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Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations of derivative type in the scattering case

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Abstract. In this paper we consider the blow-up for solutions to a weakly coupled system of semilinear damped wave equations of derivative type in the scattering case. After introducing suitable functionals proposed by Lai-Takamura for the corresponding single semilinear equation, we employ Kato's lemma to derive the blow-up result in the subcritical case. On the other hand, in the critical case an iteration procedure based on the slicing method is employed. Let us point out that we find as critical curve in the p - q plane for the pair of exponents (p, q) in the nonlinear terms the same one as for the weakly coupled system of semilinear not-damped wave equations with the same kind of nonlinearities.

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1. Introduction

In this paper we consider a weakly coupled system of wave equations with time-dependent and scattering producing damping terms and with powers of the first order time-derivatives of components of the solution as nonlinear terms (semilinear term of *derivative type*), namely,

$$\begin{cases} u_{tt} - \Delta u + b_1(t)u_t = |\partial_t v|^p, & x \in \mathbb{R}^n,\ t > 0, \\ v_{tt} - \Delta v + b_2(t)v_t = |\partial_t u|^q, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x) & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $b_1, b_2 \in \mathscr{C}([0,\infty)) \cap L^1([0,\infty))$ are nonnegative functions, ε is a positive parameter describing the size of initial data and $p, q > 1$. We will prove blow-up results for (1.1) both in the subcritical case and in the critical case. Moreover, an upper bound for the lifespan of local solutions is derived in these two cases.

The nonexitence of global in time solutions in the case without damping terms (that is, for $b_1 = b_2 = 0$) has been studied in [3, 22], while the existence part has been proved in the three dimensional and radial case in [8]. Recently, in [6, Section 8] the upper bound for the lifespan has been derived.

Summarizing the blow-up results above cited, for the weakly coupled system

$$\begin{cases} u_{tt} - \Delta u = |\partial_t v|^p, & x \in \mathbb{R}^n,\ t > 0, \\ v_{tt} - \Delta v = |\partial_t u|^q, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t, v, v_t)(0,x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x) & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

it holds the following result: under certain integral sign assumptions for the initial data, if $p, q > 1$ satisfy

$$\max\left\{\frac{p+1}{pq-1}; \frac{q+1}{pq-1}\right\} \geqslant \frac{n-1}{2},$$

then, (u, v) has to blow-up in finite time; moreover, the following upper bound estimate for the lifespan holds

$$T(\varepsilon) \leqslant \begin{cases} C\varepsilon^{-\max\{\Lambda(n,p,q),\Lambda(n,q,p)\}^{-1}} & \text{if } \Upsilon(n,p,q) > 0, \\ \exp\left(C\varepsilon^{-(pq-1)}\right) & \text{if } \Upsilon(n,p,q) = 0,\ p \neq q, \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } \Upsilon(n,p,q) = 0,\ p = q, \end{cases}$$

where

$$\Lambda(n,p,q) \doteq \frac{p+1}{pq-1} - \frac{n-1}{2} \tag{1.3}$$

and

$$\Upsilon(n,p,q) \doteq \max\{\Lambda(n,p,q), \Lambda(n,q,p)\}. \tag{1.4}$$

Let us stress that the study of the blow-up results for the system (1.2) is not a trivial generalization of the corresponding results related to Glassey's conjecture for the semilinear Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = |\partial_t u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t)(0,x) = (\varepsilon u_0, \varepsilon u_1)(x) & x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

Indeed, for the semilinear Cauchy problem (1.5) it has been proved that the critical exponent is the so-called Glassey exponent

$$p_{\mathrm{Gla}}(n) \doteq \frac{n+1}{n-1} \tag{1.6}$$

(in dimension $n = 1$ there exist solutions that cannot be prolonged for all time for any exponent $p > 1$ regardless of the smallness of initial data).

We refer to the classical results [7, 17, 12, 16, 15, 1, 4, 20, 25, 5] and references therein contained for further details.

We remark that the condition $p \leqslant p_{\mathrm{Gla}}(n)$ can be equivalently expressed as

$$\frac{1}{p-1} \geqslant \frac{n-1}{2}. \tag{1.7}$$

Therefore, because of

$$\frac{\max\{p,q\}+1}{pq-1} \geqslant \frac{1}{\max\{p,q\}-1}, \tag{1.8}$$

where the equality holds if and only if $p = q$, it may happen that the condition for (p, q), which implies the validity of a blow-up result, is satisfied even though one among p, q is greater than the Glassey exponent (of course, in the case $p \neq q$). This fact follows immediately by (1.7) and (1.8).

Recently, semilinear wave equations with scattering producing damping terms have been studied in [9, 10, 11] in the case of single equations and in [13, 14] for weakly coupled systems with power nonlinearities and mixed nonlinearities, respectively.

In this work we will study blow-up results for the weakly coupled system (1.1) in the subcritical case and in the critical case by considering the blow-up dynamic of suitable functionals, that represent a generalization of the functional introduced in [10] in order to study the semilinear wave equation with damping in the scattering case related to Glassey's conjecture.

The novelty of our results consists on the way in which methods, typically used for the semilinear classical wave equation with power nonlinearity of the solution itself, are suitably adapted to the study of (1.1). More specifically, these methods are Kato's lemma (see [18, 23, 19, 24]) and the slicing method combined with an iteration argument (see [2, 21]). These methods have been studied only for the classical semilinear wave equation with power nonlinearity $|u|^p$ and for the corresponding weakly coupled system. In this sense, it is surprising that they can be applied to study the weakly coupled system (1.1) with semilinear terms of derivative type.

Before stating the blow-up results of this paper, let us introduce a suitable notion of energy solutions.

Definition 1.1. Let $u_0, v_0 \in H^1(\mathbb{R}^n)$ and $u_1, v_1 \in L^2(\mathbb{R}^n)$. We say that (u, v) is an energy solution of (1.1) on $[0, T)$ if

$$u \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u \in L^q_{\mathrm{loc}}([0,T) \times \mathbb{R}^n),$$
$$v \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \quad \text{and} \quad \partial_t v \in L^p_{\mathrm{loc}}([0,T) \times \mathbb{R}^n)$$

satisfy $u(0,x)=\varepsilon u_0(x)$, $v(0,x)=\varepsilon v_0(x)$ in $H^1(\mathbb{R}^n)$ and the equalities

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^n}|\partial_t v(s,x)|^p\phi(s,x)\,dx\,ds=\int_{\mathbb{R}^n}\partial_t u(t,x)\phi(t,x)\,dx\\
&-\int_{\mathbb{R}^n}\varepsilon u_1(x)\phi(0,x)\,dx-\int_0^t\int_{\mathbb{R}^n}\partial_t u(s,x)\phi_s(s,x)\,dx\,ds\\
&+\int_0^t\int_{\mathbb{R}^n}\nabla u(s,x)\cdot\nabla\phi(s,x)\,dx\,ds+\int_0^t\int_{\mathbb{R}^n}b_1(s)\partial_t u(s,x)\phi(s,x)\,dx\,ds
\end{aligned}\tag{1.9}$$

and

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^n}|\partial_t u(s,x)|^q\psi(s,x)\,dx\,ds=\int_{\mathbb{R}^n}\partial_t v(t,x)\psi(t,x)\,dx\\
&-\int_{\mathbb{R}^n}\varepsilon v_1(x)\psi(0,x)\,dx-\int_0^t\int_{\mathbb{R}^n}\partial_t v(s,x)\psi_s(s,x)\,dx\,ds\\
&+\int_0^t\int_{\mathbb{R}^n}\nabla v(s,x)\cdot\nabla\psi(s,x)\,dx\,ds+\int_0^t\int_{\mathbb{R}^n}b_2(s)\partial_t v(s,x)\psi(s,x)\,dx\,ds
\end{aligned}\tag{1.10}$$

for any test functions $\phi,\psi\in\mathscr{C}_0^\infty([0,T)\times\mathbb{R}^n)$ and any $t\in[0,T)$.

Performing a further step of integrations by parts in (1.9), (1.10) and letting $t\to T$, we find that (u,v) fulfills the definition of weak solution to (1.1).

Let us state the main blow-up result for (1.1) of this paper.

Theorem 1.2. *Let b_1,b_2 be continuous, nonnegative and summable functions. Let us consider $p,q>1$ satisfying*

$$\max\left\{\frac{p+1}{pq-1},\frac{q+1}{pq-1}\right\}\geqslant\frac{n-1}{2}\,. \tag{1.11}$$

Assume that $u_0,v_0\in H^1(\mathbb{R}^n)$ and $u_1,v_1\in L^2(\mathbb{R}^n)$ are nonnegative and compactly supported in B_R functions such that $u_1,v_1\not\equiv 0$.

Let (u,v) be an energy solution of (1.1) with lifespan $T=T(\varepsilon)$ according to Definition 1.1, satisfying

$$\operatorname{supp}u,\ \operatorname{supp}v\subset\{(t,x)\in[0,T)\times\mathbb{R}^n:|x|\leqslant t+R\}. \tag{1.12}$$

Then, there exists a positive constant $\varepsilon_0=\varepsilon_0(u_0,u_1,v_0,v_1,n,p,q,b_1,b_2,R)$ such that for any $\varepsilon\in(0,\varepsilon_0]$ the solution (u,v) blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon)\leqslant\begin{cases}C\varepsilon^{-\max\{\Lambda(n,p,q),\Lambda(n,q,p)\}^{-1}} & \text{if }\ \Upsilon(n,p,q)>0,\\ \exp\left(C\varepsilon^{-(pq-1)}\right) & \text{if }\ \Upsilon(n,p,q)=0,\ p\neq q,\\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if }\ \Upsilon(n,p,q)=0,\ p=q,\end{cases} \tag{1.13}$$

holds, where C is an independent of ε, positive constant and $\Lambda(n,p,q)$ and $\Upsilon(n,p,q)$ are defined by (1.3) and (1.4).

Remark 1.3. The upper bound estimates in (1.13) for the lifespan of the solution coincide with the ones for the case $b_1 = b_2 = 0$ as we recalled in the historical overview in the first part of this introduction.

The remaining part of this paper is organized as follows: in Section 2 we recall the definition of a multiplier, that has been introduced in [9] in order to study the corresponding single semilinear wave equation with power nonlinearity, and its properties; moreover, following [10] we introduce a suitable pair of functionals related to a local solution of (1.1) and we determine certain lower bounds for these functionals; then, in Section 3 we prove Theorem 1.2 in the subcritical case by using a Kato's type lemma. Finally, in Section 4 we modify the approach in the critical case employing an iteration argument together with the slicing method.

Notations

Throughout this paper we will use the following notations: B_R denotes the ball around the origin with radius R; $f \lesssim g$ means that there exists a positive constant C such that $f \leqslant Cg$ and, analogously, for $f \gtrsim g$; finally, as in the introduction, $p_{\mathrm{Gla}}(n)$ denotes the Glassey exponent, whose definition is given in (1.6).

2. Definition of the functionals and derivation of the iteration frame

Let us recall the definition of some multipliers related to our model, which have been introduced in [9], and some of their properties as well, that will be useful throughout this paper.

Definition 2.1. Let $b_1, b_2 \in \mathscr{C}([0,\infty)) \cap L^1([0,\infty))$ be the nonnegative, time-dependent coefficients in (1.1). We define the corresponding *multipliers*

$$m_j(t) \doteq \exp\left(- \int_t^\infty b_j(\tau) d\tau \right) \qquad \text{for} \quad t \geqslant 0 \quad \text{and } j = 1, 2.$$

Due to the nonnegativity of b_1, b_2, it follows the monotonicity of m_1, m_2. Furthermore, as these coefficients are summable, we get also that these multipliers are bounded and

$$m_j(0) \leqslant m_j(t) \leqslant 1 \qquad \text{for} \quad t \geqslant 0 \quad \text{and } j = 1, 2. \tag{2.1}$$

An important property of these multipliers is the relation with the corresponding derivatives. More precisely,

$$m_j'(t) = b_j(t)\, m(t) \qquad \text{for} \quad j = 1, 2. \tag{2.2}$$

The properties described by (2.1) and (2.2) play a crucial role in the remaining part of this section, which is devoted to the introduction of a pair of functionals, whose dynamic is investigated in the proof of Theorem 1.2. This kind of functionals have been considered for a single semilinear wave equation of derivative type with a scattering-producing damping in [10].

However, before introducing the above quoted functionals, we need to derive suitable lower bound estimates for a different pair of functionals related to a local solution (u, v) of (1.1). Thus, we introduce the functionals

$$U_1(t) \doteq \int_{\mathbb{R}^n} u(t,x)\Psi(t,x)\,dx \quad \text{and} \quad V_1(t) \doteq \int_{\mathbb{R}^n} v(t,x)\Psi(t,x)\,dx, \tag{2.3}$$

where $\Psi = \Psi(t,x) \doteq e^{-t}\Phi(x)$ and

$$\Phi = \Phi(x) \doteq \begin{cases} e^{x} + e^{-x} & \text{for } \ n = 1, \\ \displaystyle\int_{\mathbb{S}^{n-1}} e^{\omega\cdot x}\,dS_\omega & \text{for } \ n \geqslant 2 \end{cases} \tag{2.4}$$

is an eigenfunction of the Laplace operator, as $\Delta\Phi = \Phi$. In order to derive lower bounds for U_1, V_1 we prove a result, which is valid even when we consider more general nonnegative nonlinearities.

Lemma 2.2. *Let $(w, \widetilde{w})$ be a local energy solution of the Cauchy problem*

$$\begin{cases} w_{tt} - \Delta w + b_1(t)w_t = G_1(t,x,w,w_t,\widetilde{w},\widetilde{w}_t), & x \in \mathbb{R}^n, \ t \in (0,T), \\ \widetilde{w}_{tt} - \Delta \widetilde{w} + b_2(t)\widetilde{w}_t = G_2(t,x,w,w_t,\widetilde{w},\widetilde{w}_t), & x \in \mathbb{R}^n, \ t \in (0,T), \\ (w,w_t,\widetilde{w},\widetilde{w}_t)(0,x) = (\varepsilon w_0, \varepsilon w_1, \varepsilon \widetilde{w}_0, \varepsilon \widetilde{w}_1)(x), & x \in \mathbb{R}^n, \end{cases}$$

where the coefficients of the damping terms $b_1, b_2 \in \mathscr{C}([0,\infty)) \cap L^1([0,\infty))$ and the nonlinear terms G_1, G_2 are nonnegative. Furthermore, we assume that $w_0, w_1, \widetilde{w}_0, \widetilde{w}_1$ are nonnegative, pairwise nontrivial and compactly supported and that $w, \widetilde{w}$ satisfy a support condition as in (1.12)*. Let $W_1, \widetilde{W_1}$ be defined by*

$$W_1(t) \doteq \int_{\mathbb{R}^n} w(t,x)\Psi(t,x)\,dx \quad \textit{and} \quad \widetilde{W_1}(t) \doteq \int_{\mathbb{R}^n} \widetilde{w}(t,x)\Psi(t,x)\,dx$$

for any $t \geqslant 0$. Then, for any $t \geqslant 0$ the following estimates hold

$$\begin{aligned} W_1(t) &\geqslant \varepsilon\,\frac{m_1(0)}{2}\int_{\mathbb{R}^n} w_0(x)\Phi(x)\,dx, \\ \widetilde{W_1}(t) &\geqslant \varepsilon\,\frac{m_2(0)}{2}\int_{\mathbb{R}^n} \widetilde{w}_0(x)\Phi(x)\,dx. \end{aligned}$$

Proof. We follow the main ideas of the proof of Lemma 3.1 in [10]. We prove the lower bound estimate for W_1, since the proof of the one for $\widetilde{W_1}$ is completely analogous. Clearly, the definition of energy solution for the considered Cauchy problem is analogous to the one given in Definition 1.1 for (1.1). The only difference consists in the assumptions on the nonlinear terms G_1, G_2, which have to be supposed in $L^1_{\text{loc}}([0,T) \times \mathbb{R}^n)$ and replace $|v_t|^p, |u_t|^q$ in (1.9), (1.10), respectively.

Thanks to the support property for w, we can apply the definition of energy solution with test functions that are not compactly supported. Hence, employing the definition of energy solution with Ψ as test function and taking the time derivative of the obtained relation, we find for any $t \in (0,T)$

$$\frac{d}{dt}\int_{\mathbb{R}^n} \partial_t w(t,x)\Psi(t,x)\,dx - \int_{\mathbb{R}^n} \Big(\partial_t w(t,x)\Psi_t(t,x) + w(t,x)\Delta\Psi(t,x)\Big)dx$$
$$+ \int_{\mathbb{R}^n} b_1(t)\partial_t w(t,x)\Psi(t,x)dx = \int_{\mathbb{R}^n} G_1(t,x)\Psi(t,x)\,dx,$$

where we denote $G_1(t,x) \equiv G_1(t,x,w(t,x),w_t(t,x),\widetilde{w}(t,x),\widetilde{w}_t(t,x))$ for the sake of brevity. Multiplying both sides of the previous equality by $m_1(t)$, we find

$$\begin{aligned}
m_1(t)\int_{\mathbb{R}^n} & G_1(t,x)\Psi(t,x)dx \\
&= m_1(t)\frac{d}{dt}\int_{\mathbb{R}^n} \partial_t w(t,x)\Psi(t,x)\,dx + m_1(t)b_1(t)\int_{\mathbb{R}^n} \partial_t w(t,x)\Psi(t,x)\,dx \\
&\quad - m_1(t)\int_{\mathbb{R}^n} \big(\partial_t w(t,x)\Psi_t(t,x) + w(t,x)\,\Delta\Psi(t,x)\big)\,dx \\
&= \frac{d}{dt}\left(m_1(t)\int_{\mathbb{R}^n} \partial_t w(t,x)\Psi(t,x)\,dx\right) \\
&\quad + m_1(t)\int_{\mathbb{R}^n} \big(\partial_t w(t,x) - w(t,x)\big)\Psi(t,x)\,dx,
\end{aligned}$$

where in the last step we used (2.2) and the properties $\Psi_t = -\Psi$ and $\Delta\Psi = \Psi$. Integrating the last relation over $[0,t]$, we find

$$\begin{aligned}
\int_0^t m_1(s)\int_{\mathbb{R}^n} & G_1(s,x)\Psi(s,x)\,dx\,ds \\
&= m_1(t)\int_{\mathbb{R}^n} \partial_t w(t,x)\Psi(t,x)\,dx - \varepsilon\, m_1(0)\int_{\mathbb{R}^n} w_1(x)\,\Phi(x)\,dx \\
&\quad + \int_0^t m_1(s)\int_{\mathbb{R}^n} \big(\partial_t w(s,x) - w(s,x)\big)\Psi(s,x)\,dx\,ds. \qquad (2.5)
\end{aligned}$$

An integration by parts with respect to t provides

$$\begin{aligned}
\int_0^t m_1(s)\int_{\mathbb{R}^n} & \partial_t w(s,x)\Psi(s,x)\,dx\,ds \\
&= m_1(t)\int_{\mathbb{R}^n} w(t,x)\Psi(t,x)\,dx - \varepsilon\, m_1(0)\int_{\mathbb{R}^n} w_0(x)\,\Phi(x)\,dx \\
&\quad - \int_0^t\int_{\mathbb{R}^n} w(s,x)\,b_1(s)\,m_1(s)\Psi(s,x)\,dx\,ds \\
&\quad + \int_0^t m_1(s)\int_{\mathbb{R}^n} w(s,x)\Psi(s,x)\,dx\,ds. \qquad (2.6)
\end{aligned}$$

Consequently, combining (2.5) and (2.6), we arrive at

$$\begin{aligned}
&\int_0^t m_1(s) \int_{\mathbb{R}^n} G_1(s,x)\Psi(s,x)\,dx\,ds \\
&\qquad + \int_0^t b_1(s)\, m_1(s) \int_{\mathbb{R}^n} w(s,x)\,\Psi(s,x)\,dx\,ds \\
&\qquad + \varepsilon\, m_1(0) \int_{\mathbb{R}^n} \big(w_0(x)+w_1(x)\big)\Phi(x)\,dx \\
&\quad = m_1(t) \int_{\mathbb{R}^n} \big(\partial_t w(t,x)\Psi(t,x) + w(t,x)\Psi(t,x)\big)\,dx \\
&\quad = m_1(t)\,\frac{d}{dt} \int_{\mathbb{R}^n} w(t,x)\Psi(t,x)\,dx + 2m_1(t) \int_{\mathbb{R}^n} w(t,x)\Psi(t,x)\,dx.
\end{aligned}$$

By the definition of W_1 and the nonnegativity of the semilinear term G_1, from the previous relation we obtain the inequality

$$m_1(t)\big(W_1'(t) + 2W_1(t)\big) \geqslant \varepsilon\, m_1(0)\, C(w_0,w_1) + \int_0^t b_1(s)\, m_1(s)\, W_1(s)\,ds$$

where $C(w_0,w_1) \doteq \int_{\mathbb{R}^n} \big(w_0(x)+w_1(x)\big)\Phi(x)\,dx$. Since the multiplier m_1 is bounded, using (2.1), we have

$$W_1'(t) + 2W_1(t) \geqslant \varepsilon\, m_1(0)\, C(w_0,w_1) + \frac{1}{m(t)} \int_0^t b_1(s)\, m_1(s)\, W_1(s)\,ds. \quad (2.7)$$

Multiplying both sides in the last estimate by e^{2t} and integrating over $[0,t]$, we get

$$\begin{aligned}
e^{2t}W_1(t) \geqslant W_1(0) &+ \frac{m_1(0)}{2}\,\varepsilon\, C(w_0,w_1)(e^{2t}-1) \\
&+ \int_0^t \frac{e^{2s}}{m(s)} \int_0^s b_1(\tau)\, m_1(\tau)\, W_1(\tau)\,d\tau\,ds. \qquad (2.8)
\end{aligned}$$

Let us prove first the positiveness of W_1 by using a comparison argument. Because we assumed the data pairwise nontrivial, at least one among w_0, w_1 is not identically 0. In the first case $w_0 \not\equiv 0$, as $w_0 \geqslant 0$ implies $W_1(0) > 0$, by continuity it holds $W_1(t) > 0$ in a right neighborhood of $t = 0$. If $t_0 > 0$ was the smallest time such that $W_1(t_0) = 0$, then, the evaluation of (2.8) in $t = t_0$ would yield a contradiction. In the second case $w_0 \equiv 0$ and $w_1 \not\equiv 0$, we can use (2.7) to find a contradiction. In fact, in this case $W_1(0) = 0$ and $W_1'(0) = \varepsilon \int_{\mathbb{R}^n} w_1(x)\Phi(x)\,dx > 0$. Thus, by continuity $W_1'(t) > 0$ for any $t \in [0,t_1)$ for a suitable $t_1 > 0$. Hence, W_1 is strictly increasing, and, in particular, positive in $(0,t_1)$. We assume by contradiction that $t_2 > t_1$ is the smallest time such that $W_1(t_2) = 0$. Then, $W_1'(t_2) \leqslant 0$. Indeed, if $W_1'(t_2)$ was positive, then, W_1 would be strictly increasing in a neighborhood of t_2, but this would be a contradiction to the definition of t_2. In fact, there would be a smaller zero, because W_1 would be negative in a left neighborhood of t_2. Finally, if we plug $W_1(t_2) = 0, W_1'(t_2) \leqslant 0$ and we use $W_1(t) > 0$ for any $t \in (0,t_2)$ in (2.7), we have a contradiction.

Also, thanks to the fact that W_1 is positive, from (2.8) we obtain

$$\begin{aligned} W_1(t) &\geqslant e^{-2t}W_1(0) + \frac{m_1(0)}{2}\,\varepsilon\, C(w_0,w_1)(1-e^{-2t}) \\ &\geqslant \frac{m_1(0)}{2}\,\varepsilon \int_{\mathbb{R}^n} w_0(x)\Phi(x)\,dx, \end{aligned}$$

which is the desired estimate. This concludes the proof. □

In particular, applying Lemma 2.2 to an energy solution (u,v) of (1.1), we find

$$U_1(t) \geqslant \frac{m_1(0)}{2}\,\varepsilon \int_{\mathbb{R}^n} u_0(x)\Phi(x)\,dx, \tag{2.9}$$

$$V_1(t) \geqslant \frac{m_2(0)}{2}\,\varepsilon \int_{\mathbb{R}^n} v_0(x)\Phi(x)\,dx \tag{2.10}$$

for any $t \geqslant 0$, where U_1 and V_1 are defined by (2.3).

Next we follow the main ideas of [10, Section 3] in order to introduce the suitable functionals for the proof of the blow-up result. Let us point out that

$$\begin{aligned} &\frac{d}{dt}\left(m_1(t) \int_{\mathbb{R}^n} \big(\partial_t u(t,x) + u(t,x)\big)\Psi(t,x)\,dx \right) \\ &\quad = b_1(t)m_1(t) \int_{\mathbb{R}^n} \big(\partial_t u(t,x) + u(t,x)\big)\Psi(t,x)\,dx \\ &\qquad + m_1(t)\frac{d}{dt} \int_{\mathbb{R}^n} \big(\partial_t u(t,x) + u(t,x)\big)\Psi(t,x)\,dx. \end{aligned} \tag{2.11}$$

Choosing $\phi \equiv \Psi$ on $\operatorname{supp} u$ in (1.9), we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,dx - \int_{\mathbb{R}^n} \varepsilon u_1(x)\Phi(x)\,dx - \int_0^t \int_{\mathbb{R}^n} \partial_t u(s,x)\Psi_s(s,x)\,dx\,ds \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \nabla u(s,x)\cdot\nabla\Psi(s,x)\,dx\,ds + \int_0^t \int_{\mathbb{R}^n} b_1(s)\partial_t u(s,x)\Psi(s,x)\,dx\,ds \\ &= \int_0^t \int_{\mathbb{R}^n} |\partial_t v(s,x)|^p\Psi(s,x)\,dx\,ds. \end{aligned}$$

Differentiating both sides of the previous equality with respect to t, we arrive at

$$\begin{aligned} \int_{\mathbb{R}^n} |\partial_t v(t,x)|^p \Psi(t,x)\,dx &= \frac{d}{dt}\int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,dx - \int_{\mathbb{R}^n} \partial_t u(t,x)\Psi_t(t,x)\,dx \\ &\quad + \int_{\mathbb{R}^n} \big(\nabla u(t,x)\cdot\nabla\Psi(t,x) + b_1(t)\partial_t u(t,x)\Psi(t,x)\big)\,dx. \end{aligned} \tag{2.12}$$

Using $\Delta\Psi = \Psi$ and $\Psi_t = -\Psi$, (2.12) yields

$$\begin{aligned} \int_{\mathbb{R}^n} |\partial_t v(t,x)|^p \Psi(t,x)\,dx &= \frac{d}{dt}\int_{\mathbb{R}^n} \big(\partial_t u(t,x) + u(t,x)\big)\Psi(t,x)\,dx \\ &\quad + b_1(t)\int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,dx. \end{aligned} \tag{2.13}$$

If we combine (2.11) and (2.13), we obtain

$$\frac{d}{dt}\bigg(m_1(t)\int_{\mathbb{R}^n}\big(\partial_t u(t,x)+u(t,x)\big)\Psi(t,x)\,dx\bigg) = b_1(t)m_1(t)\,U_1(t)+m_1(t)\int_{\mathbb{R}^n}|\partial_t v(t,x)|^p\,\Psi(t,x)\,dx, \tag{2.14}$$

where U_1 is defined by (2.3). Thanks to (2.9) we have that U_1 is nonnegative, then, integrating (2.14) over $[0,t]$, we get the estimate

$$\begin{aligned} m_1(t)\int_{\mathbb{R}^n}\big(\partial_t u(t,x)+u(t,x)\big)\Psi(t,x)\,dx &\geqslant \varepsilon\, m_1(t)\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\Phi(x)\,dx \\ &\quad+\int_0^t m_1(s)\int_{\mathbb{R}^n}|\partial_t v(s,x)|^p\,\Psi(s,x)\,dx. \end{aligned} \tag{2.15}$$

Furthermore, we may rewrite (2.12) as follows

$$\begin{aligned} \int_{\mathbb{R}^n}|\partial_t v(t,x)|^p\,\Psi(t,x)\,dx &= \frac{d}{dt}\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx \\ &\quad+b_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx \\ &\quad+\int_{\mathbb{R}^n}\big(\partial_t u(t,x)-u(t,x)\big)\Psi(t,x)\,dx. \end{aligned} \tag{2.16}$$

If we multiply both sides of (2.16) by $m_1(t)$, we find

$$\begin{aligned} \frac{d}{dt}\bigg(m_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\bigg)+m_1(t)\int_{\mathbb{R}^n}\big(\partial_t u(t,x)-u(t,x)\big)\Psi(t,x)\,dx \\ = m_1(t)\int_{\mathbb{R}^n}|\partial_t v(t,x)|^p\,\Psi(t,x)\,dx. \end{aligned} \tag{2.17}$$

Adding (2.15) and (2.17), we find

$$\begin{aligned} &\frac{d}{dt}\bigg(m_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx\bigg)+2m_1(t)\int_{\mathbb{R}^n}\partial_t u(t,x)\Psi(t,x)\,dx \\ &\quad\geqslant \varepsilon\, m_1(0)\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\Phi(x)\,dx+m_1(t)\int_{\mathbb{R}^n}|\partial_t v(t,x)|^p\,\Psi(t,x)\,dx \\ &\qquad+\int_0^t m_1(s)\int_{\mathbb{R}^n}|\partial_t v(s,x)|^p\,\Psi(s,x)\,dx. \end{aligned} \tag{2.18}$$

In a complete analogous way, one can prove

$$\begin{aligned} &\frac{d}{dt}\bigg(m_2(t)\int_{\mathbb{R}^n}\partial_t v(t,x)\Psi(t,x)\,dx\bigg)+2m_2(t)\int_{\mathbb{R}^n}\partial_t v(t,x)\Psi(t,x)\,dx \\ &\quad\geqslant \varepsilon\, m_2(0)\int_{\mathbb{R}^n}\big(v_0(x)+v_1(x)\big)\Phi(x)\,dx+m_2(t)\int_{\mathbb{R}^n}|\partial_t u(t,x)|^q\,\Psi(t,x)\,dx \\ &\qquad+\int_0^t m_2(s)\int_{\mathbb{R}^n}|\partial_t u(s,x)|^q\,\Psi(s,x)\,dx. \end{aligned} \tag{2.19}$$

Let us set the auxiliary functionals

$$
\begin{aligned}
U_2(t) &\doteq m_1(t)\int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,dx - \varepsilon\,\frac{m_1(0)}{2}\int_{\mathbb{R}^n} u_1(x)\Phi(x)\,dx \\
&\quad - \frac{1}{2}\int_0^t m_1(s)\int_{\mathbb{R}^n} |\partial_t v(s,x)|^p\Psi(t,x)\,dx\,ds\,, \\
V_2(t) &\doteq m_2(t)\int_{\mathbb{R}^n} \partial_t v(t,x)\Psi(t,x)\,dx - \varepsilon\,\frac{m_2(0)}{2}\int_{\mathbb{R}^n} v_1(x)\Phi(x)\,dx \\
&\quad - \frac{1}{2}\int_0^t m_2(s)\int_{\mathbb{R}^n} |\partial_t u(s,x)|^q\Psi(t,x)\,dx\,ds\,.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
U_2(0) &= \varepsilon\,\frac{m_1(0)}{2}\int_{\mathbb{R}^n} u_1(x)\Phi(x)\,dx \geqslant 0, \\
V_2(0) &= \varepsilon\,\frac{m_2(0)}{2}\int_{\mathbb{R}^n} v_1(x)\Phi(x)\,dx \geqslant 0.
\end{aligned}
$$

Besides, (2.18) implies

$$
\begin{aligned}
&U_2'(t) + 2U_2(t) \\
&\quad \geqslant \varepsilon\, m_1(0)\int_{\mathbb{R}^n} u_0(x)\Phi(x)\,dx + \frac{1}{2}\,m_1(t)\int_{\mathbb{R}^n} |\partial_t v(t,x)|^p\Psi(t,x)\,dx \\
&\quad \geqslant 0. \qquad (2.20)
\end{aligned}
$$

Hence, multiplying the left hand side of (2.20) by e^{2t} and integrating over $[0,t]$, we get $U_2(t) \geqslant e^{-2t}U_2(0) \geqslant 0$. Similarly, employing (2.19), we can prove that $V_2(t) \geqslant 0$ for any $t \geqslant 0$.

Therefore, as the functionals U_2, V_2 are nonnegative we may write

$$
\begin{aligned}
m_1(t)\int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,dx &\geqslant \varepsilon\,\frac{m_1(0)}{2}\int_{\mathbb{R}^n} u_1(x)\Phi(x)\,dx \\
&\quad + \frac{1}{2}\int_0^t m_1(s)\int_{\mathbb{R}^n} |\partial_t v(s,x)|^p\Psi(t,x)\,dx\,ds\,, \qquad (2.21) \\
m_2(t)\int_{\mathbb{R}^n} \partial_t v(t,x)\Psi(t,x)\,dx &\geqslant \varepsilon\,\frac{m_2(0)}{2}\int_{\mathbb{R}^n} v_1(x)\Phi(x)\,dx \\
&\quad + \frac{1}{2}\int_0^t m_2(s)\int_{\mathbb{R}^n} |\partial_t u(s,x)|^q\Psi(t,x)\,dx\,ds\,. \qquad (2.22)
\end{aligned}
$$

After the above preparatory results we can finally introduce the functionals whose dynamic is studied in order to prove Theorem 1.2. Let us define for any $t \geqslant 0$

$$\mathcal{F}(t) \doteq \varepsilon \, \frac{m_1(0)}{2} \int_{\mathbb{R}^n} u_1(x)\Phi(x)\,dx + \frac{1}{2}\int_0^t m_1(s) \int_{\mathbb{R}^n} |\partial_t v(s,x)|^p \Psi(t,x)\,dx\,ds\,, \tag{2.23}$$

$$\mathcal{G}(t) \doteq \varepsilon \frac{m_2(0)}{2} \int_{\mathbb{R}^n} v_1(x)\Phi(x)\,dx + \frac{1}{2}\int_0^t m_2(s) \int_{\mathbb{R}^n} |\partial_t u(s,x)|^q \Psi(t,x)\,dx\,ds\,. \tag{2.24}$$

In particular, (2.21) and (2.22) may be rewritten as

$$m_1(t) \int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,dx \geqslant \mathcal{F}(t), \tag{2.25}$$

$$m_2(t) \int_{\mathbb{R}^n} \partial_t v(t,x)\Psi(t,x)\,dx \geqslant \mathcal{G}(t). \tag{2.26}$$

Using Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \partial_t v(t,x)\Psi(t,x)\,dx &\leqslant \left(\int_{\mathbb{R}^n} |\partial_t v(t,x)|^p \Psi(t,x)\,dx\right)^{\frac{1}{p}} \left(\int_{B_{R+t}} \Psi(t,x)\,dx\right)^{\frac{1}{p'}} \\ &\lesssim (1+t)^{\frac{n-1}{2p'}} \left(\int_{\mathbb{R}^n} |\partial_t v(t,x)|^p \Psi(t,x)\,dx\right)^{\frac{1}{p}}, \end{aligned} \tag{2.27}$$

where in the last step we used

$$\int_{B_{R+t}} \Psi(t,x)\,dx \lesssim (1+t)^{\frac{n-1}{2}}.$$

For further details on this estimate see [24] or [10, estimate (3.5)].

Combing (2.1), (2.26) and (2.27), we finally get

$$\begin{aligned}\mathcal{F}'(t) &= \tfrac{1}{2} m_1(t) \int_{\mathbb{R}^n} |\partial_t v(t,x)|^p \Psi(t,x)\,dx \\ &\gtrsim m_1(t)\,(1+t)^{-\frac{n-1}{2}(p-1)} \left(\int_{\mathbb{R}^n} \partial_t v(t,x)\Psi(t,x)\,dx\right)^p \\ &\gtrsim m_1(t)\,(m_2(t))^{-p}\,(1+t)^{-\frac{n-1}{2}(p-1)} (\mathcal{G}(t))^p \gtrsim (1+t)^{-\frac{n-1}{2}(p-1)} (\mathcal{G}(t))^p. \end{aligned}$$

In a similar way, it is possible to show that (2.25) implies the estimate $\mathcal{G}'(t) \gtrsim (1+t)^{-\frac{n-1}{2}(q-1)} (\mathcal{F}(t))^q$.

Summarizing, throughout this section we proved the following lemma.

Lemma 2.3. *Let us assume that u_0, u_1, v_0, v_1 satisfy the assumption of Theorem 1.2. Let (u,v) be a local solution of (1.1) and let $\mathcal{F}$ and $\mathcal{G}$ be the functionals defined by (2.23) and (2.24), respectively. Then, the following estimates hold:*

$$\mathcal{F}'(t) \geqslant C(1+t)^{-\frac{n-1}{2}(p-1)} (\mathcal{G}(t))^p, \tag{2.28}$$

$$\mathcal{G}'(t) \geqslant K(1+t)^{-\frac{n-1}{2}(q-1)} (\mathcal{F}(t))^q, \tag{2.29}$$

for any $t \geqslant 0$, where C, K are positive constants depending on n, p, q, R, b_1, b_2.

Remark 2.4. In some cases it is more convenient to rewrite (2.28) and (2.29) in the integral form, namely,

$$\mathcal{F}(t) \geqslant \mathcal{F}(0) + C \int_0^t (1+s)^{-\frac{n-1}{2}(p-1)} (\mathcal{G}(s))^p \, ds, \tag{2.30}$$

$$\mathcal{G}(t) \geqslant \mathcal{G}(0) + K \int_0^t (1+s)^{-\frac{n-1}{2}(q-1)} (\mathcal{F}(s))^q \, ds, \tag{2.31}$$

for any $t \geqslant 0$.

3. Proof of Theorem 1.2: subcritical case

In this section we prove the blow-up result in the subcritical case, that is, for $p, q > 1$ satisfying

$$\max\left\{ \frac{p+1}{pq-1}, \frac{q+1}{pq-1} \right\} > \frac{n-1}{2} .$$

The main tool of the proof is the next Kato's type lemma on ordinary differential inequalities including an upper bound estimate for the lifespan, whose proof can be found in [19].

Lemma 3.1. *Let $r > 1$, $a > 0$, $b > 0$ satisfy*

$$M \doteq \frac{r-1}{2} a - \frac{b}{2} + 1 > 0.$$

Assume that $H \in \mathcal{C}^2([0,T))$ satisfies

$$H(t) \geqslant At^a \qquad \textit{for} \ \ t \geqslant T_0, \tag{3.1}$$

$$H''(t) \geqslant B(t+R)^{-b} |H(t)|^r \qquad \textit{for} \ \ t \geqslant 0, \tag{3.2}$$

$$H(0) \geqslant 0, \quad H'(0) > 0, \tag{3.3}$$

where A, B, R, T_0 are positive constants. Then, there exists a positive constant $C_0 = C_0(r, a, b, B)$ such that

$$T < 2^{\frac{2}{M}} T_1 \tag{3.4}$$

holds, provided that

$$T_1 \doteq \max\left\{ T_0, \frac{H(0)}{H'(0)}, R \right\} \geqslant C_0 A^{-\frac{r-1}{2M}}. \tag{3.5}$$

Let us consider the case in which p, q satisfy $\frac{p+1}{pq-1} - \frac{n-1}{2} > 0$. From (2.31) and Hölder's inequality, it follows

$$\begin{aligned} \mathcal{G}(t) &\gtrsim (1+t)^{-\frac{n-1}{2}(q-1)} \int_0^t (\mathcal{F}(s))^q \, ds \\ &\gtrsim (1+t)^{-\frac{n-1}{2}(q-1)-(q-1)} \left(\int_0^t \mathcal{F}(s) \, ds \right)^q . \end{aligned}$$

Plugging this lower bound for $\mathscr{G}$ in (2.28), we get

$$\mathscr{F}'(t) \gtrsim (1+t)^{-\frac{n-1}{2}(pq-1)-p(q-1)} \left(\int_0^t \mathscr{F}(s)\, ds \right)^{pq}. \tag{3.6}$$

Let us define now the functional

$$F(t) \doteq \int_0^t \mathscr{F}(\tau) d\tau.$$

Then, (3.6) is equivalent to

$$F''(t) \gtrsim (1+t)^{-\frac{n-1}{2}(pq-1)-p(q-1)} (F(t))^{pq} \qquad \text{for} \ \ t \geqslant 0. \tag{3.7}$$

Besides,

$$F(0) = 0, \qquad F'(0) = \mathscr{F}(0) = \varepsilon I_1[u_1] > 0, \tag{3.8}$$

where $I_1[u_1] \doteq \frac{m_1(0)}{2} \int_{\mathbb{R}^n} u_1(x)\Phi(x)\, dx$. Finally, since $\mathscr{F}$ is increasing,

$$F(t) \geqslant t\,\mathscr{F}(0) = A_0\varepsilon\, t \qquad \text{for} \ \ t \geqslant 0, \tag{3.9}$$

where A_0 is positive, independent of ε constant. Combining (3.7), (3.8) and (3.9), we can apply Lemma 3.1 to F with $r = pq$, $a = 1$, $b = \frac{n-1}{2}(pq-1) + p(q-1)$, $A = A_0\varepsilon$ and $R = 1$. In particular, thanks to (3.9), we can take $T_0 \doteq (A_0\varepsilon)^{-\Lambda(n,p,q)^{-1}}$.

Therefore, we may choose ε_0 sufficiently small, such that for any $\varepsilon \in (0, \varepsilon_0]$ the condition

$$T_0 \geqslant \max\left\{ \frac{F(0)}{F'(0)}, R \right\}$$

holds, due to the fact that the quantity $\frac{F(0)}{F'(0)} = 0$ does not depend on ε, and, then, using the notations of Lemma 3.1, we have $T_0 = T_1$. Hence, (3.4) implies $T \lesssim \varepsilon^{-\Lambda(n,p,q)^{-1}}$.

The treatment of the case in which p, q satisfy $\frac{q+1}{pq-1} - \frac{n-1}{2} > 0$ is totally symmetric. Indeed, by switching the role of $\mathscr{F}$ and $\mathscr{G}$ we get $T \lesssim \varepsilon^{-\Lambda(n,q,p)^{-1}}$. This completes the proof of (1.13) in the subcritical case.

4. Proof of Theorem 1.2: critical case

In this section we prove the blow-up result in the critical case

$$\max\left\{ \frac{p+1}{pq-1}, \frac{q+1}{pq-1} \right\} = \frac{n-1}{2}.$$

Differently from Section 3 in this case we will employ an iteration argument. Without loss of generality we may assume $\frac{p+1}{pq-1} = \frac{n-1}{2}$ (in the case $\frac{q+1}{pq-1} = \frac{n-1}{2}$ the proof is completely analogous, provided that we switch the roles of $\mathscr{F}$ and $\mathscr{G}$).

In order to get (1.13) in critical case, we have to consider separately the case $p \neq q$, which corresponds to $\Lambda(n,p,q) = 0 < \Lambda(n,q,p)$, from the case $p = q$, which corresponds to $\Lambda(n,p,q) = 0 = \Lambda(n,q,p)$.

Case $p \neq q$

In this case we apply the so-called slicing method (cf. [2], where this approach has been used for the first time). We introduce the sequence $\{\ell_j\}_{j\in\mathbb{N}}$ with $\ell_j \doteq 2 - 2^{-j}$. The first step of our procedure consists in proving via an inductive argument the sequence of lower bound estimates

$$\mathscr{F}(t) \geqslant C_j \left(\log\left(\frac{t}{\ell_j}\right)\right)^{a_j} \qquad \text{for any } t \geqslant \ell_j \text{ and for any } j \in \mathbb{N}, \tag{4.1}$$

where $\{a_j\}_{j\in\mathbb{N}}$ and $\{C_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative numbers that we will determine throughout this section. We point out that, due to (2.30), (4.1) is satisfied in the case $j = 0$ with

$$a_0 \doteq 0 \quad \text{and} \quad C_0 \doteq \mathscr{F}(0) = \varepsilon\, I_1[u_1].$$

Let us prove now the inductive step. We assume that (4.1) is true for some $j \geqslant 0$. Hence, plugging (4.1) in (2.31) and shrinking the domain of integration, we have for $t \geqslant \ell_{j+1} \geqslant 1$

$$\begin{aligned}
\mathscr{G}(t) &\geqslant K \int_{\ell_j}^t (1+s)^{-\frac{n-1}{2}(q-1)} C_j^q \left(\log\left(\tfrac{s}{\ell_j}\right)\right)^{a_j q} ds \\
&\geqslant K C_j^q (1+t)^{-\frac{n-1}{2}(q-1)} \int_{\ell_j}^t \left(\log\left(\tfrac{s}{\ell_j}\right)\right)^{a_j q} ds \\
&\geqslant K C_j^q (1+t)^{-\frac{n-1}{2}(q-1)} \int_{\frac{\ell_j t}{\ell_{j+1}}}^t \left(\log\left(\tfrac{s}{\ell_j}\right)\right)^{a_j q} ds \\
&\geqslant K C_j^q \left(1 - \tfrac{\ell_j}{\ell_{j+1}}\right) t\,(1+t)^{-\frac{n-1}{2}(q-1)} \left(\log\left(\tfrac{t}{\ell_{j+1}}\right)\right)^{a_j q} \\
&\geqslant K C_j^q 2^{-(j+3)} (1+t)^{-\frac{n-1}{2}(q-1)+1} \left(\log\left(\tfrac{t}{\ell_{j+1}}\right)\right)^{a_j q},
\end{aligned}$$

where in the last step we employed the inequality $1 - \frac{\ell_j}{\ell_{j+1}} \geqslant 2^{-(j+2)}$.

Using the above lower bound for $\mathscr{G}$ in (2.30), after restricting the domain of integration, for $t \geqslant \ell_{j+1}$ we arrive at

$$\begin{aligned}
\mathscr{F}(t) &\geqslant C K^p C_j^{pq} 2^{-(j+3)p} \int_{\ell_{j+1}}^t (1+s)^{-\frac{n-1}{2}(pq-1)+p} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{a_j pq} ds \\
&\geqslant C K^p C_j^{pq} 2^{-(j+3)p-1} \int_{\ell_{j+1}}^t s^{-1} \left(\log\left(\tfrac{s}{\ell_{j+1}}\right)\right)^{a_j pq} ds \\
&= C K^p C_j^{pq} 2^{-(j+3)p-1} (a_j pq + 1)^{-1} \left(\log\left(\tfrac{t}{\ell_{j+1}}\right)\right)^{a_j pq+1},
\end{aligned}$$

where in the second last step we used the condition $\Lambda(n,p,q) = 0$ to get -1 as power for term $(1+s)$ in the integral. Summarizing we proved (4.1) for $j+1$ with

$$C_{j+1} \doteq C K^p 2^{-(j+3)p-1} (a_j pq + 1)^{-1} C_j^{pq} \quad \text{and} \quad a_{j+1} \doteq a_j pq + 1.$$

In the next step we determine a lower bound for C_j. However, we need to find the explicit representation of a_j first. As $a_j = 1 + pq a_{j-1}$, applying iteratively this relation and the value of initial element of the sequence $a_0 = 0$, we find

$$a_j = \sum_{k=0}^{j-1} (pq)^k + (pq)^j a_0 = \frac{(pq)^j - 1}{pq - 1}. \tag{4.2}$$

In particular, $a_{j-1}pq + 1 = a_j \leqslant (pq-1)^{-1}(pq)^j$ implies

$$C_j \geqslant N 2^{-jp} (pq)^{-j} C_{j-1}^{pq}, \tag{4.3}$$

where $N \doteq CK^p 2^{-2p-1}(pq-1)$. Applying the logarithmic function to both sides of (4.3) and using iteratively the obtained inequality, we find

$$\begin{aligned}
\log C_j &\geqslant pq \log C_{j-1} - j \log(2^p(pq)) + \log N \\
&\geqslant (pq)^2 \log C_{j-2} - (j + (j-1)pq) \log(2^p(pq)) + (1+pq) \log N \\
&\geqslant \cdots \geqslant (pq)^j \log C_0 - \sum_{k=0}^{j-1} (j-k)(pq)^k \log(2^p(pq)) + \sum_{k=0}^{j-1} (pq)^k \log N \\
&= (pq)^j \left(\log C_0 - \frac{pq}{(pq-1)^2} \log(2^p pq) + \frac{\log N}{pq-1} \right) \\
&\quad + (j+1) \frac{\log(2^p pq)}{pq-1} + \frac{\log(2^p pq)}{(pq-1)^2} - \frac{\log N}{pq-1},
\end{aligned} \tag{4.4}$$

where in the last step we employed the formula

$$\sum_{k=0}^{j-1} (j-k)(pq)^k = \frac{1}{pq-1} \left(\frac{(pq)^{j+1} - 1}{pq-1} - (j+1) \right), \tag{4.5}$$

which can be proved by induction.

Thus, for $j \geqslant j_0 \doteq \lceil \frac{\log N}{\log(2^p pq)} - 1 - \frac{1}{pq-1} \rceil$ by (4.4) we get

$$\begin{aligned}
\log C_j &\geqslant (pq)^j \left(\log C_0 - \frac{pq}{(pq-1)^2} \log(2^p pq) + \frac{\log N}{pq-1} \right) \\
&= (pq)^j \log(D\varepsilon),
\end{aligned} \tag{4.6}$$

where $D \doteq (2^p pq)^{-\frac{pq}{(pq-1)^2}} N^{\frac{1}{pq-1}} I_1[u_1]$. Combining (4.1) and (4.6), it results for any $t \geqslant 2 \geqslant \ell_j$

$$\mathscr{F}(t) \geqslant \exp\left((pq)^j \log(D\varepsilon)\right) \left(\log\left(\tfrac{t}{\ell_j}\right) \right)^{a_j} \geqslant \exp\left((pq)^j \log(D\varepsilon)\right) \left(\log\left(\tfrac{t}{2}\right) \right)^{a_j}.$$

Since for any $t \geqslant 4$ it holds $\log\left(\frac{t}{2}\right) \geqslant \frac{1}{2} \log t$, from the above relation and (4.2) it follows

$$\mathscr{F}(t) \geqslant \left((pq)^j \log\left(2^{-\frac{1}{pq-1}} D\varepsilon \, (\log t)^{\frac{1}{pq-1}} \right) \right) \left(\log\left(\tfrac{t}{2}\right) \right)^{-\frac{1}{pq-1}}. \tag{4.7}$$

Finally, we may choose $\varepsilon_0 > 0$ so small that

$$\exp\left(2D^{-pq+1} \varepsilon_0^{-(pq-1)} \right) \geqslant 4.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp\big(2D^{-pq+1}\varepsilon^{-(pq-1)}\big)$ it results $t \geqslant 4$ and $\log\Big(2^{-\frac{1}{pq-1}} D\varepsilon\,(\log t)^{\frac{1}{pq-1}}\Big) > 0$ and, thus, letting $j \to \infty$ in (4.7) we find that the lower bound of $\mathcal{F}(t)$ blows up. Therefore, $\mathcal{F}(t)$ may be finite only for $t \leqslant \exp\big(2D^{-pq+1}\varepsilon^{-(pq-1)}\big)$. This is exactly (1.13) in the critical case $\Lambda(n, p, q) = 0$ for $p \neq q$.

Case $p = q$

In this section we consider the case $\Lambda(n, p, q) = 0 = \Lambda(n, q, p)$. In particular, we have $p = q$. Moreover, the condition $\Lambda(n, p, p) = 0$ is satisfied if and only if $p = p_{\mathrm{Gla}}(n) = \frac{n+1}{n-1}$. This implies that the powers of $(1 + s)$ in the right hand sides of (2.28) and (2.29) are exactly -1. Therefore, up to a not relevant modification of the multiplicative constants, we may reformulate the integral version of (2.28) and (2.29) as follows:

$$\mathcal{F}(t) \geqslant \mathcal{F}(0) + C \int_1^t \frac{(\mathcal{G}(s))^p}{s}\, ds, \tag{4.8}$$

$$\mathcal{G}(t) \geqslant \mathcal{G}(0) + K \int_1^t \frac{(\mathcal{F}(s))^q}{s}\, ds. \tag{4.9}$$

Due to the particular structure of the iteration frame given by (4.8), (4.9), it is not necessary to slice the time interval in this case. As in the previous section, the first step is to prove the lower bound estimates

$$\mathcal{F}(t) \geqslant C_j \big(\log t\big)^{a_j} \qquad \text{for any } t \geqslant 1 \text{ and for any } j \in \mathbb{N}, \tag{4.10}$$

where $\{a_j\}_{j\in\mathbb{N}}$ and $\{C_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative numbers that we will determine throughout the proof. According to (4.8), we see that (4.10) is true for $j = 0$ provided that $a_0 \doteq 0$ and $C_0 \doteq \varepsilon I_1[u_1]$. Let us prove now the inductive step. Plugging the lower bound (4.10) in (4.9), we find for $t \geqslant 1$

$$\mathcal{G}(t) \geqslant K C_j^q \int_1^t \frac{\big(\log s\big)^{a_j q}}{s}\, ds = K C_j^q (a_j q + 1)^{-1} \big(\log t\big)^{a_j q + 1}.$$

Combining the last inequality with (4.8), we have

$$\begin{aligned} \mathcal{F}(t) &\geqslant C K^p C_j^{pq} (a_j q + 1)^{-p} \int_1^t \frac{\big(\log s\big)^{a_j pq + p}}{s}\, ds \\ &= C K^p C_j^{pq} (a_j q + 1)^{-p} (a_j pq + p + 1)^{-1} \big(\log t\big)^{a_j pq + p + 1}. \end{aligned}$$

Also, we proved (4.10) for $j + 1$, provided that

$$a_{j+1} \doteq a_j pq + p + 1 \quad \text{and} \quad C_{j+1} \doteq C K^p C_j^{pq} (a_j q + 1)^{-p} (a_j pq + p + 1)^{-1}.$$

By iteration, we arrive at

$$a_j = (p + 1) \sum_{k=0}^{j-1} (pq)^k + (pq)^j a_0 = \frac{p + 1}{pq - 1} \big((pq)^j - 1\big) = \frac{(pq)^j - 1}{p - 1}. \tag{4.11}$$

By (4.11) we derive, in particular,

$$\begin{aligned} a_{j-1}pq+p+1 = a_j &\leqslant (p-1)^{-1}(pq)^j, \\ a_{j-1}q+1 = \tfrac{(pq)^j}{p(p-1)} - \tfrac{1}{p-1} &\leqslant (p(p-1))^{-1}(pq)^j. \end{aligned}$$

The above estimates imply

$$C_j \geqslant N\big((pq)^{p+1}\big)^{-j} C_{j-1}^{pq}, \tag{4.12}$$

where $N \doteq CK^p(p-1)^{p+1}p^p$. Applying the logarithm function to both sides of (4.12) and using in an iterative way the resulting relation, we have

$$\begin{aligned} \log C_j &\geqslant pq\log C_{j-1} - j\log(pq)^{p+1} + \log N \\ &\geqslant (pq)^2 \log C_{j-2} - (j+(j-1)pq)\log(pq)^{p+1} + (1+pq)\log N \\ &\geqslant \cdots \geqslant (pq)^j \log C_0 - \sum_{k=0}^{j-1}(j-k)(pq)^k \log(pq)^{p+1} + \sum_{k=0}^{j-1}(pq)^k \log N \\ &= (pq)^j\left(\log C_0 - \frac{pq}{(pq-1)^2}\log(pq)^{p+1} + \frac{\log N}{pq-1}\right) \\ &\quad + (j+1)\,\frac{\log(pq)^{p+1}}{pq-1} + \frac{\log(pq)^{p+1}}{(pq-1)^2} - \frac{\log N}{pq-1}, \end{aligned} \tag{4.13}$$

where we used again (4.5).

Thus, for $j \geqslant j_0 \doteq \lceil \frac{\log N}{\log(pq)^{p+1}} - 1 - \frac{1}{pq-1}\rceil$ by (4.13) we get

$$\begin{aligned} \log C_j &\geqslant (pq)^j\left(\log C_0 - \frac{pq}{(pq-1)^2}\log(pq)^{p+1} + \frac{\log N}{pq-1}\right) \\ &= (pq)^j \log(D\varepsilon), \end{aligned} \tag{4.14}$$

where $D \doteq (pq)^{-\frac{pq(p+1)}{(pq-1)^2}} N^{\frac{1}{pq-1}} I_1[u_1]$. Then, using together (4.10), (4.11) and (4.14), $t \geqslant 1$ it holds

$$\begin{aligned} \mathscr{F}(t) &\geqslant \exp\big((pq)^j \log(D\varepsilon)\big)\big(\log t\big)^{a_j} \\ &= \exp\Big((pq)^j \log\Big(D\varepsilon\,\big(\log t\big)^{\frac{1}{p-1}}\Big)\Big)\big(\log t\big)^{-\frac{1}{p-1}}. \end{aligned} \tag{4.15}$$

In conclusion, we can find $\varepsilon_0 > 0$ sufficiently small, so that

$$\exp\Big(D^{-p+1}\varepsilon_0^{-(p-1)}\Big) \geqslant 1.$$

So, for any $\varepsilon \in (0,\varepsilon_0]$ and for $t > \exp\big(D^{-p+1}\varepsilon^{-(p-1)}\big)$ we have $t \geqslant 1$ and

$$\log\Big(D\varepsilon\,\big(\log t\big)^{\frac{1}{p-1}}\Big) > 0.$$

However, taking the limit as $j \to \infty$ in (4.15), we find that the lower bound of $\mathscr{F}(t)$ blows up. Hence, $\mathscr{F}(t)$ can be finite only for $t \leqslant \exp\big(D^{-p+1}\varepsilon^{-(p-1)}\big)$. This is precisely (1.13) in the critical case $p = q = p_{\mathrm{Gla}}(n)$.

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