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A NOTE ON THE NONEXISTENCE OF GLOBAL SOLUTIONS TO THE SEMILINEAR WAVE EQUATION WITH NONLINEARITY OF DERIVATIVE-TYPE IN THE GENERALIZED EINSTEIN-DE SITTER SPACETIME

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Abstract. In this paper, we establish blow-up results for the semilinear wave equation in generalized Einstein-de Sitter spacetime with nonlinearity of derivative type. Our approach is based on the integral representation formula for the solution to the corresponding linear problem in the one-dimensional case, that we will determine through Yagdjian's Integral Transform approach. As upper bound for the exponent of the nonlinear term, we discover a Glassey-type exponent which depends both on the space dimension and on the Lorentzian metric in the generalized Einstein-de Sitter spacetime.

1. Introduction.

The aim of the present paper is to establish a blow-up result for local in time solutions to the Cauchy problem with *nonlinearity of derivative type* $|\partial_t u|^p$

$$\begin{cases} \partial_t^2 u - t^{-2k}\Delta u + 2t^{-1}\partial_t u = |\partial_t u|^p, & x \in \mathbb{R}^n,\ t>1, \\ u(1,x) = \varepsilon u_0(x), & x\in\mathbb{R}^n, \\ u_t(1,x) = \varepsilon u_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $k \in (0,1)$, $p>1$ and ε is a positive constant describing the size of Cauchy data.

In the related literature (see, for example, [3]), the differential operator with time-dependent coefficients on the left-hand side of (1.1) is called the wave operator on

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the *generalized Einstein-de Sitter spacetime*. This nomenclature is due to the fact that for $k = \frac{2}{3}$ and $n = 3$ the operator $\partial_t^2 - t^{-\frac{4}{3}}\Delta + 2t^{-1}\partial_t$ coincides with the d'Alembertian operator in Einstein-de Sitter's Lorentzian metric. We emphasize that in our setting the speed of propagation for the wave operator in the generalized Einstein-de Sitter spacetime is a decreasing and not singular function.

In recent years, many papers have been devoted to the study of blow-up results and lifespan estimates for the semilinear wave equation in the generalized Einstein - de Sitter (EdS) spacetime with power nonlinearities [3, 14] and generalizations [19, 20, 15]. More specifically, it has been conjectured that the critical exponent for the semilinear Cauchy problem with *power nonlinearity* $|u|^p$

$$\begin{cases} \partial_t^2 u - t^{-2k}\Delta u + 2t^{-1}\partial_t u = |u|^p, & x \in \mathbb{R}^n, \ t > 1, \\ u(1,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(1,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

is given by the competition between two exponents, namely,

$$\max\left\{p_0(n,k), 1 + \tfrac{2}{(1-k)n}\right\}, \tag{1.3}$$

where $p_0(n,k)$ is the greatest root of the quadratic equation

$$((1-k)n+1)(p-1)^2 + ((1-k)n - 1 - 2k)(p-1) - 4 = 0.$$

The results from the previously quoted papers have been obtained via four different approaches, namely, Kato's type lemma, iteration argument (together with a slicing procedure in the critical case), comparison argument, and test function method. We can summarize these results by saying that for local in time solutions to (1.2) it has been shown the validity of a blow-up result (under suitable sign and support assumptions for the Cauchy data) whenever the exponent of the power nonlinearity $|u|^p$ fulfills $1 < p \leqslant \max\{p_0(n,k), 1 + \frac{2}{(1-k)n}\}$. Due to these nonexistence results, it has been conjecture that the exponent in (1.3) is critical, even though the global existence of small data solutions is a completely open problem to the best of the authors' knowledge.

The goal of the present work is to establish a blow-up result for (1.1) and to determine a candidate as critical exponent somehow related to the so-called *Glassey exponent*

$$p_{\mathrm{Gla}}(n) \doteq \begin{cases} \frac{n+1}{n-1} & \text{if } n > 1, \\ \infty & \text{if } n = 1, \end{cases} \tag{1.4}$$

which is the critical exponent for the semilinear wave equation with nonlinearity of derivative type see [21, 29, 8] and references therein.

Our approach consists in a modification of Zhou's approach for the corresponding semilinear wave equation in [29]. Consequently, we will devote the first part of the paper to the proof of an integral representation formula for the solution to the linear inhomogeneous Cauchy problem associated with (1.1) in the case $n = 1$ via Yagdjian's Integral Transform approach.

We consider also a second semilinear model, that can be studied analogously with the tools of the present work. If we delete the linear damping term $2t^{-1}\partial_t u$ in (1.1), we obtain somehow a semilinear *Tricomi-type model with negative power in the speed of propagation* and nonlinearity of derivative type. Hence, with minor modifications in our blow-up argument for (1.1) we will prove a blow-up result also for the following model

$$\begin{cases} \partial_t^2 u - t^{-2k}\Delta u = |\partial_t u|^p, & x\in\mathbb{R}^n,\ t>1, \\ u(1,x)=\varepsilon u_0(x), & x\in\mathbb{R}^n, \\ u_t(1,x)=\varepsilon u_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.5}$$

where $k\in(0,1)$, $p>1$ and ε is a positive constant describing the size of Cauchy data. We will see that the upper bound for the exponent p in the blow-up result for (1.1) is a shift of the corresponding upper bound for (1.5).

Recently, the semilinear generalized Tricomi equation with nonlinearity of derivative type, namely, the Cauchy problem

$$\begin{cases} \partial_t^2 u - t^{2\ell}\Delta u = |\partial_t u|^p, & x\in\mathbb{R}^n,\ t>0, \\ u(0,x)=\varepsilon u_0(x), & x\in\mathbb{R}^n, \\ u_t(0,x)=\varepsilon u_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.6}$$

where $\ell>0$, $p>1$, $\varepsilon>0$, has been attracting a lot of attention and different approaches have been applied to establish several blow-up results for p in suitable ranges depending on n,ℓ. Among these approaches we find, on the one hand, comparison arguments based either on the fundamental solution for the operator $\partial_t^2 - t^{2\ell}\Delta$ in [10] or on the employment of a special positive solution of the corresponding homogeneous equation involving modified Bessel functions of second kind in [6], and, on the other hand, a *modified test function method* in [9]. In particular, the result that we are going to show for (1.5) is consistent with those for (1.6) in [9, 6], although we shall use a different approach to prove this result.

1.1. **Main results.** In this section, we state the main blow-up results for the semilinear models in (1.1) and (1.5), respectively.

Notations. Since the speed of propagation in both semilinear models (1.1) and (1.5) is $a_k(t)\doteq t^{-k}$, in the following it will be useful to employ the notation

$$\phi_k(t)\doteq\frac{t^{1-k}}{1-k}, \tag{1.7}$$

for the primitive of a_k vanishing at $t=0$ and the notation

$$A_k(t)\doteq\phi_k(t)-\phi_k(1)=\int_1^t a_k(\tau)\,\mathrm{d}\tau, \tag{1.8}$$

for the distance function describing the amplitude of the curved light-cone.
Moreover, we will write $f\lesssim g$ when there exists a positive constant C such that $f\leqslant Cg$. Note that the constant C may depend on n,k,p,R but not on ε. The relation $f\gtrsim g$ is analogously defined. Finally, $f\approx g$ means that both $f\lesssim g$ and $f\gtrsim g$ are satisfied.

Theorem 1.1 (Blow-up result for the semilinear wave equation in EdS spacetime)**.** *Let $n\geqslant 1$ and $k\in(0,1)$. We assume that $(u_0,u_1)\in\mathscr{C}_0^2(\mathbb{R}^n)\times\mathscr{C}_0^1(\mathbb{R}^n)$ are nonnegative functions and have supports contained in $B_R\doteq\{x\in\mathbb{R}^n : |x|<R\}$. Let us consider an exponent p in the nonlinearity of derivative type such that*

$$1<p\leqslant p_{\mathrm{Gla}}\big((1-k)n+2k+2\big), \tag{1.9}$$

where the Glassey exponent p_{Gla} is defined by (1.4).

Then, there exists $\varepsilon_0 = \varepsilon_0(n, p, k, u_0, u_1, R) > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *if* $u \in \mathscr{C}^2([1, T) \times \mathbb{R}^n)$ *is a local in time solution to* (1.1) *such that*

$$\operatorname{supp} u(t, \cdot) \subset B_{R+A_k(t)}$$

for any $t \in [1, T)$ *and with lifespan* $T = T(\varepsilon)$*, then,* u *blows up in finite time, that is,* $T < \infty$*.*

Furthermore, the following upper bound estimate for the lifespan holds

$$T(\varepsilon) \leqslant \begin{cases} C\varepsilon^{-\left(\frac{1}{p-1} - \frac{(1-k)n+2k+1}{2}\right)^{-1}} & \text{if } \ 1 < p < p_{\mathrm{Gla}}\big((1-k)n + 2k + 2\big), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } \ p = p_{\mathrm{Gla}}\big((1-k)n + 2k + 2\big), \end{cases} \tag{1.10}$$

where the positive constant C *is independent of* ε*.*

Theorem 1.2 (Blow-up result for the semilinear Tricomi-type equation)**.** *Let* $n \geqslant 1$ *and* $k \in (0, 1)$*. We assume that* $u_0 = 0$ *and that* $u_1 \in \mathscr{C}_0^1(\mathbb{R}^n)$ *is nonnegative function with support contained in* B_R *for some* $R > 0$*. Let us consider an exponent* p *in the nonlinearity of derivative type such that*

$$1 < p \leqslant p_{\mathrm{Gla}}\big((1-k)n + 2k\big), \tag{1.11}$$

where the Glassey exponent p_{Gla} *is defined by* (1.4)*.*

Then, there exists $\varepsilon_0 = \varepsilon_0(n, p, k, u_1, R) > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *if* $u \in \mathscr{C}^2([1, T) \times \mathbb{R}^n)$ *is a local in time solution to* (1.5) *such that*

$$\operatorname{supp} u(t, \cdot) \subset B_{R+A_k(t)}$$

for any $t \in [1, T)$ *and with lifespan* $T = T(\varepsilon)$*, then,* u *blows up in finite time, that is,* $T < \infty$*.*

Furthermore, the following upper bound estimate for the lifespan holds

$$T(\varepsilon) \leqslant \begin{cases} C\varepsilon^{-\left(\frac{1}{p-1} - \frac{(1-k)n+2k-1}{2}\right)^{-1}} & \text{if } \ 1 < p < p_{\mathrm{Gla}}\big((1-k)n + 2k\big), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } \ p = p_{\mathrm{Gla}}\big((1-k)n + 2k\big), \end{cases} \tag{1.12}$$

where the positive constant C *is independent of* ε*.*

Remark 1. The upper bound for p in the blow-up range (1.9) is a shift of magnitude 2 of the Glassey exponent that appears as upper bound in (1.11). This kind of phenomenon has already been observed in the semilinear model with power nonlinearity in [19, 15] for the wave equation in the generalized Einstein-de Sitter spacetime.

Remark 2. The upper bound $p_{\mathrm{Gla}}\big((1-k)n + 2k\big)$ in Theorem 1.2 is consistent with the upper bound for the semilinear generalized Tricomi with nonlinearity of derivative type (when the power in the speed of propagation is positive and the Cauchy data are assumed at the initial time $t = 0$) see e.g. [9, 6].

Remark 3. In Theorem 1.2 we required a trivial first Cauchy data ($u_0 = 0$). This assumption is due to the fact that, in general, the kernel function $K_0\big(t, x; y; 0, 0, k\big)$ is not a nonnegative function, see (2.6).

Remark 4. Let us emphasize that the support conditions in the statements of Theorems 1.1 and 1.2 can be proved extending the representation formulae from Section 2 to the multidimensional case. However, this is beyond the purposes of the present paper and we refer to the appendix in [19] for the proof of the property of finite speed of propagation.

The paper is organized as follows: in Section 2 we will derive some representation formulae for the solutions to the linear models associated with (1.1) and (1.5), respectively, by using Yagdjian's Integral Transform approach. Then, in Section 3 we will prove Theorems 1.1 and 1.2 by means of Zhou's approach from [29] for the classical semilinear wave equation with nonlinearity of derivative type. Finally, in Section 4 we provide some final remarks.

2. Integral representation formula.

In the series of papers [22, 23, 27, 28, 24, 25, 13, 26], several integral representation formulae for solutions to Cauchy problems associated with linear hyperbolic equations with variable coefficients have been derived and applied both to study the necessity and the sufficiency part concerning the problem of the global (in time) existence of solutions. The general scheme to determine an integral representation in the above cited literature is the following: the desired formula is obtained by considering the composition of two operators. On the one hand, the first operator is an integral transformation, whose kernel is determined by the time-dependent coefficients and/or by the lower-order terms in the associated partial differential operator. On the other hand, the second operator is a *solution operator for a family of parameter dependent Cauchy problems* (this step is often called, in the above quoted literature, a revised Duhamel's principle). The above mentioned solution operator associates with a given function the solution to the Cauchy problem for the classical free wave equation with the given function as first initial data and with vanishing second initial data in some special cases. This happens for example when the considered hyperbolic model is a wave equation with time-dependent speed of propagation and/or lower-order terms.

In the present section, we are going to use Yagdjian's Integral Transform approach in order to determine an explicit integral representation formula for the linear Cauchy problem

$$\begin{cases} \partial_t^2 u - t^{-2k}\partial_x^2 u + \mu\, t^{-1}\partial_t u + \nu^2\, t^{-2} u = g(t,x), & x\in\mathbb{R},\ t>1, \\ u(1,x)=u_0(x), & x\in\mathbb{R}, \\ \partial_t u(1,x)=u_1(x), & x\in\mathbb{R}, \end{cases} \tag{2.1}$$

where μ,ν^2 are nonnegative real parameters and $k\in(0,1)$. Of course, (2.1) represents a more general model than the ones which are necessary to prove the blow-up results for (1.1) and (1.5) that are stated in Section 1.1. Nonetheless, thanks to the study of the representation formula for (2.1) we will be able to derive as special cases the representation formulae for the solutions to the corresponding linear wave equation in the Einstein-de Sitter spacetime and linear Tricomi-type equation with negative power in the speed of propagation. Let us introduce the operator

$$\mathscr{L}_{k,\mu,\nu^2} \doteq \partial_t^2 - t^{-2k}\partial_x^2 + \mu\, t^{-1}\partial_t + \nu^2\, t^{-2}.$$

The quantity

$$\delta = \delta(\mu,\nu^2) \doteq (\mu-1)^2 - 4\nu^2, \tag{2.2}$$

has a crucial role in determining some properties of the fundamental solution of $\mathscr{L}_{k,\mu,\nu^2}$. In the special case $k=0$ (the so-called wave operator with scale-invariant damping and mass), it is known in the literature that the value of δ affects not only the fundamental solution of $\mathscr{L}_{0,\mu,\nu^2}$ but also the critical exponents in the treatment of semilinear Cauchy problem associated with $\mathscr{L}_{0,\mu,\nu^2}$ with power nonlinearity [11, 16, 17, 1], nonlinearity of derivative type [18], and combined nonlinearity [4, 5].

We shall see that even in the case $k \in (0,1)$ some properties of the fundamental solution of $\mathscr{L}_{k,\mu,\nu^2}$ depend strongly on the value of δ. Of course, in the general case, we will find an interplay of δ and k in the description of the fundamental solution.

We point out that, even though in this section we will focus on the case $n = 1$, analogously to what is done in [22, 23, 27, 28, 13] it is possible to extend the integral representation even to the higher dimensional case by using the spherical means and the method of descent.

Let us state now the representation formula for the solution to (2.1) in space dimension 1.

Theorem 2.1. *Let $n = 1$, $k \in (0,1)$ and let μ, ν^2 be nonnegative constants. We assume $u_0 \in \mathscr{C}^2(\mathbb{R})$, $u_1 \in \mathscr{C}^1(\mathbb{R})$ and $g \in \mathscr{C}^{0,1}_{t,x}([1,\infty)\times\mathbb{R})$. Then, a representation formula for the solution u to (2.1) is given by*

$$\begin{aligned} u(t,x) &= \frac{1}{2}\, t^{\frac{k-\mu}{2}} \big(u_0(x+A_k(t)) + u_0(x-A_k(t))\big) \\ &\quad + \int_{x-A_k(t)}^{x+A_k(t)} u_0(y) K_0\big(t,x;y;\mu,\nu^2,k\big)\,\mathrm{d}y \\ &\quad + \int_{x-A_k(t)}^{x+A_k(t)} u_1(y) K_1\big(t,x;y;\mu,\nu^2,k\big)\,\mathrm{d}y \\ &\quad + \int_0^t \int_{x-\phi_k(t)+\phi_k(b)}^{x+\phi_k(t)-\phi_k(b)} g(b,y) E\big(t,x;b,y;\mu,\nu^2,k\big)\,\mathrm{d}y\,\mathrm{d}b. \end{aligned} \tag{2.3}$$

Here the kernel function E is defined by

$$\begin{aligned} E\big(t,x;b,y;\mu,\nu^2,k\big) &\doteq c\, t^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} b^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \left((\phi_k(t)+\phi_k(b))^2-(y-x)^2\right)^{-\gamma} \\ &\quad \times \mathsf{F}\left(\gamma,\gamma;1;\frac{(\phi_k(t)-\phi_k(b))^2-(y-x)^2}{(\phi_k(t)+\phi_k(b))^2-(y-x)^2}\right), \end{aligned} \tag{2.4}$$

where

$$c = c(\mu,\nu^2,k) \doteq 2^{-\frac{\sqrt{\delta}}{1-k}}(1-k)^{-1+\frac{\sqrt{\delta}}{1-k}} \quad \text{and} \quad \gamma = \gamma(\mu,\nu^2,k) \doteq \frac{1}{2} - \frac{\sqrt{\delta}}{2(1-k)}, \tag{2.5}$$

and $\mathsf{F}(\alpha_1,\alpha_2;\beta;z)$ denotes the Gauss hypergeometric function (cf. [12, Chapter 15]*), while the kernel functions K_0, K_1 appearing in the integral terms involving the Cauchy data are given by*

$$K_0\big(t,x;y;\mu,\nu^2,k\big) \doteq \mu E\big(t,x;1,y;\mu,\nu^2,k\big) - \frac{\partial}{\partial b} E\big(t,x;b,y;\mu,\nu^2,k\big)\Big|_{b=1}, \tag{2.6}$$

$$K_1\big(t,x;y;\mu,\nu^2,k\big) \doteq E\big(t,x;1,y;\mu,\nu^2,k\big). \tag{2.7}$$

Proof. We are going to prove the representation formula in (2.3) by means of a suitable change of variables that transforms (2.1) in a linear wave equation with scale-invariant damping and mass terms and allows us to employ a result from [13]. More specifically, we perform the transformation

$$\tau \doteq t^{1-k} - 1, \quad z \doteq (1-k)x. \tag{2.8}$$

Setting

$$v(\tau,z) = u(t,x),$$

by a straightforward computation it follows that u solves (2.1) if and only if v is a solution to

$$\begin{cases} \partial_\tau^2 v - \partial_z^2 v + \frac{\mu-k}{1-k}(1+\tau)^{-1}\partial_\tau v + \frac{\nu^2}{(1-k)^2}(1+\tau)^{-2} v = f(\tau,z), & z\in\mathbb{R},\ \tau>0,\\ v(0,z) = u_0\big(\frac{z}{1-k}\big), & z\in\mathbb{R},\\ \partial_\tau v(0,z) = \frac{1}{1-k}u_1\big(\frac{z}{1-k}\big), & z\in\mathbb{R}, \end{cases}$$

where

$$f(\tau,z) \doteq (1-k)^{-2}(1+\tau)^{\frac{2k}{1-k}} g\big((1+\tau)^{\frac{1}{1-k}}, \tfrac{z}{1-k}\big).$$

According to [13, Theorem 1.1], we can represent v in the following way

$$v = \sum_{j=0}^{3} v_j, \tag{2.9}$$

where the addends $\{v_j\}_{j\in\{0,1,2,3\}}$ are given by

$$\begin{aligned} v_0(\tau,z) &\doteq \tfrac12 (1+\tau)^{-\frac{\widetilde{\mu}}{2}}\big(u_0\big(\tfrac{z+\tau}{1-k}\big) + u_0\big(\tfrac{z-\tau}{1-k}\big)\big),\\ v_1(\tau,z) &\doteq 2^{-\sqrt{\widetilde{\delta}}}\int_{z-\tau}^{z+\tau} u_0\big(\tfrac{\widetilde{y}}{1-k}\big)\widetilde{K}_0\big(\tau,z;\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big)\,\mathrm{d}\widetilde{y},\\ v_2(\tau,z) &\doteq 2^{-\sqrt{\widetilde{\delta}}}\int_{z-\tau}^{z+\tau} \Big(\tfrac{1}{1-k}u_1\big(\tfrac{\widetilde{y}}{1-k}\big) + \tfrac{\mu-k}{1-k}u_0\big(\tfrac{\widetilde{y}}{1-k}\big)\Big)\widetilde{K}_1\big(\tau,z;\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big)\,\mathrm{d}\widetilde{y},\\ v_3(\tau,z) &\doteq 2^{-\sqrt{\widetilde{\delta}}}\int_0^\tau\int_{z-\tau+\widetilde{b}}^{z+\tau-\widetilde{b}} f(\widetilde{b},\widetilde{y})\widetilde{E}\big(\tau,z;\widetilde{b},\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big)\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{b}, \end{aligned}$$

and the kernel functions are given by

$$\begin{aligned} \widetilde{E}\big(\tau,z;\widetilde{b},\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big) &\doteq (1+\tau)^{-\frac{\widetilde{\mu}}{2}+\frac{1-\sqrt{\widetilde{\delta}}}{2}}\big(1+\widetilde{b}\big)^{\frac{\widetilde{\mu}}{2}+\frac{1-\sqrt{\widetilde{\delta}}}{2}}\\ &\quad\times\big((\tau+\widetilde{b}+2)^2-(\widetilde{y}-z)^2\big)^{\frac{\sqrt{\widetilde{\delta}}-1}{2}}\\ &\quad\times \mathsf{F}\left(\tfrac{1-\sqrt{\widetilde{\delta}}}{2},\tfrac{1-\sqrt{\widetilde{\delta}}}{2};1;\frac{(\tau-\widetilde{b})^2-(\widetilde{y}-z)^2}{(\tau+\widetilde{b}+2)^2-(\widetilde{y}-z)^2}\right),\\ \widetilde{K}_0\big(\tau,z;\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big) &\doteq -\frac{\partial}{\partial\widetilde{b}}\widetilde{E}\big(\tau,z;\widetilde{b},\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big)\Big|_{\widetilde{b}=0},\\ \widetilde{K}_1\big(\tau,z;\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big) &\doteq \widetilde{E}\big(\tau,z;0,\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big), \end{aligned}$$

where $\mathsf{F}(\alpha_1,\alpha_2;\beta;z)$ is the Gauss hypergeometric function and

$$\widetilde{\mu}\doteq\frac{\mu-k}{1-k},\quad \widetilde{\nu}\doteq\frac{\nu}{1-k},\quad \widetilde{\delta}=\widetilde{\delta}(\mu,\nu^2,k)\doteq(\widetilde{\mu}-1)^2-4\widetilde{\nu}^2=\frac{\delta(\mu,\nu^2)}{(1-k)^2}.$$

In order to show the validity of (2.3), we will transform back each term in (2.9) through (2.8).

Let us begin with the function v_0. Recalling the definition of the function A_k in (1.8), since $\frac{z\pm\tau}{1-k} = x\pm A_k(t)$, we can write immediately

$$v_0(\tau,y) = \tfrac12 t^{\frac{k-\mu}{2}}\big(u_0(x+A_k(t)) + u_0(x-A_k(t))\big). \tag{2.10}$$

Let us deal with the term v_3. Using the explicit expression of f, we get

$$v_3(\tau,z) = 2^{-\sqrt{\widetilde{\delta}}}\int_0^\tau\int_{z-\tau+\widetilde{b}}^{z+\tau-\widetilde{b}} \frac{(1+\widetilde{b})^{\frac{2k}{1-k}}}{(1-k)^2} g\big((1+\widetilde{b})^{\frac{1}{1-k}},\tfrac{\widetilde{y}}{1-k}\big)\widetilde{E}\big(\tau,z;\widetilde{b},\widetilde{y};\widetilde{\mu},\widetilde{\nu}^2\big)\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{b}.$$

Carrying out the change of variables $y = (1-k)^{-1}\widetilde{y}$, $b = \big(1+\widetilde{b}\big)^{\frac{1}{1-k}}$ and the transformation (2.8) in the last integral, we arrive at

$$v_3(\tau, z) = \int_1^t \int_{x-\phi_k(t)+\phi_k(b)}^{x+\phi_k(t)-\phi_k(b)} g(b,y)\, E\big(t,x;b,y;\mu,\nu^2,k\big)\, \mathrm{d}y\, \mathrm{d}b, \tag{2.11}$$

where we used the identity

$$\widetilde{E}\big(t^{1-k}-1,(1-k)x;b^{1-k}-1,(1-k)y;\widetilde{\mu},\widetilde{\nu}^2\big) = 2^{\frac{\sqrt{\delta}}{1-k}} b^{-k} E\big(t,x;b,y;\mu,\nu^2,k\big), \tag{2.12}$$

and the definition in (2.4).

Finally, we deal with the functions v_1, v_2. Performing the change of variables $y = (1-k)^{-1}\widetilde{y}$ and using (2.8), we find

$$v_1(\tau,z) \doteq 2^{-\frac{\sqrt{\delta}}{1-k}}(1-k) \int_{x-A_k(t)}^{x+A_k(t)} u_0(y) \widetilde{K}_0\big(t^{1-k}-1,(1-k)x;(1-k)y;\widetilde{\mu},\widetilde{\nu}^2\big)\, \mathrm{d}y,$$

$$\begin{aligned} v_2(\tau,z) \doteq{}& 2^{-\frac{\sqrt{\delta}}{1-k}} \int_{x-A_k(t)}^{x+A_k(t)} u_1(y) \widetilde{K}_1\big(t^{1-k}-1,(1-k)x;(1-k)y;\widetilde{\mu},\widetilde{\nu}^2\big)\, \mathrm{d}y \\ &+ 2^{-\frac{\sqrt{\delta}}{1-k}}(\mu-k) \int_{x-A_k(t)}^{x+A_k(t)} u_0(y) \widetilde{K}_1\big(t^{1-k}-1,(1-k)x;(1-k)y;\widetilde{\mu},\widetilde{\nu}^2\big)\, \mathrm{d}y. \end{aligned}$$

By elementary computations we have

$$\begin{aligned} &\widetilde{K}_0\big(t^{1-k}-1,(1-k)x;(1-k)y;\widetilde{\mu},\widetilde{\nu}^2\big) \\ &= -\,2^{\frac{\sqrt{\delta}}{1-k}} \frac{\partial}{\partial \widetilde{b}} \Big(\big(1+\widetilde{b}\big)^{-\frac{k}{1-k}} E\big(t,x;\big(1+\widetilde{b}\big)^{\frac{1}{1-k}},y;\mu,\nu^2,k\big)\Big)\Big|_{\widetilde{b}=0}, \end{aligned}$$

and

$$\widetilde{K}_1\big(t^{1-k}-1,(1-k)x;(1-k)y;\widetilde{\mu},\widetilde{\nu}^2\big) = 2^{\frac{\sqrt{\delta}}{1-k}} E\big(t,x;1,y;\mu,\nu^2,k\big),$$

whew we used (2.12) in the second identity.

Considering the transformation $b = \big(1+\widetilde{b}\big)^{\frac{1}{1-k}}$ and using the relation

$$\left(\frac{\partial}{\partial \widetilde{b}}\right)_{\widetilde{b}=0} = \frac{\partial b}{\partial \widetilde{b}}\Big|_{\widetilde{b}=0} \left(\frac{\partial}{\partial b}\right)_{b=1} = \frac{1}{1-k}\left(\frac{\partial}{\partial b}\right)_{b=1},$$

we obtain

$$\begin{aligned} &(1-k)\widetilde{K}_0\big(t^{1-k}-1,(1-k)x;(1-k)y;\widetilde{\mu},\widetilde{\nu}^2\big) \\ &= -\,2^{\frac{\sqrt{\delta}}{1-k}} \frac{\partial}{\partial b}\Big(b^{-k} E\big(t,x;b,y;\mu,\nu^2,k\big)\Big)\Big|_{b=1} \\ &= 2^{\frac{\sqrt{\delta}}{1-k}} k E\big(t,x;1,y;\mu,\nu^2,k\big) - 2^{\frac{\sqrt{\delta}}{1-k}} \frac{\partial}{\partial b}\Big(E\big(t,x;b,y;\mu,\nu^2,k\big)\Big)\Big|_{b=1}. \end{aligned}$$

Combining the previous representations for $\widetilde{K}_0, \widetilde{K}_1$, we conclude

$$\begin{aligned} &v_1(\tau,z) + v_2(\tau,z) \\ &= \int_{x-A_k(t)}^{x+A_k(t)} u_1(y) E\big(t,x;1,y;\mu,\nu^2,k\big)\, \mathrm{d}y \\ &\quad + \int_{x-A_k(t)}^{x+A_k(t)} u_0(y) \left(\mu E\big(t,x;1,y;\mu,\nu^2,k\big) - \frac{\partial}{\partial b}\Big(E\big(t,x;b,y;\mu,\nu^2,k\big)\Big)\Big|_{b=1}\right) \mathrm{d}y. \end{aligned} \tag{2.13}$$

Summarizing, from (2.9), (2.10), (2.11) and (2.13) it follows immediately (2.3). This completes the proof. □

Corollary 1 (Representation formula in EdS spacetime)**.** *Let* $n = 1$ *and* $k \in (0, 1)$*. We assume* $u_0 \in \mathscr{C}^2(\mathbb{R})$*,* $u_1 \in \mathscr{C}^1(\mathbb{R})$ *and* $g \in \mathscr{C}^{0,1}_{t,x}([1,\infty) \times \mathbb{R})$*. Then, a representation formula for the solution* u *to the Cauchy problem associated with the linear wave equation in Einstein-de Sitter spacetime*

$$\begin{cases} \partial_t^2 u - t^{-2k}\partial_x^2 u + 2\,t^{-1}\partial_t u = g(t,x), & x \in \mathbb{R},\ t > 1, \\ u(1,x) = u_0(x), & x \in \mathbb{R}, \\ \partial_t u(1,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{2.14}$$

is given by

$$\begin{aligned} u(t,x) = {} & \frac{1}{2} t^{\frac{k}{2}-1} \big(u_0(x + A_k(t)) + u_0(x - A_k(t))\big) \\ & + \int_{x-A_k(t)}^{x+A_k(t)} u_0(y) K_0\big(t,x;y;2,0,k\big)\,\mathrm{d}y \\ & + \int_{x-A_k(t)}^{x+A_k(t)} u_1(y) K_1\big(t,x;y;2,0,k\big)\,\mathrm{d}y \\ & + \int_0^t \int_{x-\phi_k(t)+\phi_k(b)}^{x+\phi_k(t)-\phi_k(b)} g(b,y) E\big(t,x;b,y;2,0,k\big)\,\mathrm{d}y\,\mathrm{d}b. \end{aligned} \tag{2.15}$$

Here the kernel function E *is defined by*

$$\begin{aligned} E\big(t,x;b,y;2,0,k\big) \doteq {} & c\,t^{-1} b \left((\phi_k(t) + \phi_k(b))^2 - (y-x)^2\right)^{-\gamma} \\ & \times \mathsf{F}\left(\gamma, \gamma; 1; \frac{(\phi_k(t) - \phi_k(b))^2 - (y-x)^2}{(\phi_k(t) + \phi_k(b))^2 - (y-x)^2}\right), \end{aligned} \tag{2.16}$$

where

$$c = c(2,0,k) \doteq 2^{-\frac{1}{1-k}} (1-k)^{\frac{k}{1-k}} \quad \text{and} \quad \gamma = \gamma(2,0,k) \doteq -\frac{k}{2(1-k)}, \tag{2.17}$$

and $\mathsf{F}(\alpha_1, \alpha_2; \beta; z)$ *denotes the Gauss hypergeometric function, while the kernel functions* K_0, K_1 *appearing in the integral terms involving the Cauchy data are given by* (2.6) *and* (2.7)*, respectively, for the special values* $(\mu, \nu^2) = (2, 0)$*.*

Corollary 2 (Representation formula for the Tricomi-type equation with negative power)**.** *Let* $n = 1$ *and* $k \in (0, 1)$*. We assume* $u_0 \in \mathscr{C}^2(\mathbb{R})$*,* $u_1 \in \mathscr{C}^1(\mathbb{R})$ *and* $g \in \mathscr{C}^{0,1}_{t,x}([1,\infty) \times \mathbb{R})$*. Then, a representation formula for the solution* u *to the Cauchy problem associated with the linear Tricomi-type equation*

$$\begin{cases} \partial_t^2 u - t^{-2k}\partial_x^2 u = g(t,x), & x \in \mathbb{R},\ t > 1, \\ u(1,x) = u_0(x), & x \in \mathbb{R}, \\ \partial_t u(1,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{2.18}$$

is given by

$$\begin{aligned} u(t,x) = {} & \frac{1}{2} t^{\frac{k}{2}} \big(u_0(x + A_k(t)) + u_0(x - A_k(t))\big) \\ & + \int_{x-A_k(t)}^{x+A_k(t)} u_0(y) K_0\big(t,x;y;0,0,k\big)\,\mathrm{d}y \\ & + \int_{x-A_k(t)}^{x+A_k(t)} u_1(y) K_1\big(t,x;y;0,0,k\big)\,\mathrm{d}y \end{aligned}$$

$$+\int_0^t \int_{x-\phi_k(t)+\phi_k(b)}^{x+\phi_k(t)-\phi_k(b)} g(b,y) E\big(t,x;b,y;0,0,k\big)\,\mathrm{d}y\,\mathrm{d}b. \tag{2.19}$$

Here the kernel function E is defined by

$$E\big(t,x;b,y;0,0,k\big) \doteq c\,\big((\phi_k(t)+\phi_k(b))^2-(y-x)^2\big)^{-\gamma} \times \mathsf{F}\left(\gamma,\gamma;1;\frac{(\phi_k(t)-\phi_k(b))^2-(y-x)^2}{(\phi_k(t)+\phi_k(b))^2-(y-x)^2}\right), \tag{2.20}$$

where

$$c = c(0,0,k) \doteq 2^{-\frac{1}{1-k}}(1-k)^{\frac{k}{1-k}} \quad \text{and} \quad \gamma=\gamma(0,0,k) \doteq -\frac{k}{2(1-k)}, \tag{2.21}$$

and $\mathsf{F}(\alpha_1,\alpha_2;\beta;z)$ denotes the Gauss hypergeometric function, while the kernel functions K_0, K_1 appearing in the integral terms involving the Cauchy data are given by (2.6) and (2.7), respectively, for the special values $(\mu,\nu^2)=(0,0)$.

Remark 5. Note that the kernel functions in (2.16) and in (2.20) are strongly related, since the following relation holds

$$E\big(t,x;b,y;2,0,k\big) = \frac{b}{t}\,E\big(t,x;b,y;0,0,k\big).$$

Remark 6. Let us stress that representation formulae for the solutions to the wave equation in Einstein-de Sitter spacetime and to the generalized Tricomi equation (even in the case with speed of propagation with negative power) have already been established in the literature (see for example [2, 25], respectively) when the Cauchy data are prescribed at the initial time $t=0$. However, since in the present work we prescribe the Cauchy data at the initial data $t=1$ and the considered models are not invariant by time translation (due to the presence of time-dependent coefficients), the representation formulae from Corollaries 1 and 2 are not redundant and will play a crucial role in the proof of Theorems 1.1 and 1.2, respectively.

Remark 7. In the next sections we shall estimate from below the kernel function E on suitable subsets of the forward light-cone. According to this purpose, the following lower bound estimate for the Gauss hypergeometric function is very helpful

$$\mathsf{F}(\alpha,\alpha;\beta;z) \geqslant 1, \tag{2.22}$$

for any $z\in[0,1)$ and for $\alpha\in\mathbb{R},\beta>0$. The previous estimate is a direct consequence of the series expansion for $\mathsf{F}(\alpha,\alpha;\beta;z)$ (cf. [12, Equation (15.2.1)]). Furthermore, for $\alpha\in\mathbb{R},\beta>0$ such that $\beta-2\alpha>0$ we have that (2.22) is optimal, meaning that $\mathsf{F}(\alpha,\alpha;\beta;z)$ can be estimated from above by a positive constant independent of $z\in[0,1)$.

3. **Proof of the main Theorems.** In the present section, we prove the main blow-up results by using a generalization of Zhou's blow-up argument on the characteristic line $A_k(t)-z=R$. In place of the d'Alembert's formula we shall employ the integral representation formulae from Theorems 1.1 and 1.2 obtained via Yagdjian's Integral Transform approach. The main steps in the proof are inspired by the computations in [29, 18, 10].

3.1. **Proof of Theorem 1.1.** Let u be a local (in time) solution to the Cauchy problem (1.1). In order to reduce our problem to a one-dimensional problem in space, we will introduce an auxiliary function which depends on the time variable and only on the first space variable. This step is achieved by integrating u with respect to the remaining $(n-1)$ spatial variables. Thus, if we denote $x=(z,w)$ with $z\in\mathbb{R}$ and $w\in\mathbb{R}^{n-1}$, then, we deal with the function

$$\mathcal{U}(t,z)\doteq\int_{\mathbb{R}^{n-1}}u(t,z,w)\,\mathrm{d}w \qquad \text{for any} \quad t>0, z\in\mathbb{R}. \tag{3.1}$$

Hereafter, we consider just the case $n\geqslant 2$ for the sake of brevity and of readability. Nevertheless, it is possible to proceed in the exact same way for $n=1$, simply by working with u instead of introducing $\mathcal{U}$. In order to prescribe the initial values of $\mathcal{U}$, we set

$$\mathcal{U}_0(z)\doteq\int_{\mathbb{R}^{n-1}}u_0(z,w)\,\mathrm{d}w,\quad \mathcal{U}_1(z)\doteq\int_{\mathbb{R}^{n-1}}u_1(z,w)\,\mathrm{d}w \qquad \text{for any} \quad z\in\mathbb{R}. \tag{3.2}$$

According to the statement of Theorem 1.1 we require that u_0,u_1 are compactly supported with support contained in B_R. Hence, $\mathcal{U}_0,\mathcal{U}_1$ are compactly supported too, with supports contained in $(-R,R)$. Analogously, since $\operatorname{supp}u(t,\cdot)\subset B_{R+A_k(t)}$ for any $t\in(1,T)$ we get the following support condition for $\mathcal{U}$

$$\operatorname{supp}\mathcal{U}(t,\cdot)\subset(-(R+A_k(t)),R+A_k(t)) \qquad \text{for any} \quad t\in(1,T). \tag{3.3}$$

Therefore, $\mathcal{U}$ solves the following Cauchy problem

$$\begin{cases}\partial_t^2\mathcal{U}-t^{-2k}\partial_z^2\mathcal{U}+2t^{-1}\partial_t\mathcal{U}=\int_{\mathbb{R}^{n-1}}|\partial_t u(t,z,w)|^p\,\mathrm{d}w, & z\in\mathbb{R},\ t>1,\\ \mathcal{U}(1,z)=\varepsilon\,\mathcal{U}_0(z), & z\in\mathbb{R},\\ \partial_t\mathcal{U}(1,z)=\varepsilon\,\mathcal{U}_1(z), & z\in\mathbb{R}.\end{cases}$$

By Corollary 1, we obtain the representation

$$\begin{aligned}\mathcal{U}(t,z)&=2^{-1}\varepsilon t^{\frac{k}{2}-1}\big(\mathcal{U}_0(z+A_k(t))+\mathcal{U}_0(z-A_k(t))\big)\\ &\quad+\varepsilon\int_{z-A_k(t)}^{z+A_k(t)}\mathcal{U}_0(y)K_0\big(t,z;y;2,0,k\big)\,\mathrm{d}y\\ &\quad+\varepsilon\int_{z-A_k(t)}^{z+A_k(t)}\mathcal{U}_1(y)K_1\big(t,z;y;2,0,k\big)\,\mathrm{d}y\\ &\quad+\int_0^t\int_{z-\phi_k(t)+\phi_k(b)}^{z+\phi_k(t)-\phi_k(b)}\int_{\mathbb{R}^{n-1}}|\partial_t u(b,y,w)|^p\,\mathrm{d}w\,E\big(t,z;b,y;2,0,k\big)\,\mathrm{d}y\,\mathrm{d}b,\end{aligned}$$

where the kernel function E is defined by (2.16), while K_0,K_1 are defined in (2.6) and in (2.7), respectively, for $(\mu,\nu^2)=(2,0)$.

Thanks to the sign assumptions on u_0,u_1 it results that $\mathcal{U}_0,\mathcal{U}_1$ are nonnegative functions as well. Consequently, from the previous identity we get

$$\begin{aligned}\mathcal{U}(t,z)&\geqslant\varepsilon\int_{z-A_k(t)}^{z+A_k(t)}\big(\mathcal{U}_0(y)K_0\big(t,z;y;2,0,k\big)+\mathcal{U}_1(y)K_1\big(t,z;y;2,0,k\big)\big)\,\mathrm{d}y\\ &\quad+\int_0^t\int_{z-\phi_k(t)+\phi_k(b)}^{z+\phi_k(t)-\phi_k(b)}\int_{\mathbb{R}^{n-1}}|\partial_t u(b,y,w)|^p\,\mathrm{d}w\,E\big(t,z;b,y;2,0,k\big)\,\mathrm{d}y\,\mathrm{d}b\\ &\doteq\varepsilon J(t,z)+I(t,z).\end{aligned}$$

We begin with the estimate of the integral $J(t,z)$ that involves the Cauchy data.

Since

$$\gamma(2,0,k) = -k/(2-2k) < 0 \tag{3.4}$$

and $1-2\gamma(2,0,k) = 1/(1-k) > 0$ (this second condition implies in particular that the hypergeometric function in (2.16) can be controlled from above and from below by positive constants, cf. Remark 7), for any $y \in [z - A_k(t), z + A_k(t)]$ it results

$$\begin{aligned} K_1\big(t,z;y;2,0,k\big) &= E\big(t,z;1,y;2,0,k\big) \\ &= ct^{-1}\left((\phi_k(t)+\phi_k(1))^2-(y-z)^2\right)^{-\gamma} \mathsf{F}\left(\gamma,\gamma;1;\frac{(\phi_k(t)-\phi_k(1))^2-(y-z)^2}{(\phi_k(t)+\phi_k(1))^2-(y-z)^2}\right) \\ &\gtrsim t^{-1}(4\phi_k(1)\phi_k(t))^{-\gamma} \gtrsim t^{\frac{k}{2}-1}. \end{aligned} \tag{3.5}$$

Moreover, for any $y \in [z - A_k(t), z + A_k(t)]$ we may prove that

$$K_0\big(t,z;y;2,0,k\big) = 2E\big(t,z;1,y;2,0,k\big) - \frac{\partial}{\partial b}E\big(t,z;b,y;2,0,k\big)\Big|_{b=1} \gtrsim t^{\frac{k}{2}-1}. \tag{3.6}$$

Indeed, by straightforward computations we find

$$\begin{aligned} &\frac{\partial}{\partial b}E\big(t,z;b,y;2,0,k\big) \\ &= ct^{-1}\left((\phi_k(t)+\phi_k(b))^2-(y-z)^2\right)^{-\gamma}\left(1+\frac{k\phi_k(b)(\phi_k(t)+\phi_k(b))}{(\phi_k(t)+\phi_k(b))^2-(y-z)^2}\right)\mathsf{F}(\gamma,\gamma;1;\zeta) \\ &\quad + c\gamma^2 t^{-1} b\left((\phi_k(t)+\phi_k(b))^2-(y-z)^2\right)^{-\gamma}\mathsf{F}(\gamma+1,\gamma+1;2;\zeta)\frac{\partial\zeta}{\partial b}, \end{aligned}$$

where

$$\zeta = \zeta(t,z;b,y;k) \doteq \frac{(\phi_k(t)-\phi_k(b))^2-(y-z)^2}{(\phi_k(t)+\phi_k(b))^2-(y-z)^2},$$

and we used the relation

$$\mathsf{F}'(\gamma,\gamma;1;\zeta) = \gamma^2\mathsf{F}(\gamma+1,\gamma+1;2;\zeta)$$

(cf. [12, Equation (15.5.1)]). Due to

$$\frac{\partial\zeta}{\partial b}(t,z;b,y;k) = -4b^{-k}\phi_k(t)\frac{(\phi_k(t))^2-(\phi_k(b))^2-(y-z)^2}{[(\phi_k(t)+\phi_k(b))^2-(y-z)^2]^2},$$

it follows

$$\begin{aligned} \frac{\partial\zeta}{\partial b}(t,z;b,y;k)\Big|_{b=1} &= -4\phi_k(t)\frac{(\phi_k(t))^2-(\phi_k(1))^2-(y-z)^2}{[(\phi_k(t)+\phi_k(1))^2-(y-z)^2]^2} \\ &\leqslant -\frac{8\phi_k(t)\phi_k(1)(\phi_k(t)-\phi_k(1))}{[(\phi_k(t)+\phi_k(1))^2-(y-z)^2]^2} \leqslant 0, \end{aligned}$$

for any $y \in [z - A_k(t), z + A_k(t)]$. Therefore, we may neglect the influence of the term $\mathsf{F}(\gamma+1,\gamma+1;2;\zeta)$ when we estimate the kernel K_0 from below. Hence,

$$\begin{aligned} K_0\big(t,z;y;2,0,k\big) &\geqslant ct^{-1}\left((\phi_k(t)+\phi_k(1))^2-(y-z)^2\right)^{-\gamma} \\ &\quad\times\left(1-\frac{k\phi_k(1)(\phi_k(t)+\phi_k(1))}{(\phi_k(t)+\phi_k(1))^2-(y-z)^2}\right)\mathsf{F}(\gamma,\gamma;1;\zeta) \\ &\geqslant ct^{-1}\left((\phi_k(t)+\phi_k(1))^2-(y-z)^2\right)^{-\gamma}\left(1-\tfrac{k}{4}(1-\zeta)\right)\mathsf{F}(\gamma,\gamma;1;\zeta) \\ &\quad - \tfrac{ck}{(1-k)^2}t^{-1}\left((\phi_k(t)+\phi_k(1))^2-(y-z)^2\right)^{-\gamma-1}\mathsf{F}(\gamma,\gamma;1;\zeta). \end{aligned} \tag{3.7}$$

Using the following estimate

$$\frac{k}{(1-k)^2}\left((\phi_k(t)+\phi_k(1))^2-(y-z)^2\right)^{-1} \leqslant \frac{k}{(1-k)^2}\left(4\phi_k(t)\phi_k(1)\right)^{-1} = \frac{k}{4t^{1-k}} \leqslant \frac{k}{4},$$

and employing (3.4), from (3.7) for $y\in[z-A_k(t),z+A_k(t)]$ and $t\geqslant 1$ we derive

$$\begin{aligned} K_0\big(t,z;y;2,0,k\big) &\geqslant c\left(1-\tfrac{k}{2}+\tfrac{k}{4}\zeta\right)t^{-1}\left((\phi_k(t)+\phi_k(1))^2-(y-z)^2\right)^{-\gamma}\mathsf{F}(\gamma,\gamma;1;\zeta) \\ &\geqslant c\left(1-\tfrac{k}{2}\right)t^{-1}\left((\phi_k(t)+\phi_k(1))^2-A_k(t)^2\right)^{-\gamma}\mathsf{F}(\gamma,\gamma;1;\zeta) \\ &= c\left(1-\tfrac{k}{2}\right)t^{-1}\left(4\phi_k(1)\phi_k(t)\right)^{-\gamma}\mathsf{F}(\gamma,\gamma;1;\zeta) \gtrsim t^{\frac{k}{2}-1}\mathsf{F}(\gamma,\gamma;1;\zeta), \end{aligned}$$

which implies in turn (3.6) thanks to (2.22).

Therefore, combining (3.5) and (3.6), we obtain

$$J(t,z)\gtrsim \varepsilon t^{\frac{k}{2}-1}\int_{z-A_k(t)}^{z+A_k(t)}\left(\mathcal{U}_0(y)+\mathcal{U}_1(y)\right)\mathrm{d}y.$$

From now on, we will work on the characteristic line with equation $A_k(t)-z=R$ for $z\geqslant R$. For $z\geqslant R$ such that $A_k(t)-z=R$ it holds

$$[-R,R]\subset[z-A_k(t),z+A_k(t)].$$

Consequently, using the support condition $\operatorname{supp}\mathcal{U}_0,\operatorname{supp}\mathcal{U}_1\subset(-R,R)$, we conclude

$$J(t,z)\gtrsim \varepsilon t^{\frac{k}{2}-1}\int_{\mathbb{R}}\left(\mathcal{U}_0(y)+\mathcal{U}_1(y)\right)\mathrm{d}y = \varepsilon t^{\frac{k}{2}-1}\|u_0+u_1\|_{L^1(\mathbb{R}^n)}, \tag{3.8}$$

where in the last step we used Fubini's theorem.

Now we estimate the term $I(t,z)$. Clearly, the following support condition

$$\operatorname{supp}\partial_t u(t,\cdot)\subset B_{R+A_k(t)},$$

holds for any $t\in(1,T)$ due the shape of the light-cone. With respect to the $w\in\mathbb{R}^{n-1}$ variable we can express the previous support condition as follows:

$$\operatorname{supp}\partial_t u(t,z,\cdot)\subset\left\{w\in\mathbb{R}^{n-1}:|w|\leqslant\left((R+A_k(t))^2-z^2\right)^{1/2}\right\},$$

for any $t\in(1,T)$ and any $z\in\mathbb{R}$ such that $|z|\leqslant R+A_k(t)$. Then, from Hölder's inequality we derive the estimate

$$\begin{aligned} |\partial_t\mathcal{U}(b,y)| &= \Big|\int_{\mathbb{R}^{n-1}}\partial_t u(b,y,w)\,\mathrm{d}w\Big| \\ &\leqslant \left(\int_{\mathbb{R}^{n-1}}|\partial_t u(b,y,w)|^p\,\mathrm{d}w\right)^{\frac{1}{p}}\left(\operatorname{meas}_{n-1}\left(\operatorname{supp}\partial_t u(b,y,\cdot)\right)\right)^{1-\frac{1}{p}} \\ &\lesssim \left((R+A_k(b))^2-y^2\right)^{\frac{n-1}{2}(1-\frac{1}{p})}\left(\int_{\mathbb{R}^{n-1}}|\partial_t u(b,y,w)|^p\,\mathrm{d}w\right)^{\frac{1}{p}}. \end{aligned}$$

Hereafter, the unexpressed multiplicative constants will depend on n,k,R,p. From the previous inequality, we have

$$\int_{\mathbb{R}^{n-1}}|\partial_t u(b,y,w)|^p\,\mathrm{d}w \gtrsim \left((R+A_k(b))^2-y^2\right)^{-\frac{n-1}{2}(p-1)}|\partial_t\mathcal{U}(b,y)|^p,$$

which provides in turn the following estimate

$$
\begin{aligned}
I(t,z) &\gtrsim \int_1^t \int_{z-\phi_k(t)+\phi_k(b)}^{z+\phi_k(t)-\phi_k(b)} \frac{E\big(t,z;b,y;2,0,k\big)}{\big((R+A_k(b))^2-y^2\big)^{\frac{n-1}{2}(p-1)}} |\partial_t \mathcal{U}(b,y)|^p \,\mathrm{d}y\,\mathrm{d}b \\
&= \int_{z-A_k(t)}^{z+A_k(t)} \int_1^{\phi_k^{-1}(\phi_k(t)-|z-y|)} \frac{E\big(t,z;b,y;2,0,k\big)}{\big((R+A_k(b))^2-y^2\big)^{\frac{n-1}{2}(p-1)}} |\partial_t \mathcal{U}(b,y)|^p \,\mathrm{d}b\,\mathrm{d}y,
\end{aligned}
$$

where we applied Fubini's theorem in the last equality. Since we are working on the characteristic line $A_k(t)-z=R$, for $z\geqslant R$ it holds $[z-A_k(t),z+A_k(t)]\supset[R,z]$. Consequently, we can shrink the domain of integration in the last lower bound estimate for $I(t,z)$, obtaining

$$
I(t,z) \gtrsim \int_R^z \int_1^{\phi_k^{-1}(\phi_k(t)-|z-y|)} \frac{E\big(t,z;b,y;2,0,k\big)}{\big((R+A_k(b))^2-y^2\big)^{\frac{n-1}{2}(p-1)}} |\partial_t \mathcal{U}(b,y)|^p \,\mathrm{d}b\,\mathrm{d}y.
$$

On the characteristic line $A_k(t)-z=R$ and for $y\in[R,z]$, it holds

$$
\phi_k^{-1}(\phi_k(t)-|z-y|) = \phi_k^{-1}(\phi_k(t)-z+y) = \phi_k^{-1}(\phi_k(1)+R+y) = A_k^{-1}(R+y),
$$

where in the last step we used the relation

$$
A_k^{-1}(\tau) = \phi_k^{-1}(\phi_k(1)+\tau).
$$

Moreover, for $y\in[R,z]$ it holds

$$
1 \leqslant \phi_k^{-1}(\phi_k(1)+y-R) = A_k^{-1}(y-R),
$$

thanks to the monotonicity of ϕ_k^{-1}. Thus, after a further shrinking of the domain of integration, we get

$$
\begin{aligned}
I(t,z) &\gtrsim \int_R^z \int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} \frac{E\big(t,z;b,y;2,0,k\big)}{\big((R+A_k(b))^2-y^2\big)^{\frac{n-1}{2}(p-1)}} |\partial_t \mathcal{U}(b,y)|^p \,\mathrm{d}b\,\mathrm{d}y \\
&\gtrsim \int_R^z \int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} \frac{E\big(t,z;b,y;2,0,k\big)}{\big((R+A_k(A_k^{-1}(y+R)))^2-y^2\big)^{\frac{n-1}{2}(p-1)}} |\partial_t \mathcal{U}(b,y)|^p \,\mathrm{d}b\,\mathrm{d}y \\
&\gtrsim \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)} \int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} |\partial_t \mathcal{U}(b,y)|^p E\big(t,z;b,y;2,0,k\big) \,\mathrm{d}b\,\mathrm{d}y,
\end{aligned}
$$

for $z\geqslant R$ such that $A_k(t)-z=R$. Now we estimate the kernel function $E(\cdot;2,0,k)$ from below on the restricted domain of integration. Due to $1-2\gamma(2,0,k)>0$ the factor of E involving the hypergeometric function can be controlled from above and from below with a positive constant (see Remark 7).

Consequently, for $y\in[R,z]$ and $b\in[A_k^{-1}(y-R),A_k^{-1}(y+R)]$, on the characteristic $A_k(t)-z=R$ we have

$$
\begin{aligned}
&E\big(t,z;b,y;2,0,k\big) \\
&= c\,\frac{b}{t}\,\big((\phi_k(t)+\phi_k(b))^2-(y-z)^2\big)^{-\gamma}\,\mathsf{F}\left(\gamma,\gamma;1;\frac{(\phi_k(t)-\phi_k(b))^2-(y-z)^2}{(\phi_k(t)+\phi_k(b))^2-(y-z)^2}\right) \\
&\gtrsim \frac{b}{t}\,\big((\phi_k(t)+\phi_k(b))^2-(y-z)^2\big)^{-\gamma} \\
&= \frac{b}{t}\,\big((A_k(t)+A_k(b)+2\phi_k(1))^2-(y-z)^2\big)^{-\gamma} \\
&\gtrsim \frac{b}{t}\,\big((A_k(t)-R+y+2\phi_k(1))^2-(y-z)^2\big)^{-\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{b}{t}\left((z+y+2\phi_k(1))^2-(y-z)^2\right)^{-\gamma}\\
&= \frac{b}{t}\Big(4(z+\phi_k(1))(y+\phi_k(1))\Big)^{-\gamma} \gtrsim \frac{A_k^{-1}(y-R)}{t}(z+R)^{-\gamma}(y+R)^{-\gamma},
\end{aligned}
$$

where we used that $\gamma=\gamma(2,0,k)$ is a negative parameter. We remark that for $y\in[R,z]$, it holds

$$
\begin{aligned}
\frac{A_k^{-1}(y-R)}{A_k^{-1}(y+R)} &= \frac{\phi_k^{-1}(\phi_k(1)+y-R)}{\phi_k^{-1}(\phi_k(1)+y+R)} = \left(\frac{\phi_k(1)+y-R}{\phi_k(1)+y+R}\right)^{\frac{1}{1-k}}\\
&= \left(1-\frac{2R}{\phi_k(1)+y+R}\right)^{\frac{1}{1-k}} \geqslant \left(1-\frac{2R}{\phi_k(1)+2R}\right)^{\frac{1}{1-k}} = C_{R,k}\,.
\end{aligned}
$$

Since A_k^{-1} is an increasing function, from this inequality we find for $y\geqslant R$ that $A_k^{-1}(y-R)\approx A_k^{-1}(y+R)$. In particular, this fact is a consequence of A_k being the primitive of a_k vanishing for $t=1$, or in other words, $A_k^{-1}(0)=1$.

Combining the lower bound estimate for E on the domain of integration and the last inequality, we arrive at

$$
\begin{aligned}
(z+R)^{\gamma} t\, I(t,z) &\gtrsim \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma} A_k^{-1}(y+R)\int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} |\partial_t \mathcal{U}(b,y)|^p\,\mathrm{d}b\,\mathrm{d}y\\
&\gtrsim \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma+\frac{1}{1-k}} \int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} |\partial_t \mathcal{U}(b,y)|^p\,\mathrm{d}b\,\mathrm{d}y, \qquad (3.9)
\end{aligned}
$$

for $z\geqslant R$ such that $A_k(t)-z=R$.

Applying Jensen's inequality, we get

$$
\begin{aligned}
&\left|\int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} \partial_t\mathcal{U}(b,y)\,\mathrm{d}b\right|^p\\
&\leqslant \left(A_k^{-1}(y+R)-A_k^{-1}(y-R)\right)^{p-1}\int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} |\mathcal{U}_t(b,y)|^p\,\mathrm{d}b\\
&\leqslant (2R)^{p-1}\left(\max_{\tau\in[y-R,y+R]}\frac{\mathrm{d}}{\mathrm{d}\tau}A_k^{-1}(\tau)\right)^{p-1}\int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} |\partial_t\mathcal{U}(b,y)|^p\,\mathrm{d}b\\
&\lesssim (\phi_k(1)+y+R)^{\frac{k}{1-k}(p-1)}\int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)} |\partial_t\mathcal{U}(b,y)|^p\,\mathrm{d}b, \qquad (3.10)
\end{aligned}
$$

where we used $\frac{\mathrm{d}}{\mathrm{d}\tau}A_k^{-1}(\tau)=\left((1-k)(\phi_k(1)+\tau)\right)^{\frac{k}{1-k}}$.

Combining the fundamental theorem of calculus, (3.9) and (3.10), on the characteristic $A_k(t)-z=R$ we find

$$
\begin{aligned}
&(z+R)^{\gamma+\frac{1}{1-k}} I(A_k^{-1}(z+R),z)\\
&\gtrsim \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma+\frac{1}{1-k}-\frac{k}{1-k}(p-1)}\left|\int_{A_k^{-1}(y-R)}^{A_k^{-1}(y+R)}\partial_t\mathcal{U}(b,y)\,\mathrm{d}b\right|^p\mathrm{d}y\\
&= \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma+\frac{1}{1-k}-\frac{k}{1-k}(p-1)}|\mathcal{U}(A_k^{-1}(y+R),y)|^p\,\mathrm{d}y, \qquad (3.11)
\end{aligned}
$$

where in the second step we can apply the relation $\mathcal{U}(A_k^{-1}(y-R),y)=0$ due to (3.3).

Using together (3.8) and (3.11), on the characteristic line $A_k(t) - z = R$ for $z \geqslant R$, we obtain

$$(R+z)^{\gamma+\frac{1}{1-k}}\mathcal{U}(A_k^{-1}(z+R),z) \tag{3.12}$$
$$\geqslant C\varepsilon\,\|u_0+u_1\|_{L^1(\mathbb{R}^n)}+C\int_R^z (R+y)^{-(\frac{n-1}{2}+\frac{k}{1-k})(p-1)-\gamma+\frac{1}{1-k}}\,|\mathcal{U}(A_k^{-1}(y+R),y)|^p\mathrm{d}y,$$

where $C = C(n, k, R, p) > 0$ is a suitable multiplicative constant.

We may define the functional

$$U(z) \doteq (R+z)^{\gamma+\frac{1}{1-k}}\mathcal{U}(A_k^{-1}(z+R),z), \quad z \geqslant R,$$

whose dynamic will be employed to prove the blow-up result.

Rewriting (3.12) through U, we find

$$U(z) \geqslant C\varepsilon\,\|u_0+u_1\|_{L^1(\mathbb{R}^n)}+C\int_R^z (R+y)^{\left(-\frac{n-1}{2}-\frac{1+k}{1-k}-\gamma\right)(p-1)-2\gamma}|U(y)|^p\mathrm{d}y \tag{3.13}$$

for $z \geqslant R$. Finally, we apply a comparison argument to U. We define the auxiliary function G in the following way

$$G(z) \doteq M\varepsilon + C\int_R^z (R+y)^{\left(-\frac{n-1}{2}-\frac{1+k}{1-k}-\gamma\right)(p-1)-2\gamma}\,|U(y)|^p\,\mathrm{d}y \qquad \text{for } z \geqslant R,$$

where $M \doteq C\|u_0 + u_1\|_{L^1(\mathbb{R}^n)}$. (3.13) means that $U \geqslant G$. Furthermore, G fulfills the ordinary differential inequality

$$\begin{aligned} G'(z) &= C\,(R+z)^{\left(-\frac{n-1}{2}-\frac{1+k}{1-k}-\gamma\right)(p-1)-2\gamma}\,|U(z)|^p \\ &\geqslant C\,(R+z)^{\left(-\frac{n-1}{2}-\frac{1+k}{1-k}-\gamma\right)(p-1)-2\gamma}\,(G(z))^p \end{aligned}$$

and satisfies the initial condition $G(R) = M\varepsilon$. If p satisfies

$$\left(-\frac{n-1}{2}-\frac{1+k}{1-k}-\gamma\right)(p-1)-2\gamma = -1, \tag{3.14}$$

then,

$$(M\varepsilon)^{1-p} - G(z)^{1-p} \geqslant C(p-1)\log\left(\frac{R+z}{2R}\right). \tag{3.15}$$

We underline that (3.14) is equivalent to $p = p_{\mathrm{Gla}}\big((1-k)n+2k+2\big)$.

Otherwise, since $G(z) > 0$ for any $z \geqslant R$, it follows

$$\begin{aligned} (M\varepsilon)^{1-p} - G(z)^{1-p} \geqslant{}& \frac{C(1-k)}{\frac{1}{p-1}-\frac{(1-k)n+2k+1}{2}}(R+z)^{1-2\gamma-\frac{(1-k)n+2k+1}{2(1-k)}(p-1)} \\ &- \frac{C(1-k)}{\frac{1}{p-1}-\frac{(1-k)n+2k+1}{2}}(2R)^{1-2\gamma-\frac{(1-k)n+2k+1}{2(1-k)}(p-1)}. \end{aligned} \tag{3.16}$$

Thus, if $p \in \big(1, p_{\mathrm{Gla}}\big((1-k)n+2k+2\big)\big)$, then, the multiplicative factor on the right-hand side of (3.16) is positive. So, we let $\varepsilon_0 = \varepsilon_0(n, p, k, u_0, u_1, R)$ sufficiently small such that for any $\varepsilon \in (0, \varepsilon_0]$ it results

$$G(z) \geqslant \left[2(M\varepsilon)^{1-p} - \frac{C(1-k)}{\frac{1}{p-1}-\frac{(1-k)n+2k+1}{2}}(R+z)^{1-2\gamma-\frac{(1-k)n+2k+1}{2(1-k)}(p-1)}\right]^{-\frac{1}{p-1}}. \tag{3.17}$$

In the limit case $p = p_{\mathrm{Gla}}\big((1-k)n+2k+2\big)$, from (3.15) we get the estimate

$$U(z) \geqslant G(z) \geqslant \left[(M\varepsilon)^{1-p} - C(p-1)\log\left(\tfrac{R+z}{2R}\right)\right]^{-\frac{1}{p-1}},$$

that provides the blow-up in finite time of $U(z)$ and, since $A_k(t) - z = R$, the lifespan estimate

$$\log T(\varepsilon) \lesssim \varepsilon^{-(p-1)}.$$

Otherwise, in the case $p \in \big(1, p_{\mathrm{Gla}}\big((1-k)n + 2k + 2\big)\big)$, the right-hand side of the inequality in (3.17) blows up for

$$A_k(t) = R + z \approx \varepsilon^{-(1-k)\left(\frac{1}{p-1} - \frac{(1-k)n+2k+1}{2}\right)^{-1}}.$$

Therefore, G (and U, in turn) blows up and the following upper bound estimates holds

$$T(\varepsilon) \lesssim \varepsilon^{-\left(\frac{1}{p-1} - \frac{(1-k)n+2k+1}{2}\right)^{-1}}.$$

This completes the proof of Theorem 1.1.

3.2. **Proof of Theorem 1.2.** The proof of Theorem 1.2 is analogous to the one of Theorem 1.1. Let us just sketch the key points in the blow-up argument and emphasize the modifications that we have to carry out in comparison to the previous case.

Given a local solution u to (1.5), we may introduce also in this case the function $\mathcal{U} = \mathcal{U}(t, z)$ for $t \geqslant 1$, $z \in \mathbb{R}$ as in (3.1) by integrating with respect to the last $(n-1)$-space variables. Thanks to the assumption $u_0 = 0$ and Corollary 2, the following representation holds

$$\mathcal{U}(t, z) = \varepsilon \widetilde{J}(t, z) + \widetilde{I}(t, z),$$

where

$$\widetilde{J}(t, z) \doteq \int_{z - A_k(t)}^{z + A_k(t)} \mathcal{U}_1(y) K_1\big(t, z; y; 0, 0, k\big)\, \mathrm{d}y,$$

$$\widetilde{I}(t, z) \doteq \int_0^t \int_{z - \phi_k(t) + \phi_k(b)}^{z + \phi_k(t) - \phi_k(b)} \int_{\mathbb{R}^{n-1}} |\partial_t u(b, y, w)|^p \, \mathrm{d}w\, E\big(t, z; b, y; 0, 0, k\big)\, \mathrm{d}y\, \mathrm{d}b,$$

with $\mathcal{U}_1$ defined analogously as in (3.2). In order to estimate from below the terms $\widetilde{J}, \widetilde{I}$, we can proceed very similarly as in the previous proof, keeping in mind the relation pointed out in Remark 5 on the fundamental solution E. More precisely, in place of (3.8), we get

$$\widetilde{J}(t, z) \gtrsim \varepsilon\, t^{\frac{k}{2}} \|u_1\|_{L^1(\mathbb{R}^n)},$$

while instead of (3.11), we find

$$\begin{aligned} &\widetilde{I}(A_k^{-1}(z + R), z) \\ &\gtrsim (z + R)^{-\gamma} \int_R^z (R + y)^{-\frac{n-1}{2}(p-1) - \gamma - \frac{k}{1-k}(p-1)} |\mathcal{U}(A_k^{-1}(y + R), y)|^p \, \mathrm{d}y, \end{aligned}$$

on the characteristic $A_k(t) - z = R$, for $z \geqslant R$. Therefore, the functional that we have to consider in order to study the blow-up dynamic is

$$\widetilde{U}(z) \doteq (z + R)^{\gamma} \mathcal{U}(A_k^{-1}(z + R), z).$$

Consequently, the ordinary integral inequality for $\widetilde{U}$ is given by

$$\widetilde{U}(z) \geqslant \widetilde{C} \varepsilon \|u_1\|_{L^1(\mathbb{R}^n)} + \widetilde{C} \int_R^z (R + y)^{\left(-\frac{n-1}{2} - \frac{k}{1-k} - \gamma\right)(p-1) - 2\gamma} |\widetilde{U}(y)|^p \, \mathrm{d}y$$

for $z \geqslant R$, where $\widetilde{C} > 0$ is a suitable positive constant.

Finally, we observe that the power for $(y+R)$ in the last integral is equal to -1 if and only if $p = p_{\text{Gla}}((1-k)n+2k)$. Repeating the same kind of computations as in Section 3.1, we obtain the blow-up in finite time for $\widetilde{U}$ provided that the exponent of the nonlinearity satisfies $p \in (1, p_{\text{Gla}}((1-k)n+2k)]$ and the corresponding upper bound estimates for the lifespan. We concluded the proof of Theorem 1.2.

4. Final remarks.

In the present paper, we proved when $\mu \in \{0, 2\}$ blow-up results for the semilinear model

$$\begin{cases} \partial_t^2 u - t^{-2k}\Delta u + \mu t^{-1}\partial_t u = |\partial_t u|^p, & x \in \mathbb{R}^n,\ t > 1, \\ u(1,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(1,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

provided that the Cauchy data fulfill suitable sign and support assumptions and that the exponent of the nonlinear term belongs to the following range

$$1 < p \leqslant p_{\text{Gla}}((1-k)n + 2k + \mu).$$

Due to the consistency of this result with other results known in the literature for special values of the parameters k and μ (namely, for $k = 0$ and/or $\mu = 0$), we conjecture that the previous upper bound for p could be the critical exponent for (4.1). Nevertheless, in order to prove the validity of this conjecture the proof of the global (in time) existence result for small data solutions in the supercritical case $p > p_{\text{Gla}}((1-k)n + 2k + \mu)$ is necessary.

We point out that in the general case $\mu > 0$, the blow-up argument from Theorems 1.1 and 1.2 does not work sharply, especially for μ in the interval $(k, 2-k)$. In the forthcoming paper [7], we will study systematically the blow-up dynamic for (4.1) via a completely different approach. We point out that in the present paper we focus more on the geometric nature of the problem, since we proved a blow-up result working on the characteristic line $A_k(t) - z = R$. On the other hand, in [7] we work with weighted space averages of the solutions and its time derivative. In particular, the weight function is a positive solution to the corresponding adjoint linear equation

$$\psi_{tt} - t^{-2k}\Delta\psi - \partial_t(\mu t^{-1}\psi) = 0,$$

defined by means of a suitable modified Bessel function of the second kind.

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