This is a pre-print of our paper published on Discrete and Continuous Dynamical Systems - Series S,
Vol. 15, No. 12, 2022, pp. 3745-3763.
DOI: 10.3934/dcdss.2022013

Existence of minimizers for a quasilinear elliptic system of gradient type

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Abstract

The aim of this paper is to investigate the existence of weak solutions of the coupled quasilinear elliptic system of gradient type

$$(P) \qquad \begin{cases} -\mathrm{div}(a(x,u,\nabla u) + A_t(x,u,\nabla u) = g_1(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(b(x,v,\nabla v) + B_t(x,v,\nabla v) = g_2\,(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $p_1,\ p_2 > 1$ and $A(x,t,\xi),\ B(x,t,\xi)$ are $\mathcal{C}^1$–Carathéodory functions on $\Omega\times\mathbb{R}\times\mathbb{R}^N$ with partial derivatives $A_t(x,t,\xi)$, $a = \nabla_\xi A$, respectively $B_t(x,t,\xi)$, $b = \nabla_\xi B$, while $g_1(x,t,s)$, $g_2(x,t,s)$ are given Carathéodory maps defined on $\Omega\times\mathbb{R}\times\mathbb{R}$ which are partial derivatives of a function $G(x,t,s)$.

We prove that, even if the general form of the terms A and B make the variational approach more difficult, under suitable hypotheses, the functional $\mathcal{J}$, related to problem (P), admits at least one critical point in the "right" Banach space X.

The proof, which exploits the interaction between two different norms, is based on a weak version of the Cerami–Palais–Smale condition, and a suitable generalizations of the Weierstrass Theorem.

1 Introduction

This paper aims at investigating the existence of one or more solutions for the following quasilinear elliptic system with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = g_1(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(b(x,v,\nabla v)) + B_t(x,v,\nabla v) = g_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, $p_1, p_2 > 1$, functions $A(x,t,\xi)$, $B(x,t,\xi)$: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ admit partial derivatives

$$\begin{aligned} A_t(x,t,\xi) = \ &\frac{\partial A}{\partial t}(x,t,\xi), \qquad a(x,t,\xi) = \left(\frac{\partial A}{\partial \xi_1}, (x,t,\xi)..., \frac{\partial A}{\partial \xi_N}(x,t,\xi)\right), \\ B_t(x,t,\xi) = \ &\frac{\partial B}{\partial t}(x,t,\xi), \qquad b(x,t,\xi) = \left(\frac{\partial B}{\partial \xi_1}, (x,t,\xi), ..., \frac{\partial B}{\partial \xi_N}(x,t,\xi)\right), \end{aligned} \tag{1.2}$$

and a function $G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ exists such that

$$\frac{\partial G}{\partial t}(x,t,s) = g_1(x,t,s), \quad \frac{\partial G}{\partial s}(x,t,s) = g_2(x,t,s) \quad \text{for a.e. } x \in \Omega, \text{ for all } (t,s) \in \mathbb{R}^2. \tag{1.3}$$

We note that the terms $a(x,u,\nabla u)$ and $b(x,v,\nabla v)$ make the variational approach more difficult. In fact, also investigating the existence of solution for just an equation

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

requires suitable approaches such as nonsmooth tecniques or null Gâteaux derivative only along "good" directions or a suitable variational setting [4, 5, 11]. For example, a family of model problem is given by:

$$A(x,t,\xi) = \left(\sum_{i,j=1}^{N} a_{i,j}(x,t)\xi_i\xi_j\right)^{\frac{p}{2}}$$

where $p > 1$ and $(a_{i,j}(x,t))_{1\leq i,j\leq N}$ is an elliptic matrix.
In particular, if $\mathcal{A}, \mathcal{B} : \Omega \times \mathbb{R} \to \mathbb{R}$ are given functions such that:

$$a_{i,j}(x,t) = \left(\tfrac{1}{p_1}\mathcal{A}(x,t)\right)^{\frac{2}{p_1}} \delta_i^j$$

$$b_{i,j}(x,t) = \left(\tfrac{1}{p_2}\mathcal{B}(x,t)\right)^{\frac{2}{p_2}} \delta_i^j,$$

then:

$$A(x,t,\xi) = \tfrac{1}{p_1}\mathcal{A}(x,t)|\xi|^{p_1}$$

$$B(x,t,\xi) = \tfrac{1}{p_2}\mathcal{B}(x,t)|\xi|^{p_2},$$

and the problem (1.1) reduces to:

$$\begin{cases} -\text{div}(\mathcal{A}(x,u)|\nabla u|^{p_1-2}\nabla u) + \frac{1}{p_1}\mathcal{A}_t(x,u)|\nabla u|^{p_1} = g_1(x,u,v) & \text{in } \Omega, \\ -\text{div}(\mathcal{B}(x,v)|\nabla v|^{p_2-2}\nabla v) + \frac{1}{p_2}\mathcal{B}_t(x,v)|\nabla v|^{p_2} = g_2\,(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

which solutions are the critical points of its corresponding funcional

$$\mathcal{J}(u,v) = \frac{1}{p_1}\int_\Omega \mathcal{A}(x,u)|\nabla u|^{p_1}dx + \frac{1}{p_2}\int_\Omega \mathcal{B}(x,v)|\nabla v|^{p_2}dx - \int_\Omega G(x,u,v)dx,$$

under suitable growth condition of nonlinear term G (see [12]).
In the particular case, $\mathcal{A}(x,u) = A^*(x)$ and $\mathcal{B}(x,v) = B^*(x)$, problem (1.5) becomes:

$$\begin{cases} -\text{div}(A^*(x)|\nabla u|^{p_1-2}\nabla u) = g_1(x,u,v) & \text{in } \Omega, \\ -\text{div}(B^*(x)|\nabla v|^{p_2-2}\nabla v) = g_2\,(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with A^*, $B^* : \Omega \to \mathbb{R}$. If $A^*(x)$, $B^*(x)$ are measurable, bounded and strictly positive functions, then (1.6) generalizes the classical (p_1,p_2)–Laplacian system

$$\begin{cases} -\Delta_{p_1}\, u \;=\; g_1(x,u,v) & \text{in } \Omega, \\ -\Delta_{p_2}\, v \;=\; g_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The interest in (p_1,p_2) - Laplacian systems, or their generalization such as (1.1), arises from the fact that they allow to model various physical phenomena. For example, they describe problems related to the equilibrium of anisotropic media which possibly are somewhere "perfect" insulators or "perfect" conductors. So that the couple (p_1,p_2) represents the characteristic of the medium which involves a pseudoplastic fluid if $p_i < 2$, a dilatant fluid if $p_i > 2$ or a Newtonian fluid if $p_1 = 2$. Moreover, problem (1.1) arises in the theory of quasiregular and quasiconformal mappings and is useful for investigating population dynamics or the spread of microrganisms.
As a consequence, many authors have studied problem (1.7) obtaining existence results under hyphotheses of sublinear, superlinear or resonant type of the non linearity $G(x,u,v)$.
In particular, in [2] the authors found a solution of (1.7) as critical point of the related action functional

$$\mathcal{I}(u,v) = \frac{1}{p_1}\int_\Omega |\nabla u|^{p_1}dx + \frac{1}{p_2}\int_\Omega |\nabla v|^{p_2}dx - \int_\Omega G(x,u,v)dx \tag{1.8}$$

which is well defined on the space $W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ if the function G is continuous on $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}$ such that:

$$|G(x,u,v)| \le c(1 + |u|^{p_1^*} + |v|^{p_2^*}),$$

where p_1^*, p_2^* denote the critical esponents for the Soboler imbeddings and $p_1 \le p_1^*, p_2 \le p_2^*$.
Moreover, if more restrictive subcritical assumptions for g_1 and g_2 hold, functional $\mathcal{I}$ is of class C^1

and its critical points are weak solutions of (1.7).
On the other hand, the presence of coefficients $\mathcal{A}(x,u)$ and $\mathcal{B}(x,v)$ make the variational approach more difficult. Indeed, even in the simplest case $G(x,u,v) \equiv 0$ and $\mathcal{A}(x,u)$, $\mathcal{B}(x,u)$are smooth, bounded and far away from zero, the corresponding functional for the problem (1.5)

$$\frac{1}{p_1}\int_\Omega \mathcal{A}(x,u)(x,u)|\nabla u|^{p_1}dx + \frac{1}{p_2}\int_\Omega \mathcal{B}(x,u)(x,v)|\nabla v|^{p_2}dx$$

is defined in $W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$, but is Gâteaux differentiable only along directions of the Banach space $(W_0^{1,p_1}(\Omega)\cap L^\infty(\Omega)) \times (W_0^{1,p_2}(\Omega)\cap L^\infty(\Omega))$ (see [11] for the case of a single equation).
In this paper, we want to extend to our quasilinear system (1.1) the existence result stated in [2] for (p_1, p_2)- Laplacian system (1.7) with a "sublinear like" growth for the nonlinear term G.
To this aim, following the approach introduced in [4, 5] for quasilinear equations and extended in [12] to quasilinear systems, we look for solution of (1.1) as critical points of the functional:

$$\mathcal{J}(u,v) = \int_\Omega A(x,u,\nabla u)dx + \int_\Omega B(x,v,\nabla v)dx - \int_\Omega G(x,u,v)dx \tag{1.9}$$

in the Banach space $X_1 \times X_2$, where $X_i = W_0^{1,p_i}(\Omega)\cap L^\infty(\Omega)$, for $i = 1, 2$.
We note that functional $\mathcal{J}$ does not satisfy the Palais–Smale condition or its Cerami's variant since, even in the case of a single equation, a Palais–Smale sequence converging in the $W_0^{1,p}(\Omega)$ norm but unbounded in $L^\infty(\Omega)$ can be found (see, for example, [7, Example 4.3]). Therefore, by exploiting the interaction between two different norms on X, we introduce the weak Cerami–Palais–Smale condition (see Definition 2.6) and apply a suitable generalized version of Weierstrass Theorem in order to prove the existence of at least one solution (see Theorems 2.3 and 2.5).

2 Main Theorems and variational principle

From now on, let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, $N \geq 2$ and consider two functions $A, B : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ with partial derivatives $A_t(x,t,\xi), a(x,t,\xi), B_t(x,t,\xi), b(x,t,\xi)$, according to the notation (1.2) and a function $G : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with the partial derivatives $g_1(x,t,s), g_2(x,t,s)$ as in (1.3).
Assume that two real numbers $p_1, p_2 > 1$ and a radius $R \geq 1$ esist so that the following assumptions are held:

(H_0) A and B are $\mathcal{C}^1$–Carathéodory functions, i.e.,

$$\begin{aligned}
&A(\cdot,t,\xi) : x \in \Omega \mapsto A(x,t,\xi) \in \mathbb{R} \text{ is measurable for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N,\\
&A(x,\cdot,\cdot) : (t,\xi) \in \mathbb{R}\times\mathbb{R}^N \mapsto A(x,t,\xi) \in \mathbb{R} \text{ is } C^1 \text{ for a.e. } x \in \Omega;\\
&B(\cdot,t,\xi) : x \in \Omega \mapsto B(x,t,\xi) \in \mathbb{R} \text{ is measurable for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N,\\
&B(x,\cdot,\cdot) : (t,\xi) \in \mathbb{R}\times\mathbb{R}^N \mapsto B(x,t,\xi) \in \mathbb{R} \text{ is } C^1 \text{ for a.e. } x \in \Omega;
\end{aligned}$$

(H_1) some positive continuous functions $\Phi_i, \phi_i : \mathbb{R} \to \mathbb{R}, i \in \{1,2\}$ exist such that:

$$\begin{aligned}
|A_t(x,t,\xi)| &\leq \Phi_1(t) + \phi_1(t)|\xi|^{p_1} && \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \\
|a(x,t,\xi)| &\leq \Phi_2(t) + \phi_2(t)|\xi|^{p_1-1} && \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \\
|B_t(x,t,\xi)| &\leq \Phi_1(t) + \phi_1(t)|\xi|^{p_2} && \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \\
|b(x,t,\xi)| &\leq \Phi_2(t) + \phi_2(t)|\xi|^{p_2-1} && \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N;
\end{aligned}$$

(H_2) some costants μ_0 and $\sigma_1, \sigma_2 \geq 0$ exist such that

$$A(x,t,\xi) \geq \mu_0 \,(1 + |t|^{p_1\sigma_1})\, |\xi|^{p_1} \quad \text{a.e. in } \Omega, \text{ for all } |(t,\xi)| \geq R,$$
$$B(x,t,\xi) \geq \mu_0 \,(1 + |s|^{p_2\sigma_2})\, |\xi|^{p_2} \quad \text{a.e. in } \Omega, \text{ for all } |(t,\xi)| \geq R;$$

(H_3) there exists $\eta_1 > 0$ such that

$$|A(x,t,\xi)| \leq \eta_1 \quad \text{a.e. in } \Omega, \text{ if } |(t,\xi)| \leq R$$
$$|B(x,t,\xi)| \leq \eta_1 \quad \text{a.e. in } \Omega, \text{ if } |(t,\xi)| \leq R;$$

(H_4) there exists $\eta_2 > 0$ such that

$$A(x,t,\xi) \leq \eta_2 a(x,t,\xi) \cdot \xi \quad \text{a.e. in } \Omega, \text{ if } |(t,\xi)| \geq R,$$
$$B(x,t,\xi) \leq \eta_2 b(x,t,\xi) \cdot \xi \quad \text{a.e. in } \Omega, \text{ if } |(t,\xi)| \geq R;$$

(H_5) there exists $\mu_1 > 0$ such that

$$a(x,t,\xi) \cdot \xi + A_t(x,t,\xi)t \geq \mu_1 a(x,t,\xi) \cdot \xi \quad \text{a.e. in } \Omega, \text{ if } |(t,\xi)| \geq R,$$
$$b(x,t,\xi) \cdot \xi + B_t(x,t,\xi)t \geq \mu_1 b(x,t,\xi) \cdot \xi \quad \text{a.e. in } \Omega, \text{ if } |(t,\xi)| \geq R;$$

(H_6) there exists $\mu_2 > 0$ such that

$$a(x,t,\xi) \cdot \xi \geq \mu_2 |\xi|^{p_1} \quad \text{a.e. in } \Omega, \text{ if } |(t,\xi)| \leq R,$$
$$b(x,t,\xi) \cdot \xi \geq \mu_2 |\xi|^{p_2} \quad \text{a.e. in } \Omega, \text{ if } |(t,\xi)| \leq R;$$

(H_7) for all $\xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$, it is

$$[a(x,t,\xi) - a(x,t,\xi^*)] \cdot [\xi - \xi^*] > 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$
$$[b(x,t,\xi) - b(x,t,\xi^*)] \cdot [\xi - \xi^*] > 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

(G_0) G is a $\mathcal{C}^1$–Caratheodory function such that $G(\cdot, 0, 0) \in L^\infty(\Omega)$;

(G_1) some real numbers $q_i \geq 1$, $s_i \geq 0$ if $i \in \{1,2\}$, and $c > 0$ exist such that

$$|g_1(x,u,v)| \leq c\,(1 + |u|^{q_1-1} + |v|^{s_1}) \quad \text{for a.e. } x \in \Omega, \text{ all } (u,v) \in \mathbb{R}^2; \tag{2.1}$$
$$|g_2(x,u,v)| \leq c\,(1 + |u|^{s_2} + |v|^{q_2-1}) \quad \text{for a.e. } x \in \Omega, \text{ all } (u,v) \in \mathbb{R}^2, \tag{2.2}$$

(G_2)

$$g_1(x,0,0)=g_2(x,0,0)=0 \quad \text{for a.e. } x\in\Omega.$$

Remark 2.1. From condition (H_2) and (H_3) a constant $\eta_3>0$ esists such that

$$\begin{aligned} A(x,t,\xi) &\geq \mu_0(1+|t|^{p_1\sigma_1})|\xi|^{p_1}-\eta_3 \\ B(x,t,\xi) &\geq \mu_0(1+|t|^{p_2\sigma_2})|\xi|^{p_2}-\eta_3 \end{aligned} \tag{2.3}$$

for a.e. $x\in\Omega$, all $(t,\xi)\in\mathbb{R}\times\mathbb{R}^N$.
On the other hand, from (H_1), (H_3) and (H_4) direct computations imply

$$\begin{aligned} A(x,t,\xi) &\leq \eta_1+\eta_2\Phi_2(t)+\eta_2(\Phi_2(t)+\phi_2(t))|\xi|^{p_1} \\ B(x,t,\xi) &\leq \eta_1+\eta_2\Phi_2(t)+\eta_2(\Phi_2(t)+\phi_2(t))|\xi|^{p_2} \end{aligned} \tag{2.4}$$

for a.e. $x\in\Omega$, all $(t,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

Remark 2.2. From condition (G_0) - (G_1), the Main Value Theorem and standard computations we have that:

$$|G(x,u,v)|\leq c_1\left(1+|u|+|u|^{q_1}+|u||v|^{s_1}+|v|+|u|^{s_2}|v|+|v|^{q_2}\right) \tag{2.5}$$

for a.e. $x\in\Omega$, all $(u,v)\in\mathbb{R}^2$.
In fact, from (G_0) - (G_1) and the Main Value Theorem, for a.e. $x\in\Omega$ and for any $(u,v)\in\mathbb{R}^2$ there exists $\theta\in\,]0,1[$ such that

$$\begin{aligned} |G(x,u,v)| &\leq |G(x,u,v)-G(x,0,0)|+|G(x,0,0)| = |g_1(x,\theta u,\theta v)u+g_2(x,\theta u,\theta v)v|+|G(x,0,0)| \\ &\leq c\left(1+|u|^{q_1-1}+|v|^{s_1}\right)|u|+c\left(1+|u|^{s_2}+|v|^{q_2-1}\right)|v|+|G(\cdot,0,0)|_\infty \end{aligned}$$

and so (2.5) follows with $c_1=\max\{c,|G(\cdot,0,0)|_\infty\}$.

Theorem 2.3. *Assume that (H_0) - (H_7), (G_0) - (G_1) hold. If*

$$1<q_1<p_1(1+\sigma_1), \quad 1<q_2<p_2(1+\sigma_2), \tag{2.6}$$

$$\begin{aligned} 0\leq s_1 &\leq \min\left\{p_2(1+\sigma_2)\left(1-\frac{1}{p_1(1+\sigma_1)}\right), \frac{p_1p_2^*}{N}(1+\sigma_2)\left(1-\frac{1}{p_1(1+\sigma_1)}\right)\right\} \\ 0\leq s_2 &\leq \min\left\{p_1(1+\sigma_1)\left(1-\frac{1}{p_2(1+\sigma_2)}\right), \frac{p_1^*p_2}{N}(1+\sigma_1)\left(1-\frac{1}{p_2(1+\sigma_2)}\right)\right\}, \end{aligned} \tag{2.7}$$

then system (1.1) *has at least a weak bounded solution* $(\overline{u},\overline{v})$.

Remark 2.4. From direct computations the hypothesis (2.7) becomes:

$$\begin{aligned} 0\leq s_1 &\leq p_2(1+\sigma_2)\left(1-\frac{1}{p_1(1+\sigma_1)}\right), \\ 0\leq s_2 &\leq p_1(1+\sigma_1)\left(1-\frac{1}{p_2(1+\sigma_2)}\right), \end{aligned}$$

if and only if $N \leq p_1 + p_2$.
In particular, if $N \leq p_1 + p_2$ and $\sigma_1 = \sigma_2 = \sigma$, than the condition (2.7) reduces to:

$$0 \leq s_1 < \left(1 + \sigma - \frac{1}{p_1}\right) p_2, \quad 0 \leq s_2 < \left(1 + \sigma - \frac{1}{p_2}\right) p_1.$$

If the additional assumption (G_2) holds, than $(u, v) \equiv (0, 0)$ is a trivial solution of (1.1).
The next result states the existence of a non-trivial solution.

Theorem 2.5. *Assume that (H_0) - (H_7), (G_0) - (G_2) and (2.6) - (2.7) conditions hold. Moreover, suppose that a constant η_4 exists such that*

(H_8)

$$|A(x, t, \xi) \leq \eta_4 |\xi|^{p_1},$$
$$|B(x, t, \xi) \leq \eta_4 |\xi|^{p_2},$$

(G_3)

$$\liminf_{|(u,v)| \to (0,0)} \frac{G(x, u, v)}{|u|^{p_1} + |v|^{p_2}} > \eta_4 \max\{\lambda_{1,1}, \lambda_{2,1}\} \qquad \text{uniformly a.e. in } \Omega,$$

where for $i \in \{1, 2\}$ $\lambda_{i,1}$ are the first eigenvalue of $-\Delta_{p_i}$ in W_i which is characterized as

$$\lambda_{i,1} = \inf_{\xi \in W_i \setminus \{0\}} \frac{\int_\Omega |\nabla \xi|^{p_i} dx}{\int_\Omega |\xi|^{p_i} dx}, \tag{2.8}$$

and is strictly positive, simple, isolated and has a unique eigenfunction $\varphi_{i,1}$ such that

$$\varphi_{i,1} > 0 \text{ a.e. in } \Omega, \quad \varphi_{i,1} \in L^\infty(\Omega) \quad \text{and} \quad |\varphi_{i,1}|_{p_i} = 1 \tag{2.9}$$

Then, system (1.1) *has at least a weak bounded non-trivial solution.*

The proofs of theorems (2.3) and (2.5) are based on a suitable generalization of the Weierstrass Theorem.
We denote by $(X, \|\cdot\|_X)$ a Banach space with dual $(X', \|\cdot\|_{X'})$, by $(W, \|\cdot\|_W)$ another Banach space such that $X \hookrightarrow W$ continuously and by J a given $\mathcal{C}^1$ functional.
Then, we introduce a suitable weaker version of the Cerami's variant of Palais–Smale condition.
Taking $\beta \in \mathbb{R}$, we say that a sequence $(\xi_n)_n \subset X$ is a *Cerami–Palais–Smale sequence at level* β, briefly $(CPS)_\beta$*–sequence*, if

$$\lim_{n \to +\infty} J(\xi_n) = \beta \quad \text{and} \quad \lim_{n \to +\infty} \|dJ(\xi_n)\|_{X'}(1 + \|\xi_n\|_X) = 0.$$

Moreover, β is a *Cerami–Palais–Smale level*, briefly (CPS)*–level*, if there exists a $(CPS)_\beta$–sequence.
As $(CPS)_\beta$–sequences may exist which are unbounded in $\|\cdot\|_X$ but converge with respect to $\|\cdot\|_W$, we have to weaken the classical Cerami–Palais–Smale condition in a suitable way according to the ideas already developed in previous papers (see, e.g., [4, 5, 6]).

Definition 2.6. The functional J satisfies the *weak Cerami–Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ *condition*, if for every $(wCPS)_\beta$ *sequence* $(\xi_n)_n \in X$ a point $\xi \in X$ exists, such that:

(i) $\lim\limits_{n\to+\infty} \|\xi_n - \xi\|_W = 0$ (up to subsequences),

(ii) $J(\xi) = \beta, \; dJ(\xi) = 0$.

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

Condition $(wCPS)_\beta$ implies that the set of critical point of J at level β is compact with respect to $\|\cdot\|_W$, hence a Deformation Lemma and some abstract critical point Theorems con be stated (see [6]). In particular, the following Minimum Principle can be stated (for the proof, see [6, Theorem 1.6]).

Theorem 2.7. *(Minumun Principle) If $J \in \mathcal{C}^1(X, \mathbb{R})$ is bounded from below in X and $(wCPS)_\beta$ holds at level $\beta = \inf\limits_X J \in \mathbb{R}$, then J attains its infimum, i.e. $\xi_0 \in X$ exists such that $J(\xi_0) = \beta$.*

3 Variational setting and first properties

From now on, we denote by:

- meas(D) the usual Lebesgue measure of a measurable set D in $\mathbb{R}^N$;
- $L^r(\Omega)$ the Lebesgue space with norm $|\xi|_r = \left(\int_\Omega |\xi|^r dx\right)^{1/r}$ if $1 \le r < +\infty$;
- $L^\infty(\Omega)$ the space of Lebesgue–measurable and essentially bounded functions $\xi : \Omega \to \mathbb{R}$ with norm $|\xi|_\infty = \operatorname{ess\,sup}\limits_\Omega |\xi|$;
- $W_0^{1,p}(\Omega)$ the classical Sobolev space with norm $\|\xi\|_{W_0^{1,p}} = |\nabla \xi|_p$ if $1 \le p < +\infty$.

For simplicity, here and in the following we denote by $|\cdot|$ the standard norm on any Euclidean space, as the dimension of the considered vector is clear and no ambiguity occurs, and by C any strictly positive constant which arises by computation.
In order to look for weak solutions of the nonlinear problem (1.1), consider $p_1,\, p_2 > 1$ and, for $i \in \{1, 2\}$, the related Sobolev space

$$W_i = W_0^{1,p_i}(\Omega) \quad \text{with norm } \|\cdot\|_{W_i} = \|\cdot\|_{W_0^{1,p_i}}.$$

From the Sobolev Embedding Theorem, for any $r \in [1, p_i^*]$ with $p_i^* = \frac{Np_i}{N-p_i}$ if $N > p_i$, or any $r \in [1, +\infty[$ if $p_i \ge N$, W_i is continuously embedded in $L^r(\Omega)$, i.e. a positive constant $\tau_{i,r}$ exists such that

$$|\xi|_r \le \tau_{i,r} \|\xi\|_{W_i} \quad \text{for all } \xi \in W_i. \tag{3.1}$$

For simplicity, we put

$$p_i^* = +\infty \quad \text{and} \quad \frac{1}{p_i^*} = 0 \qquad \text{if } p_i \ge N.$$

Here, the notation introduced for the abstract setting at the beginning of Section 2 is referred to our problem with

$$W = W_1 \times W_2 \tag{3.2}$$

while the Banach space $(X, \|\cdot\|_X)$ is defined as

$$X = X_1 \times X_2, \tag{3.3}$$

where

$$X_1 := W_1 \cap L^\infty(\Omega) \quad \text{ and } \quad X_2 := W_2 \cap L^\infty(\Omega)$$

with the norms

$$\|u\|_{X_1} = \|u\|_{W_1} + |u|_\infty \text{ if } u \in X_1 \quad \text{ and } \quad \|v\|_{X_2} = \|v\|_{W_2} + |v|_\infty \text{ if } v \in X_2. \tag{3.4}$$

Since $(W_i, \|\cdot\|_{W_i})$ is a reflexive Banach space for both $i = 1$ and $i = 2$, so is $(W, \|\cdot\|_W)$ where, for $(u, v) \in W$ it is

$$\|(u, v)\|_W = \|u\|_{W_1} + \|v\|_{W_2}.$$

Setting

$$L := L^\infty(\Omega) \times L^\infty(\Omega) \quad \text{with } \|(u, v)\|_L = |u|_\infty + |v|_\infty,$$

we have that X in (3.3) can also be written as

$$X = W \cap L$$

and can be equipped with the norm

$$\|(u, v)\|_X = \|(u, v)\|_W + \|(u, v)\|_L = \|u\|_{X_1} + \|v\|_{X_2}.$$

By definition, for $i \in \{1, 2\}$ we have $X_i \hookrightarrow W_i$ and $X_i \hookrightarrow L^\infty(\Omega)$ with continuous embeddings.

Remark 3.1. If $i \in \{1, 2\}$ is such that $p_i > N$, then $X_i = W_i$, as $W_i \hookrightarrow L^\infty(\Omega)$. Hence, if both $p_1 > N$ and $p_2 > N$, then $X = W_1 \times W_2$ and the classical Weierstrass Theorems in [3] can be used, if required.

Now, we note that if conditions (H_0)–(H_1), (G_0)–(G_1) hold, the functional $\mathcal{J}(u, v)$ in (1.9) is well defined for all $(u, v) \in X$. Moreover, taking any (u, v), $(w, z) \in X$, the Gâteaux differential of functional $\mathcal{J}$ in (u, v) along the direction (w, z) is

$$\begin{aligned} d\mathcal{J}(u, v)[(w, z)] = &\int_\Omega a(x, u, \nabla u) \cdot \nabla w \; dx + \int_\Omega A_t(x, u, \nabla u) w dx \\ &+ \int_\Omega B(x, v, \nabla v) \cdot \nabla z \; dx + \int_\Omega B_t(x, v, \nabla v) z dx \\ &- \int_\Omega g_1(x, u, v) w \; dx - \int_\Omega g_2(x, u, v) z \; dx. \end{aligned} \tag{3.5}$$

For simplicity, we put

$$\begin{aligned} &\frac{\partial \mathcal{J}}{\partial u}(u, v) : w \in X_1 \mapsto \frac{\partial \mathcal{J}}{\partial u}(u, v)[w] = d\mathcal{J}(u, v)[(w, 0)] \in \mathbb{R}, \\ &\frac{\partial \mathcal{J}}{\partial v}(u, v) : z \in X_2 \mapsto \frac{\partial \mathcal{J}}{\partial v}(u, v)[z] = d\mathcal{J}(u, v)[(0, z)] \in \mathbb{R}. \end{aligned}$$

So, from (3.5) it follows that

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial u}(u,v)[w] = &\int_\Omega a(x,u,\nabla u)\cdot \nabla w\ dx + \int_\Omega A_t(x,u,\nabla u) w dx \\ &- \int_\Omega g_1(x,u,v) w\ dx\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial v}(u,v)[z] = &\int_\Omega b(x,v,\nabla v)\cdot \nabla z dx + \int_\Omega B_t(x,v,\nabla v) z dx \\ &- \int_\Omega g_2(x,u,v) z dx.\end{aligned} \tag{3.7}$$

Remark 3.2. Taking $(u,v) \in X$, since $d\mathcal{J}(u,v) \in X'$, then

$$\frac{\partial \mathcal{J}}{\partial u}(u,v) \in X_1', \qquad \frac{\partial \mathcal{J}}{\partial v}(u,v) \in X_2'$$

and

$$d\mathcal{J}(u,v)[(w,z)] = \frac{\partial \mathcal{J}}{\partial u}(u,v)[w] + \frac{\partial \mathcal{J}}{\partial v}(u,v)[z] \quad \forall (w,z) \in X. \tag{3.8}$$

Moreover, direct computations imply that

$$\left\|\frac{\partial \mathcal{J}}{\partial u}(u,v)\right\|_{X_1'} \le \|d\mathcal{J}(u,v)\|_{X'}, \quad \left\|\frac{\partial \mathcal{J}}{\partial v}(u,v)\right\|_{X_2'} \le \|d\mathcal{J}(u,v)\|_{X'}, \tag{3.9}$$

and

$$\|d\mathcal{J}(u,v)\|_{X'} \le \left\|\frac{\partial \mathcal{J}}{\partial u}(u,v)\right\|_{X_1'} + \left\|\frac{\partial \mathcal{J}}{\partial v}(u,v)\right\|_{X_2'}. \tag{3.10}$$

Clearly, we have

$$d\mathcal{J}(u,v) = 0 \text{ in } X \quad \iff \quad \frac{\partial \mathcal{J}}{\partial u}(u,v) = 0 \text{ in } X_1 \text{ and } \frac{\partial \mathcal{J}}{\partial v}(u,v) = 0 \text{ in } X_2.$$

The following regularity result hold:

Proposition 3.3. *Assume that conditions (H_0)–(H_4), (G_0), (G_1) hold. Let $((u_n,v_n))_n \subset X$ and $(u,v) \in X$ be such that*

$$u_n \rightharpoonup u \text{ in } W_1, \quad u_n \to u \text{ a.e. in } \Omega \text{ if } n \to \infty, \tag{3.11}$$

$$v_n \rightharpoonup v \text{ in } W_2, \quad v_n \to v \text{ a.e. in } \Omega \text{ if } n \to \infty, \tag{3.12}$$

and, moreover, $M > 0$ exists such that

$$|u_n|_\infty \le M \quad \text{and} \quad |v_n|_\infty \le M \quad \forall n \in \mathbb{N}. \tag{3.13}$$

Then,

$$\mathcal{J}(u_n,v_n) \to \mathcal{J}(u,v) \quad \text{and} \quad \|d\mathcal{J}(u_n,v_n) - d\mathcal{J}(u,v)\|_{X'} \to 0 \quad \text{as } n \to \infty.$$

Hence, $\mathcal{J}$ is a C^1 functional on X with Fréchet differential defined as in (3.5).

Proof. First of all, consider the functionals $\mathcal{J}_1, \mathcal{J}_2$ defined as:

$$\mathcal{J}_1(u) = \int_\Omega A(x, u, \nabla u)dx, \quad u \in X_1$$
$$\mathcal{J}_2(v) = \int_\Omega B(x, v, \nabla v)dx, \quad v \in X_2.$$

From assumptions (H_0)–(H_4) and Remark (2.1), arguing as in the Proof of [4, Proposition 3.1], it follows that $\mathcal{J}_1$ and $\mathcal{J}_2$ are of class $\mathcal{C}^1$ with Fréchet differential defined as

$$d\mathcal{J}_1(u)[w] = \int_\Omega a(x, u, \nabla u) \cdot \nabla w dx + \int_\Omega A_t(x, u, \nabla u)w dx, \qquad u, w \in X_1$$

and with

$$d\mathcal{J}_2(v)[z] = \int_\Omega b(x, v, \nabla v) \cdot \nabla z dx + \int_\Omega B_t(x, v, \nabla v)z dx, \qquad v, z \in X_2.$$

On the other hand, reasoning as in [10, Proposition 3.7], (G_0),(G_1), (2.7), (3.11) and (3.13) imply that the functional

$$\mathcal{J}_3(u, v) = \int_\Omega G(x, u, v)dx, \qquad (u, v) \in X$$

is of class $\mathcal{C}^1$ with Fréchet differential defined as

$$d\mathcal{J}_3(u, v)[(w, z)] = \int_\Omega g_1(x, u, v)w dx + \int_\Omega g_2(x, u, v)z dx, \qquad (w, z) \in X.$$

Hence, summing up, we conclude that $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$ is $\mathcal{C}^1$ on X with Fréchet differential as in (3.5). ☐

We point out that up to now no sub-critical growth is required for $g_1(x, u, v)$ and $g_2(x, u, v)$ since in our setting it is $X_i \hookrightarrow L^r(\Omega) \quad \forall r \geq 1$ and $i \in \{1, 2\}$.
Anyway, subcritical growth assumptions need for proving in general the weak Cerami–Palais–Smale at any level while further growth assumptions on A, B and G allows us to prove that $\mathcal{J}$ is bounded from below. To this aim, we will use the following result.

Proposition 3.4. *Assume that condition (G_0)-(G_1) hold. Then, there exist $\overline{q}_1$, $\overline{q}_2$ such that*

$$|G(x, u, v)| \leq c(1 + |u|^{\overline{q}_1} + |v|^{\overline{q}_2}). \tag{3.14}$$

Moreover if we assume (2.6) and

$$\begin{aligned} 0 \leq s_1 <& p_2(1 + \sigma_2)\left(1 - \frac{1}{p_1(1 + \sigma_1)}\right), \\ 0 \leq s_2 <& p_1(1 + \sigma_1)\left(1 - \frac{1}{p_2(1 + \sigma_2)}\right), \end{aligned} \tag{3.15}$$

it follows that

$$1 < \overline{q}_1 < p_1(1 + \sigma_1), \quad 1 < \overline{q}_2 < p_2(1 + \sigma_2), . \tag{3.16}$$

Proof. From Remark (2.2) G verifies (2.5). Now, the Young inequality implies that for any $s_3 > 1$ it is

$$|u||v|^{s_1} \leq \frac{1}{s_3}|u|^{s_3} + \left(1 - \frac{1}{s_3}\right)|v|^{s_1 \frac{s_3}{s_3-1}}. \tag{3.17}$$

We note that the constant s_3 can be choosen such that

$$1 < s_3 < p_1(1+\sigma_1), \qquad 0 \leq s_4 = s_1 \frac{s_3}{s_3 - 1} < p_2(1+\sigma_2) \tag{3.18}$$

or equivalently

$$\frac{p_2(1+\sigma_2)}{p_2(1+\sigma_2) - s_1} < s_3 < p_1(1+\sigma_1)$$

if and only if $0 \leq s_1 < p_2(1+\sigma_2)\left(1 - \dfrac{1}{p_1(1+\sigma_1)}\right)$.
Arguing in a similar way, we have that

$$|u|^{s_2}|v| \leq \left(1 - \frac{1}{s_5}\right)|u|^{s_2 \frac{s_5}{s_5-1}} + \frac{1}{s_5}|v|^{s_5} \tag{3.19}$$

where we can choose $s_5 > 1$ such that:

$$0 \leq s_2 \frac{s_5}{s_5 - 1} < p_1(1+\sigma_1) \qquad 1 < s_5 < p_2(1+\sigma_2), \tag{3.20}$$

or equivalently

$$0 \leq \frac{p_1(1+\sigma_1)}{p_1(1+\sigma_1) - s_2} < s_5 < p_2(1+\sigma_2)$$

if and only if $0 \leq s_2 < p_1(1+\sigma_1)\left(1 - \dfrac{1}{p_2(1+\sigma_2)}\right)$.

Summing up, from (2.5), (3.17) and (3.19), estimate (3.14) follows with $\overline{q}_1$, $\overline{q}_2$ where:

$$\overline{q}_1 = \max\left\{q_1, s_3, s_2 \frac{s_5}{s_5 - 1}\right\} \qquad \text{and} \qquad \overline{q}_2 = \max\left\{q_2, s_5, s_1 \frac{s_3}{s_3 - 1}\right\}. \tag{3.21}$$

Moreover, if (2.6) and (3.15) hold, then $\overline{q}_1, \overline{q}_2$ verify (3.16). □

Remark 3.5. We point out that (G_0), (G_1) imply only that G has a polinomial growth. Under the additional assumption (2.6) and (3.15), G has a "little more" than "subquadratic growth". In particular, if we take $\sigma_1 = \sigma_2 = 0$ in (H_2), G has a "subquadratic like growth", assuming that:

$$\begin{cases} 1 < q_1 < p_1, \quad 1 < q_2 < p_2, \\ \\ 0 \leq s_1 < p_2\left(1 - \frac{1}{p_1}\right), \quad 0 \leq s_2 < p_1\left(1 - \frac{1}{p_2}\right) \end{cases}$$

Remark 3.6. If $p_1 = p_2 = p$, condition (2.6) and (3.15) reduce to:

$$\begin{aligned} &1 < q_1 < p(1+\sigma_1), \quad 1 < q_2 < p(1+\sigma_2), \\ &0 \leq s_1 < p(1+\sigma_2) - \frac{1}{1+\sigma_1}, \quad 0 \leq s_2 < p(1+\sigma_1) - \frac{1}{1+\sigma_2} \end{aligned}$$

and the exponents $\overline{q}_1, \overline{q}_2$ in (3.14) verify (3.16).
In particular, if $p_1 = p_2 = 2$ and $\sigma_1 = \sigma_2 = 0$, the classical subquadratic growth for G is obtained.

4 Proof of the Main Theorem

In this section we will reply Theorem 2.7 (Minimum Principle) to our functional $\mathcal{J} : X \to \mathbb{R}$ defined as in (1.9). To this aim, we will use the following results.

Remark 4.1. Taking $(u, v) \in X$, it results $(|u|^{\sigma_1}u, |v|^{\sigma_2}v \in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega))$ as for a.e. $x \in \Omega$

$$\begin{aligned} |\nabla(|u|^{\sigma_1}u)|^{p_1} &= (1+\sigma_1)^{p_1}|u|^{p_1\sigma_1}|\nabla u|^{p_1} \\ |\nabla(|v|^{\sigma_2}v)|^{p_2} &= (1+\sigma_2)^{p_2}|u|^{p_2\sigma_2}|\nabla v|^{p_2} \end{aligned} \tag{4.1}$$

Lemma 4.2. *Assume that conditions (H_0)–(H_3), (G_0)–(G_1) hold. Then, some positive constants c_i exist such that:*

$$\begin{aligned} \mathcal{J}(u,v) \geq \mu_0\|u\|_{W^1}^{p_1} + \frac{\mu_0}{(1+\sigma_1)^{p_1}}\||u|^{\sigma_1}u\|_{W^1}^{p_1} - c_1\||u|^{\sigma_1}u\|_{W^1}^{\frac{\overline{q}_1}{1+\sigma_1}} \\ +\mu_0\|v\|_{W^2}^{p_2} + \frac{\mu_0}{(1+\sigma_2)^{p_2}}\||v|^{\sigma_2}v\|_{W^2}^{p_2} - c_2\||v|^{\sigma_2}v\|_{W^2}^{\frac{\overline{q}_2}{1+\sigma_2}} - c_3 \end{aligned} \tag{4.2}$$

Hence, if also (2.6) and (3.15) *hold, then*

$$\inf_X J(u,v) > -\infty. \tag{4.3}$$

Proof. Fixing any $(u, v) \in X$, from Remark (2.1), (3.14) and (4.1), it follows that:

$$\begin{aligned} \mathcal{J}(u,v) \geq \mu_0\int_\Omega |\nabla u|^{p_1}dx + \frac{\mu_0}{(1+\sigma_1)^{p_1}}\int_\Omega |\nabla(|u|^{\sigma_1}u)|^{p_1}dx - c\int_\Omega (|u|^{\sigma_1}u)^{\frac{\overline{q}_1}{1+\sigma_1}}dx \\ +\mu_0\int_\Omega |\nabla v|^{p_2}dx + \frac{\mu_0}{(1+\sigma_2)^{p_2}}\int_\Omega |\nabla(|v|^{\sigma_2}v)|^{p_2}dx - c\int_\Omega (|v|^{\sigma_2}v)^{\frac{\overline{q}_2}{1+\sigma_2}}dx - 2\eta_3\text{meas}(\Omega) \end{aligned}$$

then (4.2) follows from the Sobolev Imbedding Theorem.
Clearly, (4.2) implies:

$$\begin{aligned} \mathcal{J}(u,v) \geq \frac{\mu_0}{(1+\sigma_1)^{p_1}}\||u|^{\sigma_1}u\|_{W^1}^{p_1} - c_1\||u|^{\sigma_1}u\|_{W^1}^{\frac{\overline{q}_2}{1+\sigma_1}} \\ +\frac{\mu_0}{(1+\sigma_2)^{p_2}}\||v|^{\sigma_2}v\|_{W^2}^{p_2} - c_2\||v|^{\sigma_2}v\|_{W^2}^{\frac{\overline{q}_2}{1+\sigma_2}} - c_3. \end{aligned}$$

Hence, if also (2.6) and (3.15) hold, then (4.3) follows from (3.16). □

Remark 4.3. From (4.2), (2.6), (3.15) and direct computations, we obtain

$$\mathcal{J}(u,v) \geq \mu_0\|u\|_{W^1}^{p_1} + c_4\||u|^{\sigma_1}u\|_{W^1}^{p_1} + \mu_0\|v\|_{W^2}^{p_2} + c_5\||v|^{\sigma_2}v\|_{W^2}^{p_2} - c_6$$

for suitable constants c_4, c_5 and $c_6 > 0$.

In order to prove that functional $\mathcal{J}$ satislies that $(wCPS)$ condition in $\mathbb{R}$, we need the following results.

Lemma 4.4. *Fix $\sigma \geq 0$ and let $(\xi_n)_n \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a sequence such that*

$$\left(\int_\Omega (1 + |\xi_n|^{p\sigma})|\nabla \xi_n|^p dx\right)_n \text{ is bounded.}$$

Then, $\xi \in W_0^{1,p}(\Omega)$ esists such that $|u|^\sigma u \in W_0^{1,p}(\Omega)$, too, and up to subsequences, if $n \to \infty$, we have

$$\begin{aligned}
&\xi_n \rightharpoonup \xi \text{ weakly in } W_0^{1,p}(\Omega)\\
&|\xi_n|^\sigma \xi_n \longrightarrow |\xi|^\sigma \xi \text{ in } W_0^{1,p}(\Omega)\\
&\xi_n \longrightarrow \xi \text{ a.e. in } \Omega\\
&\xi_n \longrightarrow \xi \text{ strongly in } L^r(\Omega \quad \text{for each } r \in [1, p(1+\sigma)[.
\end{aligned}$$

Proof. For the proof, see [10, Lemma 3.8] □

Lemma 4.5. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and consider $\xi \in W_0^{1,p}(\Omega)$ with $p \leq N$. Suppose that $\gamma > 0$ and $k_0 \in \mathbb{N}$ exist such that*

$$\int_{\Omega_k^+} |\nabla \xi|^p dx \ \leq \ \gamma \left(\int_{\Omega_k^+} (\xi - k)^r dx\right)^{\frac{p}{r}} + \gamma \sum_{j=1}^m k^{\alpha_j} [\text{meas}(\Omega_k^+)]^{1 - \frac{p}{N} + \varepsilon_j} \quad \text{for all } k \geq k_0,$$

with $\Omega_k^+ := \{x \in \Omega : \xi(x) > k\}$ and $r, m, \alpha_j, \varepsilon_j$ are positive constants such that

$$1 \leq r < p^*, \qquad \varepsilon_j > 0, \qquad p \leq \alpha_j < \varepsilon_j p^* + p.$$

Then $\operatorname{ess\,sup}_\Omega \xi$ is bounded from above by a positive constant which can be chosen so that it depends only on $\text{meas}(\Omega), N, p, \gamma, k_0, r, m, \varepsilon_j, \alpha_j, |\xi|_{p^*}$ (eventually, $|\xi|_l$ for some $l > r$ if $p^* = +\infty$).

Proof. For the proof, see [14, Theorem II 5.1]. □

Proposition 4.6. *Assume hypotheses (H_0)–(H_7), (G_0)–(G_1) and (2.6) – (2.7) hold. Then, functional $\mathcal{J}$ satisfies condition $(wCPS)$ in $\mathbb{R}$.*

Proof. Let $\beta \in \mathbb{R}$ be fixed and consider $((u_n, v_n))_n \subset X$ a sequence such that

$$\mathcal{J}(u_n, v_n) \longrightarrow \beta \quad \text{and} \quad \|d\mathcal{J}(u_n, v_n)\|_{X'}(1 + \|(u_n, v_n)\|_X) \longrightarrow 0 \quad \text{if } n \to +\infty, \tag{4.4}$$

Our proof is divided in several steps:

1. $((u_n, v_n))_n$ is bounded in W. Hence, there exists $(u, v) \in W$ such that $(|u|^{\sigma_1}u, |v|^{\sigma_2}v) \in W$ and up to subsequences, we have that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } W, \tag{4.5}$$

$$(|u_n|^{\sigma_1}u_n, |v_n|^{\sigma_2}v_n) \rightharpoonup (|u|^{\sigma_1}u, |v|^{\sigma_2}v) \text{ weakly in } W, \tag{4.6}$$

$$(u_n, v_n) \longrightarrow (u, v) \text{ in } L^{r_1}(\Omega) \times L^{r_2}(\Omega) \text{ if } 1 \leq r_1 < p_1^*(1+\sigma_1), 1 \leq r_2 < p_2^*(1+\sigma_2), \tag{4.7}$$

$$(u_n, v_n) \longrightarrow (u, v) \text{ a.e. in } \Omega. \tag{4.8}$$

2. $(u,v) \in L$;

3. for any $k>0$, defining $T_k : \mathbb{R} \to \mathbb{R}$ such that

$$T_k t := \begin{cases} t & \text{if } |t| \le k \\ k\frac{t}{|t|} & \text{if } |t| > k \end{cases}$$

and

$$\mathcal{T}_k : (t_1,t_2) \in \mathbb{R}^2 \mapsto \mathcal{T}_k(t_1,t_2) = (T_k t_1, T_k t_2) \in \mathbb{R}^2,$$

then, if $k \ge \max\{|(u,v)|_L, R\} + 1$ (with $R \ge 1$ as in our set of hypotheses), we have

$$\|d\mathcal{J}(\mathcal{T}_k(u_n,v_n))\|_{X'} \longrightarrow 0 \tag{4.9}$$

and

$$\mathcal{J}(\mathcal{T}_k(u_n,v_n)) \longrightarrow \beta; \tag{4.10}$$

4. $\|(T_k u_n - u, T_k v_n - v)\|_W \longrightarrow 0$ and then $\|(u_n-u, v_n-v)\|_W \longrightarrow 0$ (up to subsequences);

5. $\mathcal{J}(u,v) = \beta, d\mathcal{J}(u,v) = 0$.

Step 1. Firstly, we note that (2.7) implies (3.15).
Then from (H_0)–(H_3), (2.6) and (2.7), Remark (4.3) holds. Hence, (4.1) and (4.4) imply that the sequences:

$$\left(\int_\Omega (1+|u_n|^{p_1\sigma_1})|\nabla u_n|^{p_1} dx\right)_n \text{ and } \left(\int_\Omega (1+|v_n|^{p_2\sigma_2})|\nabla v_n|^{p_2} dx\right)_n \tag{4.11}$$

are bounded and the conclusion follows from Lemma (4.4).
Step 2. Arguing by contradiction we assume that either $u \notin L^\infty(\Omega)$ or $v \notin L^\infty(\Omega)$. If $u \notin L^\infty(\Omega)$, either

$$\operatorname*{ess\,sup}_\Omega u = +\infty \tag{4.12}$$

or

$$\operatorname*{ess\,sup}_\Omega (-u) = +\infty. \tag{4.13}$$

For example, suppose that (4.12) holds. Then, for any fixed $k \in \mathbb{N}$, $k > R$ (with $R \ge 1$ as in the hypotheses), we have

$$\text{meas}(\Omega_k^+) > 0, \tag{4.14}$$

with $\Omega_k^+ := \{x \in \Omega \mid u(x) > k\}$.
Now, for any $\tilde{k} > 0$, we consider the new function $R_{\tilde{k}}^+ : \mathbb{R} \to \mathbb{R}$ such that

$$R_{\tilde{k}}^+ t = \begin{cases} 0 & \text{if } t \le \tilde{k} \\ t - \tilde{k} & \text{if } t > \tilde{k} \end{cases}.$$

Taking $\tilde{k} = k^{\sigma_1+1}$, from (4.6), it follows that

$$R^+_{k^{\sigma_1+1}}(|u_n|^{\sigma_1}u_n) \rightharpoonup R^+_{k^{\sigma_1+1}}(|u|^{\sigma_1}u) \quad \text{in } W_1.$$

Thus, by the sequentially weakly lower semicontinuity of $\|\cdot\|_{W^1}$, it follows that

$$\int_\Omega |\nabla R^+_{k^{\sigma_1+1}}(|u|^{\sigma_1}u)|^{p_1}dx \leq \liminf_{n\to+\infty}\int_\Omega |\nabla R^+_{k^{\sigma_1+1}}(|u_n|^{\sigma_1}u_n)|^{p_1}dx,$$

i.e.,

$$\int_{\Omega_k^+} |\nabla(u)^{\sigma_1+1}|^{p_1}dx \leq \liminf_{n\to+\infty}\int_{\Omega^+_{n,k}} |\nabla(u_n)^{\sigma_1+1}|^{p_1}dx, \tag{4.15}$$

as $|t|^{\sigma_1}t > k^{\sigma_1+1} \iff t > k$ with $\Omega^+_{n,k} := \{x\in\Omega \mid u_n(x) > k\}$.
On the other hand, from $\|R^+_k u_n\|_{X_1} \leq \|u_n\|_{X_1}$ (4.4) and (4.14) it follows that $n_k \in \mathbb{N}$ exists so that:

$$d\mathcal{J}(u_n,v_n)[R^+_k u_n] \ < \text{meas}(\Omega^+_k) \quad \text{ for all } n \geq n_k. \tag{4.16}$$

Taking any $k > R$ and $n\in\mathbb{N}$, from (3.6), (H_5) with $\mu_1 < 1$, (H_2) and (4.1), it follows that:

$$\begin{aligned}\frac{\partial\mathcal{J}}{\partial u}(u_n,v_n)\left[R^+_k u_n\right] &= \int_{\Omega^+_{n,k}}\left(1-\frac{k}{u_n}\right)(a(x,u_n,\nabla u_n)\cdot\nabla u_n + A_t(x,u_n,\nabla u_n)u_n)\,dx\\ &\quad+\int_{\Omega^+_{n,k}}\frac{k}{u_n}a(x,u_n,\nabla u_n)\cdot\nabla u_n dx - \int_\Omega g_1(x,u_n,v_n)R^+_k u_n dx\\ &\geq \mu_1\int_{\Omega^+_{n,k}} a(x,u_n,\nabla u_n)\cdot\nabla u_n dx - \int_\Omega g_1(x,u_n,v_n)R^+_k u_n dx\\ &\geq \mu_0\mu_1\int_{\Omega^+_{n,k}} {u_n}^{p_1\sigma_1}|\nabla u_n|^{p_1}dx - \int_\Omega g_1(x,u_n,v_n)R^+_k u_n dx =\\ &= \frac{\mu_0\mu_1}{(\sigma_1+1)^{p_1}}\int_{\Omega^+_{n,k}}|\nabla(u_n)^{\sigma_1+1}|^{p_1}dx - \int_\Omega g_1(x,u_n,v_n)R^+_k u_n dx\end{aligned}$$

which, together with (4.16), implies that

$$\int_{\Omega^+_{n,k}}|\nabla(u_n)^{\sigma_1+1}|^{p_1}dx \leq \frac{(\sigma_1+1)^{p_1}}{\mu_0\mu_1}\left(\text{meas}(\Omega^+_k) + \int_\Omega g_1(x,u_n,v_n)R^+_k u_n dx\right) \quad \text{ for all } n\geq n_k. \tag{4.17}$$

We note that, from (4.9) and (G_0), we have

$$g_1(x,u_n,v_n)R^+_k u_n \longrightarrow g_1(x,u,v)R^+_k u \quad \text{ a.e. in } \Omega.$$

Since $|R^+_k u_n(x)| \leq |u_n(x)| \quad \text{a.e.} x\in\Omega$, from (G_1) and (3.17), we have:

$$\begin{aligned}|g_1(x,u_n,v_n)R^+_k u_n| &\leq \ c(|u_n| + |u_n|^{q_1} + |u_n||v_n|^{s_1})\\ &\leq c(|u_n| + |u_n|^{q_1} + \frac{|u_n|^{s_3}}{s_3} + \frac{s_3-1}{s_3}|v_n|^{s_1\frac{s_3}{s_3-1}}\end{aligned}$$

Thus, since (3.18) and (4.7), we have that a function $h \in L^1(\Omega)$ exists such that

$$|g_1(x,u_n,v_n)R_k^+ u_n| \le \ C(|u_n| + |u_n|^{q_1} + |u_n|^{s_3} + |v_n|^{s_4}) \le h(x) \quad \text{for a.e. } x \in \Omega,$$

up to subsequences (see, e.g., [3, Theorem 4.9]).
So, by using the Dominated Convergence Theorem, we have

$$\lim_{n\to+\infty} \int_\Omega g_1(x,u_n,v_n)R_k^+ u_n dx = \int_\Omega g_1(x,u,v)R_k^+ u dx. \tag{4.18}$$

Hence, summing up, from (4.15), (4.17) and (4.18), passing to the limit, we obtain

$$\int_{\Omega_k^+} |\nabla(u)^{\sigma_1+1}|^{p_1} dx \le C\left(\text{meas}(\Omega_k^+) + \int_{\Omega_k^+} g_1(x,u,v)R_k^+ u dx\right),$$

which implies, by using again (2.1), that

$$\int_{\Omega_k^+} |\nabla(u)^{\sigma_1+1}|^{p_1} dx \le C\left(\text{meas}(\Omega_k^+) + \int_{\Omega_k^+} |u| dx + \int_{\Omega_k^+} |u|^{q_1} dx + \int_{\Omega_k^+} |u||v|^{s_1} dx\right). \tag{4.19}$$

Now, from (3.17), (3.19) Hölder inequality with conjugate exponents $\frac{p_2^*(1+\sigma_2)}{s_4} > 1$ and $\frac{p_2^*(1+\sigma_2)}{p_2^*(1+\sigma_2)-s_4}$ and direct computations, it follows that

$$\begin{aligned}\int_{\Omega_k^+} |(u)^{\sigma_1+1}||v|^{s_1} dx \le\ & \frac{1}{s_3}\int_{\Omega_k^+} |u|^{s_3} dx + \ \frac{s_3-1}{s_3}\ \int_{\Omega_k^+} |v|^{s_4} dx \\ \le\ & \frac{1}{s_3}\int_{\Omega_k^+} |u|^{s_3} dx + \ \frac{s_3-1}{s_3}\ |v|^{s_4}_{p_2^*(1+\sigma_2)}\ [\text{meas}(\Omega_k^+)]^{1-\frac{s_4}{p_2^*(1+\sigma_2)}}.\end{aligned} \tag{4.20}$$

Taking $\overline{q}_1$ as in (3.21) so from (4.19) and (4.20) it results

$$\int_{\Omega_k^+} |\nabla(u)^{\sigma_1+1}|^{p_1} dx \le C\left(\int_{\Omega_k^+} |u|^{\overline{q}_1} dx + \text{meas}(\Omega_k^+) + [\text{meas}(\Omega_k^+)]^{1-\frac{s_4}{p_2^*(1+\sigma_2)}}\right) \tag{4.21}$$

with $C = C(\|v\|_{W_2}) > 0$.
Now, if we set $\overline{u} = |u|^{\sigma_1} u$, as $\overline{u} \in W_0^{1,p_1}(\Omega)$ and $\Omega_k^+ := \{x \in \Omega \mid \overline{u}(x) > k^{\sigma_1+1}\}$, (in particular, $\overline{u} = u^{\sigma_1+1}$ in Ω_k^+), from (4.21) we obtain:

$$\int_{\Omega_k^+} |\nabla \overline{u}|^{p_1} dx \le C\left(\int_{\Omega_k^+} |\overline{u}|^{\frac{\overline{q}_1}{\sigma_1+1}} dx + \text{meas}(\Omega_k^+) + \left[\text{meas}(\Omega_k^+)]^{1-\frac{s_4}{p_2^*(1+\sigma_2)}}\right]\right).$$

At last, we note that:

$$\int_{\Omega_k^+} |\overline{u}|^{\frac{\overline{q}_1}{\sigma_1+1}} dx = \int_{\Omega_k^+} |\overline{u}-k+k|^{\frac{\overline{q}_!}{\sigma_1+1}} |dx \leq 2^{\frac{\overline{q}_1}{\sigma_1+1}-1} \left(\int_{\Omega_k^+} |\overline{u}-k|^{\frac{\overline{q}_1}{\sigma_1+1}} dx + \int_{\Omega_k^+} k^{\frac{\overline{q}_1}{\sigma_1+1}} dx \right)$$

$$= 2^{\frac{\overline{q}_1}{\sigma_1+1}-1} \left[\left(\int_{\Omega_k^+} |\overline{u}-k|^{\frac{\overline{q}_1}{\sigma_1+1}} dx \right)^{\frac{\frac{\overline{q}_1}{\sigma_1+1}-p_1}{\left(\frac{\overline{q}_1}{\sigma_1+1}\right)}} \left(\int_{\Omega_k^+} |\overline{u}-k|^{\frac{\overline{q}_1}{\sigma_1+1}} dx \right)^{\frac{p_1}{\left(\frac{\overline{q}_1}{\sigma_1+1}\right)}} \right] + 2^{\frac{\overline{q}_1}{\sigma_1+1}-1} k^{\frac{\overline{q}_1}{\sigma_1+1}} \text{meas}(\Omega_k^+)$$

$$\leq 2^{\frac{\overline{q}_1}{\sigma_1+1}-1} \left[\left(|\overline{u}|_{\frac{\overline{q}_1}{\sigma_1+1}} \right)^{\frac{\overline{q}_1}{\sigma_1+1}-p_1} \left(\int_{\Omega_k^+} |\overline{u}-k|^{\frac{\overline{q}_1}{\sigma_1+1}} dx \right)^{\frac{p_1}{\left(\frac{\overline{q}_1}{\sigma_1+1}\right)}} \right] + 2^{\frac{\overline{q}_1}{\sigma_1+1}-1} k^{\frac{\overline{q}_1}{\sigma_1+1}} \text{meas}(\Omega_k^+).$$

Thus, from (3.16) we obtain:

$$\int_{\Omega_k^+} |\nabla \overline{u}|^{p_1} dx \leq C \left(\left(\int_{\Omega_k^+} |\overline{u}-k|^{\overline{q}_1} dx \right)^{\frac{p_1}{\left(\frac{\overline{q}_1}{\sigma_1+1}\right)}} + k^{p_1} \text{meas}(\Omega_k^+) + \text{meas}(\Omega_k^+)^{1-\frac{s_4}{p_2{}^*(1+\sigma_2)}} \right),$$

with $C = C(\|u\|_{W_1}, \|v\|_{W_2}) > 0$.
Using the notations of Lemma (4.5), we note that, from (2.6) it is:

$$1 \leq r = \tfrac{\overline{q}_1}{\sigma_1+1} < p_1{}^*,$$

$$k^{p_1} \text{meas}(\Omega_k^+) = k^{p_1} \left(\text{meas}(\Omega_k^+) \right)^{1-\frac{p_1}{N}+\epsilon_1}$$

with:

$$\epsilon_1 = \frac{p_1}{N} > 0, \qquad p_1 = \alpha_1 < \epsilon_1 p_1{}^* + p_1.$$

At last, it results:

$$\left(\text{meas}(\Omega_k^+ \right)^{1-\frac{s_4}{p_2^*(1+\sigma_2)}} = \left(\text{meas}(\Omega_k^+ \right)^{1-\frac{p_1}{N}+\epsilon_2},$$

where $\epsilon_2 = \frac{p_1}{N} - \frac{s_4}{p_2^*(1+\sigma_2)} > 0$ i.e. $s_4 < \frac{p_1}{N} p_2^*(1+\sigma_2)$, as (2.7) holds.
Indeed, since $s_4 = s_1 \frac{s_3}{s_3-1}$, we can choose s_3 such that

$$\begin{cases} s_3 \leq p_1(1+\sigma_1) \\ s_4 \leq \frac{p_1}{N} p_2^*(1+\sigma_2) \end{cases} \tag{4.22}$$

if and only if it results

$$s_1 < \tfrac{p_1 p_2^*}{N}(1+\sigma_2) \left(1 - \tfrac{1}{p_1(1+\sigma_1)} \right),$$

which is true from (2.7).
So, as $k \geq 1$ implies $1 \leq k^{p_1}$, Lemma (4.5) applies and yields a contradiction to (4.12).
Now, suppose that (4.13) holds which implies that, fixing any $k \in \mathbb{N}$, $k \geq R$, it is

$$\text{meas}(\Omega_k^-) > 0 \quad \text{with } \Omega_k^- = \{x \in \Omega : u(x) < -k\}.$$

In this case, by replacing function R_k^+ with $R_k^- : \mathbb{R} \to \mathbb{R}$ such that

$$R_k^- t := \begin{cases} 0 & \text{if } t \geq -k \\ t+k & \text{if } t < -k \end{cases},$$

we can reason as above so to apply again Lemma (4.5) which yields a contradiction to (4.13). Then, it has to be $u \in L^\infty(\Omega)$.
Similar arguments but considering $\frac{\partial \mathcal{J}}{\partial v}(u_n, v_n)$ and $R_k^+ v_n$, respectively $R_k^- v_n$, and the related sets, allow us to prove that it has to be also $v \in L^\infty(\Omega)$.
Step 3. Fixing k as required in this step, define the functions

$$R_k : t \in \mathbb{R} \ \mapsto \ R_k t = t - T_k t = \begin{cases} 0 & \text{if } |t| \leq k \\ t - k\frac{t}{|t|} & \text{if } |t| > k \end{cases} \in \mathbb{R},$$

$$\mathcal{R}_k : (t_1, t_2) \in \mathbb{R}^2 \mapsto \mathcal{R}_k(t_1, t_2) = (R_k t_1, R_k t_2) \in \mathbb{R}^2,$$

and the sets

$$\Omega_{n,k}^u := \{x \in \Omega : |u_n(x)| > k\}, \quad \Omega_{n,k}^v := \{x \in \Omega : |v_n(x)| > k\} \qquad \text{for any } n \in \mathbb{N}.$$

By definition, we have that

$$\|\mathcal{T}_k(u_n, v_n)\|_X \leq \|(u_n, v_n)\|_X \quad \text{and} \quad \|\mathcal{R}_k(u_n, v_n)\|_X \leq \|(u_n, v_n)\|_X \qquad \text{for all } n \in \mathbb{N}; \tag{4.23}$$

furthermore, we note that

$$\mathcal{T}_k(u, v) = (u, v) \quad \text{and} \quad \mathcal{R}_k(u, v) = (0, 0) \qquad \text{for a.e. } x \in \Omega.$$

Then, from (4.5)–(4.7) it follows that

$$\begin{aligned} &\mathcal{T}_k(u_n, v_n) \rightharpoonup (u, v) \ \text{ weakly in } W, \\ &\mathcal{T}_k(u_n, v_n) \longrightarrow (u, v) \ \text{ in } L^{r_1}(\Omega) \times L^{r_2}(\Omega) \quad \text{for any } (r_1, r_2) \in [1, p_1^*(1+\sigma_1)[\times[1, p_2^*(1+\sigma_2)[, \\ &\mathcal{T}_k(u_n, v_n) \longrightarrow (u, v) \ \text{ a.e. in } \Omega, \end{aligned} \tag{4.24}$$

and also, again from (4.6) and (4.7), we have that

$$\mathcal{R}_k(u_n, v_n) \longrightarrow (0, 0) \ \text{ in } L^{r_1}(\Omega) \times L^{r_2}(\Omega) \quad \text{for any } (r_1, r_2) \in [1, p_1^*[\times[1, p_2^*[, \tag{4.25}$$

$$\text{meas}(\Omega_{n,k}^u) \longrightarrow 0 \quad \text{and} \quad \text{meas}(\Omega_{n,k}^v) \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{4.26}$$

Furthermore, (3.9), (4.4) and (4.23) imply that

$$\left\| \frac{\partial \mathcal{J}}{\partial u}(u_n, v_n) \right\|_{X_1'} \|R_k u_n\|_{X_1} \longrightarrow 0 \quad \text{and} \quad \left\| \frac{\partial \mathcal{J}}{\partial v}(u_n, v_n) \right\|_{X_2'} \|R_k v_n\|_{X_2} \longrightarrow 0. \tag{4.27}$$

Now, by reasonig as in the proof of *Step 2* but replacing $R_k^+ u_n$ and $R_k^+ v_n$ with $R_k u_n$, respectively $R_k v_n$, from (4.27) we have that

$$\begin{aligned}\varepsilon_n =\ & \frac{\partial \mathcal{J}}{\partial u}(u_n, v_n)[R_k u_n] + \int_\Omega g_1(x, u_n, v_n) R_k u_n dx \\ & \geq \mu_1 \int_{\Omega^u_{n,k}} a(x, u_n, \nabla u_n) \nabla u_n dx \geq \mu_1 \mu_2 \int_{\Omega^u_{n,k}} |\nabla u_n|^{p_1} dx,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\varepsilon_n =\ & \frac{\partial \mathcal{J}}{\partial v}(u_n, v_n)[R_k v_n] + \int_\Omega g_2(x, u_n, v_n) R_k v_n dx \\ & \geq \mu_1 \int_{\Omega^v_{n,k}} b(x, v_n, \nabla v_n) \nabla v_n dx \geq \mu_1 \mu_2 \int_{\Omega^v_{n,k}} |\nabla v_n|^{p_2} dx,\end{aligned} \tag{4.29}$$

as the same arguments used for proving (4.18) apply so that from (4.25) we obtain

$$\int_\Omega g_1(x, u_n, v_n) R_k u_n dx \longrightarrow 0 \quad \text{and} \quad \int_\Omega g_2(x, u_n, v_n) R_k v_n dx \longrightarrow 0.$$

Whence, (4.28) and (4.29) imply not only that

$$\int_{\Omega^u_{n,k}} |\nabla u_n|^{p_1} dx \longrightarrow 0 \quad \text{and} \quad \int_{\Omega^v_{n,k}} |\nabla v_n|^{p_2} dx \longrightarrow 0,$$

i.e.

$$\|R_k u_n\|_{W_1} \longrightarrow 0 \quad \text{and} \quad \|R_k v_n\|_{W_2} \longrightarrow 0, \tag{4.30}$$

but also

$$\int_{\Omega^u_{n,k}} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx \longrightarrow 0 \quad \text{and} \quad \int_{\Omega^v_{n,k}} b(x, v_n, \nabla v_n) \cdot \nabla v_n dx \longrightarrow 0. \tag{4.31}$$

Now, in order to prove (4.9), from (3.10) it is enough to verify that

$$\left\| \frac{\partial \mathcal{J}}{\partial u}(\mathcal{T}_k(u_n, v_n)) \right\|_{X_1'} \longrightarrow 0 \quad \text{and} \quad \left\| \frac{\partial \mathcal{J}}{\partial v}(\mathcal{T}_k(u_n, v_n)) \right\|_{X_2'} \longrightarrow 0. \tag{4.32}$$

To this aim, let $w \in X_1$, $z \in X_2$ be such that $\|w\|_{X_1} = 1$, $\|z\|_{X_2} = 1$. Then, direct computations imply that

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial u}(\mathcal{T}_k(u_n, v_n))[w] = & \frac{\partial \mathcal{J}}{\partial u}(u_n, v_n)[w] - \int_{\Omega^u_{n,k}} a(x, u_n, \nabla u_n) \cdot \nabla w dx \\ & - \int_{\Omega^u_{n,k}} A_t(x, u_n, \nabla u_n) w dx \\ & + \int_{\Omega^u_{n,k}} (a(x, T_k u_n, 0) \cdot \nabla w + A_t(x, T_k u_n, 0) w)\, dx \\ & + \int_\Omega (G_u(x, u_n, v_n) - G_u(x, T_k u_n, T_k v_n)) w dx\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial v}(\mathcal{T}_k(u_n,v_n))[z] = \frac{\partial \mathcal{J}}{\partial v}(u_n,v_n)[z] &- \int_{\Omega^v_{n,k}} b(x,v_n,\nabla v_n)\cdot \nabla z dx \\ &- \int_{\Omega^v_{n,k}} B_t(x,v_n,\nabla v_n) z dx \\ &+ \int_{\Omega^v_{n,k}} \left(b(x,T_k v_n,0)\cdot \nabla z + B_t(x,T_k v_n,0) z\right) dx \\ &+ \int_\Omega (g_2(x,u_n,v_n) - g_2(x,T_k u_n,T_k v_n)) w dx,\end{aligned} \tag{4.34}$$

where by (4.26) and the boundedness of $a(x,T_k u_n,0), A_t(x,T_k u_n,0)$ in $\Omega^u_{n,k}$ and of $b(x,T_k v_n,0), B_t(x,T_k v_n,0)$ in $\Omega^v_{n,k}$ (which follow from (H_1)), it is:

$$\int_{\Omega^u_{n,k}} \left(a(x,T_k u_n,0)\cdot \nabla w + A_t(x,T_k u_n,0) w\right) dx \longrightarrow 0$$

and

$$\int_{\Omega^v_{n,k}} \left(b(x,T_k v_n,0)\cdot \nabla z + B_t(x,T_k v_n,0) z\right) dx \longrightarrow 0$$

uniformly with respect to w and z respectively.
From (3.9) and (4.4) we have that

$$\left|\frac{\partial \mathcal{J}}{\partial u}(u_n,v_n)[w]\right| \le \left\|\frac{\partial \mathcal{J}}{\partial u}(u_n,v_n)\right\|_{X_1'} \longrightarrow 0, \quad \left|\frac{\partial \mathcal{J}}{\partial v}(u_n,v_n)[z]\right| \le \left\|\frac{\partial \mathcal{J}}{\partial v}(u_n,v_n)\right\|_{X_2'} \longrightarrow 0.$$

On the other hand, it results

$$\begin{aligned}\left|\int_\Omega (g_1(x,u_n,v_n) - g_1(x,T_k u_n,T_k v_n)) w dx\right| &\le \int_\Omega |g_1(x,u_n,v_n) - g_1(x,u,v)| dx \\ &\quad + \int_\Omega |g_1(x,T_k u_n,T_k v_n) - g_1(x,u,v)| dx, \\ \left|\int_\Omega (g_2(x,u_n,v_n) - g_2(x,T_k u_n,T_k v_n)) z dx\right| &\le \int_\Omega |g_2(x,u_n,v_n) - g_2(x,u,v)| dx \\ &\quad + \int_\Omega |g_2(x,T_k u_n,T_k v_n) - g_2(x,u,v)| dx.\end{aligned}$$

We note that from (G_0) and (4.8), respectively (4.24), we have

$$\begin{aligned}&g_1(x,u_n,v_n) \longrightarrow g_1(x,u,v) \quad \text{and} \quad g_2(x,u_n,v_n) \longrightarrow g_2(x,u,v) \qquad \text{a.e. in } \Omega, \\ &g_1(x,T_k u_n,T_k v_n) \longrightarrow g_1(x,u,v) \quad \text{and} \quad g_2(x,T_k u_n,T_k v_n) \longrightarrow g_2(x,u,v) \qquad \text{a.e. in } \Omega,\end{aligned}$$

while from (G_1) and Young inequality, direct computations imply that

$$|g_1(x,u_n,v_n)| \le C(1 + |u_n|^{q_1} + |v_n|^{s_4}) \quad \text{and} \quad |g_2(x,u_n,v_n)| \le C(1 + |u_n|^{s_6} + |v_n|^{q_2}) \qquad \text{a.e. in } \Omega,$$

with s_4 as in (4.22) and since $s_6 = s_2 \frac{s_5}{s_5-1}$, we can choose s_5 such that

$$\begin{cases} 1 < s_5 < p_2(1+\sigma_2) \\ 0 \le s_6 < \frac{p_2}{N} p_1^*(1+\sigma_1) \end{cases} \tag{4.35}$$

if and only if it results, respectively for s_4 and s_6,:

$$s_1 < \frac{p_1 p_2^*}{N}(1+\sigma_2)\left(1 - \frac{1}{p_1(1+\sigma_1)}\right) \tag{4.36}$$

and

$$s_2 < \frac{p_1^* p_2}{N}(1+\sigma_1)\left(1 - \frac{1}{p_2(1+\sigma_2)}\right) \tag{4.37}$$

which are true from (2.7).
Moreover,

$$|g_1(x, T_k u_n, T_k v_n)| \le C \quad \text{and} \quad |g_2(x, T_k u_n, T_k v_n)| \le C \qquad \text{a.e. in } \Omega.$$

Thus, (4.22), (4.35), (4.7) and [3, Theorem 4.9] allow us to apply the Dominated Convergence Theorem so that we obtain

$$\int_\Omega |g_1(x, u_n, v_n) - g_1(x, u, v)| dx \longrightarrow 0, \quad \int_\Omega |g_1(x, T_k u_n, T_k v_n) - g_1(x, u, v)| dx \longrightarrow 0,$$
$$\int_\Omega |g_2(x, u_n, v_n) - g_2(x, u, v)| dx \longrightarrow 0, \quad \int_\Omega |g_2(x, T_k u_n, T_k v_n) - g_2(x, u, v)| dx \longrightarrow 0.$$

So, summing up, from (4.33) and (4.34), all the previous limits imply that

$$\begin{aligned} \left|\frac{\partial \mathcal{J}}{\partial u}(\mathcal{T}_k(u_n, v_n))[w]\right| &\le \varepsilon_{k,n} + \left|\int_{\Omega^u_{n,k}} (a(x, u_n, \nabla u_n)\cdot \nabla w + A_t(x, u_n, \nabla u_n) w) \ dx\right|, \\ \left|\frac{\partial \mathcal{J}}{\partial v}(\mathcal{T}_k(u_n, v_n))[z]\right| &\le \varepsilon_{k,n} + \left|\int_{\Omega^v_{n,k}} (b(x, v_n, \nabla v_n)\cdot \nabla z + B_t(x, v_n, \nabla v_n) z) \ dx\right|, \end{aligned} \tag{4.38}$$

where both $(\varepsilon_{k,n})_n$ represent suitable infinitesimal sequences independent of w, respectively z.
Using the same notation introduced in *Step 2*, we evaluate the last integrals in (4.38) by reasoning as in the proof of *Step 3* in [5, Proposition 4.6] but taking new test functions and passing to the limit in

$$\frac{\partial \mathcal{J}}{\partial u}(\mathcal{T}_k(u_n, v_n))[w R_k^+ u_n], \quad \frac{\partial \mathcal{J}}{\partial u}(\mathcal{T}_k(u_n, v_n))[w R_{k-1}^+ u_n],$$
$$\frac{\partial \mathcal{J}}{\partial u}(\mathcal{T}_k(u_n, v_n))[w R_k^- u_n], \quad \frac{\partial \mathcal{J}}{\partial u}(\mathcal{T}_k(u_n, v_n))[w R_{k-1}^- u_n],$$

respectively

$$\frac{\partial \mathcal{J}}{\partial v}(\mathcal{T}_k(u_n, v_n))[z R_k^+ v_n], \quad \frac{\partial \mathcal{J}}{\partial v}(\mathcal{T}_k(u_n, v_n))[z R_{k-1}^+ v_n],$$
$$\frac{\partial \mathcal{J}}{\partial v}(\mathcal{T}_k(u_n, v_n))[z R_k^- v_n], \quad \frac{\partial \mathcal{J}}{\partial v}(\mathcal{T}_k(u_n, v_n))[z R_{k-1}^- v_n].$$

Hence, we are able to claim that (4.32) hold.
At last, direct computations imply that

$$\mathcal{J}(\mathcal{T}_k(u_n,v_n)) = \mathcal{J}(u_n,v_n) - \int_{\Omega^u_{n,k}} A(x,u_n,\nabla u_n)dx - \int_{\Omega^v_{n,k}} B(x,v_n,\nabla v_n)dx$$
$$+ \int_\Omega (G(x,u_n,v_n) - G(x,T_k u_n, T_k v_n))dx,$$

where from (**??**), (**??**), (4.7), (4.8), (4.24), and again the Dominated Convergence Theorem we have

$$\int_\Omega (G(x,u_n,v_n) - G(x,T_k u_n, T_k v_n))dx \longrightarrow 0.$$

Thus, (4.10) follows from (4.4) and (4.31).
Step 4. By following some ideas introduced in [1] and considering the real map $\psi(t) = te^{\eta t^2}$, where η can be fixed in a suitable way, in particular by applying the same arguments developed in the proof of [8, Proposition 3.6] and in the proof of *Step 4* of [5, Proposition 4.6] in order to estimate $\frac{\partial \mathcal{J}}{\partial u}(\mathcal{T}_k(u_n,v_n))[\psi(T_k u_n - u)]$, we prove that

$$\|T_k u_n - u\|_{W_1} \to 0. \tag{4.39}$$

Moreover, reasonig in the same way but considering $\frac{\partial \mathcal{J}}{\partial v}(\mathcal{T}_k(u_n,v_n))[\psi(T_k v_n - v)]$, we have also

$$\|T_k v_n - v\|_{W_2} \to 0. \tag{4.40}$$

Then, condition (i) follows from (4.30), (4.39) and (4.40).
Step 5. By means of Proposition 3.3 applied to the uniformly bounded sequence $(\mathcal{T}_k(u_n,v_n))_n$, from (4.24), (4.39) and (4.40) it follows that

$$\mathcal{J}(\mathcal{T}_k(u_n,v_n)) \longrightarrow \mathcal{J}(u,v) \quad \text{and} \quad \|d\mathcal{J}(\mathcal{T}_k(u_n,v_n)) - d\mathcal{J}(u,v)\|_{X'} \longrightarrow 0.$$

Hence, (ii) follows from (4.9) and (4.10). □

Proof of Theorem 2.3. The proof follows from the Minimum Principle (Theorem 2.7), since we have proved that the functional $\mathcal{J}$ is bounded from below in X (Lemma 4.2) and satisfies condition $(wCPS)$ in $\mathbb{R}$ (Lemma 4.6). Thus, $\mathcal{J}$ admits a minimum point in X, which is a weak bounded solution if the system (1.1). □

Proof of Theorem 2.5. Consider $\overline{\lambda} \in \mathbb{R}$ such that

$$\liminf_{|(u,v)|\to(0,0)} \frac{G(x,u,v)}{|u|^{p_1} + |v|^{p_2}} > \overline{\lambda} > \eta_4 \max\{\lambda_{1,1}, \lambda_{2,1}\}. \tag{4.41}$$

Then, fixing any $\epsilon > 0$, a constant $\delta > 0$ exists so that $|u|^{p_1} + |v|^{p_2} < \delta$, it is

$$\frac{G(x,u,v)}{|u|^{p_1} + |v|^{p_2}} > \left(\overline{\lambda} - \epsilon\right) > \eta_4 \max\{\lambda_{1,1}, \lambda_{2,1}\}. \tag{4.42}$$

Thus, it follows that

$$G(x, t\varphi_{1,1}, t\varphi_{2,1}) > \left(\overline{\lambda} - \epsilon\right)\left(|t\varphi_{1,1}|^{p_1} + |t\varphi_{2,1}|^{p_2}\right) \tag{4.43}$$

if

$$|t\varphi_{1,1}|^{p_1} + |t\varphi_{2,1}|^{p_2} < \delta, \tag{4.44}$$

and since $|t\varphi_{1,1}|^{p_1} + |t\varphi_{2,1}|^{p_2} < |t|^{p_1}|\varphi_{1,1}|_\infty + |t|^{p_2}|t\varphi_{2,1}|_\infty < \delta$, it results that

$$G(x, t\varphi_{1,1}, t\varphi_{2,1}) > \left(\overline{\lambda} - \epsilon\right)\left(|t|^{p_1}\varphi_{1,1}|^{p_1} + |t|^{p_2}|\varphi_{2,1}|^{p_2}\right) \tag{4.45}$$

if and only if

$$\begin{cases} |t|^{p_1} < \frac{\delta}{2|\varphi_{1,1}|_\infty} \\ |t|^{p_2} < \frac{\delta}{2|\varphi_{2,1}|_\infty}. \end{cases} \tag{4.46}$$

Then, from (4.41) and (4.46) it results:

$$\mathcal{J}(t\varphi_{1,1}, t\varphi_{2,1}) \leq \eta_4 t^{p_1}\lambda_{1,1}|\varphi_{1,1}|_{p_1}{}^{p_1} + \eta_4 t^{p_2}\lambda_{2,1}|\varphi_{2,1}|_{p_2}{}^{p_2} - \int_\Omega G(x, t\varphi_{1,1}, t\varphi_{2,1})dx$$

$$\leq \eta_4 t^{p_1}\lambda_{1,1}|\varphi_{1,1}|_{p_1}{}^{p_1} + \eta_4 t^{p_2}\lambda_{2,1}|\varphi_{2,1}|_{p_2}{}^{p_2} - (\overline{\lambda} - \epsilon)\left(t^{p_1}|\varphi_{1,1}|_{p_1}{}^{p_1} + t^{p_2}|\varphi_{2,1}|_{p_2}{}^{p_2}\right) < 0.$$

Thus, the conclusion follows. □

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