1)}$$

if $m \leq n$ and

$$\chi_m^{H_p}(NF_n) = \emptyset$$

otherwise.

The conclusion, in this case, is different than in the case of pointed cocommutative Hopf algebra. Indeed

an H_p-variety generated by a null-filiform Leibniz algebra

always satisfies the Specht property.

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