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## ABSTRACT

In the present paper, we prove the blow-up in finite time for local solutions of a semilinear Cauchy problem associated with a wave equation in anti-de Sitter spacetime in the critical case. According to this purpose, we combine a result for ordinary differential inequalities with an iteration argument by using an explicit integral representation formula for the solution to a linear Cauchy problem associated with the wave equation in anti-de Sitter spacetime in one space dimension.

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## I. INTRODUCTION

In the previous work,<sup>1</sup> we studied the following semilinear Cauchy problem associated with a semilinear wave equation in anti-de Sitter spacetime:

$$\begin{cases} \partial_t^2 v - c^2 e^{2Ht} \Delta v + b \partial_t v + m^2 v = f(t, v), & x \in \mathbb{R}^n, t \in (0, T), \\ v(0, x) = \varepsilon v_0(x), & x \in \mathbb{R}^n, \\ \partial_t v(0, x) = \varepsilon v_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where  $c, H$  are positive constants,  $b, m^2$  are non-negative real parameters satisfying  $b^2 \geq 4m^2$ ,  $\varepsilon > 0$  is a parameter describing the size of initial data, and  $T = T(\varepsilon) \in (0, \infty]$  is the lifespan of a classical solution  $v$  (i.e., the maximal existence time), and the nonlinear term is given by

$$f(t, v) \doteq \Gamma(t) \left( \int_{\mathbb{R}^n} |v(t, y)|^p dy \right)^\beta |v|^p, \quad (1.2)$$

where  $p > 1$ ,  $\beta \geq 0$ , and  $\Gamma = \Gamma(t)$  is a suitable positive function. In cosmology, the constant  $H$  is the so-called Hubble constant,  $m$  is the mass of a particle, and  $b$  is taken equal to the space dimension  $n$  [cf. Eq. (0.6) in Ref. 2].

We recall that the assumption  $b^2 \geq 4m^2$  guarantees that the damping term  $b \partial_t v$  is somehow dominant over the mass term  $m^2 v$ ; cf. Ref. 3 (Subsections 1.1–1.3).

More precisely, considering as the  $\Gamma$  factor in (1.2),

$$\Gamma(t) \doteq \mu e^{0t} (1+t)^\zeta \quad (1.3)$$

for some  $\mu > 0$  and  $\varrho, \zeta \in \mathbb{R}$ , we have established the following threshold values for  $\varrho$ :

$$\varrho_{\text{crit}}(n, H, b, m^2, \beta, p) \doteq \begin{cases} \frac{1}{2}(b - \sqrt{b^2 - 4m^2})((\beta + 1)p - 1) + nH(\beta + 1)(p - 1) & \text{if } n \leq N, \\ \frac{1}{2}(b + nH)((\beta + 1)p - 1) + nH - (n - 1)H(\beta + 1) - \frac{H}{p} & \text{if } n > N, \end{cases} \quad (1.4)$$

where

$$N = N(H, b, m^2, p) \doteq \frac{2}{p} + \frac{\sqrt{b^2 - 4m^2}}{H}.$$

While for the case  $n \leq N$ , we provided a full picture on blow-up results for local weak solutions to (1.1) for  $\varrho \geq \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$  (cf. Ref. 1, Theorems 1.6–1.8), when  $n > N$ , only the case  $\varrho > \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$  was investigated; see Ref. 1, Theorem 1.9.

The aim of the present paper is to study the blow-up for local solutions to (1.1) under suitable sign conditions for the Cauchy data in the threshold case  $\varrho = \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$  for  $n > N$  and to derive the corresponding upper bound estimates for the lifespan.

Furthermore, since we consider the limit case  $\varrho = \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$ , we have to prescribe a lower bound for the power of the polynomial factor in (1.3), namely,

$$\zeta > -\frac{1}{p}. \quad (1.5)$$

The threshold case that we treat in the present work is somehow a blow-up result for a critical case. Consequently, the approach that we use to prove the blow-up in finite time of the spatial average  $V = V(t)$  of a local solution  $v$  to (1.1) is inspired by the one in Ref. 4 for the derivation of the sharp upper bound estimate for the lifespan of a local solution to the semilinear wave equation in the critical case when  $n \geq 4$ . As in Ref. 5, when  $n \geq 2$ , we use the Radon transform with respect to the spatial variables to handle the problem as it was in one space dimension.

The main difficulty in our argument will be the derivation of a sequence of lower bound estimates for the nonlinear term  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ . In particular, we will derive an iteration frame for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  combining two estimates involving the Radon transform of  $v(t, \cdot)$ . A fundamental tool for this kind of argument is provided by Yagdjian's integral transform approach. Indeed, we will make use of an explicit integral representation formula for the solution to a linear one-dimensional wave equation in anti-de Sitter spacetime in order to derive one of the aforementioned inequalities involving the Radon transform of  $v(t, \cdot)$ .

After deriving this sequence of lower bound estimates for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ , we will derive in turn a sequence of lower bound estimates for  $V$  with an additional polynomial factor. Combining these lower bound estimates for  $V$  with a comparison argument for an ordinary differential equation (ODE) with "critical" exponential growth, we will be able to derive the desired upper bound estimates for the lifespan.

We point out that the speed of propagation, namely, the function  $a(t) \doteq c e^{Ht}$ , is exponentially increasing in the previous semilinear wave equation. Moreover, the amplitude of the forward light-cone is given by

$$A(t) \doteq \int_0^t a(\tau) \, d\tau = \frac{c}{H}(e^{Ht} - 1). \quad (1.6)$$

This means that, considering smooth solutions, if we assume  $v_0$  and  $v_1$  compactly supported in  $B_R \doteq \{x \in \mathbb{R}^n : |x| \leq R\}$ , given a local solution  $v$  to (1.1), we have that

$$\text{supp } v(t, \cdot) \subset B_{R+A(t)} \quad \text{for any } t \in (0, T). \quad (1.7)$$

For the proof of this support condition, one can use the property of finite speed of propagation or, alternatively, the explicit representation formulas from the series of works by Galstian and Yagdjian in Refs. 6–8.

In this second part of the Introduction, we provide a short summary of the results in the literature for wave models with a non-flat and time-dependent metric in the spacetime.

In the case of de Sitter spacetime, i.e., for  $H < 0$  in (1.1), the wave equation was considered by several authors. We recall the integral representation formulas (and their applications) established by Yagdjian and Galstian in Refs. 2 and 9–14 and the global existence results for semilinear wave models in Refs. 3 and 15. Concerning blow-up results, we recall the blow-up result with a pure imaginary mass term in (1.1), namely, when we replace  $m$  with  $im$ , for both de Sitter and anti-de Sitter spacetime in Ref. 16 (Proposition 1.1). Moreover, in Ref. 17, a blow-up result is proved in a de Sitter-type spacetime when  $m = 0$ . Finally, in Ref. 1, we slightly improved some result from Ref. 9 for the semilinear wave equation in de Sitter spacetime with the same nonlinear term as in (1.2), further providing the lifespan estimates; see Ref. 1, Theorems 1.2–1.4.

For anti-de Sitter spacetime, we refer to Refs. 6–8, 18, and 19, where among other things,  $L^p - L^q$  estimates are derived for the solutions to the corresponding linear Cauchy problem and the existence of global solutions is proved assuming the smallness of the Cauchy data in suitable Sobolev spaces and that the parameters  $\varrho, p$  in the nonlinear term belong to a suitable range (according to our notations, when  $\beta = 0$

and  $\zeta = 0$ ). Moreover, as we have already mentioned above, in Ref. 1, some blow-up results for (1.1) have been proved together with the corresponding upper bound estimates for the lifespan.

Finally, for the wave equation in Einstein–de Sitter spacetime [that is, for the d'Alembert operator  $\square_{\text{EdS}} = \partial_t^2 - t^{-2k}\Delta + bt^{-1}\partial_t$ , with  $k \in (0, 1)$  and  $b \geq 0$ ], we cite Refs. 20 and 21 for the linear model, Refs. 22–27 for the semilinear model with power nonlinearity  $|v|^p$ , and the recent studies<sup>28–30</sup> for the semilinear model with nonlinearity of derivative type  $|\partial_t v|^p$ .

It is interesting to compare our approach in this paper to deal with a critical case in comparison to those in Refs. 23–25 for the treatment of the corresponding critical cases in Einstein–de Sitter spacetime. Indeed, while the critical case in Ref. 25 is studied by using the approach from Ref. 31, in Refs. 23 and 24, the method from Ref. 32 is adapted to the case with time-dependent coefficients. However, in both cases, the employment of the techniques from Refs. 31 and 32 to the case with time-dependent coefficients produces some restrictive assumptions: in Ref. 25, an upper bound is prescribed for the size of the ball containing the supports of the Cauchy data, while in Ref. 23, a restriction on the multiplicative constant in the damping term is prescribed. We emphasize that with the approach of the present work, no restriction of this kind (either on the size of the support for the data or on the range for the multiplicative constants in the lower order terms) appears.

### A. Main results

Before stating the main theorem for (1.1), we recall the notion of weak solutions to (1.1) that has been employed in Ref. 1. We stress that although we call them weak solutions, actually, for these solutions, more regularity than for usual distributional solutions is required with respect to the time variable in order to handle a space average that is a  $\mathcal{C}^2$  function with respect to  $t$ . Indeed, in Ref. 1, we worked with the larger class of solutions that can be considered when employing the spatial average for proving the blow-up in finite time.

*Definition I.1.* Let  $v_0, v_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$  such that  $\text{supp } v_0, \text{supp } v_1 \subset B_R$  for some  $R > 0$ . We say that

$$v \in \mathcal{C}^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n)) \text{ such that } f(t, v) \in L^1_{\text{loc}}((0, T) \times \mathbb{R}^n),$$

where the definition of the nonlinear term  $f(t, u)$  is given in (1.2), is a *weak solution* to (1.1) on  $[0, T]$  if  $v$  fulfills the support condition (1.7) and the integral identity

$$\begin{aligned} & \int_{\mathbb{R}^n} \partial_t v(t, x) \varphi(t, x) \, dx - \int_{\mathbb{R}^n} v(t, x) \varphi_t(t, x) \, dx + b \int_{\mathbb{R}^n} v(t, x) \varphi(t, x) \, dx + \int_0^t \int_{\mathbb{R}^n} v(s, x) (\varphi_{ss}(s, x) - c^2 e^{2Hs} \Delta \varphi(s, x) - b \varphi_s(s, x) + m^2 \varphi(s, x)) \, dx \, ds \\ & = \varepsilon \int_{\mathbb{R}^n} v_1(x) \varphi(0, x) \, dx + \varepsilon \int_{\mathbb{R}^n} v_0(x) (b \varphi(0, x) - \varphi_t(0, x)) \, dx + \int_0^t \Gamma(s) \left( \int_{\mathbb{R}^n} |v(s, y)|^p \, dy \right)^\beta \int_{\mathbb{R}^n} |v(s, x)|^p \varphi(s, x) \, dx \, ds \end{aligned} \quad (1.8)$$

holds for any  $t \in (0, T)$  and any test function  $\varphi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R}^n)$ .

We point out explicitly that in the present paper, we work with classical solutions to (1.1) since we need to employ an integral representation formula that requires a pointwise evaluation of the Cauchy data and of the nonlinear term. Nonetheless, it is clear that classical solutions to (1.1) are, in particular, weak solutions according to Definition I.1. We emphasize that it was necessary to recall Definition I.1 since in what follows, we are going to use some results from Ref. 1 (see Sec. II) that have been obtained for weak solutions in the aforementioned sense.

**Theorem I.2.** Let  $n \geq 1$ ,  $c, H > 0$ , and  $b, m^2 \geq 0$  such that  $b^2 \geq 4m^2$ . Let us assume  $\beta \geq 0$  and  $p > 1$  such that

$$\frac{n}{2} - \frac{\sqrt{b^2 - 4m^2}}{2H} > \frac{1}{p}. \quad (1.9)$$

For  $\varrho = \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$  and  $\zeta > -\frac{1}{p}$ , we consider

$$\Gamma(t) \doteq \mu e^{\varrho_{\text{crit}}(n, H, b, m^2, \beta, p)t} (1+t)^\zeta \quad (1.10)$$

for some  $\mu > 0$  in the term  $f(t, v)$  given by (1.2).

Let us assume that  $(v_0, v_1) \in \mathcal{C}_0^2(\mathbb{R}) \times \mathcal{C}_0^1(\mathbb{R})$  are non-negative and nontrivial functions with supports contained in  $B_R$  for some  $R > 0$ . Let  $v \in \mathcal{C}^2([0, T] \times \mathbb{R}^n)$  be a classical solution to the Cauchy problem (1.1) with lifespan  $T = T(\varepsilon)$ .

Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, \zeta, v_0, v_1, R)$  such that for any  $\varepsilon \in (0, \varepsilon_0]$ , the classical solution  $v$  blows up in finite time. Furthermore, the following upper bound estimates for the lifespan hold:

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{p((\beta+1)p-1)}{1+\varphi}} & \text{if } \varsigma \in \left(-\frac{1}{p}, 0\right], \\ C\varepsilon^{-p((\beta+1)p-1)} & \text{if } \varsigma \in \left(0, \frac{1}{p}\right), \\ C\varepsilon^{-\frac{(\beta+1)p-1}{\varsigma}} & \text{if } \varsigma \in \left[\frac{1}{p}, \infty\right), \end{cases} \quad (1.11)$$

where the positive constant  $C$  is independent of  $\varepsilon$ .

This paper is organized as follows: in Sec. II, we briefly recall some estimates derived in Ref. 1; in Sec. III, we prove Theorem I.2. More precisely, this proof is split into the following intermediate steps: in Subsection III A, we derive a comparison argument for an *ordinary differential inequality* (ODI) with “critical” exponential growth; in Subsection III B, we derive an integral representation formula for the solution to the one-dimensional linear Cauchy problem associated with the wave equation in anti-de Sitter spacetime; in Subsection III C, we derive the crucial iteration frame for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ ; and in Subsection III D, we use this iteration frame to derive a sequence of lower bound estimates for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ . Hence, we, in turn, use such a sequence to derive a sequence of lower bound estimates for the spatial average of the local solution in Subsection III E and complete the proof in the case  $\varsigma \in (-\frac{1}{p}, 0]$ . Finally, in Subsection III F, we also conclude the proof for the case  $\varsigma > 0$ .

## II. PRELIMINARY RESULTS

Before beginning with the Proof of Theorem I.2, we recall some estimates that are proved in Sec. 3 of Ref. 1.

### A. Iteration frame for the spatial average

Let  $v$  be a local classical solution to (1.1). In particular, by using the property of finite speed of propagation, we have that  $v$  satisfies the support condition (1.7). We set

$$V(t) \doteq \int_{\mathbb{R}^n} v(t, x) \, dx \quad \text{for } t \in (0, T).$$

In Ref. 1 (Subsection 3.1), we proved the identity

$$V''(t) + bV'(t) + m^2V(t) = \Gamma(t) \left( \int_{\mathbb{R}^n} |v(t, x)|^p \, dx \right)^{\beta+1}. \quad (2.1)$$

We underline that (2.1) is obtained by choosing a suitable cut-off function in (1.8), which localizes the forward light-cone on the strip  $[0, t] \times \mathbb{R}^n$ . Hence, by factorizing the differential operator  $\partial_t^2 + b\partial_t + m^2$  in (2.1), we then derived the following inequality for  $V$ :

$$V(t) \geq e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{\alpha_1 \tau} \Gamma(\tau) \left( \|v(\tau, \cdot)\|_{L^p(\mathbb{R}^n)}^p \right)^{\beta+1} \, d\tau \, ds \quad (2.2)$$

for  $t \geq 0$ , where  $\alpha_1, \alpha_2$  are the roots of the quadratic equation  $\alpha^2 - b\alpha + m^2 = 0$ .

The estimate in (2.2) is very important since it will be used in Subsection III E to derive a sequence of lower bound estimates for  $V(t)$  from the sequence of lower bound estimates for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  derived in Subsection III D.

### B. First lower bound estimate for $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$

In Ref. 1 [cf. Eq. (3.20) from Subsection 3.2], by working with a weighted spatial average of  $v$ , with the weight function given by a suitable positive solution of the linear adjoint equation with separable variables, we derived the following lower bound estimates:

$$\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p \geq \tilde{B} \varepsilon^p e^{\left[-\frac{1}{2}(b+H)p + (n-1)H\left(1-\frac{p}{2}\right)\right]t} \quad (2.3)$$

for  $t \geq 0$  and for a suitable constant  $\tilde{B} = \tilde{B}(n, c, H, b, m^2, p, v_0, v_1) > 0$ .

## III. PROOF OF THEOREM I.2

The proof of the results in the critical case  $\varrho = \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$  when we are in the case  $n > N$  is more delicate than the ones for  $n \leq N$  seen in Ref. 1. Roughly speaking, the functional  $V$  is no longer sufficient to show the blow-up in finite time of a local solution. The tools

that we are going to use in this section are inspired by the ones for the treatment of the critical case for the semilinear wave equation in the flat case. In particular, we are going to combine the approaches from Refs. 4 and 5 with some ideas from Ref. 33 for the treatment of a semilinear wave equation with time-dependent coefficients.

### A. ODI comparison argument in the critical case

We state and prove a Kato-type lemma for exponentially growing functions in the “critical case.” This result is the counterpart in our framework of Lemma 2.1 in Ref. 4.

*Lemma III.1.* Let  $b, m^2$  be non-negative real numbers such that  $b^2 \geq 4m^2$ . We set

$$\alpha_1 \doteq \frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4m^2} \quad \text{and} \quad \alpha_2 \doteq \frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4m^2}.$$

Let  $q > 1, k_0, k_1 \in \mathbb{R}$  such that

$$k_0 + (q - 1)k_1 = 0, \tag{3.1}$$

$$k_1 + \alpha_1 \geq 0. \tag{3.2}$$

Suppose that  $G \in \mathcal{C}^2([0, T])$  satisfies

$$G''(t) + bG'(t) + m^2G(t) \geq B e^{k_0 t} |G(t)|^q \quad \text{for } t \in [0, T], \tag{3.3}$$

$$G(t) \geq K e^{k_1 t} \quad \text{for } t \in [T_0, T], \tag{3.4}$$

$$G(0), G'(0) \geq 0, \quad \text{and} \quad \alpha_1 G(0) + G'(0) > 0 \tag{3.5}$$

with  $T_0 \in [0, T)$  and for some positive constants  $B, K$ . Let us define

$$T_1 \doteq \begin{cases} \max\{T_0, \tilde{T}_0, (k_1 + \alpha_1)^{-1}\} & \text{if } k_1 + \alpha_1 > 0, \\ \max\{T_0, \tilde{T}_0\} & \text{if } k_1 + \alpha_1 = 0, \end{cases} \tag{3.6}$$

$$K_0 \doteq \begin{cases} \left(\frac{q+1}{B}\right)^{\frac{1}{q-1}} \left(\frac{k_1 + \alpha_1}{1 - e^{-\vartheta}}\right)^{\frac{2}{q-1}} & \text{if } k_1 + \alpha_1 > 0, \\ \left(\frac{q+1}{B}\right)^{\frac{1}{q-1}} (\kappa\vartheta)^{-\frac{2}{q-1}} & \text{if } k_1 + \alpha_1 = 0, \end{cases}$$

where  $\vartheta \in (0, \frac{q-1}{2})$  and  $\kappa \in (0, T_0)$  are arbitrarily chosen, and

$$\tilde{T}_0 = \tilde{T}_0(b, m^2, G(0), G'(0)) \doteq \begin{cases} \frac{1}{\alpha_1 - \alpha_2} \ln\left(1 + \frac{(\alpha_1 - \alpha_2)G(0)}{\alpha_1 G(0) + G'(0)}\right) & \text{if } b^2 > 4m^2, \\ \frac{G(0)}{\frac{b}{2}G(0) + G'(0)} & \text{if } b^2 = 4m^2. \end{cases}$$

If the multiplicative constant on the right-hand side of (3.4) satisfies  $K \geq K_0$ , then the lifespan of  $G$  is finite and fulfills  $T \leq 2T_1$ .

*Remark 1.* Since  $q > 1$ , the two conditions (3.1) and (3.2) immediately imply that

$$k_0 - \alpha_1(q - 1) \leq 0. \tag{3.7}$$

We will also see that the condition in (3.7) on  $k_0, k_1$ , along with (3.1) and (3.2), is fundamental for the Proof of Lemma III.1.

*Remark 2.* As we are going to see in the Proof of Lemma III.1, we may assume without loss of generality that the coefficient  $k_1$  appearing in the lower bound for  $G$  in (3.4) satisfies

$$k_1 \geq -\alpha_2 \geq -\alpha_1,$$

where the equality  $-\alpha_2 = -\alpha_1$  in the last inequality holds only for  $b^2 = 4m^2$ . Indeed, it is possible to replace the lower bound for  $G$  in (3.4) with (3.9) below in order to get  $k_1 \geq -\alpha_2$ .

In particular, it makes sense to consider the limit case  $k_1 + \alpha_1 = 0$  [and, consequently, to modify accordingly  $T_1$  and  $K_0$  as in (3.6)] only in the balanced case  $b^2 = 4m^2$ .

We point out explicitly that condition (3.2) in our application of Lemma III.1 will be always satisfied thanks to (1.9).

*Proof.* By contradiction, we assume that  $G(t)$  is defined for any  $t \in [0, 2T_1]$ . We will show that this fact is not compatible with the choice  $K \geq K_0$ .

Let us begin by proving that  $G$  is actually non-negative for any  $t \in [0, T)$ . By using the factorization

$$e^{-\alpha_2 t} \frac{d}{dt} \left( e^{(\alpha_2 - \alpha_1)t} \frac{d}{dt} (e^{\alpha_1 t} G(t)) \right) = G''(t) + b G'(t) + m^2 G(t) \geq 0, \tag{3.8}$$

straightforward computations lead to the following lower bound estimate for  $G(t)$ :

$$G(t) \geq G_{\text{lin}}(t) \doteq \begin{cases} \frac{\alpha_1 e^{-\alpha_2 t} - \alpha_2 e^{-\alpha_1 t}}{\alpha_1 - \alpha_2} G(0) + \frac{e^{-\alpha_2 t} - e^{-\alpha_1 t}}{\alpha_1 - \alpha_2} G'(0) & \text{if } \alpha_1 \neq \alpha_2, \\ \left(1 + \frac{b}{2}t\right) e^{-\frac{b}{2}t} G(0) + t e^{-\frac{b}{2}t} G'(0) & \text{if } \alpha_1 = \alpha_2. \end{cases} \tag{3.9}$$

See Ref. 1 (Sec. 2.1) for a detailed derivation of the previous kind of inequality for a function  $G$  satisfying the ordinary differential inequality in (3.3).

Let us now introduce the further time-dependent function  $F(t) \doteq e^{\alpha_1 t} G(t)$ . By the previous considerations, it results in  $F(t) \geq e^{\alpha_1 t} G_{\text{lin}}(t) \geq 0$ . From (3.3), it follows that

$$F''(t) + (b - 2\alpha_1)F'(t) \geq B e^{(k_0 - \alpha_1(q-1))t} (F(t))^q. \tag{3.10}$$

By using (3.8), which can be rewritten as

$$e^{-\alpha_2 t} \frac{d}{dt} \left( e^{(\alpha_2 - \alpha_1)t} F'(t) \right) \geq 0,$$

we get easily that

$$F'(t) \geq e^{(\alpha_1 - \alpha_2)t} F'(0) = e^{(\alpha_1 - \alpha_2)t} (G'(0) + \alpha_1 G(0)) > 0, \tag{3.11}$$

thanks to the last sign assumption for the initial values of  $G$  in (3.5). Therefore, we multiply both sides of the inequality in (3.10) by  $F'(t)$ , arriving at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( (F'(t))^2 \right) &\geq (\alpha_1 - \alpha_2) (F'(t))^2 + \frac{B e^{(k_0 - \alpha_1(q-1))t}}{q+1} \frac{d}{dt} \left( (F(t))^{q+1} \right) \\ &\geq \frac{B e^{(k_0 - \alpha_1(q-1))t}}{q+1} \frac{d}{dt} \left( (F(t))^{q+1} \right), \end{aligned} \tag{3.12}$$

where in the second estimate, we used the fact that  $\alpha_1 \geq \alpha_2$ . Integrating both sides of the previous inequality over  $[0, t]$ , we obtain

$$\begin{aligned} (F'(t))^2 - (F'(0))^2 &\geq \frac{2B}{q+1} \int_0^t e^{(k_0 - \alpha_1(q-1))\tau} \frac{d}{d\tau} \left( (F(\tau))^{q+1} \right) d\tau \\ &\geq \frac{2B e^{(k_0 - \alpha_1(q-1))t}}{q+1} \left[ (F(t))^{q+1} - (F(0))^{q+1} \right], \end{aligned}$$

where in the second step, we used (3.7). By straightforward computations, we find

$$F(t) - F(0) \geq e^{\alpha_1 t} G_{\text{lin}}(t) - G(0) = \begin{cases} (\alpha_1 G(0) + G'(0)) \frac{e^{(\alpha_1 - \alpha_2)t} - 1}{\alpha_1 - \alpha_2} & \text{if } \alpha_1 \neq \alpha_2, \\ \left(\frac{b}{2}G(0) + G'(0)\right)t & \text{if } \alpha_1 = \alpha_2, \end{cases} \tag{3.13}$$

so in both cases, it holds that  $F(t) > F(0) \geq 0$  for  $t > 0$ . Hence,

$$\begin{aligned} (F'(t))^2 &\geq \frac{2Be^{(k_0-\alpha_1(q-1))t}}{q+1} (F(t))^q \left[ F(t) - F(0) \left( \frac{F(0)}{F(t)} \right)^q \right] \\ &\geq \frac{2Be^{(k_0-\alpha_1(q-1))t}}{q+1} (F(t))^q [F(t) - F(0)]. \end{aligned} \tag{3.14}$$

Next, we show that  $F(t) \geq 2F(0)$  for  $t \geq \tilde{T}_0$ . For  $\alpha_1 \neq \alpha_2$ , from (3.13), we have

$$F(t) - 2F(0) \geq (\alpha_1 G(0) + G'(0)) \frac{e^{(\alpha_1-\alpha_2)t} - 1}{\alpha_1 - \alpha_2} - G(0),$$

and then,  $F(t) \geq 2F(0)$  is satisfied, provided that

$$\frac{e^{(\alpha_1-\alpha_2)t} - 1}{\alpha_1 - \alpha_2} \geq \frac{G(0)}{\alpha_1 G(0) + G'(0)} \iff t \geq \frac{1}{\alpha_1 - \alpha_2} \ln \left( 1 + \frac{(\alpha_1 - \alpha_2)G(0)}{\alpha_1 G(0) + G'(0)} \right) = \tilde{T}_0.$$

For  $\alpha_1 = \alpha_2$ , the situation is even simpler since

$$F(t) - 2F(0) \geq \left( \frac{b}{2} G(0) + G'(0) \right) t - G(0).$$

Hence, for  $t \geq \tilde{T}_0$ , it follows from (3.14) that

$$(F'(t))^2 \geq \frac{Be^{(k_0-\alpha_1(q-1))t}}{q+1} (F(t))^{q+1}.$$

From the last inequality, we get

$$F'(t) \geq \sqrt{\frac{B}{q+1}} e^{\frac{1}{2}(k_0-\alpha_1(q-1))t} (F(t))^{\frac{q+1}{2}}.$$

Now, we multiply both sides of the previous inequality by  $(F(t))^{-(1+\vartheta)}$  for some  $\vartheta \in (0, \frac{q-1}{2})$ , obtaining

$$\frac{F'(t)}{(F(t))^{1+\vartheta}} = \frac{d}{dt} \left( -\frac{1}{\vartheta(F(t))^\vartheta} \right) \geq \sqrt{\frac{B}{q+1}} e^{\frac{1}{2}(k_0-\alpha_1(q-1))t} (F(t))^{\frac{q-1}{2}-\vartheta}.$$

Integrating both sides of the last inequality over  $[T_1, t]$  results in

$$\frac{1}{\vartheta} \left( \frac{1}{(F(T_1))^\vartheta} - \frac{1}{(F(t))^\vartheta} \right) \geq \sqrt{\frac{B}{q+1}} \int_{T_1}^t e^{\frac{1}{2}(k_0-\alpha_1(q-1))\tau} (F(\tau))^{\frac{q-1}{2}-\vartheta} d\tau. \tag{3.15}$$

The next step will consist in determining a suitable upper (respectively, lower) bound estimate for the left-hand (respectively, right-hand) side of (3.15). In order to derive both these estimates, we employ (3.4) to establish the following lower bound estimate for  $F$ :

$$F(t) \geq Ke^{(\alpha_1+k_1)t} \quad \text{for } t \geq T_0. \tag{3.16}$$

Therefore, employing (3.16), for  $t \geq T_1$ , we find, on the one hand,

$$\frac{1}{\vartheta} \left( \frac{1}{(F(T_1))^\vartheta} - \frac{1}{(F(t))^\vartheta} \right) < \frac{1}{\vartheta} \frac{1}{(F(T_1))^\vartheta} \leq \frac{e^{-\vartheta(k_1+\alpha_1)T_1}}{\vartheta K^\vartheta}. \tag{3.17}$$

In this last part of the proof, we have to consider separately the case  $k_1 + \alpha_1 > 0$  and the case  $k_1 + \alpha_1 = 0$ . For  $k_1 + \alpha_1 > 0$ , we have

$$\begin{aligned} \sqrt{\frac{B}{q+1}} \int_{T_1}^t e^{\frac{1}{2}(k_0 - \alpha_1(q-1))\tau} (F(\tau))^{\frac{q-1}{2} - \vartheta} d\tau &\geq \sqrt{\frac{B}{q+1}} \frac{K^{\frac{q-1}{2}}}{K^\vartheta} \int_{T_1}^t \exp\left\{\left[\frac{1}{2}(k_0 + (q-1)k_1) - \vartheta(k_1 + \alpha_1)\right]\tau\right\} d\tau \\ &= \sqrt{\frac{B}{q+1}} \frac{K^{\frac{q-1}{2}}}{K^\vartheta} \int_{T_1}^t \exp(-\vartheta(k_1 + \alpha_1)\tau) d\tau \\ &= \sqrt{\frac{B}{q+1}} \frac{K^{\frac{q-1}{2}}}{\vartheta K^\vartheta} (k_1 + \alpha_1)^{-1} \left(e^{-\vartheta(k_1 + \alpha_1)T_1} - e^{-\vartheta(k_1 + \alpha_1)t}\right), \end{aligned} \tag{3.18}$$

where in the second step, we used the threshold condition (3.1). Hence, combining (3.15), (3.17), and (3.18) and using the fact that  $T_1 \geq (k_1 + \alpha_1)^{-1}$ , we have for  $t = 2T_1$ ,

$$\begin{aligned} 1 &> \sqrt{\frac{B}{q+1}} K^{\frac{q-1}{2}} (k_1 + \alpha_1)^{-1} \left(1 - e^{-\vartheta(k_1 + \alpha_1)(t - T_1)}\right) \\ &\geq \sqrt{\frac{B}{q+1}} K^{\frac{q-1}{2}} (k_1 + \alpha_1)^{-1} \left(1 - e^{-\vartheta}\right) = \left(\frac{K}{K_0}\right)^{\frac{q-1}{2}}, \end{aligned}$$

which contradicts condition  $K \geq K_0$  for  $k_1 + \alpha_1 > 0$ . On the other hand, for  $k_1 + \alpha_1 = 0$ , we have

$$\begin{aligned} \sqrt{\frac{B}{q+1}} \int_{T_1}^t e^{\frac{1}{2}(k_0 - \alpha_1(q-1))\tau} (F(\tau))^{\frac{q-1}{2} - \vartheta} d\tau &\geq \sqrt{\frac{B}{q+1}} \frac{K^{\frac{q-1}{2}}}{K^\vartheta} \int_{T_1}^t \exp(-\vartheta(k_1 + \alpha_1)\tau) d\tau \\ &= \sqrt{\frac{B}{q+1}} \frac{K^{\frac{q-1}{2}}}{K^\vartheta} (t - T_1). \end{aligned} \tag{3.19}$$

Analogously as before, we combine (3.15), (3.17), and (3.19), obtaining for  $t = \kappa + T_1 \leq 2T_1$  with  $\kappa \in (0, T_0)$ ,

$$1 > \sqrt{\frac{B}{q+1}} K^{\frac{q-1}{2}} \kappa \vartheta = \left(\frac{K}{K_0}\right)^{\frac{q-1}{2}},$$

which contradicts condition  $K \geq K_0$  for  $k_1 + \alpha_1 = 0$ . This completes the proof.  $\square$

*Remark 3.* In the previous proof, we neglected in (3.12) the influence of the term  $(\alpha_1 - \alpha_2)(F'(t))^2$  in the intermediate steps that lead to (3.14). If we used (3.11) to estimate this term from below and we kept the resulting term till the estimate (3.14), by having ignored the other term that appears on the right-hand side of (3.14), we would obtain  $(F'(t))^2 \geq e^{2(\alpha_1 - \alpha_2)t}$  that would imply in turn  $F(t) \geq e^{(\alpha_1 - \alpha_2)t}$  for  $t$  sufficiently large. In particular, by comparing this lower bound for  $F$  with the one in (3.16), we see that the latter is stronger, provided that  $k_1 \geq -\alpha_2$ .

As we pointed out in Remark 2, we may always assume without loss of generality that  $k_1 \geq -\alpha_2$  and, consequently, that (3.16) provides the best lower bound estimate for  $F$ .

*Remark 4.* We point out that in the literature, there are other ODI comparison arguments for functions with exponential growth; see, for example, Lemma 3.2 in Ref. 9. Lemma III.1 differs from the previously cited lemma in Ref. 9 since, on the one hand, it deals with an ordinary differential inequality with both first- and zero-order terms and, on the other hand, it concerns only the threshold case [meaning that we just consider the case in which (3.1) is fulfilled]. Additionally, in Lemma III.1, an upper bound is provided for the lifespan.

## B. Integral representation formula for the 1D linear Cauchy problem

We derive now an integral representation formula for the solution of the following linear inhomogeneous Cauchy problem in one space dimension:

$$\begin{cases} \partial_t^2 v - c^2 e^{2Ht} \partial_x^2 v + b \partial_t v + m^2 v = g(t, x), & x \in \mathbb{R}, t > 0, \\ v(0, x) = v_0(x), & x \in \mathbb{R}, \\ \partial_t v(0, x) = v_1(x), & x \in \mathbb{R}. \end{cases} \tag{3.20}$$

Similar representation formulas are well-established in the literature by Yagdjian and Galstian and Yagdjian (cf. the Introduction) in both de Sitter and anti-de Sitter spacetimes for the Klein–Gordon equation, normalizing  $c = 1 = H$ . By means of a suitable change of variables first and, then, employing the so-called dissipative transformation, we will easily derive a representation formula for the solution of (3.20) by using Ref. 7 (Theorems 3 and 4).

Let us consider the change of variables,

$$\tau \doteq Ht, \quad y \doteq \frac{H}{c}x. \tag{3.21}$$

Elementary computations show that  $v$  solves the following Cauchy problem with respect to  $(\tau, y)$ :

$$\begin{cases} \partial_\tau^2 v - e^{2\tau} \partial_y^2 v + \frac{b}{H} \partial_\tau v + \frac{m^2}{H^2} v = \frac{1}{H^2} g\left(\frac{\tau}{H}, \frac{cy}{H}\right), & y \in \mathbb{R}, \tau > 0, \\ v(0, y) = v_0\left(\frac{cy}{H}\right), & y \in \mathbb{R}, \\ \partial_\tau v(0, x) = \frac{1}{H} v_1\left(\frac{cy}{H}\right), & y \in \mathbb{R}. \end{cases} \tag{3.22}$$

Applying the transformation  $v(\tau, y) \doteq e^{-\frac{b\tau}{2H}} w(\tau, y)$ , we find that  $v$  solves the Cauchy problem in (3.22) if and only if  $w$  solves

$$\begin{cases} \partial_\tau^2 w - e^{2\tau} \partial_y^2 w - \frac{b^2 - 4m^2}{4H^2} w = \frac{e^{\frac{b\tau}{2H}}}{H^2} g\left(\frac{\tau}{H}, \frac{cy}{H}\right), & y \in \mathbb{R}, \tau > 0, \\ w(0, y) = v_0\left(\frac{cy}{H}\right), & y \in \mathbb{R}, \\ \partial_\tau w(0, x) = \frac{b}{2H} v_0\left(\frac{cy}{H}\right) + \frac{1}{H} v_1\left(\frac{cy}{H}\right), & y \in \mathbb{R}. \end{cases} \tag{3.23}$$

Let us set  $G(\tau, y) \doteq \frac{e^{\frac{b\tau}{2H}}}{H^2} g\left(\frac{\tau}{H}, \frac{cy}{H}\right)$ ,  $w_0(y) \doteq v_0\left(\frac{cy}{H}\right)$ ,  $w_1(y) \doteq \frac{b}{2H} v_0\left(\frac{cy}{H}\right) + \frac{1}{H} v_1\left(\frac{cy}{H}\right)$ , and  $v \doteq \frac{\sqrt{b^2 - 4m^2}}{2H}$ .

According to Theorems 3 and 4 from Ref. 7,  $w$  admits the following representation:

$$\begin{aligned} w(\tau, y) &= \int_0^\tau \int_{y-e^\tau+e^\sigma}^{y+e^\tau-e^\sigma} \tilde{E}(\tau, y; \sigma, y_0; \nu) G(\sigma, y_0) dy_0 d\sigma + \frac{1}{2} e^{-\frac{\tau}{2}} (w_0(y + e^\tau - 1) + w_0(y - e^\tau + 1)) \\ &\quad + \int_{y-e^\tau+1}^{y+e^\tau-1} \tilde{K}_0(\tau, y; y_0; \nu) w_0(y_0) dy_0 + \int_{y-e^\tau+1}^{y+e^\tau-1} \tilde{K}_1(\tau, y; y_0; \nu) w_1(y_0) dy_0, \end{aligned} \tag{3.24}$$

with the kernel functions given by

$$\tilde{E}(\tau, y; \sigma, y_0; \nu) \doteq 2^{-2\nu} e^{-\nu(\tau+\sigma)} ((e^\tau + e^\sigma)^2 - (y - y_0)^2)^{-\frac{1}{2} + \nu} F\left(\frac{1}{2} - \nu, \frac{1}{2} - \nu; 1; \frac{(e^\tau - e^\sigma)^2 - (y - y_0)^2}{(e^\tau + e^\sigma)^2 - (y - y_0)^2}\right), \tag{3.25}$$

$$\tilde{K}_0(\tau, y; y_0; \nu) \doteq -\frac{\partial}{\partial \sigma} \tilde{E}(\tau, y; \sigma, y_0; \nu)|_{\sigma=0}, \tag{3.26}$$

$$\tilde{K}_1(\tau, y; y_0; \nu) \doteq \tilde{E}(\tau, y; 0, y_0; \nu), \tag{3.27}$$

where  $F(\frac{1}{2} - \nu, \frac{1}{2} - \nu; 1; \cdot)$  denotes the Gauss hypergeometric function.

Inverting the change of variables in (3.21), we may rewrite the four addends in the representation for  $w$  in (3.24) in a more convenient way. Let us begin with the double integral involving the source term,

$$\begin{aligned} \int_0^\tau \int_{y-e^\tau+e^\sigma}^{y+e^\tau-e^\sigma} \tilde{E}(\tau, y; \sigma, y_0; \nu) G(\sigma, y_0) dy_0 d\sigma &= \frac{1}{H^2} \int_0^\tau \int_{y-e^\tau+e^\sigma}^{y+e^\tau-e^\sigma} \tilde{E}(\tau, y; \sigma, y_0; \nu) e^{\frac{b\sigma}{2H}} g\left(\frac{\sigma}{H}, \frac{cy_0}{H}\right) dy_0 d\sigma \\ &= \frac{1}{c} \int_0^{\frac{\tau}{H}} \int_{\frac{c}{H}(y-e^\tau+e^{Hs})}^{\frac{c}{H}(y+e^\tau-e^{Hs})} \tilde{E}\left(\tau, y; Hs, \frac{Hz}{c}; \nu\right) e^{\frac{b}{2} s} g(s, z) dz ds \\ &= \frac{1}{c} \int_0^t \int_{x-\frac{c}{H}(e^{Ht}-e^{Hs})}^{x+\frac{c}{H}(e^{Ht}-e^{Hs})} e^{\frac{b}{2} s} \tilde{E}\left(Ht, \frac{Hx}{c}; Hs, \frac{Hz}{c}; \nu\right) g(s, z) dz ds. \end{aligned} \tag{3.28}$$

By (3.21) and the definition for  $w_0$ , we can easily express the second addend in (3.24) as follows:

$$\begin{aligned} \frac{1}{2} e^{-\frac{\tau}{2}} (w_0(y + e^\tau - 1) + w_0(y - e^\tau + 1)) &= \frac{1}{2} e^{-\frac{Ht}{2}} (v_0(x + \frac{c}{H}(e^{Ht} - 1)) + v_0(x - \frac{c}{H}(e^{Ht} - 1))) \\ &= \frac{1}{2} e^{-\frac{Ht}{2}} (v_0(x + A(t)) + v_0(x - A(t))). \end{aligned} \tag{3.29}$$

Finally, we rewrite together the remaining integrals in (3.24). By using the definition of  $w_0, w_1$ , and (3.21), we get

$$\begin{aligned} & \int_{y=e^{-\tau}+1}^{y+e^{\tau}-1} (\tilde{K}_0(\tau, y; y_0; \nu) w_0(y_0) + \tilde{K}_1(\tau, y; y_0; \nu) w_1(y_0)) dy_0 \\ &= \int_{y=e^{-\tau}+1}^{y+e^{\tau}-1} \left( \tilde{K}_0(\tau, y; y_0; \nu) v_0\left(\frac{cy_0}{H}\right) + \frac{1}{H} \tilde{K}_1(\tau, y; y_0; \nu) \left( \frac{b}{2} v_0\left(\frac{cy_0}{H}\right) + v_1\left(\frac{cy_0}{H}\right) \right) \right) dy_0 \\ &= \frac{1}{c} \int_{\frac{c}{H}(y-e^{-\tau}+1)}^{\frac{c}{H}(y+e^{\tau}-1)} \left( H \tilde{K}_0\left(\tau, y; \frac{Hz}{c}; \nu\right) + \frac{b}{2} \tilde{K}_1\left(\tau, y; \frac{Hz}{c}; \nu\right) \right) v_0(z) + \tilde{K}_1\left(\tau, y; \frac{Hz}{c}; \nu\right) v_1(z) dz \\ &= \frac{1}{c} \int_{x-\frac{c}{H}(e^{Ht}-1)}^{x+\frac{c}{H}(e^{Ht}-1)} \left( H \tilde{K}_0\left(Ht, \frac{Hx}{c}; \frac{Hz}{c}; \nu\right) + \frac{b}{2} \tilde{K}_1\left(Ht, \frac{Hx}{c}; \frac{Hz}{c}; \nu\right) \right) v_0(z) dz + \frac{1}{c} \int_{x-\frac{c}{H}(e^{Ht}-1)}^{x+\frac{c}{H}(e^{Ht}-1)} \tilde{K}_1\left(Ht, \frac{Hx}{c}; \frac{Hz}{c}; \nu\right) v_1(z) dz. \end{aligned}$$

We more carefully analyze the kernel for the first data, that is,

$$\begin{aligned} H \tilde{K}_0\left(Ht, \frac{Hx}{c}; \frac{Hz}{c}; \nu\right) + \frac{b}{2} \tilde{K}_1\left(Ht, \frac{Hx}{c}; \frac{Hz}{c}; \nu\right) &= -H \frac{\partial}{\partial \sigma} \tilde{E}\left(Ht, \frac{Hx}{c}; \sigma, \frac{Hz}{c}; \nu\right) \Big|_{\sigma=0} + \frac{b}{2} \tilde{E}\left(Ht, \frac{Hx}{c}; 0, \frac{Hz}{c}; \nu\right) \\ &= -\frac{\partial}{\partial s} \tilde{E}\left(Ht, \frac{Hx}{c}; Hs, \frac{Hz}{c}; \nu\right) \Big|_{s=0} + \frac{b}{2} \tilde{E}\left(Ht, \frac{Hx}{c}; 0, \frac{Hz}{c}; \nu\right) \\ &= -\frac{\partial}{\partial s} \left( e^{\frac{b}{2}s} \tilde{E}\left(Ht, \frac{Hx}{c}; Hs, \frac{Hz}{c}; \nu\right) \right) \Big|_{s=0} + b \tilde{E}\left(Ht, \frac{Hx}{c}; 0, \frac{Hz}{c}; \nu\right), \end{aligned}$$

where in the second equality, we used the chain rule  $H \frac{\partial}{\partial \sigma} \Big|_{\sigma=0} = \frac{\partial}{\partial s} \Big|_{s=0}$  for  $\sigma = Hs$ . Consequently,

$$\begin{aligned} & \int_{y=e^{-\tau}+1}^{y+e^{\tau}-1} (\tilde{K}_0(\tau, y; y_0; \nu) w_0(y_0) + \tilde{K}_1(\tau, y; y_0; \nu) w_1(y_0)) dy_0 \\ &= \frac{1}{c} \int_{x-\frac{c}{H}(e^{Ht}-1)}^{x+\frac{c}{H}(e^{Ht}-1)} \left( -\frac{\partial}{\partial s} \left( e^{\frac{b}{2}s} \tilde{E}\left(Ht, \frac{Hx}{c}; Hs, \frac{Hz}{c}; \nu\right) \right) \Big|_{s=0} + b \tilde{E}\left(Ht, \frac{Hx}{c}; 0, \frac{Hz}{c}; \nu\right) \right) v_0(z) dz + \frac{1}{c} \int_{x-\frac{c}{H}(e^{Ht}-1)}^{x+\frac{c}{H}(e^{Ht}-1)} \tilde{E}\left(Ht, \frac{Hx}{c}; 0, \frac{Hz}{c}; \nu\right) v_1(z) dz. \end{aligned} \tag{3.30}$$

Using the inverse transformation  $v(t, x) = e^{-\frac{b}{2}t} w(\tau, y)$  in (3.24) and combining (3.28)–(3.30), we conclude

$$\begin{aligned} v(t, x) &= \frac{1}{c} \int_0^t \int_{x-\frac{c}{H}(e^{Ht}-e^{Hs})}^{x+\frac{c}{H}(e^{Ht}-e^{Hs})} e^{-\frac{b}{2}(t-s)} \tilde{E}\left(Ht, \frac{Hx}{c}; Hs, \frac{Hz}{c}; \nu\right) g(s, z) dz ds \\ &\quad + \frac{1}{2} e^{-\frac{b+H}{2}t} \left[ v_0\left(x + \frac{c}{H}(e^{Ht}-1)\right) + v_0\left(x - \frac{c}{H}(e^{Ht}-1)\right) \right] \\ &\quad + \frac{1}{c} \int_{x-\frac{c}{H}(e^{Ht}-1)}^{x+\frac{c}{H}(e^{Ht}-1)} \left( -\frac{\partial}{\partial s} + b \right) \left( e^{-\frac{b}{2}(t-s)} \tilde{E}\left(Ht, \frac{Hx}{c}; Hs, \frac{Hz}{c}; \nu\right) \right) \Big|_{s=0} v_0(z) dz \\ &\quad + \frac{1}{c} \int_{x-\frac{c}{H}(e^{Ht}-1)}^{x+\frac{c}{H}(e^{Ht}-1)} e^{-\frac{b}{2}t} \tilde{E}\left(Ht, \frac{Hx}{c}; 0, \frac{Hz}{c}; \nu\right) v_1(z) dz. \end{aligned}$$

Summarizing the computations that we have just done and using definitions (3.25)–(3.27), we proved the following result.

**Lemma III.2.** *Let  $n = 1, c, H > 0$ , and  $b, m^2 \geq 0$  such that  $b^2 \geq 4m^2$ . Let us assume  $v_0 \in \mathcal{C}^2(\mathbb{R})$ ,  $v_1 \in \mathcal{C}^1(\mathbb{R})$ , and  $g \in \mathcal{C}([0, \infty), \mathcal{C}^1(\mathbb{R}))$ . Then, the classical solution  $v$  to the Cauchy problem (3.20) can be represented as follows:*

$$\begin{aligned} v(t, x) &= \int_0^t \int_{x-A(t)+A(s)}^{x+A(t)-A(s)} E(t, x; s, z; c, H, b, m^2) g(s, z) dz ds + \frac{1}{2} e^{-\frac{(b+H)t}{2}} [v_0(x + A(t)) + v_0(x - A(t))] \\ &\quad + \int_{x-A(t)}^{x+A(t)} K_0(t, x; z; c, H, b, m^2) v_0(z) dz + \int_{x-A(t)}^{x+A(t)} K_1(t, x; z; c, H, b, m^2) v_1(z) dz, \end{aligned}$$

where  $A(t) = \frac{c}{H}(e^{Ht} - 1)$ , and the kernel functions are given by

$$E(t, x; s, z; c, H, b, m^2) \doteq \frac{1}{H} \left( \frac{2c}{H} \right)^{-2\nu} e^{-(\frac{b}{2} + \nu H)t} e^{(\frac{b}{2} - \nu H)s} \left( \left( \frac{c}{H} (e^{Ht} + e^{Hs}) \right)^2 - (x - z)^2 \right)^{-\frac{1}{2} + \nu} F \left( \frac{1}{2} - \nu, \frac{1}{2} - \nu; 1; \frac{\left( \frac{c}{H} (e^{Ht} - e^{Hs}) \right)^2 - (x - z)^2}{\left( \frac{c}{H} (e^{Ht} + e^{Hs}) \right)^2 - (x - z)^2} \right), \quad (3.31)$$

$$K_0(t, x; z; c, H, b, m^2) \doteq - \frac{\partial}{\partial s} E(t, x; s, z; c, H, b, m^2) \Big|_{s=0} + bE(t, x; 0, z; c, H, b, m^2), \quad (3.32)$$

$$K_1(t, x; z; c, H, b, m^2) \doteq E(t, x; 0, z; c, H, b, m^2), \quad (3.33)$$

and the parameter in the hypergeometric function is  $\nu = \frac{\sqrt{b^2 - 4m^2}}{2H}$ .

*Remark 5.* In (3.31), we could replace  $\nu$  with  $-\nu$  thanks to the following property of the hypergeometric function:

$$F(a_1, a_2, b; \zeta) = (1 - \zeta)^{b - (a_1 + a_2)} F(b - a_1, b - a_2, b; \zeta).$$

See, for example, Eq. (15.8.1) in Ref. 34. However, we preferred the definition provided in (3.31) since in this way, we have no singular behavior for the hypergeometric function as  $\zeta \rightarrow 1^+$  when  $\nu > 0$ .

*Remark 6.* It is clear that the kernel functions  $E$  and  $K_1$  are non-negative on the forward light-cone and on the base of the forward light-cone, respectively.

Now, we want to show that  $K_0(t, x; z; c, H, b, m^2) \geq 0$  for any  $z$  such that  $|x - z| \leq \frac{c}{H}(e^{Ht} - 1)$ .

Let us express in a more explicit way the kernel function  $K_0$  in (3.32). For the sake of brevity, we introduce the notation

$$\mathcal{E}(t, x; s, z; c, H, b, m^2) \doteq e^{(\frac{b}{2} - \nu H)s} \left( \left( \frac{c}{H} (e^{Ht} + e^{Hs}) \right)^2 - (x - z)^2 \right)^{-\frac{1}{2} + \nu} F \left( \frac{1}{2} - \nu, \frac{1}{2} - \nu; 1; \zeta \right),$$

where

$$\zeta = \zeta(t, x; s, z; c, H) \doteq \frac{\left( \frac{c}{H} (e^{Ht} - e^{Hs}) \right)^2 - (x - z)^2}{\left( \frac{c}{H} (e^{Ht} + e^{Hs}) \right)^2 - (x - z)^2}.$$

Clearly,

$$K_0(t, x; z; c, H, b, m^2) = \frac{1}{H} \left( \frac{2c}{H} \right)^{-2\nu} e^{-(\frac{b}{2} + \nu H)t} \left( - \frac{\partial}{\partial s} \Big|_{s=0} + b \right) \mathcal{E}(t, x; 0, z; c, H, b, m^2).$$

Direct computations show that

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial s}(t, x; s, z; c, H, b, m^2) &= \left( \frac{b}{2} - \nu H \right) \mathcal{E}(t, x; s, z; c, H, b, m^2) + \frac{(2\nu - 1)H \left( \frac{c}{H} \right)^2 (e^{Ht} + e^{Hs}) e^{Hs}}{\left( \frac{c}{H} (e^{Ht} + e^{Hs}) \right)^2 - (x - z)^2} \mathcal{E}(t, x; s, z; c, H, b, m^2) \\ &\quad + \left( \frac{1}{2} - \nu \right)^2 e^{(\frac{b}{2} - \nu H)s} \left( \left( \frac{c}{H} (e^{Ht} + e^{Hs}) \right)^2 - (x - z)^2 \right)^{-\frac{1}{2} + \nu} F \left( \frac{3}{2} - \nu, \frac{3}{2} - \nu; 2; \zeta \right) \frac{\partial \zeta}{\partial s}, \end{aligned}$$

where we used the recursive identity  $F'(a_1, a_2; b; \zeta) = \frac{a_1 a_2}{b} F(a_1 + 1, a_2 + 1; b + 1; \zeta)$  for the derivative of the hypergeometric function [cf. Eq. (15.5.1) in Ref. 34].

Since

$$\begin{aligned} \frac{\partial \zeta}{\partial s}(t, x; s, z; c, H) &= 4 \frac{c^2}{H} e^{H(t+s)} \frac{(x - z)^2 + \left( \frac{c}{H} \right)^2 (-e^{2Ht} + e^{2Hs})}{\left[ \left( \frac{c}{H} (e^{Ht} + e^{Hs}) \right)^2 - (x - z)^2 \right]^2} \\ &\leq 4 \frac{c^2}{H} e^{H(t+s)} \frac{2 \left( \frac{c}{H} \right)^2 e^{Hs} (e^{Hs} - e^{Ht})}{\left[ \left( \frac{c}{H} (e^{Ht} + e^{Hs}) \right)^2 - (x - z)^2 \right]^2} \leq 0 \end{aligned}$$

for  $|x - z| \leq \frac{c}{H}(e^{Ht} - e^{Hs})$  and  $s \in [0, t]$ , we find

$$\begin{aligned}
 & -\frac{\partial \mathcal{E}}{\partial s}(t, x; s, z; c, H, b, m^2) + b\mathcal{E}(t, x; s, z; c, H, b, m^2) \\
 & \geq \left(\frac{b}{2} + \nu H\right)\mathcal{E}(t, x; s, z; c, H, b, m^2) + \frac{(1 - 2\nu)H\left(\frac{c}{H}\right)^2(e^{Ht} + e^{Hs})e^{Hs}}{\left(\frac{c}{H}(e^{Ht} + e^{Hs})\right)^2 - (x - z)^2}\mathcal{E}(t, x; s, z; c, H, b, m^2).
 \end{aligned} \tag{3.34}$$

For  $\nu \in [0, \frac{1}{2}]$ , the right-hand side of (3.34) is non-negative, so  $K_0(t, x; z; c, H, b, m^2) \geq 0$  for  $|x - z| \leq \frac{c}{H}(e^{Ht} - 1)$ . On the other hand, for  $\nu > \frac{1}{2}$ , we use in (3.34) the upper bound estimate

$$\frac{\left(\frac{c}{H}\right)^2(e^{Ht} + e^{Hs})e^{Hs}}{\left(\frac{c}{H}(e^{Ht} + e^{Hs})\right)^2 - (x - z)^2} \leq \frac{(e^{Ht} + e^{Hs})e^{Hs}}{4e^{H(t+s)}} \leq \frac{(1 + e^{-H(t-s)})}{4} \leq \frac{1}{2}$$

for  $|x - z| \leq \frac{c}{H}(e^{Ht} - e^{Hs})$  and  $s \in [0, t]$  to derive the lower bound estimate

$$\begin{aligned}
 -\frac{\partial \mathcal{E}}{\partial s}(t, x; s, z; c, H, b, m^2) + b\mathcal{E}(t, x; s, z; c, H, b, m^2) & \geq \left(\frac{b}{2} + \nu H + \frac{1}{2}(1 - 2\nu)H\right)\mathcal{E}(t, x; s, z; c, H, b, m^2) \\
 & \geq \frac{1}{2}(b + H)\mathcal{E}(t, x; s, z; c, H, b, m^2) \geq 0
 \end{aligned}$$

for  $|x - z| \leq \frac{c}{H}(e^{Ht} - e^{Hs})$  and  $s \in [0, t]$ . For  $s = 0$ , the previous inequality provides the non-negativity of  $K_0$  on the domain of integration.

### C. Iteration frame for $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ via the Radon transform

The idea to apply the Radon transform to somehow reduce the problem to a one-dimensional one when  $n \geq 2$  was introduced in the study of the critical case for the semilinear wave equation in the flat case in Ref. 5. Here, we will follow the main ideas from Ref. 33 (Sec. 5) to derive an iteration frame for the nonlinear term  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ . In particular, the representation formula obtained via Yagdjian's integral transform approach will have a crucial role in the explicit representation of the Radon transform of  $v$ . We emphasize that the case  $n = 1$  can be considered as well; however, rather than working with the Radon transform of  $v$ , it is sufficient to work simply with  $v$  (cf. Remark 9 for further details).

We begin with the following remark: without loss of generality, we may assume that a local solution is radially symmetric with respect to  $x$ . Indeed, when  $v$  is not radial, it is possible to consider

$$\tilde{v}(t, r) \doteq \int_{\mathbb{S}^{n-1}} v(t, r\omega) d\sigma_\omega \quad \text{for } t \in [0, T], r \geq 0.$$

Let us clarify the meaning of this statement. By Jensen's inequality, we have

$$\int_{\mathbb{S}^{n-1}} |v(t, r\omega)|^p d\sigma_\omega \geq \left| \int_{\mathbb{S}^{n-1}} v(t, r\omega) d\sigma_\omega \right|^p,$$

that is,  $\overline{|v|^p}(t, r) \geq |\tilde{v}(t, r)|^p$ . Similarly, by Fubini's theorem, we have

$$\int_{\mathbb{R}^n} |v(t, y)|^p dy = \omega_n \int_0^\infty \int_{\mathbb{S}^{n-1}} |v(t, r\omega)|^p d\sigma_\omega r^{n-1} dr \geq \omega_n \int_0^\infty |\tilde{v}(t, r)|^p r^{n-1} dr = \|\tilde{v}(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p,$$

where  $\omega_n$  denotes the  $(n - 1)$ -dimensional measure of the unit sphere of  $\mathbb{R}^n$ .

Consequently, combining the previous inequality, from (1.2), we have

$$\overline{f(t, v)} = \Gamma(t) \left( \int_{\mathbb{R}^n} |v(t, y)|^p dy \right)^\beta \overline{|v|^p} \geq \Gamma(t) \|\tilde{v}(t, \cdot)\|_{L^p(\mathbb{R}^n)}^{\beta p} |\tilde{v}|^p = f(t, \tilde{v}).$$

Since the fundamental solution  $E$  defined in (3.31) is non-negative on the forward light-cone and the averages with respect to the space variables of  $v$  and  $\tilde{v}$  are equal, the inequality  $\overline{f(t, v)} \geq f(t, \tilde{v})$ , which we just proved, allows us to assume without loss of generality that  $v$  is radially symmetric in the proof of the blow-up result.

Let us recall the definition of Radon transform of  $v(t, \cdot)$  when  $n \geq 2$ . Given  $\rho \in \mathbb{R}$  and  $\xi \in \mathbb{R}^n$ ,  $|\xi| = 1$ , we have

$$\mathcal{R}[v](t, \rho, \xi) \doteq \int_{\{x \in \mathbb{R}^n: x \cdot \xi = \rho\}} v(t, x) d\sigma_x = \int_{\{x \in \mathbb{R}^n: x \cdot \xi = 0\}} v(t, \rho\xi + x) d\sigma_x, \tag{3.35}$$

where  $d\sigma_x$  is the Lebesgue measure on the corresponding hyperplanes. Since  $v(t, \cdot)$  is radially symmetric with respect to  $x$ , it turns out that  $\mathcal{R}[v]$  does not actually depend on  $\xi$ , and moreover,

$$\mathcal{R}[v](t, \rho) = \omega_{n-1} \int_{|\rho|}^{\infty} v(t, r) (r^2 - \rho^2)^{\frac{n-3}{2}} r dr. \tag{3.36}$$

Indeed, using polar coordinates  $x = \omega r_1$  with  $r_1 = |x|$  and  $\omega \in \mathbb{S}^{n-1}$  such that  $\omega \cdot \xi = 0$ , we obtain

$$\begin{aligned} \mathcal{R}[v](t, \rho, \xi) &= \int_{\{x \in \mathbb{R}^n: x \cdot \xi = 0\}} v(t, \rho\xi + x) d\sigma_x = \int_0^{\infty} \int_{\substack{\omega \in \mathbb{S}^{n-1} \\ \omega \cdot \xi = 0}} v\left(t, \sqrt{\rho^2 + r_1^2}\right) d\sigma_{\omega} r_1^{n-2} dr_1 \\ &= \omega_{n-1} \int_0^{\infty} v\left(t, \sqrt{\rho^2 + r_1^2}\right) r_1^{n-2} dr_1. \end{aligned}$$

Hence, employing the change of variables  $r = (\rho^2 + r_1^2)^{1/2}$  in the last integral, we find (3.36).

Using the identity  $\mathcal{R}[\Delta v] = \partial_{\rho}^2 \mathcal{R}[v]$  (see Lemma 2.1 in Chap. 1 of Ref. 35) and the linearity of  $\mathcal{R}$ , we find that  $\mathcal{R}[v]$  satisfies the following one-dimensional Cauchy problem:

$$\begin{cases} \partial_t^2 \mathcal{R}[v] - c^2 e^{2Ht} \partial_{\rho}^2 \mathcal{R}[v] + b \partial_t \mathcal{R}[v] + m^2 \mathcal{R}[v] = \mathcal{R}[f(t, v)], & \rho \in \mathbb{R}, t \in (0, T), \\ \mathcal{R}[v](0, \rho) = \varepsilon \mathcal{R}[v_0](\rho), & \rho \in \mathbb{R}, \\ \partial_t \mathcal{R}[v](0, \rho) = \varepsilon \mathcal{R}[v_1](\rho), & \rho \in \mathbb{R}. \end{cases}$$

From Subsection III B follows the following integral representation formula:

$$\mathcal{R}[v](t, \rho) = \varepsilon (\mathcal{R}[v])_{\text{lin}}(t, \rho) + (\mathcal{R}[v])_{\text{nl}}(t, \rho)$$

with

$$\begin{aligned} (\mathcal{R}[v])_{\text{lin}}(t, \rho) &\doteq \frac{1}{2} e^{-\frac{(b+H)t}{2}} (\mathcal{R}[v_0](\rho + A(t)) + \mathcal{R}[v_0](\rho - A(t))) \\ &\quad + \int_{\rho-A(t)}^{\rho+A(t)} K_0(t, \rho; \eta; c, H, b, m^2) \mathcal{R}[v_0](\eta) d\eta \\ &\quad + \int_{\rho-A(t)}^{\rho+A(t)} K_1(t, \rho; \eta; c, H, b, m^2) \mathcal{R}[v_1](\eta) d\eta, \\ (\mathcal{R}[v])_{\text{nl}}(t, \rho) &\doteq \int_0^t \int_{\rho-A(t)+A(s)}^{\rho+A(t)-A(s)} E(t, \rho; s, \eta; c, H, b, m^2) \mathcal{R}[f(\cdot, v)](s, \eta) d\eta ds, \end{aligned}$$

where the definitions of the kernel functions are given in (3.31)–(3.33), respectively. Thanks to the assumptions on the Cauchy data in the statement of Theorem I.2 and Remark 6, we have that  $(\mathcal{R}[v])_{\text{lin}}$  is a non-negative function. Therefore, we have

$$\mathcal{R}[v](t, \rho) \geq (\mathcal{R}[v])_{\text{nl}}(t, \rho).$$

We point out that  $\mathcal{R}$  acts only on the factors in the nonlinear term  $f(t, v)$  that depend on the space variable, that is,

$$\mathcal{R}[f(\cdot, v)](t, \rho) = \Gamma(t) \|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^{\beta p} \mathcal{R}[|v|^p](t, \rho).$$

Thus,

$$\mathcal{R}[v](t, \rho) \geq \int_0^t \Gamma(s) \|v(s, \cdot)\|_{L^p(\mathbb{R}^n)}^{\beta p} \int_{\rho-A(t)+A(s)}^{\rho+A(t)-A(s)} E(t, \rho; s, \eta; c, H, b, m^2) \mathcal{R}[|v|^p](s, \eta) d\eta ds. \tag{3.37}$$

From the support condition (1.7) for  $v$ , it follows that

$$\text{supp } \mathcal{R}[v](t, \cdot) \subset [-(R + A(t)), R + A(t)] \quad \text{for any } t \in [0, T].$$

Indeed, for  $|\rho| > R + A(t)$ , from the second representation in (3.35), we have that

$$\mathcal{R}[v](t, \rho, \xi) = \int_{\{x \in \mathbb{R}^n : x \cdot \xi = 0\}} v(t, \rho \xi + x) d\sigma_x = 0$$

due to the fact that on the hyperplane where we are integrating it holds that

$$|\rho \xi + x|^2 = \rho^2 + |x|^2 \geq \rho^2 \implies |\rho \xi + x| > R + A(t)$$

and, consequently, the considered hyperplane as empty intersection with the support of  $v$ .

In a completely analogous way, we have that  $\text{supp } \mathcal{R}[|v|^p](t, \cdot) \subset [-(R + A(t)), R + A(t)]$  for any  $t \in [0, T]$ .

In the next step, we shrink the domain of integration with respect to  $s$  in (3.37) so that the support of  $\mathcal{R}[|v|^p](s, \eta)$  is a subset of the  $\eta$ -domain of integration. In other words, we look for  $s \in [0, t]$  such that

$$\begin{aligned} [-(R + A(s)), R + A(s)] &\subset [\rho - A(t) + A(s), \rho + A(t) - A(s)] \\ &\iff 2A(s) \leq A(t) - |\rho| - R \\ &\iff s \leq s_0(t, \rho, R) \doteq A^{-1}\left(\frac{1}{2}(A(t) - |\rho| - R)\right). \end{aligned}$$

We point out that  $s_0 \geq 0$  if and only if  $|\rho| \leq A(t) - R$ .

Therefore, for  $|\rho| \leq A(t) - R$ , we obtain from (3.37),

$$\mathcal{R}[v](t, \rho) \geq \int_0^{s_0} \Gamma(s) \|v(s, \cdot)\|_{L^p(\mathbb{R}^n)}^{\beta p} \int_{-(R+A(s))}^{R+A(s)} E(t, \rho; s, \eta; c, H, b, m^2) \mathcal{R}[|v|^p](s, \eta) d\eta ds. \quad (3.38)$$

The next step is to estimate the kernel function  $E$  in the right-hand side of the last inequality on the shrunk  $\eta$ -interval of integration. First of all, from the Taylor expansion of the hypergeometric function,

$$F\left(\frac{1}{2} - \nu, \frac{1}{2} - \nu; 1; \zeta\right) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2} - \nu)_k}{(1)_k k!} \zeta^k \quad \text{for } \zeta \in [0, 1),$$

where  $(a)_0 \doteq 1$  and  $(a)_k \doteq a(a+1) \cdots (a+k-1)$  denotes the so-called Pochhammer symbol, we see immediately that we can estimate from below the factor involving the hypergeometric function in  $E(t, \rho; s, \eta; c, H, b, m^2)$  by the constant function 1. Furthermore, since the two exponential terms in (3.31) are independent of  $\eta$ , the only factor that we actually have to estimate from below for  $\eta \in [-(R + A(s)), R + A(s)]$  is  $((\frac{c}{H}(e^{Ht} + e^{Hs}))^2 - (\rho - \eta)^2)^{-\frac{1}{2} + \nu}$ . Note that we have to proceed in a different way in order to get such a lower bound estimate depending on whether  $\nu$  is smaller or greater than  $1/2$ .

Hereafter, for the sake of brevity, we use the notation  $\phi(t) \doteq \frac{c}{H} e^{Ht}$ . In particular, we may express the amplitude of the forward light-cone as follows  $A(t) = \phi(t) - \phi(0)$ .

Let us begin with the case  $\nu \in [0, \frac{1}{2}]$ . Let us prove that in this case, the following upper bound estimate holds:

$$\phi(t) + \phi(s) - \rho + \eta \leq 2(\phi(t) - \rho) \quad (3.39)$$

for  $s \in [0, s_0]$  and  $\eta \in [-(R + A(s)), R + A(s)]$ .

Clearly, (3.39) is equivalent to require that  $\phi(t) - \rho - \eta \geq \phi(s)$ . We check the validity of this inequality for  $s \in [0, s_0]$  and  $\eta \in [-(R + A(s)), R + A(s)]$  through a chain of inequalities,

$$\begin{aligned} \phi(t) - \rho - \eta &\geq \phi(t) - \rho - R - A(s) \geq \phi(t) - \rho - R - A(s_0) \\ &= A(t) + \phi(0) - \rho - R - \frac{1}{2}(A(t) - |\rho| - R) \geq \frac{1}{2}(A(t) - |\rho| - R) + \phi(0) \\ &\geq A(s_0) + \phi(0) \geq A(s) + \phi(0) = \phi(s), \end{aligned}$$

where we used twice condition  $A(s) \leq A(s_0)$  and the identity  $2A(s_0) = A(t) - |\rho| - R$ .

In a completely analogous way, one proves that

$$\phi(t) + \phi(s) + \rho - \eta \leq 2(\phi(t) + \rho)$$

for  $s \in [0, s_0]$  and  $\eta \in [-(R + A(s)), R + A(s)]$ .

Therefore, combining (3.39) and the last inequality, when  $\nu \leq \frac{1}{2}$ , we can estimate

$$((\phi(t) + \phi(s))^2 - (\rho - \eta)^2)^{-\frac{1}{2} + \nu} \geq 2^{-1+2\nu} (\phi^2(t) - \rho^2)^{-\frac{1}{2} + \nu} \tag{3.40}$$

for  $s \in [0, s_0]$  and  $\eta \in [-(R + A(s)), R + A(s)]$ .

On the other hand, for  $\nu \geq \frac{1}{2}$ , from the lower bound estimates,

$$\begin{aligned} \phi(t) + \phi(s) - \rho + \eta &\geq \phi(t) + \phi(s) - \rho - R - A(s) \geq \phi(t) - \rho + \phi(0) - R, \\ \phi(t) + \phi(s) + \rho - \eta &\geq \phi(t) + \phi(s) + \rho - R - A(s) \geq \phi(t) + \rho + \phi(0) - R \end{aligned} \tag{3.41}$$

for  $s \in [0, s_0]$  and  $\eta \in [-(R + A(s)), R + A(s)]$ , it follows that

$$((\phi(t) + \phi(s))^2 - (\rho - \eta)^2)^{-\frac{1}{2} + \nu} \geq ((\phi(t) + \phi(0) - R)^2 - \rho^2)^{-\frac{1}{2} + \nu}. \tag{3.42}$$

Combining (3.40) and (3.42), we conclude that the kernel function in (3.38) can be estimate from below in the following way:

$$E(t, \rho; s, \eta; c, H, b, m^2) \gtrsim e^{-(\frac{b}{2} + \nu H)t} e^{(\frac{b}{2} - \nu H)s} ((\phi(t) + R_1)^2 - \rho^2)^{-\frac{1}{2} + \nu} \tag{3.43}$$

for  $s \in [0, s_0]$  and  $\eta \in [-(R + A(s)), R + A(s)]$ , where

$$R_1 \doteq \begin{cases} 0 & \text{if } \nu \leq \frac{1}{2}, \\ \phi(0) - R & \text{if } \nu > \frac{1}{2}. \end{cases} \tag{3.44}$$

*Remark 7.* Note that for  $R \leq \phi(0)$ , in (3.43), we might consider  $R_1 = 0$  even for  $\nu > \frac{1}{2}$ .

*Remark 8.* In the previous considerations, we estimate from below the hypergeometric function by a constant. In the limit case  $b^2 = 4m^2$  (that is, for  $\nu = 0$ ), we might think to employ the asymptotic estimate  $F(\frac{1}{2}, \frac{1}{2}; 1; \zeta) \sim -\ln(1 - \zeta)$  as  $\zeta \rightarrow 1^-$  in order to improve this lower bound estimate. However, for  $s \in [0, s_0]$  and  $\eta \in [-(R + A(s)), R + A(s)]$ , setting

$$\zeta = \zeta(t, \rho; s, \eta; c, H) \doteq \frac{(\frac{c}{H}(e^{Ht} - e^{Hs}))^2 - (\rho - \eta)^2}{(\frac{c}{H}(e^{Ht} + e^{Hs}))^2 - (\rho - \eta)^2},$$

we have

$$\begin{aligned} -\ln(1 - \zeta) &= \ln\left(\frac{(\phi(t) + \phi(s))^2 - (\rho - \eta)^2}{4\phi(t)\phi(s)}\right) \geq \ln\left(\frac{(\phi(t) + \phi(0) - R)^2 - \rho^2}{4\phi(t)\phi(s_0)}\right) \\ &= \ln\left(\frac{\phi(t) + \phi(0) - R + \rho}{2\phi(t)}\right), \end{aligned}$$

where we used (3.41) and  $\phi(s_0) = \frac{1}{2}(\phi(t) + \phi(0) - \rho - R)$ . Therefore,  $-\ln(1 - \zeta)$  does not provide an improvement in the lower bound estimate since for large  $t$ , the argument of the logarithmic term on the right-hand side of the last inequality can be only estimated by a constant for  $\rho \in [0, A(t) - R]$  (this is the actual range that we will consider for  $\rho$  at the end of this subsection).

Now, we plug the lower bound estimate from (3.43) in (3.38). For  $|\rho| \leq A(t) - R$ , we have

$$\mathcal{R}[\nu](t, \rho) \geq e^{-(\frac{b}{2} + \nu H)t} ((\phi(t) + R_1)^2 - \rho^2)^{-\frac{1}{2} + \nu} \int_0^{s_0} \Gamma(s) e^{(\frac{b}{2} - \nu H)s} \|\nu(s, \cdot)\|_{L^p(\mathbb{R}^n)}^{(\beta+1)p} ds, \tag{3.45}$$

where we used the support condition for  $\mathcal{R}[|v|^p]$  and Fubini's theorem to obtain

$$\int_{-(R+A(s))}^{R+A(s)} \mathcal{R}[|v|^p](s, \eta) \, d\eta \, ds = \int_{\mathbb{R}} \mathcal{R}[|v|^p](s, \eta) \, d\eta \, ds = \|v(s, \cdot)\|_{L^p(\mathbb{R}^n)}^p.$$

The inequality in (3.45) is the first crucial estimate to obtain the iteration frame for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ . The next step is to determine a lower bound for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  with  $\mathcal{R}[v]$  appearing in a nonlinear term on the right-hand side.

In order to derive this inequality, we will follow the approach from Ref. 33 (Sec. 5). We introduce the operator

$$\mathcal{F}(h)(\tau) \doteq |A(t) + R - \tau|^{-\frac{n-1}{2}} \int_{\tau}^{A(t)+R} h(r) |r - \tau|^{\frac{n-3}{2}} \, dr$$

for any  $\tau \in \mathbb{R}$  and any  $h \in L^p(\mathbb{R})$ .

In Ref. 33 (Sec. 5), it is proved that  $\mathcal{F} \in \mathcal{L}(L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}))$  for any  $p \in (1, \infty)$  and  $n \geq 2$ . Even though the function  $A(t)$  in Ref. 33 is a polynomial function (more precisely,  $A(t) = \frac{1}{\ell+1}(t^{\ell+1} - 1)$  for some  $\ell \geq 0$ ), the proof of this result is actually independent of the explicit expression of  $A(t)$  and it can be repeated verbatim in our case with  $A(t) = \frac{c}{H}(e^{Ht} - 1)$ .

We consider now the function

$$h(t, r) \doteq \begin{cases} |v(t, r)| r^{\frac{n-1}{p}} & \text{if } r \geq 0, \\ 0 & \text{if } r < 0. \end{cases}$$

By the boundedness of the operator  $\mathcal{F}$  on  $L^p(\mathbb{R})$ , we have that  $\|\mathcal{F}(h)(t, \cdot)\|_{L^p(\mathbb{R})} \lesssim \|h(t, \cdot)\|_{L^p(\mathbb{R})}$  holds uniformly with respect to  $t \in [0, T]$ .

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} |v(t, x)|^p \, dx &= \omega_n \int_0^\infty |v(t, r)|^p r^{n-1} \, dr = \omega_n \|h(t, \cdot)\|_{L^p(\mathbb{R})}^p \\ &\geq \|\mathcal{F}(h)(t, \cdot)\|_{L^p(\mathbb{R})}^p = \int_{\mathbb{R}} |\mathcal{F}(h)(t, \rho)|^p \, d\rho \\ &= \int_{\mathbb{R}} |A(t) + R - \rho|^{-\frac{n-1}{2}p} \left| \int_{\rho}^{A(t)+R} |v(t, r)| r^{\frac{n-1}{p}} |r - \rho|^{\frac{n-3}{2}} \, dr \right|^p \, d\rho \\ &\geq \int_0^{A(t)+R} (A(t) + R - \rho)^{-\frac{n-1}{2}p} \left( \int_{\rho}^{A(t)+R} |v(t, r)| r^{\frac{n-1}{p}} (r - \rho)^{\frac{n-3}{2}} \, dr \right)^p \, d\rho. \end{aligned} \tag{3.46}$$

We have seen that  $\mathcal{R}[v]$  is a non-negative function by using an explicit integral representation. Moreover, by using the monotonicity of  $\mathcal{R}$  and (3.36), we get

$$\begin{aligned} 0 \leq \mathcal{R}[v](t, \rho) &\leq \mathcal{R}[|v|](t, \rho) = \omega_{n-1} \int_{|\rho|}^{A(t)+R} |v(t, r)| (r - |\rho|)^{\frac{n-3}{2}} (r + |\rho|)^{\frac{n-3}{2}} r \, dr \\ &\leq 2^{\frac{n-3}{2}} \omega_{n-1} \int_{|\rho|}^{A(t)+R} |v(t, r)| (r - |\rho|)^{\frac{n-3}{2}} r^{\frac{n-1}{2}} \, dr. \end{aligned} \tag{3.47}$$

Clearly, for  $\rho \in [0, A(t) + R]$  and  $r \in [\rho, A(t) + R]$ , we have

$$r^{\frac{n-1}{2}} \geq (A(t) + R)^{-(n-1)\left[\frac{1}{p} - \frac{1}{2}\right]_-} r^{\frac{n-1}{2}} \rho^{(n-1)\left[\frac{1}{p} - \frac{1}{2}\right]_+},$$

where  $\left[\frac{1}{p} - \frac{1}{2}\right]_{\pm}$  denote the positive and negative part of  $\frac{1}{p} - \frac{1}{2}$ , respectively.

Combining (3.46) and (3.47) and the above inequality, we have

$$\begin{aligned} \|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p &\geq (A(t) + R)^{-(n-1)\left[1 - \frac{\ell}{2}\right]_-} \int_0^{A(t)+R} \frac{\rho^{(n-1)\left[1 - \frac{\ell}{2}\right]_+}}{(A(t) + R - \rho)^{\frac{n-1}{2}p}} \left( \int_{\rho}^{A(t)+R} |v(t, r)| r^{\frac{n-1}{2}} (r - \rho)^{\frac{n-3}{2}} \, dr \right)^p \, d\rho \\ &\geq (A(t) + R)^{-(n-1)\left[1 - \frac{\ell}{2}\right]_-} \int_0^{A(t)+R} \frac{\rho^{(n-1)\left[1 - \frac{\ell}{2}\right]_+}}{(A(t) + R - \rho)^{\frac{n-1}{2}p}} (\mathcal{R}[v](t, \rho))^p \, d\rho. \end{aligned} \tag{3.48}$$

Finally, from (3.45) and (3.48), we obtain for  $t \geq A^{-1}(R)$ , the desired iteration frame

$$\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p \geq \frac{K e^{-(\frac{b}{2} + \nu H)t}}{(A(t) + R)^{(n-1)[1-\frac{\beta}{2}]_-}} \int_0^{A(t)-R} \frac{\rho^{(n-1)[1-\frac{\beta}{2}]_+}}{(A(t) + R - \rho)^{\frac{n-1}{2}p}} ((\phi(t) + R_1)^2 - \rho^2)^{(-\frac{1}{2} + \nu)p} \times \left( \int_0^{A^{-1}(\frac{1}{2}(A(t) - \rho - R))} \Gamma(s) e^{(\frac{b}{2} - \nu H)s} \|v(s, \cdot)\|_{L^p(\mathbb{R}^n)}^{(\beta+1)p} ds \right)^p \quad (3.49)$$

for a suitable positive multiplicative constant  $K = K(n, H, b, m^2)$ . Needless to say, in (3.49), only one between the factors  $(A(t) + R)^{-(n-1)[1-\frac{\beta}{2}]_-}$  and  $\rho^{(n-1)[1-\frac{\beta}{2}]_+}$  is actually present for  $p \neq 2$ . Nevertheless, we will do the computations formally as if both were present in order to simultaneously consider the cases  $p \in (1, 2)$  and  $p \geq 2$ .

*Remark 9.* Let us explicitly underline that (3.49) is also true for  $n = 1$ . First, (3.45) can be obtained exactly as we did for  $n \geq 2$ , with the only difference that Yagdjian integral representation formula is applied now directly to  $v$ , that is,

$$v(t, \rho) \geq e^{-(\frac{b}{2} + \nu H)t} ((\phi(t) + R_1)^2 - \rho^2)^{-\frac{1}{2} + \nu} \int_0^{s_0} \Gamma(s) e^{(\frac{b}{2} - \nu H)s} \|v(s, \cdot)\|_{L^p(\mathbb{R})}^{(\beta+1)p} ds. \quad (3.50)$$

On the other hand, for  $n = 1$ , (3.48) can be replaced by the trivial inequality

$$\|v(t, \cdot)\|_{L^p(\mathbb{R})}^p \geq \int_0^{A(t)+R} |v(t, \rho)|^p d\rho. \quad (3.51)$$

Hence, combining (3.50) and (3.51), we conclude the validity of (3.49) for  $n = 1$  too.

#### D. Iteration argument for $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$

Our next goal is to derive a sequence of lower bound estimates for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  through the iteration frame (3.49). In order to reach this target, we use an inductive argument. For the ease of readability, we emphasize in what follows the base case and the inductive step of this proof by induction.

##### 1. Base case

The starting point of our iteration procedure is given by (2.3).

Let us now derive a first lower bound estimate for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  with an additional polynomial growing factor. Plugging (2.3) in (3.49), for  $t \geq A^{-1}(R)$ , we get

$$\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p \geq K \widetilde{B}^q e^{pq} e^{-(\frac{b}{2} + \nu H)t} (A(t) + R)^{-(n-1)[1-\frac{\beta}{2}]_-} \int_0^{A(t)-R} \frac{\rho^{(n-1)[1-\frac{\beta}{2}]_+}}{(A(t) + R - \rho)^{\frac{n-1}{2}p}} ((\phi(t) + R_1)^2 - \rho^2)^{(-\frac{1}{2} + \nu)p} (I_0(t, \rho))^p d\rho, \quad (3.52)$$

where, henceforth,  $q \doteq (\beta + 1)p$  and

$$I_0(t, \rho) \doteq \int_0^{A^{-1}(\frac{1}{2}(A(t) - \rho - R))} \Gamma(s) e^{(\frac{b}{2} - \nu H - \frac{1}{2}(b+H)q + (n-1)H(\beta+1) - \frac{(n-1)}{2}Hq)s} ds.$$

We recall that  $\Gamma(s) = \mu(1+s)^s e^{\varrho_{\text{crit}} s}$ , being  $\varrho_{\text{crit}}$  defined by (1.4). Therefore, using the actual value of  $\Gamma(s)$ , we have

$$I_0(t, \rho) = \int_0^{A^{-1}(\frac{1}{2}(A(t) - \rho - R))} \mu(1+s)^s e^{(\frac{b}{2} - \nu - \frac{1}{p})Hs} ds,$$

where we used

$$\begin{aligned} \varrho_{\text{crit}} + \frac{b}{2} - \nu H - \frac{1}{2}(b+H)q + (n-1)H(\beta+1) - \frac{(n-1)}{2}Hq &= \frac{1}{2}(b+nH)(q-1) + nH - \frac{H}{p} + \frac{b}{2} - \nu H - \frac{1}{2}(b+H)q - \frac{(n-1)}{2}Hq \\ &= \frac{nH}{2} - \nu H - \frac{H}{p} \end{aligned} \quad (3.53)$$

for the coefficient in the exponential term.

Hereafter, we consider only the case  $\zeta \leq 0$ . The complementary case  $\zeta > 0$  will be discussed at the end of Sec. III (cf. Subsection III F). Therefore,

$$I_0(t, \rho) \geq \mu (1+t)^\zeta \int_0^{A^{-1}(\frac{1}{2}(A(t)-\rho-R))} e^{(\frac{n}{2}-\nu-\frac{1}{p})Hs} ds$$

$$= \frac{\mu}{(\frac{n}{2}-\nu-\frac{1}{p})H} (1+t)^\zeta \left( \left( \frac{H}{2c} \right)^{\frac{n}{2}-\nu-\frac{1}{p}} \left( A(t) - \rho - R + \frac{2c}{H} \right)^{\frac{n}{2}-\nu-\frac{1}{p}} - 1 \right),$$

where we used the analytic expression of the inverse function of  $A$  [defined in (1.6)], which is given by

$$A^{-1}(\sigma) = \frac{1}{H} \ln \left( \frac{H\sigma}{c} + 1 \right). \tag{3.54}$$

Let  $b_0 > 0$  be a fixed parameter. For  $\rho \leq A(t) - R - b_0 \frac{c}{H}$ , we may estimate  $I_0(t, \rho)$  from below as follows:

$$I_0(t, \rho) \geq \widehat{B} (1+t)^\zeta \left( A(t) - \rho - R + \frac{2c}{H} \right)^{\frac{n}{2}-\nu-\frac{1}{p}},$$

where  $\widehat{B} \doteq \mu \left( \frac{n}{2} - \nu - \frac{1}{p} \right)^{-1} H^{-1} \left( 1 - \left( \frac{b_0}{2} + 1 \right)^{-\frac{n}{2} + \nu + \frac{1}{p}} \right) \left( \frac{H}{2c} \right)^{\frac{n}{2} - \nu - \frac{1}{p}}$ . Plugging the last lower bound for  $I_0(t, \rho)$  in (3.52), we get

$$\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p \geq K \widehat{B}^q \widehat{B}^p e^{pq} e^{-(\frac{1}{2} + \nu H)pt} (A(t) + R)^{-(n-1)[1-\frac{\ell}{2}]_-} (1+t)^{Sp} J_0(t) \tag{3.55}$$

for  $t \geq A^{-1}(R + b_0 \frac{c}{H})$ , where

$$J_0(t) \doteq \int_0^{A(t)-R-b_0 \frac{c}{H}} \rho^{(n-1)[1-\frac{\ell}{2}]_+} \frac{((\phi(t) + R_1)^2 - \rho^2)^{(-\frac{1}{2} + \nu)p}}{(A(t) + R - \rho)^{\frac{n-1}{2}p}} \left( A(t) - \rho - R + \frac{2c}{H} \right)^{\frac{n}{2}p - \nu p - 1} d\rho.$$

The next step consists in estimating from below the integral  $J_0(t)$ .

First, we consider the factor  $((\phi(t) + R_1)^2 - \rho^2)^{(-\frac{1}{2} + \nu)p}$ . Recalling that the value of  $R_1$  depends on the range for  $\nu$  [cf. (3.44) for the definition of  $R_1$ ], we derive a lower bound for this factor separately in the case  $\nu \leq \frac{1}{2}$  and in the case  $\nu > \frac{1}{2}$ .

For  $\nu \leq \frac{1}{2}$ , since the power  $(-\frac{1}{2} + \nu)p$  is nonpositive, we consider an upper bound for  $(\phi(t) + R_1)^2 - \rho^2$ . For  $\rho \in [0, A(t) - R]$ , since  $R_1 = 0$  in this case, we have

$$\phi(t) + R_1 + \rho \leq \phi(t) + A(t) - R = 2\phi(t) - \frac{c}{H} - R \leq 2\phi(t)$$

and

$$\phi(t) + R_1 - \rho = A(t) - \rho + \frac{c}{H} \leq \begin{cases} A(t) - \rho - R + \frac{2c}{H} & \text{if } R \leq \frac{c}{H}, \\ A(t) - \rho + R & \text{if } R > \frac{c}{H}. \end{cases}$$

Thus, for  $\nu \leq \frac{1}{2}$  and  $\rho \in [0, A(t) - R]$ , we obtain

$$((\phi(t) + R_1)^2 - \rho^2)^{(-\frac{1}{2} + \nu)p} \geq \begin{cases} \left( \frac{2c}{H} \right)^{(-\frac{1}{2} + \nu)p} e^{(-\frac{1}{2} + \nu)Hpt} (A(t) - \rho - R + \frac{2c}{H})^{(-\frac{1}{2} + \nu)p} & \text{if } R \leq \frac{c}{H}, \\ \left( \frac{2c}{H} \right)^{(-\frac{1}{2} + \nu)p} e^{(-\frac{1}{2} + \nu)Hpt} (A(t) - \rho + R)^{(-\frac{1}{2} + \nu)p} & \text{if } R > \frac{c}{H}. \end{cases} \tag{3.56}$$

We consider now  $\nu > \frac{1}{2}$ . In this case, we determine a lower bound for  $(\phi(t) + R_1)^2 - \rho^2$ . For  $\rho \in [0, A(t) - R]$ , since  $R_1 = \frac{c}{H} - R$  in this case, we have

$$\phi(t) + R_1 + \rho \geq \phi(t) + \frac{c}{H} - R \geq \begin{cases} \phi(t) & \text{if } R \leq \frac{c}{H}, \\ \frac{1}{2}\phi(t) & \text{if } R > \frac{c}{H} \text{ and } t \geq A^{-1}\left(2R - \frac{3c}{H}\right), \end{cases}$$

and  $\phi(t) + R_1 - \rho = A(t) - \rho - R + \frac{2c}{H}$ .

We emphasize that the lower bound for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  that we are going to prove will be valid for  $t \geq A^{-1}(a_0R + b_0 \frac{c}{H})$  and for suitable  $a_0 \geq 2$  and  $b_0 > 0$ . In particular, we will use the inequality  $\phi(t) + R_1 + \rho \geq \frac{1}{2}\phi(t)$  when  $R > \frac{c}{H}$  without specifying the further condition on  $t$ .

Hence, for  $v > \frac{1}{2}$  and  $\rho \in [0, A(t) - R]$ , we proved that

$$((\phi(t) + R_1)^2 - \rho^2)^{(-\frac{1}{2}+\nu)p} \geq \begin{cases} \left(\frac{c}{H}\right)^{(-\frac{1}{2}+\nu)p} e^{(-\frac{1}{2}+\nu)H\rho t} (A(t) - \rho - R + \frac{2c}{H})^{(-\frac{1}{2}+\nu)p} & \text{if } R \leq \frac{c}{H}, \\ \left(\frac{c}{2H}\right)^{(-\frac{1}{2}+\nu)p} e^{(-\frac{1}{2}+\nu)H\rho t} (A(t) - \rho - R + \frac{2c}{H})^{(-\frac{1}{2}+\nu)p} & \text{if } R > \frac{c}{H}. \end{cases} \tag{3.57}$$

Next, we consider the factor  $(A(t) + R - \rho)^{-\frac{n-1}{2}p}$  in  $J_0(t)$ . Note that in the case  $v \leq \frac{1}{2}$  when  $R > \frac{c}{H}$ , from (3.56), we see that we have to consider actually the factor  $(A(t) + R - \rho)^{(-\frac{n}{2}+\nu)p}$ . From (1.9), it follows that  $(-\frac{n}{2} + \nu)p < 0$ . Hence, in both cases, we are interested in an upper bound estimate for the term  $A(t) + R - \rho$ . By straightforward computations, we get that

$$A(t) - \rho - R + \frac{2c}{H} \geq \frac{a_0 - 1}{a_0 + 1} (A(t) - \rho + R) \quad \text{for any } \rho \leq A(t) - a_0R + (a_0 + 1) \frac{c}{H}. \tag{3.58}$$

Combining (3.56)–(3.58) and shrinking the domain of integration in  $J_0(t)$ , we get

$$J_0(t) \geq \bar{B} e^{(-\frac{1}{2}+\nu)H\rho t} \int_0^{A(t)-a_0R-b_0\frac{c}{H}} \rho^{(n-1)[1-\frac{\rho}{2}]_+} \left(A(t) - \rho - R + \frac{2c}{H}\right)^{-1} d\rho$$

for  $t \geq A^{-1}(a_0R + b_0 \frac{c}{H})$ , where

$$\bar{B} \doteq \begin{cases} \left(\frac{2c}{H}\right)^{(-\frac{1}{2}+\nu)p} \left(\frac{a_0 - 1}{a_0 + 1}\right)^{\frac{n-1}{2}p} & \text{if } v \leq \frac{1}{2} \text{ and } R \leq \frac{c}{H}, \\ \left(\frac{2c}{H}\right)^{(-\frac{1}{2}+\nu)p} \left(\frac{a_0 - 1}{a_0 + 1}\right)^{\frac{n}{2}p} & \text{if } v \leq \frac{1}{2} \text{ and } R > \frac{c}{H}, \\ \left(\frac{c}{H}\right)^{(-\frac{1}{2}+\nu)p} \left(\frac{a_0 - 1}{a_0 + 1}\right)^{\frac{n-1}{2}p} & \text{if } v > \frac{1}{2} \text{ and } R \leq \frac{c}{H}, \\ \left(\frac{c}{2H}\right)^{(-\frac{1}{2}+\nu)p} \left(\frac{a_0 - 1}{a_0 + 1}\right)^{\frac{n-1}{2}p} & \text{if } v > \frac{1}{2} \text{ and } R > \frac{c}{H}. \end{cases} \tag{3.59}$$

Then, we further shrink the domain of integration in the right-hand side of the last inequality by increasing the bottom of the interval of integration from 0 to  $\delta(A(t) - a_0R - b_0 \frac{c}{H})$  for some  $\delta \in (0, 1)$ , obtaining for  $t \geq A^{-1}(a_0R + b_0 \frac{c}{H})$ ,

$$\begin{aligned} J_0(t) &\geq \bar{B} e^{(-\frac{1}{2}+\nu)H\rho t} \int_{\delta(A(t)-a_0R-b_0\frac{c}{H})}^{A(t)-a_0R-b_0\frac{c}{H}} \rho^{(n-1)[1-\frac{\rho}{2}]_+} \left(A(t) - \rho - R + \frac{2c}{H}\right)^{-1} d\rho \\ &\geq \frac{\bar{B} e^{(-\frac{1}{2}+\nu)H\rho t}}{\delta^{-(n-1)[1-\frac{\rho}{2}]_+}} (A(t) - a_0R - b_0 \frac{c}{H})^{(n-1)[1-\frac{\rho}{2}]_+} \int_{\delta(A(t)-a_0R-b_0\frac{c}{H})}^{A(t)-a_0R-b_0\frac{c}{H}} \frac{d\rho}{A(t) - \rho - R + \frac{2c}{H}} \\ &= \frac{\bar{B} e^{(-\frac{1}{2}+\nu)H\rho t}}{\delta^{-(n-1)[1-\frac{\rho}{2}]_+}} (A(t) - a_0R - b_0 \frac{c}{H})^{(n-1)[1-\frac{\rho}{2}]_+} \ln \left( \frac{A(t) + \frac{\delta a_0 - 1}{1 - \delta} R + \frac{\delta b_0 + 2}{1 - \delta} \frac{c}{H}}{\frac{a_0 - 1}{1 - \delta} R + \frac{b_0 + 2}{1 - \delta} \frac{c}{H}} \right). \end{aligned} \tag{3.60}$$

We emphasize that a more restrictive range for  $\delta$  is going to be prescribed in the inductive step.

Finally, plugging (3.60) in (3.55), we have proved the following result.

**Lemma III.3.** *Keeping the same notations as before, we assume that  $\delta \in (0, 1)$  and that the real parameters  $a_0, b_0$  fulfill*

$$a_0 \geq 2 \quad \text{and} \quad b_0 > 0. \tag{3.61}$$

Then, for  $t \geq A^{-1}(a_0R + b_0 \frac{c}{H})$ , we have

$$\begin{aligned} \|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p &\geq B_0 \epsilon^{pq} (1+t)^{c p} e^{-\frac{1}{2}(b+H)\rho t} (A(t) + R)^{-(n-1)[1-\frac{\rho}{2}]_+} \\ &\quad \times (A(t) - a_0R - b_0 \frac{c}{H})^{(n-1)[1-\frac{\rho}{2}]_+} \ln \left( \frac{A(t) + \frac{\delta a_0 - 1}{1 - \delta} R + \frac{\delta b_0 + 2}{1 - \delta} \frac{c}{H}}{\frac{a_0 - 1}{1 - \delta} R + \frac{b_0 + 2}{1 - \delta} \frac{c}{H}} \right), \end{aligned} \tag{3.62}$$

where  $B_0 = B_0(n, c, H, b, m^2, p, \mu, \nu_0, \nu_1, R, a_0, b_0, \delta) \doteq \delta^{(n-1)[1-\frac{\rho}{2}]_+} K \bar{B}^q \bar{B}^p \bar{B}$ .

The previous lower bound estimates for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  in Lemma III.3 will be the *base case* of the inductive argument.

We recall that the only assumptions on the parameters  $a_0$  and  $b_0$  that we did in order to obtain (3.62) are just the ones in (3.61). In the following, however, we will require a further condition on  $a_0$ ; cf. (3.63).

We stress that, since the amplitude function  $A$  for the light-cone grows exponentially, the logarithmic term in (3.62) provides, together with  $(1+t)^{sp}$ , a polynomially increasing factor. This factor constitutes the improvement with respect to the estimate in (2.3), provided that  $\varsigma \in (-\frac{1}{p}, 0]$ .

## 2. Inductive step

The main goal of Subsection III D is to prove that  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  satisfies the sequence of lower bound estimates given in the next lemma.

*Lemma III.4.* Keeping the same notations as before, we assume that  $\delta \in (\frac{1}{2}, 1)$  and that the real parameters  $a_0, b_0$  fulfill

$$a_0 \geq \max\left\{2, \frac{1}{2\delta - 1}\right\} \quad \text{and} \quad b_0 > 0, \tag{3.63}$$

and we consider the sequences  $\{a_j\}_{j \in \mathbb{N}}, \{b_j\}_{j \in \mathbb{N}}$  whose terms for  $j \in \mathbb{N}, j \geq 1$  are defined as

$$a_j \doteq (a_0 - 1) \left(\frac{4}{1 - \delta}\right)^j + 1, \tag{3.64}$$

$$b_j \doteq (b_0 + 2) \left(\frac{4}{1 - \delta}\right)^j - 2. \tag{3.65}$$

Then, for  $t \geq A^{-1}(a_j R + b_j \frac{c}{H})$ , we have

$$\begin{aligned} \|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p &\geq B_j \varepsilon^{pq^{j+1}} (1+t)^{sp \frac{q^{j+1}-1}{q-1}} e^{-\frac{1}{2}(b+H)pt} (A(t) + R)^{-(n-1)[1-\frac{\rho}{2}]} \\ &\quad \times (A(t) - a_j R - b_j \frac{c}{H})^{(n-1)[1-\frac{\rho}{2}]_+} \left( \ln \left( \frac{A(t) + \frac{\delta a_j - 1}{1-\delta} R + \frac{\delta b_j + 2}{1-\delta} \frac{c}{H}}{\frac{a_j - 1}{1-\delta} R + \frac{b_j + 2}{1-\delta} \frac{c}{H}} \right) \right)^{\frac{q^{j+1}-1}{q-1}}, \end{aligned} \tag{3.66}$$

where  $\{B_j\}_{j \in \mathbb{N}}$  is a sequence of positive real numbers whose terms satisfy the recursive relation

$$B_{j+1} \doteq D q^{-(j+1)} B_j^q \tag{3.67}$$

for any  $j \in \mathbb{N}$  and for a suitable positive constant  $D$  depending on  $n, c, H, b, m^2, \beta, p, a_0, \delta$ .

*Proof.* Clearly, we have already proved (3.66) for  $j = 0$ , namely, (3.62).

We are now going to prove (3.66) by induction. Let us assume that (3.66) holds for some  $j$ , with  $j \geq 0$ . We will prove (3.66) for  $j + 1$ , determining the value of  $B_{j+1}$  in terms of  $B_j$ . According to this goal, we plug (3.66) into the iteration frame in (3.49), obtaining

$$\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p \geq K B_j^q \varepsilon^{pq^{j+2}} e^{-(\frac{b}{2} + vH)pt} (A(t) + R)^{-(n-1)[1-\frac{\rho}{2}]_+} J_{j+1}(t), \tag{3.68}$$

where

$$J_{j+1}(t) \doteq \int_0^{A(t) - (2a_j + 1)R - 2b_j \frac{c}{H}} \rho^{(n-1)[1-\frac{\rho}{2}]_+} \frac{((\phi(t) + R_1)^2 - \rho^2)^{(-\frac{1}{2} + v)p}}{(A(t) + R - \rho)^{\frac{n-1}{2}p}} (I_{j+1}(t, \rho))^p d\rho \tag{3.69}$$

for  $t \geq A^{-1}((2a_j + 1)R + 2b_j \frac{c}{H})$  and

$$\begin{aligned} I_{j+1}(t, \rho) &\doteq \int_{A^{-1}(a_j R + b_j \frac{c}{H})}^{A^{-1}(\frac{1}{2}(A(t) - \rho - R))} \Gamma(s) (1+s)^{\varsigma \frac{q^{j+2}-q}{q-1}} e^{(\frac{b}{2} - vH - \frac{b+H}{2}q)s} \frac{(A(s) - a_j R - b_j \frac{c}{H})^{(n-1)(\beta+1)[1-\frac{\rho}{2}]_+}}{(A(s) + R)^{(n-1)(\beta+1)[1-\frac{\rho}{2}]_+}} \\ &\quad \times \left( \ln \left( \frac{A(s) + \frac{\delta a_j - 1}{1-\delta} R + \frac{\delta b_j + 2}{1-\delta} \frac{c}{H}}{\frac{a_j - 1}{1-\delta} R + \frac{b_j + 2}{1-\delta} \frac{c}{H}} \right) \right)^{(\beta+1) \frac{q^{j+1}-1}{q-1}} ds. \end{aligned} \tag{3.70}$$

Note that in (3.69), we shrank the  $\rho$ -domain of integration in order to have a nonempty  $s$ -domain of integration when using (3.66) to obtain  $I_{j+1}(t, \rho)$ .

We begin by deriving a lower bound estimate for  $I_{j+1}(t, \rho)$ . By using (3.53), we can rewrite the first three factors in  $I_{j+1}(t, \rho)$  as follows:

$$\begin{aligned} \Gamma(s) (1+s)^{\frac{q^{j+2}-q}{q-1}} e^{\left(\frac{b}{2}-vH-\frac{b+H}{2}q\right)s} &= \mu(1+s)^{\zeta+\frac{q^{j+2}-q}{q-1}} e^{\left(\rho_{crit}+\frac{b}{2}-vH-\frac{b+H}{2}q\right)s} \\ &= \mu(1+s)^{\frac{q^{j+2}-1}{q-1}} e^{\left(\frac{n}{2}-v-\frac{1}{p}\right)Hs} e^{-(n-1)H(\beta+1)\left(1-\frac{\rho}{2}\right)s}. \end{aligned}$$

By straightforward computations, we obtain that  $A(s) - a_j R - b_j \frac{c}{H} \geq \frac{1}{2} \phi(s)$  if and only if  $s \geq A^{-1}(2a_j R + (2b_j + 1) \frac{c}{H})$ , while  $A(s) + R \leq 2\phi(s)$  if and only if  $s \geq A^{-1}(R - \frac{2c}{H})$ . Therefore, shrinking the domain of integration in (3.70) to  $[A^{-1}(2a_j R + (2b_j + 1) \frac{c}{H}), A^{-1}(\frac{1}{2}(A(t) - \rho - R))]$  and working with  $\rho \in [0, A(t) - (4a_j + 1)R - 2(2b_j + 1) \frac{c}{H}]$  in (3.69), we may estimate

$$\frac{(A(s) - a_j R - b_j \frac{c}{H})^{(n-1)(\beta+1)[1-\frac{\rho}{2}]_+}}{(A(s) + R)^{(n-1)(\beta+1)[1-\frac{\rho}{2}]_-}} \geq 2^{-(n-1)(\beta+1)|1-\frac{\rho}{2}|} \left(\frac{c}{H} e^{Hs}\right)^{(n-1)(\beta+1)(1-\frac{\rho}{2})}.$$

Hence, for  $\rho \in [0, A(t) - (4a_j + 1)R - 2(2b_j + 1) \frac{c}{H}]$ , we have

$$I_{j+1}(t, \rho) \geq 2^{-(n-1)(\beta+1)|1-\frac{\rho}{2}|} \mu\left(\frac{c}{H}\right)^{(n-1)(\beta+1)(1-\frac{\rho}{2})} (1+t)^{\frac{q^{j+2}-1}{q-1}} \tilde{I}_{j+1}(t, \rho), \tag{3.71}$$

where

$$\tilde{I}_{j+1}(t, \rho) \doteq \int_{A^{-1}(2a_j R + (2b_j + 1) \frac{c}{H})}^{A^{-1}(\frac{1}{2}(A(t) - \rho - R))} e^{\left(\frac{n}{2}-v-\frac{1}{p}\right)Hs} \left(\ln\left(\frac{A(s) + \frac{\delta a_j - 1}{1-\delta} R + \frac{\delta b_j + 2}{1-\delta} \frac{c}{H}}{\frac{a_j - 1}{1-\delta} R + \frac{b_j + 2}{1-\delta} \frac{c}{H}}\right)\right)^{(\beta+1) \frac{q^{j+1}-1}{q-1}} ds. \tag{3.72}$$

In order to get a lower bound estimate for  $\tilde{I}_{j+1}(t, \rho)$ , we further shrink the domain of integration with respect to  $s$ . Since the inequality  $A^{-1}(\frac{1}{4}(A(t) - \rho - R)) \geq A^{-1}(2a_j R + (2b_j + 1) \frac{c}{H})$  holds for  $\rho \in [0, A(t) - (8a_j + 1)R - 4(2b_j + 1) \frac{c}{H}]$ , we may reduce the domain of integration in (3.72) to  $[A^{-1}(\frac{1}{4}(A(t) - \rho - R)), A^{-1}(\frac{1}{2}(A(t) - \rho - R))]$  for  $\rho$  in this interval.

Thus, for  $\rho \in [0, A(t) - (8a_j + 1)R - 4(2b_j + 1) \frac{c}{H}]$ , we get

$$\begin{aligned} \tilde{I}_{j+1}(t, \rho) &\geq \int_{A^{-1}(\frac{1}{4}(A(t) - \rho - R))}^{A^{-1}(\frac{1}{2}(A(t) - \rho - R))} e^{\left(\frac{n}{2}-v-\frac{1}{p}\right)Hs} \left(\ln\left(\frac{A(s) + \frac{\delta a_j - 1}{1-\delta} R + \frac{\delta b_j + 2}{1-\delta} \frac{c}{H}}{\frac{a_j - 1}{1-\delta} R + \frac{b_j + 2}{1-\delta} \frac{c}{H}}\right)\right)^{(\beta+1) \frac{q^{j+1}-1}{q-1}} ds \\ &\geq \left(\ln\left(\frac{A(t) - \rho + \left(\frac{4(\delta a_j - 1)}{1-\delta} - 1\right)R + \frac{4(\delta b_j + 2)}{1-\delta} \frac{c}{H}}{\frac{4(a_j - 1)}{1-\delta} R + \frac{4(b_j + 2)}{1-\delta} \frac{c}{H}}\right)\right)^{(\beta+1) \frac{q^{j+1}-1}{q-1}} \hat{I}_{j+1}(t, \rho) \end{aligned} \tag{3.73}$$

with

$$\begin{aligned} \hat{I}_{j+1}(t, \rho) &\doteq \int_{A^{-1}(\frac{1}{4}(A(t) - \rho - R))}^{A^{-1}(\frac{1}{2}(A(t) - \rho - R))} e^{\left(\frac{n}{2}-v-\frac{1}{p}\right)Hs} ds \\ &= \frac{(A(t) - \rho - R + \frac{2c}{H})^{\frac{n}{2}-v-\frac{1}{p}}}{\left(\frac{n}{2} - v - \frac{1}{p}\right)H\left(\frac{2c}{H}\right)^{\frac{n}{2}-v-\frac{1}{p}}} \left(1 - e^{\left(\frac{n}{2}-v-\frac{1}{p}\right)H(A^{-1}(\frac{1}{4}(A(t) - \rho - R)) - A^{-1}(\frac{1}{2}(A(t) - \rho - R)))}\right), \end{aligned} \tag{3.74}$$

where we used (3.54) to get

$$e^{\left(\frac{n}{2}-v-\frac{1}{p}\right)HA^{-1}(\frac{1}{2}(A(t) - \rho - R))} = \left(\frac{H}{2c}\right)^{\frac{n}{2}-v-\frac{1}{p}} \left(A(t) - \rho - R + \frac{2c}{H}\right)^{\frac{n}{2}-v-\frac{1}{p}}.$$

We estimate now the last factor in (3.74) for  $\rho \in [0, A(t) - (8a_j + 1)R - 4(2b_j + 1) \frac{c}{H}]$  as follows:

$$\begin{aligned}
 & 1 - \exp\left(\left(\frac{n}{2} - \nu - \frac{1}{p}\right)H\left(A^{-1}\left(\frac{1}{4}(A(t) - \rho - R)\right) - A^{-1}\left(\frac{1}{2}(A(t) - \rho - R)\right)\right)\right) \\
 &= 1 - \exp\left(\left(\frac{n}{2} - \nu - \frac{1}{p}\right)\left(\ln\left(\frac{\frac{H}{4c}(A(t) - \rho - R) + 1}{\frac{H}{2c}(A(t) - \rho - R) + 1}\right)\right)\right) \\
 &= 1 - \exp\left(\left(\frac{n}{2} - \nu - \frac{1}{p}\right)\left(\ln\left(\frac{1}{2} + \frac{\frac{c}{H}}{A(t) - \rho - R + \frac{2c}{H}}\right)\right)\right) \\
 &\geq 1 - \exp\left(\left(\frac{n}{2} - \nu - \frac{1}{p}\right)\left(\ln\left(\frac{1}{2} + \frac{\frac{c}{H}}{8a_jR + 2(4b_j + 3)\frac{c}{H}}\right)\right)\right). \tag{3.75}
 \end{aligned}$$

Let us denote by  $d_j = d_j(n, c, H, \nu, p, R, a_0, b_0)$  the right-hand side of (3.75). Since  $\{d_j\}_{j \in \mathbb{N}}$  is an increasing and bounded sequence, we can simply estimate the left-hand side of (3.75) from below with  $d_0$ . Summarizing, we proved that

$$\widehat{I}_{j+1}(t, \rho) \geq \frac{d_0}{\left(\frac{n}{2} - \nu - \frac{1}{p}\right)H\left(\frac{2c}{H}\right)^{\frac{n}{2} - \nu - \frac{1}{p}}} \left(A(t) - \rho - R + \frac{2c}{H}\right)^{\frac{n}{2} - \nu - \frac{1}{p}}.$$

Hence, combining (3.73) and the last inequality, we conclude that

$$\widetilde{I}_{j+1}(t, \rho) \geq \widetilde{C}\left(A(t) - \rho - R + \frac{2c}{H}\right)^{\frac{n}{2} - \nu - \frac{1}{p}} \left(\ln\left(\frac{A(t) - \rho + \left(\frac{4(\delta a_j - 1) - 1}{1 - \delta}\right)R + \frac{4(\delta b_j + 2)\frac{c}{H}}{1 - \delta}}{\frac{4(a_j - 1)}{1 - \delta}R + \frac{4(b_j + 2)\frac{c}{H}}{1 - \delta}}\right)\right)^{(\beta + 1)\frac{q^{j+1} - 1}{q - 1}}$$

for  $\rho \in [0, A(t) - (8a_j + 1)R - 4(2b_j + 1)\frac{c}{H}]$ , where  $\widetilde{C} \doteq \left(\frac{n}{2} - \nu - \frac{1}{p}\right)^{-1} H^{-1}\left(\frac{H}{2c}\right)^{\frac{n}{2} - \nu - \frac{1}{p}} d_0$ .

Then, plugging the last lower bound for  $\widetilde{I}_{j+1}(t, \rho)$  in (3.71), we arrive at

$$I_{j+1}(t, \rho) \geq \widehat{C}(1 + t)^{\zeta\frac{q^{j+2} - 1}{q - 1}} \left(A(t) - \rho - R + \frac{2c}{H}\right)^{\frac{n}{2} - \nu - \frac{1}{p}} \left(\ln\left(\frac{A(t) - \rho + \left(\frac{4(\delta a_j - 1) - 1}{1 - \delta}\right)R + \frac{4(\delta b_j + 2)\frac{c}{H}}{1 - \delta}}{\frac{4(a_j - 1)}{1 - \delta}R + \frac{4(b_j + 2)\frac{c}{H}}{1 - \delta}}\right)\right)^{(\beta + 1)\frac{q^{j+1} - 1}{q - 1}}$$

for  $\rho \in [0, A(t) - (8a_j + 1)R - 4(2b_j + 1)\frac{c}{H}]$ , where  $\widehat{C} \doteq 2^{-(n-1)(\beta+1)|1-\frac{\rho}{H}|} \mu\left(\frac{c}{H}\right)^{(n-1)(\beta+1)(1-\frac{\rho}{H})} \widetilde{C}$ .

Next, after shrinking the domain of integration with respect to  $\rho$  as we described in the previous steps, we plug the obtained lower bound for  $I_{j+1}(t, \rho)$  in (3.69), obtaining

$$\begin{aligned}
 J_{j+1}(t) &\geq \widehat{C}^p (1 + t)^{\zeta p \frac{q^{j+2} - 1}{q - 1}} \int_0^{A(t) - (8a_j + 1)R - 4(2b_j + 1)\frac{c}{H}} \rho^{(n-1)[1-\frac{\rho}{H}]_+} \frac{((\phi(t) + R_1)^2 - \rho^2)^{(-\frac{1}{2} + \nu)p}}{(A(t) + R - \rho)^{\frac{n-1}{2}p}} \left(A(t) - \rho - R + \frac{2c}{H}\right)^{\left(\frac{n}{2} - \nu\right)p - 1} \\
 &\quad \times \left(\ln\left(\frac{A(t) - \rho + \left(\frac{4(\delta a_j - 1) - 1}{1 - \delta}\right)R + \frac{4(\delta b_j + 2)\frac{c}{H}}{1 - \delta}}{\frac{4(a_j - 1)}{1 - \delta}R + \frac{4(b_j + 2)\frac{c}{H}}{1 - \delta}}\right)\right)^{\frac{q^{j+2} - q}{q - 1}} d\rho.
 \end{aligned}$$

Since the sequences  $\{a_j\}_{j \in \mathbb{N}}$ ,  $\{b_j\}_{j \in \mathbb{N}}$  are strictly increasing, we may apply the same estimates for the factors in the middle line of the previous inequality that we used in the base case  $j = 0$ , namely, (3.56)–(3.58), arriving at

$$\begin{aligned}
 J_{j+1}(t) &\geq \widetilde{B}\widehat{C}^p (1 + t)^{\zeta p \frac{q^{j+2} - 1}{q - 1}} e^{(-\frac{1}{2} + \nu)Hpt} \int_0^{A(t) - (8a_j + 1)R - 4(2b_j + 1)\frac{c}{H}} \frac{\rho^{(n-1)[1-\frac{\rho}{H}]_+}}{(A(t) - \rho - R + \frac{2c}{H})} \\
 &\quad \times \left(\ln\left(\frac{A(t) - \rho + \left(\frac{4(\delta a_j - 1) - 1}{1 - \delta}\right)R + \frac{4(\delta b_j + 2)\frac{c}{H}}{1 - \delta}}{\frac{4(a_j - 1)}{1 - \delta}R + \frac{4(b_j + 2)\frac{c}{H}}{1 - \delta}}\right)\right)^{\frac{q^{j+2} - q}{q - 1}} d\rho \tag{3.76}
 \end{aligned}$$

for  $t \geq A^{-1}\left((8a_j + 1)R + 4(2b_j + 1)\frac{c}{H}\right)$ , where  $\widetilde{B}$  is defined in (3.59). We denote by  $\widetilde{J}_{j+1}(t)$  the  $\rho$ -integral in the left-hand side of the last inequality.

We proceed with the lower bound estimate for  $\tilde{J}_{j+1}(t)$  by shrinking the domain of integration. Consequently, for  $t \geq A^{-1}((8a_j + 1)R + 4(2b_j + 1)\frac{c}{H})$ , we obtain

$$\begin{aligned} \tilde{J}_{j+1}(t) &\geq \int_{\delta(A(t)-(8a_j+1)R-4(2b_j+1)\frac{c}{H})}^{A(t)-(8a_j+1)R-4(2b_j+1)\frac{c}{H}} \frac{\rho^{(n-1)[1-\frac{\rho}{H}]_+}}{\delta(A(t)-(8a_j+1)R-4(2b_j+1)\frac{c}{H})} \left( \ln \left( \frac{A(t) - \rho + \left(\frac{4(\delta a_j - 1)}{1-\delta} - 1\right)R + \frac{4(\delta b_j + 2)}{1-\delta} \frac{c}{H}}{\frac{4(a_j-1)}{1-\delta}R + \frac{4(b_j+2)}{1-\delta} \frac{c}{H}} \right) \right)^{\frac{j+2-q}{q-1}} d\rho \\ &\geq \delta^{(n-1)[1-\frac{\rho}{H}]_+} \left( A(t) - (8a_j + 1)R - 4(2b_j + 1)\frac{c}{H} \right) \widehat{J}_{j+1}(t), \end{aligned} \tag{3.77}$$

where  $\widehat{J}_{j+1}(t)$  denotes the  $\rho$ -integral with the remaining factors.

Our goal now is to derive a lower bound estimate for  $\widehat{J}_{j+1}(t)$  in a such way that the power of the logarithmic factor is increased by 1. According to this purpose, we decrease the argument of the logarithmic term as follows:

$$\begin{aligned} \frac{A(t) - \rho + \left(\frac{4(\delta a_j - 1)}{1-\delta} - 1\right)R + \frac{4(\delta b_j + 2)}{1-\delta} \frac{c}{H}}{\frac{4(a_j-1)}{1-\delta}R + \frac{4(b_j+2)}{1-\delta} \frac{c}{H}} &\geq \frac{A(t) - \rho + \left(\frac{4(\delta a_j - 1)}{1-\delta} - 1 - 4a_j\right)R + \left(\frac{4(\delta b_j + 2)}{1-\delta} - 4(b_j + 1)\right) \frac{c}{H}}{\frac{4(a_j-1)}{1-\delta}R + \frac{4(b_j+2)}{1-\delta} \frac{c}{H}} \\ &= \frac{A(t) - \rho + \frac{4(2\delta-1)a_j-5+\delta}{1-\delta}R + \frac{4(2\delta-1)b_j+4+4\delta}{1-\delta} \frac{c}{H}}{\frac{4(a_j-1)}{1-\delta}R + \frac{4(b_j+2)}{1-\delta} \frac{c}{H}}, \end{aligned}$$

and we use the inequality

$$A(t) - \rho - R + \frac{2c}{H} \leq A(t) - \rho + \frac{4(2\delta - 1)a_j - 5 + \delta}{1 - \delta}R + \frac{4(2\delta - 1)b_j + 4 + 4\delta}{1 - \delta} \frac{c}{H}. \tag{3.78}$$

In particular, we can guarantee the validity of (3.78) if we assume that  $\delta > \frac{1}{2}$  and that  $a_j, b_j$  fulfill the following conditions:

$$a_j \geq \frac{1}{2\delta - 1}, \quad b_j \geq -\frac{(1 + 3\delta)}{2(2\delta - 1)}. \tag{3.79}$$

The condition for  $b_j$  in (3.79) is trivially true for any  $j \in \mathbb{N}$  since  $\{b_j\}_{j \in \mathbb{N}}$  is a sequence of positive real numbers. On the other hand, the condition for  $a_j$  in (3.79) is satisfied for any  $j \in \mathbb{N}$  if and only if it is satisfied for  $j = 0$ , being  $\{a_j\}_{j \in \mathbb{N}}$  an increasing sequence. Thanks to the condition  $a_0 \geq \frac{1}{2\delta-1}$  in (3.63), it follows the validity of (3.79), which guarantees in turn that (3.78) holds true. In conclusion, we proceed with the estimate from below for  $\widehat{J}_{j+1}(t)$  as follows:

$$\begin{aligned} \widehat{J}_{j+1}(t) &\geq \int_{\delta(A(t)-(8a_j+1)R-4(2b_j+1)\frac{c}{H})}^{A(t)-(8a_j+1)R-4(2b_j+1)\frac{c}{H}} \left( A(t) - \rho + \frac{4(2\delta - 1)a_j - 5 + \delta}{1 - \delta}R + \frac{4(2\delta - 1)b_j + 4 + 4\delta}{1 - \delta} \frac{c}{H} \right)^{-1} \\ &\quad \times \left( \ln \left( \frac{A(t) - \rho + \frac{4(2\delta-1)a_j-5+\delta}{1-\delta}R + \frac{4(2\delta-1)b_j+4+4\delta}{1-\delta} \frac{c}{H}}{\frac{4(a_j-1)}{1-\delta}R + \frac{4(b_j+2)}{1-\delta} \frac{c}{H}} \right) \right)^{\frac{j+2-q}{q-1}} d\rho \\ &= \frac{q-1}{q^{j+2}-1} \left( \ln \left( \frac{A(t) + \left(\frac{4(2\delta-1)a_j-5+\delta}{(1-\delta)^2} + \frac{\delta(8a_j+1)}{1-\delta}\right)R + \left(\frac{4(2\delta-1)b_j+4+4\delta}{(1-\delta)^2} + \frac{4\delta(2b_j+1)}{1-\delta}\right) \frac{c}{H}}{\frac{4(a_j-1)}{(1-\delta)^2}R + \frac{4(b_j+2)}{(1-\delta)^2} \frac{c}{H}} \right) \right)^{\frac{j+2-1}{q-1}}, \end{aligned}$$

$t \geq A^{-1}((8a_j + 1)R + 4(2b_j + 1)\frac{c}{H})$ . Having in mind (3.66) for  $j + 1$ , according to the terms appearing in the denominator of the argument of the logarithmic term in the previous estimate, we set

$$\frac{a_{j+1} - 1}{1 - \delta} = \frac{4(a_j - 1)}{(1 - \delta)^2}, \quad \frac{b_{j+1} + 2}{1 - \delta} = \frac{4(b_j + 2)}{(1 - \delta)^2}.$$

The previous two conditions are equivalent to define

$$a_{j+1} = \frac{4}{1 - \delta}a_j - \frac{3 + \delta}{1 - \delta}, \quad b_{j+1} = \frac{4}{1 - \delta}b_j + \frac{2(3 + \delta)}{1 - \delta}. \tag{3.80}$$

Using iteratively (3.80), we find exactly the representations for  $a_j$  and  $b_j$  given in (3.64) and (3.65), respectively.

Moreover, by straightforward computations, we find that the conditions

$$\begin{cases} \frac{4(2\delta - 1)a_j - 5 + \delta}{(1 - \delta)^2} + \frac{\delta(8a_j + 1)}{1 - \delta} \geq \frac{\delta a_{j+1} - 1}{1 - \delta}, \\ \frac{4(2\delta - 1)b_j + 4 + 4\delta}{(1 - \delta)^2} + \frac{4\delta(2b_j + 1)}{1 - \delta} \geq \frac{\delta b_{j+1} + 2}{1 - \delta} \end{cases}$$

are satisfied if and only if the inequalities in (3.79) hold. Therefore, thanks to (3.63), we can continue the lower bound estimate for  $\widehat{J}_{j+1}(t)$ , obtaining  $t \geq A^{-1}((8a_j + 1)R + 4(2b_j + 1)\frac{c}{H})$ ,

$$\widehat{J}_{j+1}(t) \geq (q - 1)q^{-(j+2)} \left( \ln \left( \frac{A(t) + \frac{\delta a_{j+1} - 1}{1 - \delta} R + \frac{\delta b_{j+1} + 2}{1 - \delta} \frac{c}{H}}{\frac{a_{j+1} - 1}{1 - \delta} R + \frac{b_{j+1} + 2}{1 - \delta} \frac{c}{H}} \right) \right)^{\frac{q^{j+2} - 1}{q - 1}}.$$

Combining the lower bound estimates for  $\widehat{J}_{j+1}(t)$  and  $\widetilde{J}_{j+1}(t)$ , we get

$$\widetilde{J}_{j+1}(t) \geq \delta^{(n-1)[1 - \frac{\ell}{2}]_+} (q - 1)q^{-(j+2)} \left( A(t) - a_{j+1}R - b_{j+1}\frac{c}{H} \right)^{(n-1)[1 - \frac{\ell}{2}]_+} \left( \ln \left( \frac{A(t) + \frac{\delta a_{j+1} - 1}{1 - \delta} R + \frac{\delta b_{j+1} + 2}{1 - \delta} \frac{c}{H}}{\frac{a_{j+1} - 1}{1 - \delta} R + \frac{b_{j+1} + 2}{1 - \delta} \frac{c}{H}} \right) \right)^{\frac{q^{j+2} - 1}{q - 1}},$$

where we used the relations  $a_{j+1} \geq 8a_j + 1$ ,  $b_{j+1} \geq 4(2b_j + 1)$  [which turn out to be equivalent to the conditions in (3.79)] to lower the first time-dependent factor on the right-hand side that comes from (3.77).

Finally, putting together the last estimate for  $\widetilde{J}_{j+1}(t)$ , (3.76), and (3.68), we conclude

$$\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p \geq B_{j+1} \varepsilon^{p q^{j+2}} (1 + t)^{c p \frac{q^{j+2} - 1}{q - 1}} e^{-\frac{1}{2}(b+H)pt} \frac{(A(t) - a_{j+1}R - b_{j+1}\frac{c}{H})^{(n-1)[1 - \frac{\ell}{2}]_+}}{(A(t) + R)^{(n-1)[1 - \frac{\ell}{2}]_-}} \times \left( \ln \left( \frac{A(t) + \frac{\delta a_{j+1} - 1}{1 - \delta} R + \frac{\delta b_{j+1} + 2}{1 - \delta} \frac{c}{H}}{\frac{a_{j+1} - 1}{1 - \delta} R + \frac{b_{j+1} + 2}{1 - \delta} \frac{c}{H}} \right) \right)^{\frac{q^{j+2} - 1}{q - 1}}, \quad (3.81)$$

where  $B_{j+1}$  is defined in (3.67) with  $D = D(n, c, H, b, m^2, \beta, p, a_0, \delta) \doteq K\widehat{B}\widehat{C}^p \delta^{(n-1)[1 - \frac{\ell}{2}]_+} (q - 1)q^{-1}$ .

Hence, we proved (3.66) for  $j + 1$ , which is exactly (3.81) with  $B_{j+1}$  given by (3.67).  $\square$

Finally, for our future considerations, it is convenient to derive an explicit representation of  $\ln B_j$ . Applying the logarithmic function to both sides of (3.67) and using the resulting identity in an iterative way, we find

$$\begin{aligned} \ln B_j &= q \ln B_{j-1} - j \ln q + \ln D = q^2 \ln B_{j-2} - (j + (j - 1)q) \ln q + (1 + q) \ln D \\ &= \dots = q^j \ln B_0 - \left( \sum_{k=0}^{j-1} (j - k)q^k \right) \ln q + \left( \sum_{k=0}^{j-1} q^k \right) \ln D \\ &= q^j \ln B_0 - \frac{1}{q - 1} \left( \frac{q^{j+1} - 1}{q - 1} - (j + 1) \right) \ln q + \frac{q^j - 1}{q - 1} \ln D \\ &= q^j \left( \ln B_0 - \frac{q \ln q}{(q - 1)^2} + \frac{\ln D}{q - 1} \right) + (j + 1) \frac{\ln q}{q - 1} + \frac{\ln q}{(q - 1)^2} - \frac{\ln D}{q - 1} \\ &= q^j \ln E + (j + 1) \frac{\ln q}{q - 1} + \frac{\ln q}{(q - 1)^2} - \frac{\ln D}{q - 1}, \end{aligned} \quad (3.82)$$

where  $E \doteq B_0 q^{-q/(q-1)^2} D^{1/(q-1)}$ .

### E. Improved lower bound estimates for the spatial average of the solution

In Subsection III D 2, we derived the sequence of lower bound estimates for  $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$  in (3.66). Now, we are going to use (3.66) to derive a sequence of lower bound estimates for the spatial average  $V(t)$ .

Indeed, plugging (3.66) in (2.2), we have

$$V(t) \geq B_j^{\beta+1} \varepsilon^{q^{j+2}} e^{-\alpha_2 t} \int_{A^{-1}(a_j R + b_j \frac{c}{H})}^t e^{(\alpha_2 - \alpha_1)s} \int_{A^{-1}(a_j R + b_j \frac{c}{H})}^s e^{(\alpha_1 - \frac{1}{2}(b+H)q)\tau} \Gamma(\tau) \\ \times (1 + \tau) \zeta^{\frac{j+2-q}{q-1}} \frac{(A(\tau) - a_j R - b_j \frac{c}{H})^{(n-1)(\beta+1)[1-\frac{\ell}{2}]_+}}{(A(\tau) + R)^{(n-1)(\beta+1)[1-\frac{\ell}{2}]_-}} \left( \ln \left( \frac{A(\tau) + \frac{\delta a_j - 1}{1-\delta} R + \frac{\delta b_j + 2}{1-\delta} \frac{c}{H}}{\frac{a_j - 1}{1-\delta} R + \frac{b_j + 2}{1-\delta} \frac{c}{H}} \right) \right)^{(\beta+1) \frac{j+1-1}{q-1}} d\tau ds$$

for  $t \geq A^{-1}(a_j R + b_j \frac{c}{H})$ .

Since  $A(\tau) - a_j R - b_j \frac{c}{H} \geq \frac{1}{2}\phi(\tau)$  for  $\tau \geq A^{-1}(2a_j R + (2b_j + 1)\frac{c}{H})$  and  $A(\tau) + R \leq 2\phi(\tau)$ , provided that  $\tau \geq A^{-1}(R - \frac{2c}{H})$ , we have that

$$\frac{(A(\tau) - a_j R - b_j \frac{c}{H})^{(n-1)(\beta+1)[1-\frac{\ell}{2}]_+}}{(A(\tau) + R)^{(n-1)(\beta+1)[1-\frac{\ell}{2}]_-}} \geq 2^{-(n-1)(\beta+1)|1-\frac{\ell}{2}|} \left( \frac{c}{H} \right)^{(n-1)(\beta+1)(1-\frac{\ell}{2})} e^{-(n-1)H(\beta+1)(1-\frac{\ell}{2})\tau}$$

for  $\tau \geq A^{-1}(2a_j R + (2b_j + 1)\frac{c}{H})$ . Therefore,

$$V(t) \geq 2^{-(n-1)(\beta+1)|1-\frac{\ell}{2}|} \mu \left( \frac{c}{H} \right)^{(n-1)(\beta+1)(1-\frac{\ell}{2})} B_j^{\beta+1} \varepsilon^{q^{j+2}} e^{-\alpha_2 t} \int_{A^{-1}(2a_j R + (2b_j + 1)\frac{c}{H})}^t e^{(\alpha_2 - \alpha_1)s} \\ \times \int_{A^{-1}(2a_j R + (2b_j + 1)\frac{c}{H})}^s (1 + \tau) \zeta^{\frac{j+2-1}{q-1}} e^{(\frac{n}{2} + \nu - \frac{1}{p})H\tau} \left( \ln \left( \frac{A(\tau) + \frac{\delta a_j - 1}{1-\delta} R + \frac{\delta b_j + 2}{1-\delta} \frac{c}{H}}{\frac{a_j - 1}{1-\delta} R + \frac{b_j + 2}{1-\delta} \frac{c}{H}} \right) \right)^{(\beta+1) \frac{j+1-1}{q-1}} d\tau ds$$

for  $t \geq A^{-1}(2a_j R + (2b_j + 1)\frac{c}{H})$ , where we used

$$e^{(\alpha_1 - \frac{1}{2}(b+H)q + (n-1)(\beta+1)(1-\frac{\ell}{2}))\tau} \Gamma(\tau) (1 + \tau) \zeta^{\frac{j+2-q}{q-1}} \\ = \mu (1 + \tau) \zeta^{\frac{j+2-1}{q-1}} e^{(\alpha_1 + \varrho_{\text{crit}} - \frac{1}{2}(b+H)q + (n-1)H(\beta+1)(1-\frac{\ell}{2}))\tau} \\ = \mu (1 + \tau) \zeta^{\frac{j+2-1}{q-1}} e^{(\alpha_1 + \frac{\nu H}{2} - \frac{b}{2} - \frac{H}{p})\tau} = \mu (1 + \tau) \zeta^{\frac{j+2-1}{q-1}} e^{(\frac{n}{2} + \nu - \frac{1}{p})H\tau}$$

with  $\varrho_{\text{crit}}$  given by (1.4), and for  $\alpha_1, \alpha_2$ , we employ the same notations as in Lemma III.1, that is,  $\alpha_1 = \frac{b}{2} + \nu H$  and  $\alpha_2 = \frac{b}{2} - \nu H$ . Since we are working with  $\zeta \leq 0$ , we have

$$V(t) \geq \tilde{N} B_j^{\beta+1} \varepsilon^{q^{j+2}} (1 + t) \zeta^{\frac{j+2-1}{q-1}} e^{-\alpha_2 t} \int_{A^{-1}(2a_j R + (2b_j + 1)\frac{c}{H})}^t e^{(\alpha_2 - \alpha_1)s} \times \int_{A^{-1}(2a_j R + (2b_j + 1)\frac{c}{H})}^s e^{(\frac{n}{2} + \nu - \frac{1}{p})H\tau} \left( \ln \left( \frac{A(\tau) + \frac{\delta a_j - 1}{1-\delta} R + \frac{\delta b_j + 2}{1-\delta} \frac{c}{H}}{\frac{a_j - 1}{1-\delta} R + \frac{b_j + 2}{1-\delta} \frac{c}{H}} \right) \right)^{(\beta+1) \frac{j+1-1}{q-1}} d\tau ds \quad (3.83)$$

for  $t \geq A^{-1}(2a_j R + (2b_j + 1)\frac{c}{H})$ , where  $\tilde{N} \doteq 2^{-(n-1)(\beta+1)|1-\frac{\ell}{2}|} \mu \left( \frac{c}{H} \right)^{(n-1)(\beta+1)(1-\frac{\ell}{2})}$ . We now focus on the lower bound estimate for the  $\tau$ -integral in the right-hand side of the last estimate. For  $s \geq A^{-1}(4a_j R + 2(2b_j + 1)\frac{c}{H})$ , we may shrink the domain of integration to  $[A^{-1}(A(s)/2), s]$ , obtaining

$$M(s) \doteq \int_{A^{-1}(\frac{A(s)}{2})}^s e^{(\frac{n}{2} + \nu - \frac{1}{p})H\tau} \left( \ln \left( \frac{A(\tau) + \frac{\delta a_j - 1}{1-\delta} R + \frac{\delta b_j + 2}{1-\delta} \frac{c}{H}}{\frac{a_j - 1}{1-\delta} R + \frac{b_j + 2}{1-\delta} \frac{c}{H}} \right) \right)^{(\beta+1) \frac{j+1-1}{q-1}} d\tau \\ \geq \left( \ln \left( \frac{A(s) + \frac{2(\delta a_j - 1)}{1-\delta} R + \frac{2(\delta b_j + 2)}{1-\delta} \frac{c}{H}}{\frac{2(a_j - 1)}{1-\delta} R + \frac{2(b_j + 2)}{1-\delta} \frac{c}{H}} \right) \right)^{(\beta+1) \frac{j+1-1}{q-1}} \int_{A^{-1}(\frac{A(s)}{2})}^s e^{(\frac{n}{2} + \nu - \frac{1}{p})H\tau} d\tau \\ = \frac{e^{(\frac{n}{2} + \nu - \frac{1}{p})Hs}}{\left( \frac{n}{2} + \nu - \frac{1}{p} \right) H} \left( \ln \left( \frac{A(s) + \frac{2(\delta a_j - 1)}{1-\delta} R + \frac{2(\delta b_j + 2)}{1-\delta} \frac{c}{H}}{\frac{2(a_j - 1)}{1-\delta} R + \frac{2(b_j + 2)}{1-\delta} \frac{c}{H}} \right) \right)^{(\beta+1) \frac{j+1-1}{q-1}} \left( 1 - e^{-\left( \frac{n}{2} + \nu - \frac{1}{p} \right) H \left( s - A^{-1}(\frac{A(s)}{2}) \right)} \right).$$

By elementary computations, we find

$$s - A^{-1}\left(\frac{A(s)}{2}\right) = -\frac{1}{H} \ln(1 + e^{-Hs}) + \frac{1}{H} \ln 2$$

so that we may estimate for  $s \geq A^{-1}(4a_j R + 2(2b_j + 1)\frac{c}{H})$  the last factor in the lower bound for  $M(s)$  as follows:

$$\begin{aligned} 1 - e^{-\left(\frac{n}{2} + \nu - \frac{1}{p}\right)H\left(s - A^{-1}\left(\frac{A(s)}{2}\right)\right)} &= 1 - \left(\frac{1 + e^{-Hs}}{2}\right)^{\left(\frac{n}{2} + \nu - \frac{1}{p}\right)} \\ &\geq 1 - \left(\frac{1}{2} + \frac{c}{2H}\left(4a_j R + (4b_j + 3)\frac{c}{H}\right)^{-1}\right)^{\left(\frac{n}{2} + \nu - \frac{1}{p}\right)} \doteq \gamma_j. \end{aligned}$$

Clearly,  $\{\gamma_j\}_{j \in \mathbb{N}}$  is an increasing sequence of positive real numbers and  $\lim_{j \rightarrow \infty} \gamma_j = 1 - 2^{-\left(\frac{n}{2} + \nu - \frac{1}{p}\right)}$ . Combining the previous considerations, we proved that the  $s$ -integral in (3.83) can be estimate from below by

$$\tilde{M}(t) \doteq \frac{\gamma_j}{\left(\frac{n}{2} + \nu - \frac{1}{p}\right)H} \int_{A^{-1}(4a_j R + 2(2b_j + 1)\frac{c}{H})}^t e^{\left(\frac{n}{2} - \nu - \frac{1}{p}\right)Hs} \left( \ln \left( \frac{A(s) + \frac{2(\delta a_j - 1)R + \frac{2(\delta b_j + 2)c}{1-\delta}}{1-\delta}}{\frac{2(a_j - 1)R + \frac{2(b_j + 2)c}{1-\delta}}{1-\delta}} \right) \right)^{(\beta+1)\frac{q^{j+1}-1}{q-1}} ds,$$

where we used  $\alpha_2 - \alpha_1 = -2\nu H$ .

For  $t \geq A^{-1}(8a_j R + 4(2b_j + 1)\frac{c}{H})$ , we increase the bottom of the interval of integration so that the domain of the integral is reduced to  $[A^{-1}(A(t)/2), t]$ . Hence, for  $t \geq A^{-1}(8a_j R + 4(2b_j + 1)\frac{c}{H})$ , we have

$$\tilde{M}(t) \geq \frac{\gamma_j}{\left(\frac{n}{2} + \nu - \frac{1}{p}\right)H} \left( \ln \left( \frac{A(t) + \frac{4(\delta a_j - 1)R + \frac{4(\delta b_j + 2)c}{1-\delta}}{1-\delta}}{\frac{4(a_j - 1)R + \frac{4(b_j + 2)c}{1-\delta}}{1-\delta}} \right) \right)^{(\beta+1)\frac{q^{j+1}-1}{q-1}} \int_{A^{-1}\left(\frac{A(t)}{2}\right)}^t e^{\left(\frac{n}{2} - \nu - \frac{1}{p}\right)Hs} ds.$$

We can estimate the integral in the last inequality for  $\tilde{M}(t)$  analogously to the corresponding term that appeared in the lower bound estimate for  $M(s)$ , that is, for  $t \geq A^{-1}(8a_j R + 4(2b_j + 1)\frac{c}{H})$ ,

$$\int_{A^{-1}\left(\frac{A(t)}{2}\right)}^t e^{\left(\frac{n}{2} - \nu - \frac{1}{p}\right)Hs} ds \geq \frac{\tilde{\gamma}_j}{\left(\frac{n}{2} - \nu - \frac{1}{p}\right)H} e^{\left(\frac{n}{2} - \nu - \frac{1}{p}\right)Ht},$$

where

$$\tilde{\gamma}_j \doteq 1 - \left(\frac{1}{2} + \frac{c}{2H}\left(8a_j R + (8b_j + 5)\frac{c}{H}\right)^{-1}\right)^{\left(\frac{n}{2} - \nu - \frac{1}{p}\right)}.$$

Summarizing, for  $t \geq A^{-1}(8a_j R + 4(2b_j + 1)\frac{c}{H})$ , we proved that

$$V(t) \geq \tilde{N} \gamma_j \tilde{\gamma}_j B_j^{\beta+1} \varepsilon^{q^{j+2}} e^{\left(\frac{nH}{2} - \frac{b}{2} - \frac{H}{p}\right)t} (1+t)^{\zeta \frac{q^{j+2}-1}{q-1}} \left( \ln \left( \frac{A(t) + \frac{4(\delta a_j - 1)R + \frac{4(\delta b_j + 2)c}{1-\delta}}{1-\delta}}{\frac{4(a_j - 1)R + \frac{4(b_j + 2)c}{1-\delta}}{1-\delta}} \right) \right)^{(\beta+1)\frac{q^{j+1}-1}{q-1}},$$

where  $\tilde{N} \doteq \tilde{N} \left( \left(\frac{n}{2} + \nu - \frac{1}{p}\right) \left(\frac{n}{2} - \nu - \frac{1}{p}\right) H^2 \right)^{-1}$ .

*Remark 10.* In the previous lower bound estimate for  $V(t)$ , the logarithmic term provides an improvement of the lower bound in comparison to what we would obtain if we worked with the same approach as in the proof of Ref. 1 (Theorem 1.9).

Since  $a_j \geq 2$  for any  $j \in \mathbb{N}$  and  $\delta \in (\frac{1}{2}, 1)$ , we find that  $\delta a_j - 1 \geq 0$  for any  $j \in \mathbb{N}$ , and consequently,

$$V(t) \geq \tilde{N} \gamma_j \tilde{\gamma}_j B_j^{\beta+1} \varepsilon^{q^{j+2}} e^{\left(\frac{nH}{2} - \frac{b}{2} - \frac{H}{p}\right)t} (1+t)^{\zeta \frac{q^{j+2}-1}{q-1}} \left( \ln \left( \frac{A(t)}{\frac{4(a_j - 1)R + \frac{4(b_j + 2)c}{1-\delta}}{1-\delta}} \right) \right)^{(\beta+1)\frac{q^{j+1}-1}{q-1}}$$

for  $t \geq A^{-1}(8a_j R + 4(2b_j + 1)\frac{c}{H})$ .

For  $t \geq A^{-1} \left( 2^4 \left( \frac{a_j - 1}{1 - \delta} R + \frac{b_j + 2}{1 - \delta} \frac{c}{H} \right)^2 \right)$ , it holds the following inequality:

$$\ln \left( \frac{A(t)}{4 \frac{(a_j - 1)}{1 - \delta} R + 4 \frac{(b_j + 2)}{1 - \delta} \frac{c}{H}} \right) \geq \frac{1}{2} \ln A(t).$$

Furthermore, the condition  $t \geq \frac{2}{H} \ln \left( \frac{H}{c} + 1 \right)$  implies that  $\ln A(t) \geq \frac{Ht}{2}$ , while for  $t \geq 1$ , we may estimate  $(1 + t)^c \geq 2^c t^c$ .  
Let us introduce

$$\sigma_j \doteq \max \left\{ A^{-1} \left( 8a_j R + 4(2b_j + 1) \frac{c}{H} \right), A^{-1} \left( 2^4 \left( \frac{a_j - 1}{1 - \delta} R + \frac{b_j + 2}{1 - \delta} \frac{c}{H} \right)^2 \right), \frac{2}{H} \ln \left( \frac{H}{c} + 1 \right), 1 \right\}. \quad (3.84)$$

Combining the previous estimates, for  $t \geq \sigma_j$ , we obtained the following lower bound estimate for  $V(t)$ :

$$V(t) \geq N Q^{q^{j+1}} \gamma_j \tilde{\gamma}_j B_j^{\beta+1} \varepsilon^{q^{j+2}} t^{-\frac{c+\beta+1}{q-1}} t^{\frac{q^{j+1}}{q-1} (\beta+1+cq)} e^{\left( \frac{nH}{2} - \frac{b}{2} - \frac{H}{p} \right) t},$$

where  $N \doteq 2^{-\frac{c}{q-1}} \tilde{N} \left( \frac{H}{4} \right)^{-\frac{\beta+1}{q-1}}$  and  $Q \doteq 2^{\frac{cq}{q-1}} \left( \frac{H}{4} \right)^{\frac{\beta+1}{q-1}}$ .  
Let us denote by

$$K_j(t, \varepsilon) \doteq N Q^{q^{j+1}} \gamma_j \tilde{\gamma}_j B_j^{\beta+1} \varepsilon^{q^{j+2}} t^{-\frac{c+\beta+1}{q-1}} t^{\frac{q^{j+1}}{q-1} (\beta+1+cq)}$$

the factor that multiplies the exponential term  $e^{\left( \frac{nH}{2} - \frac{b}{2} - \frac{H}{p} \right) t}$  in the previous lower bound for  $V(t)$ . With this notation, we can simply write

$$V(t) \geq K_j(t, \varepsilon) e^{\left( \frac{nH}{2} - \frac{b}{2} - \frac{H}{p} \right) t} \quad \text{for } t \geq \sigma_j. \quad (3.85)$$

In addition, by using the support condition (1.7) and Hölder's inequality, from (2.1), we find that  $V$  satisfies

$$\begin{aligned} V''(t) + bV'(t) + m^2V(t) &\geq \Gamma(t) e^{-nH(\beta+1)(p-1)t} (V(t))^q \\ &\geq (1+t)^c e^{(c_{\text{crit}} - nH(\beta+1)(p-1))t} (V(t))^q \\ &= (1+t)^c e^{-\left( \frac{nH}{2} - \frac{b}{2} - \frac{H}{p} \right) (q-1)t} (V(t))^q \quad \text{for } t \geq 0, \end{aligned} \quad (3.86)$$

where we used again (1.4) in the last step.

We point out that Lemma III.1 does not cover the case when a polynomial factor appears on right-hand side of the ODI in (3.3). For this reason, we consider the next lemma, whose proof is completely analogous to the one of Lemma III.1.

*Lemma III.5.* Let  $b, m^2$  be non-negative real numbers such that  $b^2 \geq 4m^2$ . We consider the same notations for  $\alpha_1$  and  $\alpha_2$  as in the statement of Lemma III.1.

Let  $q > 1, k_0, k_1 \in \mathbb{R}$  satisfying (3.1) and (3.2), and  $\ell_0 < 0, \ell_1 > 0$  such that

$$\ell_0 + (q-1)\ell_1 \geq 0. \quad (3.87)$$

Suppose that  $G \in \mathcal{C}^2([0, T])$  satisfies

$$G''(t) + bG'(t) + m^2G(t) \geq B(1+t)^{\ell_0} e^{k_0 t} |G(t)|^q \quad \text{for } t \in [0, T], \quad (3.88)$$

$$G(t) \geq K(1+t)^{\ell_1} e^{k_1 t} \quad \text{for } t \in [T_0, T), \quad (3.89)$$

and (3.5), with  $T_0 \in [0, T)$  and for some positive constants  $B, K$ . Let us define  $T_1$  and  $K_0$  as in (3.6), where  $\vartheta \in (0, \frac{q-1}{2})$  is arbitrarily chosen so that  $2\vartheta\ell_1 \leq \ell_0 + (q-1)\ell_1$ .

If  $K \geq K_0$ , then the lifespan of  $G$  is finite and fulfills  $T \leq 2T_1$ .

*Remark 11.* We emphasize that thanks to the factor  $K_j(t, \varepsilon)$ , we have a lot of freedom in the choice of the parameter  $\ell_1$  in (3.89).

In order to control more accurately  $K_j(t, \varepsilon)$ , we rewrite it as follows:

$$\begin{aligned} K_j(t, \varepsilon) &= \exp\left\{q^{j+1}\left(\ln \varepsilon^q + \ln t \frac{\beta+1+\zeta q}{q-1} + \ln Q\right) + (\beta+1) \ln B_j + \ln(\gamma_j \tilde{\gamma}_j) - \frac{\zeta+\beta+1}{q-1} \ln t + \ln N\right\} \\ &= \exp\left\{q^{j+1}\left(\ln \varepsilon^q + \ln t \frac{\beta+1+\zeta q}{q-1} + \ln\left(QE^{\frac{1}{p}}\right)\right) + (j+1) \frac{(\beta+1) \ln q}{q-1} + \ln(\gamma_j \tilde{\gamma}_j) - \frac{\zeta+\beta+1}{q-1} \ln t + \ln N + \frac{(\beta+1) \ln q}{(q-1)^2} - \frac{(\beta+1) \ln D}{q-1}\right\}, \end{aligned}$$

where we used (3.82) in the second equality. Let us introduce the function

$$L(t, \varepsilon) \doteq \ln\left(\varepsilon^q QE^{\frac{1}{p}} t \frac{\beta+1+\zeta q}{q-1}\right). \tag{3.90}$$

We point out that  $L(t, \varepsilon) \geq 1$  if and only if

$$t \geq T_0(\varepsilon) \doteq E_1 \varepsilon^{-\frac{p(q-1)}{1+\zeta p}},$$

where  $E_1 \doteq \left(eQ^{-1}E^{-\frac{1}{p}}\right)^{\frac{q-1}{\beta+1+\zeta q}}$ .

Our goal is to apply Lemma III.1 or Lemma III.5 to  $V$  (depending on whether  $\zeta = 0$  or  $\zeta < 0$ ). We underline that  $V$  is twice continuously differentiable [see Eq. (3.1) in Ref. 1]. Clearly, (3.86) and (3.85) correspond to conditions (3.88) and (3.89), respectively, with  $\ell_0 = \zeta$ . Concerning  $\ell_1$ , we have some freedom in its choice thanks to the factor  $K_j(t, \varepsilon)$ . More precisely, we choose  $\ell_1 > 0$  such that  $\zeta + (q-1)\ell_1 > 0$ .

In particular, for  $t \geq \max\{T_0(\varepsilon), \sigma_j\}$ , it holds that

$$K_j(t, \varepsilon)(1+t)^{-\ell_1} \geq \exp\left\{q^{j+1} + (\beta+1)(j+1) \frac{\ln q}{q-1} + \ln(\gamma_j \tilde{\gamma}_j) - \frac{\zeta+\beta+1}{q-1} \ln t - \ell_1 \ln(1+t) + \ln N + \frac{(\beta+1) \ln q}{(q-1)^2} - \frac{(\beta+1) \ln D}{q-1}\right\}.$$

Let us now introduce the family of intervals  $\{\mathcal{I}(j)\}_{j \in \mathbb{N}}$ , where  $\mathcal{I}(j) \doteq [\sigma_j, \sigma_{j+1}]$ .

From (3.84), we see that there exists  $j_0 = j_0(R, \frac{c}{H}, a_0, b_0, \delta) \in \mathbb{N}$  such that for  $j \geq j_0$ , we have  $\sigma_j = A^{-1}\left(2^4\left(\frac{a_j-1}{1-\delta}R + \frac{b_j+2}{1-\delta}\frac{c}{H}\right)^2\right)$ . For  $j \geq j_0$  and for any  $t \in \mathcal{I}(j)$  such that  $t \geq T_0(\varepsilon)$ , we have

$$\begin{aligned} K_j(t, \varepsilon)(1+t)^{-\ell_1} &\geq \exp\left\{q^{j+1} + (\beta+1)(j+1) \frac{\ln q}{q-1} + \ln(\gamma_j \tilde{\gamma}_j) + \ln N + \frac{(\beta+1) \ln q}{(q-1)^2} - \frac{(\beta+1) \ln D}{q-1} \right. \\ &\quad \left. - \left(\frac{\zeta+\beta+1}{q-1} + \ell_1\right) \ln\left(1 + A^{-1}\left(2^4\left(\frac{a_{j+1}-1}{1-\delta}R + \frac{b_{j+1}+2}{1-\delta}\frac{c}{H}\right)^2\right)\right)\right\}. \end{aligned} \tag{3.91}$$

We consider now  $K_0$  and  $\tilde{T}_0(b, m^2, v_0, v_1)$  introduced in Lemmas III.1 and III.5 relative to  $V$ . More precisely, we use this lemma with  $k_1 = \frac{nH}{2} - \frac{b}{2} - \frac{H}{p}$ ,  $k_0 = -(q-1)k_1$ ,  $\ell_0 = \zeta$ , and  $\ell_1$  as discussed above. We stress that condition (3.2) is exactly (1.9) in this case.

Since the lower bound for  $K_j(t, \varepsilon)$  in (3.91) is divergent as  $j \rightarrow \infty$ , we can fix an index  $J = J(n, c, H, R, b, m^2, p, \beta, \sigma, \mu, a_0, b_0, \delta) \in \mathbb{N}$ , satisfying  $J \geq j_0$  in a such way that for any  $t \geq A^{-1}\left(2^4\left(\frac{a_J-1}{1-\delta}R + \frac{b_J+2}{1-\delta}\frac{c}{H}\right)^2\right)$  such that  $t \geq T_0(\varepsilon)$ , it holds that  $K_J(t, \varepsilon) \geq K_0$ .

We may now fix  $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, v_0, v_1, R, a_0, b_0, \delta, J)$  such that

$$T_0(\varepsilon_0) \geq \max\left\{A^{-1}\left(2^4\left(\frac{a_J-1}{1-\delta}R + \frac{b_J+2}{1-\delta}\frac{c}{H}\right)^2\right), \tilde{T}_0\left(\left(\frac{n}{2} - v - \frac{1}{p}\right)H\right)^{-1}\right\} \tag{3.92}$$

due to the fact that all terms on the right-hand side are independent of  $\varepsilon$ . Consequently, for any  $\varepsilon \in (0, \varepsilon_0]$ , since  $T_0(\varepsilon) \geq T_0(\varepsilon_0)$ , we have  $V(t) \geq K_0(1+t)^{\ell_1} e^{\left(\frac{nH}{2} - \frac{b}{2} - \frac{H}{p}\right)t}$  for any  $t \geq T_0(\varepsilon)$ . In conclusion, Lemmas III.1 and III.5 provide the following upper bound for the lifespan  $T(\varepsilon)$  of  $V$ :

$$T(\varepsilon) \leq 2 \max\left\{T_0(\varepsilon), \tilde{T}_0\left(\left(\frac{n}{2} - v - \frac{1}{p}\right)H\right)^{-1}\right\} \leq 2T_0(\varepsilon) = 2E_1 \varepsilon^{-\frac{p(q-1)}{1+\zeta p}},$$

where in the second inequality, we used (3.92). This concludes the Proof of Theorem I.2 for  $\zeta \in (-\frac{1}{p}, 0]$ .

## F. Upper bound estimates for the lifespan when $\zeta > 0$

In the previous proof, we showed the validity of (1.11) when  $\zeta \in (-\frac{1}{p}, 0]$ . Of course, we can repeat the same argument as before when  $\zeta > 0$ , obtaining the same upper bound estimate for the lifespan as in the case  $\zeta = 0$ , that is,  $T(\varepsilon) \lesssim \varepsilon^{-p(q-1)}$ . On the other hand, when  $\zeta > 0$  and  $\varrho = \varrho_{\text{crit}}$ , we may use the same approach as in the proof of Ref. 1 (Theorem 1.9) to prove the blow-up in finite time of  $V$  and the lifespan estimate  $T(\varepsilon) \lesssim \varepsilon^{-\frac{q-1}{\zeta}}$ . Comparing the last two lifespan estimates when  $\zeta > 0$ , we conclude the validity of (1.11).

## IV. FINAL REMARKS AND OPEN PROBLEMS

We explicitly point out that we expect that the lifespan estimate in (1.11) is not sharp when  $\zeta > 0$ .

We also stress that when  $n > N$  in the double limit case  $\varrho = \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$  and  $\zeta = -\frac{1}{p}$ , we are not able to prove the blow-up in finite time of  $V$ . This is due to the fact that the function  $L(t, \varepsilon)$  does no longer depend on  $t$  as the power for  $t$  is 0 in (3.90) when  $\zeta = -\frac{1}{p}$ .

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

**Alessandro Palmieri:** Conceptualization (equal); Formal analysis (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Hiroyuki Takamura:** Conceptualization (equal); Formal analysis (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

## DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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