

Asymptotic profiles and critical exponents for a semilinear damped plate equation with time-dependent coefficients

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Abstract. In this paper we study the asymptotic profile (as $t \rightarrow \infty$) of the solution to the Cauchy problem for the linear plate equation

$$u_{tt} + \Delta^2 u - \lambda(t)\Delta u + u_t = 0$$

when $\lambda = \lambda(t)$ is a decreasing function, assuming initial data in the energy space and verifying a moment condition. For sufficiently small data, we find the critical exponent for global solutions to the corresponding problem with power nonlinearity

$$u_{tt} + \Delta^2 u - \lambda(t)\Delta u + u_t = |u|^p.$$

In order to do that, we assume small data in the energy space and, possibly, in L^1 . In this latter case, we also determinate the asymptotic profile of the solution to the semilinear problem for supercritical power nonlinearities.

Keywords: Asymptotic profile, Diffusion phenomena, Critical exponents, Plate equation

Declaration

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1. Introduction

In this paper, we study the asymptotic profile of the solutions to the Cauchy problem for the damped plate equation

$$\begin{cases} u_{tt} + \Delta^2 u - \lambda(t)\Delta u + u_t = 0, & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where $\lambda = \lambda(t)$ is a decreasing function.

Fourth-order evolution partial differential equations as in (1) arise in problems of solid mechanics as, for example, in the theory of thin plates and beams. Also, in particular formulations of problems related with the Navier-Stokes equations (see [36]) appear elliptic equations of fourth-order. Models to study the vibrations of thin plates ($n = 2$) given by the full von Kármán system have been studied by several authors, in particular, see [6, 25, 34].

The equation in (1) represents a plate equation with a time-dependent decreasing coefficient, under the action of a damping term. In space dimension $n = 1$, it may be considered as a linearized model of the Woinovsky-Krieger equation with damping

$$u_{tt} + \partial_x^4 u - \lambda(\|u_x(t, \cdot)\|_{L^2})\partial_x^2 u + u_t = 0,$$

which represents the vibration of an extensible beam proposed in [41]. The nonlinear term represents the change in the tension of the beam due to its extensibility [1].

The energy for problem (1) is given by

$$E(t) = \frac{1}{2} \|u_t(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|\Delta u(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \lambda(t) \|\nabla u(t, \cdot)\|_{L^2}^2, \quad (2)$$

and it dissipates as $t \rightarrow \infty$, as a consequence of

$$E'(t) = -\|u_t(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \lambda'(t) \|\nabla u(t, \cdot)\|_{L^2}^2 \leq 0.$$

We discuss the possible asymptotic profiles for the solution to (1) as $t \rightarrow \infty$, under the moment condition

$$u_0, u_1 \in L^1, \quad M = \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) dx \neq 0. \quad (3)$$

We define the diffusive-type kernels

$$G_2(t, x) = \mathcal{F}^{-1} e^{-t|\xi|^4}, \quad (4)$$

$$G_1(\Lambda(t), x) = \mathcal{F}^{-1} e^{-\Lambda(t)|\xi|^2} = (4\pi\Lambda(t))^{-n/2} e^{-\frac{|x|^2}{4\Lambda(t)}}, \quad (5)$$

where

$$\Lambda(t) = \int_0^t \lambda(\tau) d\tau.$$

The definition of diffusive-type kernels is motivated by the fact that G_2 and G_1 are the fundamental solutions to two diffusive problems: the kernel G_2 solves the fourth order diffusive equation

$$u_t + \Delta^2 u = 0,$$

and the kernel $G_1(\Lambda(t), x)$ solves the heat equation with time-dependent coefficient $\lambda(t)$:

$$u_t - \lambda(t)\Delta u = 0.$$

Here G_1 is the classical Gauss-Weierstrass kernel. We also define

$$G(t, x) = \mathcal{F}^{-1} e^{-t|\xi|^4 - \Lambda(t)|\xi|^2} = (G_2(t, \cdot) * G_1(\Lambda(t), \cdot))(x). \quad (6)$$

Here $\mathcal{F}\varphi$ denotes the Fourier transform of a function φ in $\mathbb{R}^n$, $\mathcal{F}^{-1}$ denote the inverse of the Fourier transformation, and we write $\hat{\varphi} = \mathcal{F}\varphi$. If $\varphi = \varphi(t, x)$, we always intend $\mathcal{F}\varphi(t, \xi) = (\mathcal{F}\varphi(t, \cdot))(\xi)$.

The asymptotic profile of the solution to (1) as $t \rightarrow \infty$ may be described by one of the three kernels in (4), (5) or (6), according to the behavior of $\lambda(t)$ for sufficiently large t . In this paper, we assume that $\lambda(t)$ has a polynomial speed

$$\lambda(t) = \frac{\mu}{(1+t)^\ell}, \quad \mu > 0, \quad \ell > 0, \quad (7)$$

but this assumption may be easily relaxed to consider more general functions with a prescribed behavior for sufficiently large t . As a consequence,

$$\Lambda(t) = \frac{\mu}{1-\ell} ((1+t)^{1-\ell} - 1), \quad \text{if } \ell \neq 1,$$

whereas $\Lambda(t) = \mu \log(1+t)$ if $\ell = 1$. We remark that $\Lambda \in L^1$ if $\ell > 1$.

We will distinguish the asymptotic profile of the solution to (1) according to three cases: $\ell > 1/2$, $\ell \in (0, 1/2)$, and the limit case $\ell = 1/2$.

Theorem 1. *Let $n \geq 1$. Assume $u_0, u_1 \in L^2$, together with (3), and let $\lambda = \lambda(t)$ be as in (7). Let $u \in \mathcal{C}([0, \infty), H^2) \cap \mathcal{C}^1([0, \infty), L^2)$ be the solution to (1). Then we may distinguish three cases.*

- *If*

$$2\ell > 1, \quad (8)$$

then the asymptotic profile of u is described by (4) in L^2 , namely,

$$\|u(t, \cdot) - MG_2(t, \cdot)\|_{L^2} = o(t^{-\frac{n}{8}}), \quad t \rightarrow \infty. \quad (9)$$

• If

$$2\ell < 1, \quad (10)$$

then the asymptotic profile of u is described by (5) in L^2 , namely,

$$\|u(t, \cdot) - MG_1(\Lambda(t), \cdot)\|_{L^2} = o(\Lambda(t)^{-\frac{n}{4}}), \quad t \rightarrow \infty. \quad (11)$$

• In the limit case

$$2\ell = 1, \quad (12)$$

the asymptotic profile of u is described by (6) in L^2 , namely,

$$\|u(t, \cdot) - MG(t, \cdot)\|_{L^2} = o(t^{-\frac{n}{8}}), \quad t \rightarrow \infty. \quad (13)$$

By the scaling properties of the Fourier transform, one gets

$$\|G_2(t, \cdot)\|_{L^2} = t^{-\frac{n}{8}} \|G_2(1, \cdot)\|_{L^2},$$

$$\|G_1(\Lambda(t), \cdot)\|_{L^2} = \Lambda(t)^{-\frac{n}{4}} \|G_1(\Lambda(1), \cdot)\|_{L^2},$$

for any $t > 0$. The right-hand sides of the two equalities are finite. In view of this information, thanks to the reverse triangle inequality, estimates (9) and (11) in Theorem 1 describe the asymptotic profile of the solution to (1) when $2\ell \neq 1$. The case $2\ell = 1$ is more complicated since the asymptotic profile $MG(t, \cdot)$ of the solution to (1) is not scale-invariant so we do not have a norm equality, but still

$$\|G(t, \cdot)\|_{L^2} \approx t^{-\frac{n}{8}} \|G(1, \cdot)\|_{L^2}, \quad \text{as } t \rightarrow \infty.$$

In view of this latter, estimate (13) in Theorem 1 describes the asymptotic profile of the solution to (1) when $2\ell = 1$.

Remark 1. If the moment condition in (3) is dropped, that is,

$$u_0, u_1 \in L^1, \quad M = \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) dx = 0,$$

then estimates (9), (11) and (13) are still valid, in the sense that

$$\|u(t, \cdot)\|_{L^2} = o(t^{-\max\{\frac{n}{8}, \frac{n}{4}(1-\ell)\}}),$$

but they no longer determinate the asymptotic profile of u . Assuming additional regularity on u_0, u_1 (for instance, $u_0, u_1 \in L^1(|x|^\gamma dx)$, for a sufficiently large $\gamma > 0$), it is possible to describe the “secondary” asymptotic profiles of u , when $M = 0$. By “secondary asymptotic profile”, we mean that

$$\|u(t, \cdot) - M^\sharp \cdot \nabla G_2(t, \cdot)\|_{L^2} = o(t^{-\frac{n+2}{8}}), \quad t \rightarrow \infty,$$

if $M = 0$ and (8) holds, where the components of the first-order moment $M^\sharp$ are given by (see, for instance, [9])

$$M_j^\sharp = \int_{\mathbb{R}^n} x_j (u_0(x) + u_1(x)) dx.$$

Similarly if (10) or (12) holds. On the other hand, the assumption $M = 0$ has a different influence on the semilinear problem (21). Indeed, the secondary asymptotic profiles, which are relevant for the linear problem, do not come into play to describe the asymptotic profile of the solution to the semilinear problem (21), as we show in Section 7.

The asymptotic profile for the damped evolution equation with constant coefficients

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + u_t = 0, & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (14)$$

has been studied for a general $\sigma > 0$ by G. Karch [24] (see also [19, 27, 30]), who proved that it is given by $MG_\sigma(t, x)$, with $G_\sigma(t, \cdot) = \mathcal{F}^{-1}e^{-t|\xi|^{2\sigma}}$. The fact that the asymptotic profile for damped evolution equations as in (14) is given by diffusive-type kernels, under a moment condition on the initial data, like (3), is a consequence of the *diffusion phenomenon*. The diffusion phenomenon means that the solution to (14) may be approximated by the solution to the corresponding heat-type equation with initial data $u_0 + u_1$. Namely, by $e^{-t(-\Delta)^\sigma}(u_0 + u_1)$. This latter phenomenon holds, more in general, for evolution equations with *effective damping* (according to the classification introduced by the authors in [12]) and possibly with time-dependent coefficients:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\theta u_t = 0, & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (15)$$

In particular, in the case of constant coefficients

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\theta u_t = 0, & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (16)$$

the damping is effective, in the sense that the diffusion phenomenon holds for (15), if $2\theta < \sigma$. In [10] the authors showed that for $\theta > 0$ a *double diffusion phenomenon* holds, when the L^p norm of the solution to (16) is considered, with $p \in [1, \infty]$, inspired by the estimates obtained in [29]. That is, one has to consider two different heat-type equations to approximate the solution to (16), according to the space dimension and to the L^p space considered. On the other hand, when $2\theta > \sigma$, the asymptotic profile to (16) changes completely, since the fundamental solution may be approximated by considering the convolution between a diffusive-type kernel and the fundamental solution of an evolution operator [23]. The presence of the oscillations has a strong influence on $L^p - L^q$ estimates, in particular on $L^1 - L^1$

estimates (see [14, 35]). In the case of the wave equation with viscoelastic or strong damping ($\theta = \sigma = 1$), the profile has been investigated in [21].

The diffusion phenomenon for (1) has been investigated in [8], in the case of constant coefficient $\lambda(t) = \mu$, and in [38], in space dimension $n = 1$, with $\ell < 1/2$. Further details on the diffusion phenomenon for (1) are provided by Lemma 3 in Section 4.

As a consequence of the diffusion phenomenon, we may derive optimal long time decay estimates for the solution to (1) and its energy, under different assumptions for the initial data.

Theorem 2. *Let $n \geq 1$ and $\lambda = \lambda(t)$ be as in (7). Assume $(u_0, u_1) \in \mathcal{A} := (L^\eta \cap H^2) \times (L^\eta \cap L^2)$, for some $\eta \in [1, 2]$. Let $u \in \mathcal{C}([0, \infty), H^2) \cap \mathcal{C}^1([0, \infty), L^2)$ be the solution to (1). Then we have the following decay estimates*

$$\|u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\max\{1-\ell, \frac{1}{2}\} \frac{n}{2} (\frac{1}{\eta} - \frac{1}{2})} \|(u_0, u_1)\|_{\mathcal{A}}, \quad (17)$$

$$\|u_t(t, \cdot)\|_{L^2} \lesssim (1+t)^{-1-\max\{1-\ell, \frac{1}{2}\} \frac{n}{2} (\frac{1}{\eta} - \frac{1}{2})} \|(u_0, u_1)\|_{\mathcal{A}}, \quad (18)$$

$$\|\Delta u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\max\{1-\ell, \frac{1}{2}\} (\frac{n}{2} (\frac{1}{\eta} - \frac{1}{2}) + 1)} \|(u_0, u_1)\|_{\mathcal{A}}. \quad (19)$$

The optimality of decay estimates (17) and (19) in Theorem 2 is guaranteed by Lemma 3 in Section 4. For the sake of brevity, we will not discuss the optimality of estimate (18), but we expect this estimate to be optimal, as well.

Estimates (17)-(18)-(19) may be written in a compact form as

$$\|\partial_t^j \Delta^k u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-j-\max\{1-\ell, \frac{1}{2}\} (\frac{n}{2} (\frac{1}{\eta} - \frac{1}{2}) + k)} \|(u_0, u_1)\|_{\mathcal{A}}, \quad j+k=0, 1. \quad (20)$$

As a consequence of (17) and (19), we get

$$\|\nabla u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\max\{1-\ell, \frac{1}{2}\} (\frac{n}{2} (\frac{1}{\eta} - \frac{1}{2}) + \frac{1}{2})} \|(u_0, u_1)\|_{\mathcal{A}}.$$

By using Theorem 2, one may estimate the energy for the solution to (1) given in (2), according to the different scenarios. Due to

$$(1+t)^{-2\max\{1-\ell, \frac{1}{2}\}} \leq (1+t)^{-\ell-\max\{1-\ell, \frac{1}{2}\}} \iff \ell \leq 1/2,$$

we obtain

$$E(t) \lesssim (E(0) + \|u_0\|_{L^\eta}^2 + \|u_1\|_{L^\eta}^2) \times \begin{cases} (1+t)^{-1-\frac{n}{2}(\frac{1}{\eta}-\frac{1}{2})} & \text{if } \ell \geq 1/2, \\ (1+t)^{-1-(1-\ell)n(\frac{1}{\eta}-\frac{1}{2})} & \text{if } \ell \in (0, 1/2]. \end{cases}$$

Thanks to the estimates obtained for the solution to the linear Cauchy problem (1), we may investigate the critical exponent of small data global solutions for the following problem with power nonlinearity:

$$\begin{cases} u_{tt} + \Delta^2 u - \lambda(t)\Delta u + u_t = |u|^p, & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (21)$$

Indeed, the diffusion phenomenon for evolution equations with effective damping as in (16) makes the study of global small data solutions for the associated problem with power nonlinearity $|u|^p$ (or even $|u_t|^p$, when $\theta > 0$) analogous to the study for the heat-type equation [11, 13, 17, 33]. The critical exponent for the classical damped wave ($\sigma = 1$ in (14)) with power nonlinearity $|u|^p$ was first determined by A. Matsumura [28] in space dimension $n = 1, 2$, and by G. Todorova and B. Yordanov [37] in space dimension $n \geq 3$. The critical exponent for global small data solutions to semilinear evolution equations with effective time-dependent damping has been widely investigated in these years [2, 3, 7, 16, 26, 31, 32, 39].

In the study of global small data solutions for (21) some new interesting effects appear. There is competition between two critical exponents, one related to the fourth-order, constant coefficient term $\Delta^2 u$, which becomes dominant when $\ell \geq 1/2$, and the other related to the second-order, time-dependent term $\lambda(t)\Delta u$, which is the Fujita exponent modified by the vanishing speed of $\lambda(t)$. Moreover, since the equation still has the regularity of the plate, we are able to obtain a global existence result, assuming only small data in the energy space, in any space dimension $n \geq 1$, for sufficiently slowly decaying speed of $\lambda(t)$ (Theorem 5).

Let $\lambda(t) = \mu(1+t)^{-\ell}$, with $\ell \in \mathbb{R}$. By the test function method (see [15, 18]), we know that global-in-time solutions to (21) cannot exist if

$$1 < p \leq \bar{p} = \min \left\{ 1 + \frac{4}{n}, 1 + \frac{2}{n(1-\ell)_+} \right\} = \begin{cases} 1 + 4/n & \text{if } \ell \in [1/2, +\infty), \\ 1 + 2/(n(1-\ell)) & \text{if } \ell \in (-\infty, 1/2], \end{cases}$$

in space dimension $n \geq 1$, provided that we make a suitable sign assumption on the data (analogous to the moment condition (3)), namely,

$$\int_{\mathbb{R}^n} (u_0 + u_1)(x) dx > 0.$$

We show that small data global solutions exist for $p > \bar{p}$, in low space dimension, when $\ell > 0$. The exponent is optimal, since in the subcritical case and critical case, the nonexistence result holds. We don't discuss the case of high space dimension, since it requires more sophisticated estimates (for instance, weighted energy estimates, as in [37]) to deal with power nonlinearities $p \in (1, 2)$. We also don't discuss the case of $\ell < 0$, since it requires a different approach to prove Theorem 2.

We mention that the local-in-time existence of the solution to the semilinear problem (21) may be derived by standard arguments in the energy space, by using the energy estimates obtained for the linear problem (see, for instance, [5]). In particular, assuming initial data in $H^2 \times L^2$, as in Theorem 5, local-in-time energy solutions exist for any $p \in [1, \infty)$ if $n \leq 4$ and for any $p \in [1, 1 + 4/(n-4)]$ if $n \geq 5$.

Theorem 3. *Let $n = 1, 2, 3, 4$, $\lambda = \mu(1+t)^{-\ell}$, and $p > \bar{p}$. Assume that $\ell > 0$ if $n = 1, 2$, $\ell \geq 1/3$ if $n = 3$ or $\ell \geq 1/2$ if $n = 4$. Then there exists $\varepsilon > 0$ such that for any initial data*

$$(u_0, u_1) \in \mathcal{A} = (H^2 \cap L^1) \times (L^2 \cap L^1), \quad \text{with} \quad \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon, \quad (22)$$

there is a (unique) global-in-time energy solution $u \in C([0, \infty), H^2) \cap C^1([0, \infty), L^2)$ to (21). Moreover, it satisfies the following long time decay estimates

$$\|\partial_t^j \Delta^k u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-j-\max\{1-\ell, \frac{1}{2}\}(\frac{n}{4}+k)} \|(u_0, u_1)\|_{\mathcal{A}}, \quad j+k = 0, 1. \quad (23)$$

Estimates (23) are the same as (20), with $\eta = 1$. Thanks to Theorem 3, we may also describe the asymptotic profile of the solution to (21), for sufficiently small data in the energy space and in L^1 , in space dimension $n \leq 4$.

Theorem 4. *Let $n = 1, 2, 3, 4$, $\lambda = \mu(1+t)^{-\ell}$, and $p > \bar{p}$. Assume that $\ell > 0$ if $n = 1, 2$, $\ell \geq 1/3$ if $n = 3$ or $\ell \geq 1/2$ if $n = 4$. Assume initial data as in (22), with $\varepsilon > 0$ as provided by Theorem 3.*

Then the (unique) global-in-time energy solution $u \in \mathcal{C}([0, \infty), H^2) \cap C^1([0, \infty), L^2)$ to (21) has its asymptotic profile described by (4) or, respectively, (5), or (6), in L^2 , if (8) or, respectively, (10), or (12) holds, provided that $M \neq 0$, where now M is defined by

$$M = \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) dx + \int_0^\infty \int_{\mathbb{R}^n} |u(t, x)|^p dx dt. \quad (24)$$

More precisely, (9) or, respectively, (11), or (13) holds.

If we remove the assumption that the initial data are in L^1 and we only assume that they are in the energy space, then the critical exponent is modified into

$$\tilde{p} = \min \left\{ 1 + \frac{8}{n}, 1 + \frac{4}{n(1-\ell)_+} \right\} = \begin{cases} 1 + 8/n & \text{if } \ell \in [1/2, +\infty), \\ 1 + 4/(n(1-\ell)) & \text{if } \ell \in (-\infty, 1/2]. \end{cases}$$

For the classical damped wave equation, this phenomenon has been investigated in [22].

In particular, global solutions cannot exist, in any space dimension, if $1 < p < \tilde{p}$, provided that we assume

$$(u_0 + u_1)(x) \geq \varepsilon |x|^{-\frac{n}{2}} (\log |x|)^{-1},$$

for some $\varepsilon > 0$, and for sufficiently large $|x|$ (see [13]). We may prove an existence result of global solution for small data in the energy space, in space dimension $n \leq 7$ if $\ell \geq 1/2$, and in space dimension $n < 4/\ell$ if $\ell \in (0, 1/2)$. In the limit case $\ell = 0$, the result holds in any space dimension $n \geq 1$.

Theorem 5. *Let $\lambda = \mu(1+t)^{-\ell}$, and $p > \tilde{p}$. Assume that $\ell > 0$ if $n \leq 7$, or $\ell \in (0, 4/n)$ if $n \geq 8$. Moreover, assume that $p \leq 1 + 4/(n-4)$ if $n \geq 5$. Then there exists $\varepsilon > 0$ such that for any initial data*

$$(u_0, u_1) \in H^2 \times L^2, \quad \text{with} \quad \|(u_0, u_1)\|_{H^2 \times L^2} \leq \varepsilon, \quad (25)$$

there is a (unique) global-in-time energy solution $u \in \mathcal{C}([0, \infty), H^2) \cap C^1([0, \infty), L^2)$ to (21). Moreover, it satisfies the following long time decay estimates

$$\|\partial_t^j \Delta^k u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-j-k \max\{1-\ell, \frac{1}{2}\}} \|(u_0, u_1)\|_{H^2 \times L^2}, \quad j+k = 0, 1. \quad (26)$$

We mention that in Theorem 5 a peculiar property of the solution to the linear problem (1) is heavily used. Even if the asymptotic profile of its solution is described by $MG_1(\Lambda(t), \cdot)$, for $\ell \in (0, 1/2)$, the regularity of the solution is still determined by the higher order terms of the equation, i.e., by $\partial_t^2 u + \Delta^2 u$. This dichotomy, which is typical for the plate equation in which two different powers of the Laplace

operator appears (Δ^2 and $-\Delta$), remains valid also in the constant coefficients case ($\lambda = \mu$, i.e. $\ell = 0$), and allows us to obtain an existence result valid in any space dimension in Theorem 5, for a sufficiently slowly decaying speed of $\lambda(t)$. We also mention that the critical exponent $p = \tilde{p}$ may be included in the global existence result in Theorem 5, following the ideas in [20].

1.1. Discussion of a model with a different power nonlinearity

Our interest for studying the critical exponent for a power nonlinearity $|u|^p$ is related to the variety of the mathematical properties which appears when the decay rate obtained for the linear problem is employed to find the critical exponent for the nonlinear problem. However, problems with different nonlinearities may have an interest from a physical background. The following problem

$$\begin{cases} u_{tt} + \Delta^2 u + u_t = \nabla \cdot ((\lambda(t) + |\nabla u|^{p-1}) \nabla u), & t \geq 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (27)$$

is related to Falk's model for the thermomechanical phase transitions in shape memory alloys, in space dimension $n = 1$, that is,

$$\begin{cases} u_{tt} + u_{xxxx} + u_t = \partial_x ((\lambda(t) + |u_x|^{p-1}) u_x), & t \geq 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}. \end{cases} \quad (28)$$

In this model, shape memory alloys modify their shapes according to the temperature. At low temperature, an alloy behaves like a plastic body, whereas at high temperature the behaviour is pseudoelastic (see [4, chapter 5]).

We have the following result, analogous to Theorems 3 and 4.

Theorem 6. *Let $\lambda = \mu(1+t)^{-\ell}$, for some $\ell > 0$, and assume that $p \geq 2$ if $\ell \in (0, 1/2)$ or $p > 2$ if $\ell \geq 1/2$. Then there exists $\varepsilon > 0$ such that for any initial data*

$$(u_0, u_1) \in \mathcal{A} = (H^2 \cap L^1) \times (L^2 \cap L^1), \quad \text{with} \quad \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon, \quad (29)$$

there is a (unique) global-in-time energy solution $u \in C([0, \infty), H^2) \cap C^1([0, \infty), L^2)$ to (28). Moreover, it satisfies the following long time decay estimates

$$\|\partial_t^j \partial_x^k u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-j-\max\{1-\ell, \frac{1}{2}\}(\frac{1}{4}+\frac{k}{2})} \|(u_0, u_1)\|_{\mathcal{A}}, \quad 2j+k=0,1,2, \quad (30)$$

and its asymptotic profile described by (4) or, respectively, (5), or (6), in L^2 , if (8) or, respectively, (10), or (12) holds, provided that $M \neq 0$, where now M is defined by

$$M = \int_{-\infty}^{+\infty} (u_0(x) + u_1(x)) dx + \int_0^\infty \int_{-\infty}^{+\infty} \partial_x (|u_x|^{p-1} u_x) dx dt. \quad (31)$$

More precisely, (9) or, respectively, (11), or (13) holds.

2. Representation of the fundamental solution of the problem

We perform the Fourier transform of (1) with respect to the x variable to obtain

$$\begin{cases} \hat{u}_{tt} + \hat{u}_t + (|\xi|^4 + \lambda(t)|\xi|^2)\hat{u} = 0, \\ (\hat{u}(0, \xi), \hat{u}_t(0, \xi)) = (\hat{u}_0(\xi), \hat{u}_1(\xi)), \quad \xi \in \mathbb{R}^n. \end{cases} \quad (32)$$

Applying the change of variable

$$\psi(t, \xi) = \hat{u}(t, \xi)e^{\frac{t}{2}}, \quad (33)$$

problem (32) reads as

$$\begin{cases} \psi_{tt} + m(t, \xi)\psi = 0, \\ (\psi(0, \xi), \psi_t(0, \xi)) = \left(\hat{u}_0(\xi), \frac{\hat{u}_0(\xi)}{2} + \hat{u}_1(\xi)\right), \quad \xi \in \mathbb{R}^n, \end{cases} \quad (34)$$

where

$$m(t, \xi) = |\xi|^4 + \lambda(t)|\xi|^2 - \frac{1}{4}.$$

In the following, we will separately study the solution at low and high frequencies, since the asymptotic behaviour of the solution is determined at low frequencies, while its regularity is determined at high frequencies. To divide the two zones, we fix a constant K such that

$$\frac{1}{2} < K < \frac{1}{\sqrt{2}},$$

and we denote

$$B_K \doteq \{\xi \in \mathbb{R}^n : |\xi| < K\}, \quad B_K^c \doteq \{\xi \in \mathbb{R}^n : |\xi| \geq K\}.$$

For any $t \geq s \geq 0$, we denote

$$W(t, \xi) = E(t, s, \xi) W(s, \xi), \quad \text{where} \quad W(t, \xi) = \begin{pmatrix} (1 + |\xi|^2)\psi(t, \xi) \\ \psi_t(t, \xi) \end{pmatrix},$$

and we separately estimate the fundamental solution $E(t, s, \xi)$ to the problem for our micro-energy $W(t, \xi)$, in B_K and in B_K^c . Namely,

$$\begin{cases} \partial_t E(t, s, \xi) = \begin{pmatrix} 0 & 1 + |\xi|^2 \\ -(1 + |\xi|^2)^{-1} m(t, \xi) & 0 \end{pmatrix} E(t, s, \xi), & t \geq s, \\ E(s, s, \xi) = \text{Id}. \end{cases}$$

The choice of the bound $K < 1/\sqrt{2}$ is motivated by the following property.

Remark 2. For any $\xi \in B_K$, it holds

$$-\frac{1}{4} \leq m(t, \xi) = -\left(\frac{1}{4} - |\xi|^4 - \lambda(t)|\xi|^2\right) < -\left(\frac{1}{4} - K^4 - \lambda(t)K^2\right).$$

In particular,

$$m(t, \xi) \approx -\frac{1}{4}, \quad \text{for any } t \geq t_0, \text{ for a sufficiently large } t_0, \quad (35)$$

as a consequence of $\lambda(t) \rightarrow 0$ as $t \rightarrow \infty$, and due to $K < 1/\sqrt{2}$.

To simplify our analysis, we define the “equivalent” micro-energies W^{low} in B_K and W^{high} in B_K^c :

$$W^{\text{low}}(t, \xi) = \begin{pmatrix} \psi(t, \xi)/2 \\ \psi_t(t, \xi) \end{pmatrix}, \quad W^{\text{high}}(t, \xi) = \begin{pmatrix} i|\xi|^2 \psi(t, \xi) \\ \psi_t(t, \xi) \end{pmatrix},$$

and, consequently, the fundamental solutions $E^{\text{low}}(t, s, \xi)$ and $E^{\text{high}}(t, s, \xi)$, such that

$$W^{\text{low}}(t, \xi) = E^{\text{low}}(t, s, \xi) W^{\text{low}}(s, \xi), \quad W^{\text{high}}(t, \xi) = E^{\text{high}}(t, s, \xi) W^{\text{high}}(s, \xi).$$

We first derive an estimate and a representation formula for $E^{\text{low}}(t, s, \xi)$.

Lemma 1. The fundamental solution verifies the estimate

$$|E^{\text{low}}(t, s, \xi)| \lesssim e^{d(t, s, \xi)}, \quad \text{for any } \xi \in B_K \text{ and } t \geq s \geq 0,$$

where

$$d(t, s, \xi) = \frac{t-s}{2} - (t-s)|\xi|^4 - (\Lambda(t) - \Lambda(s))|\xi|^2. \quad (36)$$

For $t \geq t_0$, we define

$$h(t, \xi) = \sqrt{-m(t, \xi)} = \sqrt{\frac{1}{4} - |\xi|^4 - \lambda(t)|\xi|^2}, \quad (37)$$

and we set

$$H(t, \xi) = \begin{pmatrix} 2h(t, \xi) & 0 \\ 0 & 1 \end{pmatrix}, \quad (38)$$

and

$$e(t, s, \xi) \doteq \left(\frac{h(t, \xi)}{h(s, \xi)}\right)^{\frac{1}{2}} \exp\left(\int_s^t h(\tau, \xi) d\tau\right). \quad (39)$$

Then, for all $\xi \in B_K$, for any $t \geq s \geq t_0$, $E^{\text{low}}(t, s, \xi)$ can be represented in the form

$$E^{\text{low}} = e(t, s, \xi) H(t, \xi)^{-1} Q(t, s, \xi) H(s, \xi),$$

where $Q(t, s, \xi)$ is uniformly bounded. Moreover, for any (s, ξ) , there exists

$$Q(\infty, s, \xi) = \lim_{t \rightarrow \infty} Q(t, s, \xi),$$

and it verifies

$$|Q(t, s, \xi) - Q(\infty, s, \xi)| \lesssim (1+t)^{-\ell} |\xi|^2 + e^{-C(t-s)}. \quad (40)$$

Remark 3. We remark that

$$e(t, s, \xi) \approx \exp \left(\int_s^t h(\tau, \xi) d\tau \right), \quad (41)$$

uniformly in B_K , as a consequence of Remark 2. By using the trivial property

$$\sqrt{x+y} \leq \sqrt{x} + \frac{y}{2\sqrt{x}}, \quad \text{for any } x \geq 0 \text{ and } y \geq -x, \quad (42)$$

we may estimate

$$\int_s^t h(\tau, \xi) d\tau \leq d(t, s, \xi),$$

so that we obtain $e(t, s, \xi) \lesssim e^{d(t, s, \xi)}$, with d as in (36).

Proof. To prove our statement, we may assume that $s \geq t_0$, where t_0 is as in (35), since the set $[0, t_0] \times \bar{B}_K$ is compact. Indeed, for any $s \in [0, t_0]$, the function $e^{d(t, s, \xi)}$ behaves as the function $e^{d(t, t_0, \xi)}$, due to the fact that the difference $d(t, t_0, \xi) - d(t, s, \xi) = d(t_0, s, \xi)$ is uniformly bounded with respect to $s \in B_K$.

For any $\xi \in B_K$ and $t \geq s \geq t_0$, the fundamental solution E^{low} solves the system

$$\begin{cases} \partial_t E^{\text{low}} = A(t, \xi) E^{\text{low}}, & t \geq s, \\ E^{\text{low}}(s, s, \xi) = \text{Id}, \end{cases} \quad (43)$$

where

$$A = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} (|\xi|^4 + \lambda(t)|\xi|^2).$$

We define

$$E^\dagger(t, s, \xi) = H(t, \xi) E^{\text{low}}(t, s, \xi) H(s, \xi)^{-1},$$

where $h(t, \xi)$ and $H(t, \xi)$ are as in (37) and (38). We remark that $H(t, \xi), H(t, \xi)^{-1}$ are uniformly bounded in B_K , due to $h(t, \xi) \approx \frac{1}{2}$, see (35).

Then $E^\dagger(t, s, \xi)$ solves the following problem:

$$\begin{cases} \partial_t E^\dagger = h(t, \xi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} E^\dagger + \frac{\partial_t h(t, \xi)}{h(t, \xi)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} E^\dagger, \\ E^\dagger(s, s, \xi) = \text{Id}, \end{cases}$$

Now we diagonalize the principal part of the system above by defining $\tilde{E} = PE^\dagger P^{-1}$, where

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad (44)$$

are constant, unitary matrices, so that $\tilde{E}$ solves the problem

$$\begin{cases} \partial_t \tilde{E} = h(t, \xi) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{E} + \frac{\partial_t h(t, \xi)}{2h(t, \xi)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tilde{E}, \quad t \geq s \\ \tilde{E}(s, s, \xi) = \text{Id}. \end{cases}$$

Let us write $\tilde{E}(t, s, \xi)$ in the form

$$\tilde{E}(t, s, \xi) = e(t, s, \xi) E^\ddagger(t, s, \xi)$$

where e is as in (39). Now $E^\ddagger$ solves the problem

$$\begin{cases} \partial_t E^\ddagger = h(t, \xi) \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} E^\ddagger + R_1 E^\ddagger, \quad t \geq s \\ E^\ddagger(s, s, \xi) = \text{Id}, \end{cases} \quad (45)$$

where

$$R_1 = \frac{\partial_t h(t, \xi)}{2h(t, \xi)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{satisfies } |R_1(t, \xi)| \lesssim (1+t)^{-1-\ell} |\xi|^2.$$

Here and in the following by $|R|$ we denote the maximum norm of the entries of a matrix. Then, we may estimate

$$\int_s^t |R_1(\tau, \xi)| d\tau \leq C |\xi|^2 \int_s^t (1+\tau)^{-1-\ell} d\tau \leq C' |\xi|^2 (1+s)^{-\ell} \leq C'.$$

Therefore, recalling that h is nonnegative, $E^\ddagger$ is uniformly bounded. Following as in [12], one may verify that

$$\exists \lim_{t \rightarrow \infty} E^\ddagger(t, s, \xi) =: E^\ddagger(\infty, s, \xi)$$

and that

$$|E^\dagger(t, s, \xi) - E^\dagger(\infty, s, \xi)| \lesssim (1+t)^{-\ell} |\xi|^2 + e^{-C(t-s)}. \quad (46)$$

Indeed, setting

$$D_0(t, s, \xi) \doteq \begin{pmatrix} e^{-2 \int_s^t h(\tau, \xi)} & 0 \\ 0 & 1 \end{pmatrix}, \quad E^\dagger = D_0 E^\circ,$$

we find that

$$\partial_t E^\circ = D_0(t, s, \xi)^{-1} R_1(t, \xi) D_0(t, s, \xi) E^\circ.$$

Formally,

$$\begin{aligned} E^\circ(t, s, \xi) &= I + \int_s^t D_0(\tau, s, \xi)^{-1} R_1(\tau, \xi) D_0(\tau, s, \xi) d\tau \\ &\quad + \int_s^t D_0(\tau, s, \xi)^{-1} R_1(\tau, \xi) D_0(\tau, s, \xi) \int_s^\tau D_0(\sigma, s, \xi)^{-1} R_1(\sigma, \xi) D_0(\sigma, s, \xi) d\sigma d\tau + \dots \end{aligned}$$

Due the fact that D_0^{-1} is not bounded, we do not have the boundedness of E° itself, so we cannot use this representation. However,

$$\begin{aligned} E^\dagger(t, s, \xi) &= D_0(t, s, \xi) E^\circ(t, s, \xi) \\ &= D_0(t, s, \xi) + \int_s^t D_0(t, \tau, \xi) R_1(\tau, \xi) D_0(\tau, s, \xi) d\tau \\ &\quad + \int_s^t D_0(t, \tau, \xi) R_1(\tau, \xi) \int_s^\tau D_0(\tau, \sigma, \xi) R_1(\sigma, \xi) D_0(\sigma, s, \xi) d\sigma d\tau + \dots \end{aligned}$$

where we used $D_0(t, s, \xi) = D_0(t, \tau, \xi) D_0(\tau, s, \xi)$, $D_0(\tau, s, \xi) = D_0(\tau, \sigma, \xi) D_0(\sigma, s, \xi)$, and so on. This leads to obtain

$$\begin{aligned} E^\dagger(\infty, s, \xi) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left(I + \int_s^\infty R_1(\tau, \xi) D_0(\tau, s, \xi) d\tau \right. \\ &\quad \left. + \int_s^\infty R_1(\tau, \xi) \int_s^\tau D_0(\tau, \sigma, \xi) R_1(\sigma, \xi) D_0(\sigma, s, \xi) d\sigma d\tau + \dots \right), \end{aligned}$$

and (46).

We now have that

$$Q(t, s, \xi) = P^{-1} E^\dagger(t, s, \xi) P,$$

is uniformly bounded in B_K and

$$|Q(t, s, \xi) - Q(\infty, s, \xi)| \lesssim |E^\dagger(t, s, \xi) - E^\dagger(\infty, s, \xi)| \lesssim (1+t)^{-\ell} |\xi|^2 + e^{-C(t-s)}.$$

This concludes the proof. $\square$

Now we consider the fundamental solution at high frequencies.

Lemma 2. *There exists a positive constant $c < 1/2$, depending on K , such that the fundamental solution E^{high} verifies the estimate*

$$|E^{\text{high}}(t, s, \xi)| \lesssim \exp(c(t-s)), \quad \text{for any } \xi \in B_K^c.$$

Proof. For any $\xi \in B_K^c$ and $t \geq s \geq t_0$, the fundamental solution E^{high} solves the system

$$\begin{cases} \partial_t E^{\text{high}} = A(t, \xi) E^{\text{high}}, & t \geq s, \\ E^{\text{high}}(s, s, \xi) = \text{Id}, \end{cases} \quad (47)$$

where

$$A(t, \xi) = \begin{pmatrix} 0 & i|\xi|^2 \\ im|\xi|^{-2} & 0 \end{pmatrix}.$$

We diagonalize the principal part of the system (47). We set $\tilde{E} = P E^{\text{high}} P^{-1}$, with P given by (44), so that

$$\partial_t \tilde{E} = i|\xi|^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{E} + R_0(t, \xi) \tilde{E},$$

and

$$R_0 = \frac{i}{2} \left(-\lambda(t) + \frac{1}{4} |\xi|^{-2} \right) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

We may write $E^{\text{high}} = P^{-1} D(t, s, \xi) Q(t, s, \xi) P$, where

$$D(t, s, \xi) = \begin{pmatrix} e^{-i|\xi|^2(t-s)} & 0 \\ 0 & e^{i|\xi|^2(t-s)} \end{pmatrix},$$

and $Q(t, s, \xi)$ is the solution to:

$$\begin{cases} \partial_t Q = R_2(t, s, \xi) Q, & t \geq s \\ Q(s, s, \xi) = I, \end{cases}$$

where $R_2 = D(s, t, \xi) R_0 D(t, s, \xi)$, since we used the property $D^{-1}(t, s, \xi) = D(s, t, \xi)$. It is clear that $D(t, s, \xi)$ is unitary, so it remains to prove that

$$|Q(t, s, \xi)| \lesssim e^{c(t-s)},$$

for some $c < 1/2$, depending on K . However,

$$|Q(t, s, \xi)| \leq \exp \left(\frac{1}{2} \int_s^t \lambda(\tau) + \frac{1}{4} |\xi|^{-2} d\tau \right) \leq \exp \left(\frac{\Lambda(t) - \Lambda(s)}{2} + \frac{K^{-2}}{8} (t - s) \right),$$

so the proof follows by recalling that $K > 1/2$, and noticing that $\Lambda(t) - \Lambda(s) \leq \varepsilon(t - s)$, for sufficiently large s , due to

$$\Lambda(t) - \Lambda(s) = \int_s^t \lambda(\tau) d\tau \leq \varepsilon \int_s^t 1 d\tau,$$

for sufficiently large s . $\square$

3. Proof of Theorem 2

By using Lemmas 1 and 2, one may easily prove Theorem 2.

Proof. Estimates (17), (18) and (19) trivially hold at short time, since problem (1) is well-posed in L^2 , namely,

$$\|\partial_t^j \Delta^k u(t, \cdot)\|_{L^2} \leq C \|(u_0, u_1)\|_{\mathcal{A}}, \quad j + k = 0, 1, \quad t \in [0, 1].$$

Therefore, we shall only prove the decay estimates for $t \geq 1$. Let u be the solution to (1). By Plancherel's theorem,

$$\|\Delta^k \partial_t^j u(t, \cdot)\|_{L^2}^2 = \| |\xi|^{2k} \partial_t^j \hat{u}(t, \cdot) \|_{L^2(|\xi| \leq K)}^2 + \| |\xi|^{2k} \partial_t^j \hat{u}(t, \cdot) \|_{L^2(|\xi| \geq K)}^2,$$

and we may now use Lemma 1 to estimate the first term, and Lemma 2 to estimate the second one, recalling (see (33)) that

$$\hat{u}(t, \xi) = e^{-\frac{t}{2}} \psi(t, \xi).$$

For any $\eta \in [1, 2]$, we define $\eta' = \eta/(\eta - 1)$, its Hölder conjugate, and $r \in [2, \infty]$ by

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{\eta'} = \frac{1}{\eta} - \frac{1}{2}. \quad (48)$$

Using Lemma 1 with $s = 0$, thanks to

$$e^{d(t, 0, \xi) - \frac{t}{2}} = e^{-t|\xi|^4 - \Lambda(t)|\xi|^2},$$

we obtain

$$\begin{aligned} \| |\xi|^{2k} \hat{u}(t, \cdot) \|_{L^2(|\xi| \leq K)} &\lesssim \| |\xi|^{2k} e^{-t|\xi|^4 - \Lambda(t)|\xi|^2} \|_{L^r} (\|\hat{u}_0\|_{L^{\eta'}} + \|\hat{u}_1\|_{L^{\eta'}}) \\ &\lesssim t^{-\max\{1-\ell, \frac{1}{2}\}(\frac{n}{2r} + k)} (\|u_0\|_{L^\eta} + \|u_1\|_{L^\eta}), \end{aligned}$$

for $k = 0, 1$. To obtain a sharp estimate for $\hat{u}_t$, we shall refine the estimate in B_K , following the ideas in [40]. Setting $\varphi = \hat{u}_t(t, \xi)$, this latter solves the first-order equation with inhomogeneous term

$$\varphi_t + \varphi = -(|\xi|^4 + \lambda(t)|\xi|^2)\hat{u}(t, \xi),$$

that is,

$$\varphi(t, \xi) = \varphi(0, \xi)e^{-t} - \int_0^t e^{-(t-\tau)}(|\xi|^4 + \lambda(\tau)|\xi|^2)\hat{u}(\tau, \xi) d\tau.$$

Using the estimate for $\hat{u}(t, \xi)$ in B_K and integrating by parts, we derive

$$|\varphi(t, \xi)| \lesssim |\hat{u}_1(\xi)| e^{-t} + (|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|) I(t, \xi),$$

where

$$\begin{aligned} I(t, \xi) &= \int_0^t e^{-(t-\tau)}(|\xi|^4 + \lambda(\tau)|\xi|^2) e^{-\tau|\xi|^4 - \Lambda(\tau)|\xi|^2} d\tau \\ &= \int_0^t \partial_\tau(e^{-(t-\tau)}(|\xi|^4 + \lambda(\tau)|\xi|^2) e^{-\tau|\xi|^4 - \Lambda(\tau)|\xi|^2}) d\tau \\ &= e^{-(t-\tau)}(|\xi|^4 + \lambda(\tau)|\xi|^2) e^{-\tau|\xi|^4 - \Lambda(\tau)|\xi|^2} \Big|_0^t + J(t, \xi), \\ J(t, \xi) &= - \int_0^t e^{-(t-\tau)} \partial_\tau((|\xi|^4 + \lambda(\tau)|\xi|^2) e^{-\tau|\xi|^4 - \Lambda(\tau)|\xi|^2}) d\tau. \end{aligned}$$

For a sufficiently large t and for $|\xi| \leq K$, we may estimate $J \leq I/2$, so that

$$|\hat{u}_t(t, \xi)| = |\varphi(t, \xi)| \lesssim |\hat{u}_1(\xi)| e^{-t} + (|\xi|^4 + \lambda(t)|\xi|^2) e^{-t|\xi|^4 - \Lambda(t)|\xi|^2} (|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|).$$

The presence of the additional term $(|\xi|^4 + \lambda(t)|\xi|^2)$ leads to the additional decay rate t^{-1} for $\hat{u}_t$ with respect to the decay rate for $\hat{u}$:

$$\|\hat{u}_t(t, \cdot)\|_{L^2(|\xi| \leq K)} \lesssim t^{-1 - \max\{1-\ell, \frac{1}{2}\}} \frac{n}{2r} (\|u_0\|_{L^\eta} + \|u_1\|_{L^\eta}),$$

for $t \geq t_0 > 1$. Indeed, if $\ell \geq 1/2$, then

$$\|(|\xi|^4 + \lambda(t)|\xi|^2) e^{-t|\xi|^4} \|_{L^r} \lesssim t^{-\frac{1}{2}(\frac{n}{2r})} (t^{-1} + t^{-\ell - \frac{1}{2}}),$$

whereas, if $\ell \in (0, 1/2)$, then

$$\|(|\xi|^4 + \lambda(t)|\xi|^2) e^{-\Lambda(t)|\xi|^2} \|_{L^r} \lesssim t^{-(1-\ell)(\frac{n}{2r})} (t^{-2(1-\ell)} + t^{-1}).$$

The proof of Theorem 2 follows thanks to the exponential decay

$$\| |\xi|^{2k} \partial_t^j \hat{u}(t, \cdot) \|_{L^2(|\xi| \geq K)} \lesssim e^{(c-1/2)t} (\|u_0\|_{H^{2(j+k)}} + \|u_1\|_{L^2}),$$

where $c \in (0, 1/2)$ is as in Lemma 2. $\square$

4. Proof of Theorem 1

The asymptotic profiles given in (4) and (5) are, respectively, the fundamental solutions to the Cauchy problems for diffusive equations given by

$$\begin{cases} v_t + (-\Delta)^2 v = 0, & t \geq 0, x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (49)$$

and

$$\begin{cases} v_t - \lambda(t)\Delta v = 0, & t \geq 0, x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (50)$$

Also, (6) is the fundamental solution to the following Cauchy problem for a diffusive equation:

$$\begin{cases} v_t + \Delta^2 v - \lambda(t)\Delta v = 0, & t \geq 0, x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (51)$$

Since we are interested in the fundamental solution of the Cauchy problems (49), (50) and (51), we will prove Theorem 1 as a corollary of the following results.

Lemma 3. *Let $n \geq 1$ and $\lambda = \lambda(t)$ be as in (7). Assume $(u_0, u_1) \in \mathcal{A} = (H^2 \cap L^\eta) \times (L^2 \cap L^\eta)$, for some $\eta \in [1, 2]$.*

Let $u \in \mathcal{C}([0, \infty), H^2) \cap \mathcal{C}^1([0, \infty), L^2)$ be the solution to (1). Then we have the following estimate

$$\|\Delta^k(u - v)(t, \cdot)\|_{L^2} = o\left(t^{-\max\{1-\ell, \frac{1}{2}\}} \left(\frac{n}{2} \left(\frac{1}{\eta} - \frac{1}{2}\right) + k\right)\right) \|(u_0, u_1)\|_{\mathcal{A}}, \quad t \rightarrow \infty, \quad k = 0, 1, \quad (52)$$

where v is the solution to (51) with initial data $v_0 = u_0 + u_1$.

We postpone the proof of Lemma 3 to the end of the section.

The proof of Theorem 1 is based on Lemma 3 with $\eta = 1$, and on the following result.

Lemma 4. *Assume that $h \in L^1$, and let*

$$M = \int_{\mathbb{R}^n} h(x) dx.$$

Then, for every $p \in [1, \infty]$ and $\kappa > 0$, it holds

$$\|e^{-t(-\Delta)^\kappa} h - M \mathcal{F}^{-1} e^{-t|\xi|^{2\kappa}}\|_{L^p} = o\left(t^{-\frac{n}{2\kappa} \left(1 - \frac{1}{p}\right)}\right),$$

where the action of $e^{-t(-\Delta)^\kappa}$ may be defined by $e^{-t(-\Delta)^\kappa} h(x) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\kappa}} \hat{h}(\xi))$.

The statement of Lemma 4 can be found in [24, Lemma 3.2], but, for the ease of reading, we provide an alternative, more direct proof.

Proof. For the sake of brevity, we fix $G(t, x) = \mathcal{F}^{-1} e^{-t|\xi|^{2\kappa}}$, so that we shall prove the following estimate:

$$\|G(t, \cdot) * h - MG(t, \cdot)\|_{L^p} = o\left(t^{-\frac{n}{2\kappa}\left(1-\frac{1}{p}\right)}\right).$$

By interpolation, it is sufficient to prove the statement for $p = \infty$ and for $p = 1$. We first consider $p = \infty$ and we define

$$g(\xi) = \hat{h}(\xi) - M = \hat{h}(\xi) - \hat{h}(0).$$

Then g is continuous and bounded, since $\hat{h} \in \mathcal{C}_0$. Due to the continuity of g ,

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : (|\xi| \leq \delta \Rightarrow |g(\xi)| < \varepsilon),$$

since $g(0) = 0$. Then, we may estimate

$$\|G(t, \cdot) * h - MG(t, \cdot)\|_{L^\infty} \leq \|\hat{G}(t, \cdot)g\|_{L^1} \leq \varepsilon \|\hat{G}(t, \cdot)\|_{L^1(|\xi| \leq \delta)} + e^{-t\delta^{2\kappa}/2} \int_{|\xi| \geq \delta} e^{-t|\xi|^{2\kappa}/2} |g(\xi)| d\xi.$$

It is clear that

$$\|\hat{G}(t, \cdot)\|_{L^1(|\xi| \leq \delta)} \leq \|\hat{G}(t, \cdot)\|_{L^1} = t^{-\frac{n}{2\kappa}} \|\hat{G}(1, \cdot)\|_{L^1}.$$

On the other hand, the integral may be estimated by

$$\int_{|\xi| \geq \delta} e^{-t|\xi|^{2\kappa}/2} |g(\xi)| d\xi \leq (\max_{|\xi| \geq \delta} |g(\xi)|) \int_{|\xi| \geq \delta} e^{-t|\xi|^{2\kappa}/2} d\xi \leq C t^{-\frac{n}{2\kappa}}.$$

For a sufficiently large t , it holds $e^{-t\delta^{2\kappa}/2} C \leq \varepsilon$. Therefore, for a fixed h and for any $\varepsilon > 0$, for sufficiently large t , we proved that

$$\|\hat{G}(t, \cdot)g\|_{L^1} \leq 2\varepsilon t^{-\frac{n}{2\kappa}},$$

and this concludes the proof of the statement for $p = \infty$.

Now let $p = 1$. We may write

$$(G(t, \cdot) * h)(x) - MG(t, x) = \int_{\mathbb{R}^n} (G(t, x - y) - G(t, x)) h(y) dy.$$

Then

$$\|G(t, \cdot) * h - MG(t, \cdot)\|_{L^1} \leq \int_{\mathbb{R}^n} |h(y)| \int_{\mathbb{R}^n} |G(t, x - y) - G(t, x)| dx dy.$$

Let us fix $\varepsilon > 0$. Due to the fact that $h \in L^1$, there exists $A > 0$ such that

$$\int_{|y| \geq A} |h(y)| dy \leq \varepsilon.$$

On the other hand, since $G(t, x)$ is self-similar, we have

$$\lim_{t \rightarrow \infty} \max_{|y| \leq A} |G(t, x - y) - G(t, x)| = 0,$$

for any fixed $x \in \mathbb{R}^n$. By Lebesgue's dominated convergence theorem, we derive that

$$\max_{|y| \leq A} \int_{\mathbb{R}^n} |G(t, x - y) - G(t, x)| dx \leq \varepsilon,$$

for a sufficiently large t . As a consequence, we derive

$$\int_{\mathbb{R}^n} |h(y)| \int_{\mathbb{R}^n} |G(t, x - y) - G(t, x)| dx dy \leq 2 \|G(t, \cdot)\|_{L^1} \varepsilon + \varepsilon \|h\|_{L^1} = \varepsilon (2 \|G(1, \cdot)\|_{L^1} + \|h\|_{L^1}),$$

for sufficiently large t . This concludes the proof. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. If (8) holds, then we combine the following steps:

- we apply Lemma 3 with $k = 0$;
- we use Plancherel and Riemann-Lebesgue theorem to show that the solution v to (51) verifies the estimate

$$\begin{aligned} \|v(t, \cdot) - e^{-t\Delta^2} v_0\|_{L^2} &\leq (2\pi)^{-n} \|e^{-t|\xi|^4} (e^{-\Lambda(t)|\xi|^2} - 1)\|_{L_\xi^2} \|v_0\|_{L^1} \\ &\leq C_n \|\Lambda(t)|\xi|^2 e^{-t|\xi|^4}\|_{L_\xi^2} \|v_0\|_{L^1} \\ &= C_n t^{-\frac{n}{8} - \frac{1}{2}} \Lambda(t) \|\xi|^2 e^{-|\xi|^4}\|_{L_\xi^2} \|v_0\|_{L^1} = o(t^{-\frac{n}{8}}), \end{aligned}$$

where the last estimate is a consequence of assumption (8);

- we apply Lemma 4, with $\kappa = 2$, $p = 2$ and $h = v_0 = u_0 + u_1$.

Then we conclude the proof of (9) by triangle inequality.

On the other hand, if (10) holds, we may proceed as before and show that the solution v to (51) verifies the estimate

$$\|v(t, \cdot) - e^{\Lambda(t)\Delta} v_0\|_{L^2} = o(\Lambda(t)^{-\frac{n}{4}}).$$

By the change of variables $s = \Lambda(t)$, applying Lemma 4, with $\kappa = 1$, $p = 2$ and $h = v_0 = u_0 + u_1$, we derive

$$\|e^{\Lambda(t)\Delta} v_0 - M \mathcal{F}^{-1} e^{-\Lambda(t)|\xi|^2}\|_{L^2} = \|e^{s\Delta} v_0 - M \mathcal{F}^{-1} e^{-s|\xi|^2}\|_{L^2} = o(\Lambda(t)^{-\frac{n}{4}}),$$

and this concludes the proof of (11).

If we assume (12), then we first use the estimate

$$\|e^{\Lambda(t)\Delta} f(t, \cdot)\|_{L^2} \leq \|f(t, \cdot)\|_{L^2},$$

then we apply again Lemma 4, with $\kappa = 2$ and $p = 2$, to estimate

$$\begin{aligned} \|e^{-t\Delta^2 + \Lambda(t)\Delta} v_0 - M \mathcal{F}^{-1} e^{-t|\xi|^4 - \Lambda(t)|\xi|^2}\|_{L^2} &= \|e^{\Lambda(t)\Delta} (e^{-t\Delta^2} v_0 - M \mathcal{F}^{-1} e^{-t|\xi|^4})\|_{L^2} \\ &\leq \|e^{-t\Delta^2} v_0 - M \mathcal{F}^{-1} e^{-t|\xi|^4}\|_{L^2} = o(t^{-\frac{n}{8}}). \end{aligned}$$

This concludes the proof of (13). $\square$

At high frequencies, we have an exponential decay (in time) for the diffusive-type kernels and for the solution to (1). Hence we may localize our analysis to low frequencies.

The solution for the Cauchy problem

$$\hat{u}_t + \hat{u}_t = 0, \quad \hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}(0, \xi) = \hat{u}_1(\xi),$$

for $\xi = 0$ is given by

$$\hat{u}(t, 0) = \hat{u}_0(0) + (1 - e^{-t})\hat{u}_1(0).$$

On the other hand, for a fixed $\xi \in B_K$, we may write

$$\left(\frac{1}{2}\psi, \psi_t\right)^\top(t, \xi) = e(t, 0, \xi) H(t, \xi)^{-1} Q(t, 0, \xi) H(0, \xi) \left(\frac{\hat{u}_0(\xi)}{2}, \frac{\hat{u}_0(\xi)}{2} + \hat{u}_1(\xi)\right)^\top,$$

for any $t \geq t_0$. Hence

$$\hat{u}(t, \xi) = e^{-\frac{t}{2}} \psi(t, \xi) = 2e^{-\frac{t}{2}} e(t, 0, \xi) e_0^\top H(t, \xi)^{-1} Q(t, 0, \xi) H(0, \xi) \left(\frac{\hat{u}_0(\xi)}{2}, \frac{\hat{u}_0(\xi)}{2} + \hat{u}_1(\xi)\right)^\top$$

where $e_0 = (1, 0)^\top$. Thanks to $e^{-\frac{t}{2}} e(t, 0, 0) = 1$ and $H(t, 0) = I$ we conclude

$$\hat{u}(t, 0) = 2e_0^\top Q(t, 0, 0) \left(\frac{\hat{u}_0(0)}{2}, \frac{\hat{u}_0(0)}{2} + \hat{u}_1(0)\right)^\top \rightarrow \hat{u}_0(0) + \hat{u}_1(0), \quad t \rightarrow \infty.$$

We are now ready to prove Lemma 3.

Proof of Lemma 3. Let v be the solution to (51) with initial data $v_0 = u_0 + u_1$. For any $t \geq t_0$, with sufficiently large t_0 , we may write

$$\hat{u}(t, \xi) - \hat{v}(t, \xi) = e^{-t|\xi|^4 - \Lambda(t)|\xi|^2} \left(\mathcal{A}(t, \xi) \left(\frac{\hat{u}_0(\xi)}{2}, \frac{\hat{u}_0(\xi)}{2} + \hat{u}_1(\xi)\right)^\top - (\hat{u}_0(\xi) + \hat{u}_1(\xi)) \right),$$

where

$$\mathcal{A}(t, \xi) = 2 e^{-\frac{1}{2}t + t|\xi|^4 + \Lambda(t)|\xi|^2} e(t, 0, \xi) e_0^\top H(t, \xi)^{-1} Q(t, 0, \xi) H(0, \xi).$$

The solution for the system (45) at $\xi = 0$ is given by

$$E^\dagger(t, 0, 0) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\hat{u}_0 + \hat{u}_1 = 2e_0^\top Q(\infty, 0, 0) \left(\frac{\hat{u}_0(\xi)}{2}, \frac{\hat{u}_0(\xi)}{2} + \hat{u}_1(\xi) \right)^\top, \quad Q(\infty, 0, 0) = P^{-1} E^\dagger(\infty, 0, 0) P.$$

Therefore

$$\hat{u}(t, \xi) - \hat{v}(t, \xi) = 2e_0^\top e^{-t|\xi|^4 - \Lambda(t)|\xi|^2} \mathcal{B}(t, \xi) \left(\frac{\hat{u}_0(\xi)}{2}, \frac{\hat{u}_0(\xi)}{2} + \hat{u}_1(\xi) \right)^\top$$

with

$$\mathcal{B}(t, \xi) = e^{-\frac{1}{2}t + t|\xi|^4 + \Lambda(t)|\xi|^2} e(t, 0, \xi) H(t, \xi)^{-1} Q(t, 0, \xi) H(0, \xi) - Q(\infty, 0, 0).$$

We may now estimate

$$\left| \frac{\sqrt{h(t, \xi)}}{\sqrt{h(0, \xi)}} - 1 \right| \approx |\sqrt{h(t, \xi)} - \sqrt{h(0, \xi)}| \approx |h(t, \xi) - h(0, \xi)| \approx |m(0, \xi) - m(t, \xi)| \lesssim |\xi|^2,$$

and

$$|e^{-\frac{1}{2}t + t|\xi|^4 + \Lambda(t)|\xi|^2} e(t, 0, \xi) - 1| \leq |\xi|^2 + \int_0^t (|\xi|^4 + \lambda(\tau)|\xi|^2)^2 d\tau.$$

Moreover,

$$|H(t, \xi)^{-1} - \text{Id}| \approx |2h(t, \xi) - 1| \approx (|\xi|^4 + \lambda(t)|\xi|^2),$$

and, using (40), we get

$$\begin{aligned} |Q(t, 0, \xi) - Q(\infty, 0, 0)| &\leq |Q(t, 0, \xi) - Q(\infty, 0, \xi)| + |Q(\infty, 0, \xi) - Q(\infty, 0, 0)| \\ &\lesssim (1+t)^{-\ell} |\xi|^2 + e^{-Ct} + |E^\dagger(\infty, 0, \xi) - E^\dagger(\infty, 0, 0)| \\ &\lesssim (1+t)^{-\ell} |\xi|^2 + e^{-Ct} + |\xi|^2. \end{aligned}$$

The last inequality is a consequence of

$$E^\dagger(\infty, 0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \int_0^\infty |R_1(\tau, \xi)| d\tau \lesssim |\xi|^2.$$

Therefore,

$$|\mathcal{B}(t, \xi)| \lesssim |\xi|^2 + e^{-Ct} + \int_0^t (|\xi|^4 + \lambda(\tau)|\xi|^2)^2 d\tau,$$

and we conclude that

$$\| |\xi|^k (\hat{u} - \hat{v})(t, \cdot) \|_{L^2(|\xi| \leq K)} = o\left(t^{-\max\{1-\ell, \frac{1}{2}\}} \left(\frac{n}{2} \left(\frac{1}{\eta} - \frac{1}{2}\right) + k\right)\right) \| (u_0, u_1) \|_{\mathcal{A}}, \quad t \rightarrow \infty.$$

This concludes the proof of Lemma 3. $\square$

5. Proof of Theorems 3 and 5

To prove Theorems 3 and 5, we rely on Duhamel's principle. However, since the equation in (1) is not invariant by time-translations, we shall extend the estimates in Theorem 2 to the following family of parameter-dependent Cauchy problems (see [16]):

$$\begin{cases} u_{tt} + \Delta^2 u - \lambda(t)\Delta u + u_t = 0, & t \geq s, x \in \mathbb{R}^n \\ u(s, x) = 0, & x \in \mathbb{R}^n, \\ u_t(s, x) = f(s, x), & x \in \mathbb{R}^n. \end{cases} \quad (53)$$

Lemma 5. *Let $n \geq 1$, $s \geq 0$, and $\lambda = \lambda(t)$ be as in (7). Assume $f(s, \cdot) \in L^\eta \cap L^2$, for some $\eta \in [1, 2]$. Let $u \in \mathcal{C}([0, \infty), H^2) \cap \mathcal{C}^1([0, \infty), L^2)$ be the solution to (53). Then we have the following decay estimates*

$$\|\partial_t^j \Delta^k u(t, \cdot)\|_{L^2} \lesssim (1+t-s)^{-j-\frac{n}{4}(\frac{1}{\eta}-\frac{1}{2})-\frac{k}{2}} \|f(s, \cdot)\|_{L^\eta \cap L^2}, \quad j+k=0, 1 \quad (54)$$

if $\ell \geq 1/2$, and

$$\|\partial_t^j \Delta^k u(t, \cdot)\|_{L^2} \lesssim (1+t-s)^{-j(1+\Lambda(t)-\Lambda(s))-\left(\frac{n}{2}(\frac{1}{\eta}-\frac{1}{2})+k\right)} \|f(s, \cdot)\|_{L^\eta \cap L^2}, \quad j+k=0, 1, \quad (55)$$

if $\ell \in (0, 1/2)$.

We remark that

$$\Lambda(t) - \Lambda(s) = \frac{\mu}{1-\ell} ((1+t)^{1-\ell} - (1+s)^{1-\ell}) \gtrsim \frac{t-s}{(1+t)^\ell} \quad \text{if } \ell \in (0, 1). \quad (56)$$

Proof. It is sufficient to follow the proof of Theorem 2. We may assume $t \geq t_0$ and $t - s \geq 1$ and use Lemma 1, recalling that

$$e^{d(t,s,\xi) - \frac{t-s}{2}} = e^{-(t-s)|\xi|^4 - (\Lambda(t) - \Lambda(s))|\xi|^2}$$

but we now distinguish two cases. If $\ell \geq 1/2$, then we obtain

$$\begin{aligned} \||\xi|^{2k} \hat{u}(t, \cdot)\|_{L^2(|\xi| \leq K)} &\lesssim \||\xi|^{2k} e^{-(t-s)|\xi|^4} \|_{L^r} \|\hat{f}(s, \cdot)\|_{L^{r'}} \\ &\lesssim (t-s)^{-\frac{n}{4r} - \frac{k}{2}} \|f(s, \cdot)\|_{L^n}, \end{aligned}$$

for $k = 0, 1$ and r given by (48). On the other hand, if $\ell \in (0, 1/2)$, we estimate

$$\begin{aligned} \||\xi|^{2k} \hat{u}(t, \cdot)\|_{L^2(|\xi| \leq K)} &\lesssim \||\xi|^{2k} e^{-(\Lambda(t) - \Lambda(s))|\xi|^2} \|_{L^r} \|\hat{f}(s, \cdot)\|_{L^{r'}} \\ &\lesssim (\Lambda(t) - \Lambda(s))^{-\left(\frac{n}{2r} + k\right)} \|f(s, \cdot)\|_{L^n}. \end{aligned}$$

To obtain a sharp estimate for $\hat{u}_t$, we shall refine the estimate in B_K . Setting $\varphi = \hat{u}_t(t, \xi)$, we now have

$$\varphi(t, \xi) = \varphi(s, \xi) e^{-(t-s)} - \int_s^t e^{-(t-\tau)} (|\xi|^4 + \lambda(\tau)|\xi|^2) \hat{u}(\tau, \xi) d\tau.$$

Using the estimate for $\hat{u}(t, \xi)$ in B_K and integrating by parts, we derive

$$|\varphi(t, \xi)| \lesssim |\hat{f}(s, \xi)| (e^{-(t-s)} + I), \quad t \geq \max\{t_0, s + 1\},$$

where

$$I = \int_s^t e^{-(t-\tau)} (|\xi|^4 + \lambda(\tau)|\xi|^2) e^{-(\tau-s)|\xi|^4 - (\Lambda(\tau) - \Lambda(s))|\xi|^2} d\tau.$$

Proceeding as in the proof of Theorem 2, for a sufficiently large $t - s$ and for $|\xi| \leq K$, we get

$$|\hat{u}_t(t, \xi)| = |\varphi(t, \xi)| \lesssim |\hat{f}(s, \xi)| (e^{-(t-s)} + (|\xi|^4 + \lambda(t)|\xi|^2) e^{-(t-s)|\xi|^4 - (\Lambda(t) - \Lambda(s))|\xi|^2}).$$

The presence of the additional term $(|\xi|^4 + \lambda(t)|\xi|^2)$ leads to the additional decay rate $(t - s)^{-1}$ for $\hat{u}_t$ with respect to the decay rate for $\hat{u}$. Indeed:

$$\begin{aligned} \||\xi|^4 e^{-(t-s)|\xi|^4} \|_{L^r} &\lesssim (t-s)^{-\frac{n}{4r} - 1}, \\ \lambda(t) \||\xi|^2 e^{-(\Lambda(t) - \Lambda(s))|\xi|^2} \|_{L^r} &\lesssim (1+t)^{-\ell} (\Lambda(t) - \Lambda(s))^{-1 - \frac{n}{2r}} \lesssim (t-s)^{-1} (\Lambda(t) - \Lambda(s))^{-\frac{n}{2r}}, \end{aligned}$$

where we used (56) if $\ell \in (0, 1)$.

The proof follows once again thanks to the exponential decay at high frequencies (Lemma 2). $\square$

To derive the existence and uniqueness of the global solution to (21), for any $T > 0$, we employ a classical contraction argument in the space

$$X(T) = \mathcal{C}([0, T], H^2) \cap \mathcal{C}^1([0, T], L^2),$$

equipped with the norm induced by the estimates for the linear problem (1) which we obtained in Theorem 2. That is,

$$\|u\|_{X(T)} = \max_{t \in [0, T]} \sum_{j+k=0}^1 (1+t)^{j+\max\{1-\ell, \frac{1}{2}\}(\frac{n}{4}+k)} \|\partial_t^j \Delta^k u(t, \cdot)\|_{L^2}, \quad (57)$$

if we prove Theorem 3, and

$$\|u\|_{X(T)} = \max_{t \in [0, T]} \sum_{j+k=0}^1 (1+t)^{j+k \max\{1-\ell, \frac{1}{2}\}} \|\partial_t^j \Delta^k u(t, \cdot)\|_{L^2}, \quad (58)$$

if we prove Theorem 5.

In particular, thanks to Theorem 2, the solution to (1), with initial data as in (22) or, respectively, as in (25), is in $X(T)$ and

$$\|u\|_{X(T)} \leq C \varepsilon,$$

uniformly with respect to T .

We define the operator F such that, for any $u \in X(T)$,

$$Fu(t, x) \doteq \int_0^t K(t, s, x) *_{(x)} |u(s, x)|^p ds, \quad (59)$$

where $K(t, s, x)$ is the fundamental solution to (53). Then we prove the estimates

$$\|Fu\|_{X(T)} \leq C \|u\|_{X(T)}^p, \quad (60)$$

$$\|Fu - Fv\|_{X(T)} \leq C \|u - v\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}), \quad (61)$$

where $C > 0$ does not depend on T . By standard arguments (Duhamel's principle, contraction argument), estimates (60)-(61) lead to the existence of a unique global-in-time solution to (21), for sufficiently small $\varepsilon > 0$.

We first prove Theorem 5.

Proof of Theorem 5. Let $u \in X(T)$. Taking into account of the norm in (58), we may estimate

$$\|u(s, \cdot)\|_{L^2} \lesssim \|u\|_{X(T)}, \quad \|\Delta u(s, \cdot)\|_{L^2} \lesssim \|u\|_{X(T)} (1+s)^{-\max\{(1-\ell), \frac{1}{2}\}},$$

for any $s \in [0, T]$. Thanks to the Gagliardo-Nirenberg inequality, we obtain

$$\|u(s, \cdot)\|_{L^q} \lesssim \|u\|_{X(T)} (1+s)^{-\max\{(1-\ell), \frac{1}{2}\} \frac{n}{2} (\frac{1}{2} - \frac{1}{q})}, \quad (62)$$

for any $q \in [2, \infty)$ if $n \leq 4$, and for any $q \in [2, 2n/(n-4)]$ if $n \geq 5$.

We first assume $\ell \geq 1/2$ and we prove (60). For $j+k = 0, 1$, thanks to Lemma 5 with $\eta = 2$ and $f(s, x) = |u(s, x)|^p$, we may estimate

$$\begin{aligned} \|\partial_t^j \Delta^k F u(t, \cdot)\|_{L^2} &\lesssim \int_0^t (1+t-s)^{-j-\frac{k}{2}} \| |u(s, \cdot)|^p \|_{L^2} ds \\ &= \int_0^t (1+t-s)^{-j-\frac{k}{2}} \|u(s, \cdot)\|_{L^{2p}}^p ds \\ &\lesssim \|u\|_{X(T)}^p \int_0^t (1+t-s)^{-j-\frac{k}{2}} (1+s)^{-\frac{n}{8}(p-1)} ds \end{aligned}$$

thanks to (62) (we used the assumption that $2p \leq 2n/(n-4)$ if $n \geq 5$). Due to the fact that

$$j + \frac{k}{2} \leq 1 < \frac{n}{8}(p-1),$$

for $j+k = 0, 1$, thanks to the assumption of $p > \tilde{p} = 1 + 8/n$, the latter integral may be estimated by

$$\|u\|_{X(T)}^p \int_0^t (1+t-s)^{-j-\frac{k}{2}} (1+s)^{-\frac{n}{8}(p-1)} ds \lesssim \|u\|_{X(T)}^p (1+t)^{-j-\frac{k}{2}}.$$

Here and in the following, we use the well-known estimate

$$\int_0^t (1+t-s)^{-a} (1+s)^{-b} ds \lesssim (1+t)^{-a}, \quad a \leq 1 < b. \quad (63)$$

This concludes the proof of (60). In order to prove (61), we use the inequality

$$|u|^p - |v|^p \lesssim |u-v| (|u|^{p-1} + |v|^{p-1}),$$

and the Hölder inequality

$$\begin{aligned} \| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{L^2} &\lesssim \| |u(s, \cdot) - v(s, \cdot)| (|u(s, \cdot)|^{p-1} + |v(s, \cdot)|^{p-1}) \|_{L^2} \\ &\leq \|u(s, \cdot) - v(s, \cdot)\|_{L^{2p}} \| |u(s, \cdot)|^{p-1} + |v(s, \cdot)|^{p-1} \|_{L^{2p'}} \\ &\lesssim \|u(s, \cdot) - v(s, \cdot)\|_{L^{2p}} (\|u(s, \cdot)\|_{L^{2p}}^{p-1} + \|v(s, \cdot)\|_{L^{2p}}^{p-1}) \\ &\lesssim \|u-v\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}) (1+s)^{-\frac{n}{8}(p-1)}, \end{aligned}$$

to derive

$$\begin{aligned}
& \|\partial_t^j \Delta^k (Fu - Fv)(t, \cdot)\|_{L^2} \\
& \lesssim \int_0^t (1+t-s)^{-j-\frac{k}{2}} \| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{L^2} ds \\
& \lesssim \|u - v\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}) \int_0^t (1+t-s)^{-j-\frac{k}{2}} (1+s)^{-\frac{n}{8}(p-1)} ds \\
& \lesssim \|u - v\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}) (1+t)^{-j-\frac{k}{2}}.
\end{aligned}$$

Now let $\ell \in (0, 1/2)$. We only prove (60), being the proof of (61) analogous. Thanks to (56), we may now estimate

$$\begin{aligned}
\|\partial_t^j \Delta^k Fu(t, \cdot)\|_{L^2} & \lesssim (1+t)^{k\ell} \int_0^t (1+t-s)^{-j-k} \| |u(s, \cdot)|^p \|_{L^2} ds \\
& = (1+t)^{k\ell} \int_0^t (1+t-s)^{-j-k} \|u(s, \cdot)\|_{L^{2p}}^p ds \\
& \lesssim \|u\|_{X(T)}^p (1+t)^{k\ell} \int_0^t (1+t-s)^{-j-k} (1+s)^{-\frac{n(1-\ell)}{4}(p-1)} ds,
\end{aligned}$$

where we used (62). Due to the fact that

$$j+k \leq 1 < \frac{n(1-\ell)}{4}(p-1),$$

for $j+k = 0, 1$, thanks to the assumption of $p > \tilde{p} = 1 + 4/(n(1-\ell))$, recalling (63), the latter integral may be estimated by

$$\|u\|_{X(T)}^p (1+t)^{k\ell} \int_0^t (1+t-s)^{-j-k} (1+s)^{-\frac{n(1-\ell)}{4}(p-1)} ds \lesssim \|u\|_{X(T)}^p (1+t)^{-j-k(1-\ell)}.$$

This concludes the proof. $\square$

We are now ready to prove Theorem 3.

Proof of Theorem 3. We only prove (60), being the proof of (61) analogous.

Let $n \leq 4$ and $u \in X(T)$. Taking into account of the norm in (57), we may estimate

$$\|\Delta^k u(s, \cdot)\|_{L^2} \lesssim \|u\|_{X(T)} (1+s)^{-\max\{(1-\ell), \frac{1}{2}\}(\frac{n}{4}+k)},$$

for any $s \in [0, T]$. Thanks to the Gagliardo-Nirenberg inequality, we obtain

$$\|u(s, \cdot)\|_{L^q} \lesssim \|u\|_{X(T)} (1+s)^{-\max\{(1-\ell), \frac{1}{2}\} \frac{n}{2} (1-\frac{1}{q})}, \quad (64)$$

for any $q \in [2, \infty)$.

We first assume $\ell \geq 1/2$. We employ Lemma 5 with $\eta = 1$ for $s \in [0, t/2]$ and Lemma 5 with $\eta = 2$ for $s \in [t/2, t]$, so that, for $j + k = 0, 1$, we may estimate

$$\begin{aligned}
 \|\partial_t^j \Delta^k F u(t, \cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{8}-j-\frac{k}{2}} \| |u(s, \cdot)|^p \|_{L^1 \cap L^2} ds \\
 &\quad + \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} \| |u(s, \cdot)|^p \|_{L^2} ds \\
 &= \int_0^{t/2} (1+t-s)^{-\frac{n}{8}-j-\frac{k}{2}} \|u(s, \cdot)\|_{L^p \cap L^{2p}}^p ds \\
 &\quad + \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} \|u(s, \cdot)\|_{L^{2p}}^p ds \\
 &\lesssim \|u\|_{X(T)}^p \int_0^{t/2} (1+t-s)^{-\frac{n}{8}-j-\frac{k}{2}} (1+s)^{-\frac{n}{4}(p-1)} ds \\
 &\quad + \|u\|_{X(T)}^p \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} (1+s)^{-\frac{n}{4}(p-1/2)} ds \\
 &\approx \|u\|_{X(T)}^p (1+t)^{-\frac{n}{8}-j-\frac{k}{2}} \int_0^{t/2} (1+s)^{-\frac{n}{4}(p-1)} ds \\
 &\quad + \|u\|_{X(T)}^p (1+t)^{-\frac{n}{8}-\frac{n}{4}(p-1)} \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} ds,
 \end{aligned}$$

thanks to (64), and to the property $1+t-s \approx 1+t$ for $s \in [0, t/2]$ and $1+s \approx 1+t$ for $s \in [t/2, t]$. Due to the fact that

$$j + \frac{k}{2} \leq 1 < \frac{n}{4}(p-1),$$

for $j + k = 0, 1$, thanks to the assumption of $p > \bar{p} = 1 + 4/n$, the two integrals may be estimated by

$$\begin{aligned}
 \|u\|_{X(T)}^p (1+t)^{-\frac{n}{8}-j-\frac{k}{2}} \int_0^{t/2} (1+s)^{-\frac{n}{4}(p-1)} ds &\lesssim \|u\|_{X(T)}^p (1+t)^{-\frac{n}{8}-j-\frac{k}{2}}, \\
 \|u\|_{X(T)}^p (1+t)^{-\frac{n}{8}-\frac{n}{4}(p-1)} \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} ds &\lesssim \|u\|_{X(T)}^p (1+t)^{-\frac{n}{8}-j-\frac{k}{2}}.
 \end{aligned}$$

This concludes the proof for $\ell \geq 1/2$. Now let $\ell \in (0, 1/2)$. We employ again Lemma 5 with $\eta = 1$ for $s \in [0, t/2]$ and Lemma 5 with $\eta = 2$ for $s \in [t/2, t]$. Moreover:

- for $s \in [t/2, t]$, we use (56), that is, $\Lambda(t) - \Lambda(s) \gtrsim (t-s)(1+t)^{-\ell}$, and $1+s \approx 1+t$;
- for $s \in [0, t/2]$, we use $\Lambda(t) - \Lambda(s) \approx \Lambda(t)$, and $1+t-s \approx 1+t$.

Therefore, we get

$$\begin{aligned}
\|\partial_t^j \Delta^k F u(t, \cdot)\|_{L^2} &\lesssim (1+t)^{-j-(1-\ell)(\frac{n}{4}+k)} \int_0^{t/2} \|u(s, \cdot)\|_{L^p \cap L^{2p}}^p ds \\
&\quad + (1+t)^{k\ell} \int_{t/2}^t (1+t-s)^{-j-k} \|u(s, \cdot)\|_{L^{2p}}^p ds \\
&\lesssim \|u\|_{X(T)}^p (1+t)^{-j-(1-\ell)(\frac{n}{4}+k)} \int_0^{t/2} (1+s)^{-(1-\ell)\frac{n}{2}(p-1)} ds \\
&\quad + \|u\|_{X(T)}^p (1+t)^{k\ell-(1-\ell)\frac{n}{2}(p-1/2)} \int_{t/2}^t (1+t-s)^{-j-k} ds
\end{aligned}$$

where we used (64). Due to the fact that

$$j+k \leq 1 < \frac{n(1-\ell)}{2}(p-1),$$

for $j+k=0, 1$, thanks to the assumption of $p > \bar{p} = 1 + 2/(n(1-\ell))$, the both two integrals may be estimated by

$$\|u\|_{X(T)}^p (1+t)^{-j-(1-\ell)(\frac{n}{4}+k)}.$$

This concludes the proof. $\square$

6. Proof of Theorem 4

Before proving Theorem 4, we need to obtain for problem (53), a result which is analogous to Lemma 3 for problem (1). Let v be the solution to

$$\begin{cases} v_t + (-\Delta)^2 v - \lambda(t) \Delta v = 0, & t \geq 0, x \in \mathbb{R}^n, \\ v(s, x) = f(s, x), & x \in \mathbb{R}^n. \end{cases} \quad (65)$$

Lemma 6. *Let $n \geq 1$ and $\lambda = \lambda(t)$ be as in (7). Assume $f \in L^2 \cap L^\eta$, for some $\eta \in [1, 2]$.*

Let $u \in \mathcal{C}([0, \infty), H^2) \cap \mathcal{C}^1([0, \infty), L^2)$ be the solution to (53). Then we have the following estimates

$$\|\Delta^k(u-v)(t, \cdot)\|_{L^2} \lesssim o((t-s)^{-\frac{n}{4}(\frac{1}{\eta}-\frac{1}{2})-\frac{k}{2}}) \|f(s, \cdot)\|_{L^2 \cap L^\eta}, \quad t \rightarrow \infty, \quad k=0, 1, \quad (66)$$

for $\ell \geq 1/2$, and

$$\|\Delta^k(u-v)(t, \cdot)\|_{L^2} \lesssim \left((1+\Lambda(t)-\Lambda(s))^{-\frac{n}{2}(\frac{1}{\eta}-\frac{1}{2})-k-\delta} \right) \|f(s, \cdot)\|_{L^2 \cap L^\eta}, \quad t \rightarrow \infty, \quad k=0, 1, \quad (67)$$

for some sufficiently small $\delta > 0$, if $\ell \in (0, 1/2)$.

We wrote (67) including an explicit polynomial decay gain $(1 + \Lambda(t) - \Lambda(s))^{-\delta}$, since later we plan to use property (56) to prove our results.

Proof. Applying the change of variable $\psi(t, \xi) = \hat{u}(t, \xi)e^{\frac{t}{2}}$, as in (33), problem (53) reads as

$$\begin{cases} \psi_{tt} + m(t, \xi)\psi = 0, & t \geq s \\ (\psi(s, \xi), \psi_t(s, \xi)) = \left(0, e^{\frac{s}{2}}\hat{f}(s, \xi)\right), & \xi \in \mathbb{R}^n, \end{cases} \quad (68)$$

where

$$m(t, \xi) = |\xi|^4 + \lambda(t)|\xi|^2 - \frac{1}{4}.$$

For a fixed $\xi \in B_K$, we may write

$$\left(\frac{1}{2}\psi, \psi_t\right)(t, \xi) = e(t, s, \xi) H(t, \xi)^{-1} Q(t, s, \xi) H(s, \xi) \left(0, e^{\frac{s}{2}}\hat{f}(s, \xi)\right)^\top,$$

for any $t \geq t_0$. Hence

$$\hat{u}(t, \xi) = e^{-\frac{t}{2}}\psi(t, \xi) = 2e^{-\frac{t}{2}}e(t, s, \xi)e_0^\top H(t, \xi)^{-1} Q(t, s, \xi) H(s, \xi) \left(0, e^{\frac{s}{2}}\hat{f}(s, \xi)\right)^\top.$$

Thanks to $e^{-\frac{t}{2}}e(t, s, 0) = e^{-\frac{s}{2}}$ and $H(t, 0) = I$ we conclude

$$\hat{u}(t, 0) = 2e^{-\frac{s}{2}}e_0^\top Q(t, s, 0) \left(0, e^{\frac{s}{2}}\hat{f}(s, \xi)\right)^\top \rightarrow \hat{f}(s, 0), \quad t \rightarrow \infty.$$

Hence we may write

$$\hat{u}(t, \xi) - \hat{v}(t, \xi) = 2e^{-\frac{s}{2}}e_0^\top e^{-(t-s)|\xi|^4 - (\Lambda(t) - \Lambda(s))|\xi|^2} \mathcal{B}(t, \xi) \left(0, e^{\frac{s}{2}}\hat{f}(s, \xi)\right)^\top$$

with

$$\mathcal{B}(t, \xi) = e^{-\frac{1}{2}(t-s) + (t-s)|\xi|^4 + (\Lambda(t) - \Lambda(s))|\xi|^2} e(t, s, \xi) H(t, \xi)^{-1} Q(t, s, \xi) H(s, \xi) - Q(\infty, s, 0).$$

We may now estimate

$$\begin{aligned} & |e^{-\frac{1}{2}(t-s) + (t-s)|\xi|^4 + (\Lambda(t) - \Lambda(s))|\xi|^2} e(t, s, \xi) - 1| \\ & \leq |\xi|^2 + \int_s^t (|\xi|^4 + \lambda(\tau)|\xi|^2)^2 d\tau, \end{aligned}$$

and

$$\begin{aligned} |Q(t, s, \xi) - Q(\infty, s, 0)| & \lesssim (1+t)^{-\ell} |\xi|^2 + e^{-C(t-s)} + |E^\dagger(\infty, s, \xi) - E^\dagger(\infty, s, 0)| \\ & \lesssim (1+t)^{-\ell} |\xi|^2 + e^{-C(t-s)} + |\xi|^2. \end{aligned}$$

Therefore,

$$|\mathcal{B}(t, \xi)| \lesssim |\xi|^2 + e^{-C(t-s)} + \int_s^t (|\xi|^4 + \lambda(\tau)|\xi|^2)^2 d\tau,$$

and we conclude that

$$\| |\xi|^{2k} (\hat{u}(t, \cdot) - \hat{v}(t, \cdot)) \|_{L^2(|\xi| \leq K)} = o \left((1+t-s)^{-\frac{n}{4}(\frac{1}{\eta}-\frac{1}{2})-\frac{k}{2}} \|f(s, \cdot)\|_{L^\eta} \right),$$

if $\ell \geq 1/2$, and

$$\| |\xi|^{2k} (\hat{u}(t, \cdot) - \hat{v}(t, \cdot)) \|_{L^2(|\xi| \leq K)} \lesssim (1 + \Lambda(t) - \Lambda(s))^{-\frac{n}{2}(\frac{1}{\eta}-\frac{1}{2})-k-\delta} \|f(s, \cdot)\|_{L^\eta},$$

for a sufficiently small $\delta > 0$, if $\ell \in (0, 1/2)$. $\square$

We are now ready to prove Theorem 4.

Proof of Theorem 4. The proof of this theorem is very similar to the arguments employed by G. Karch in [24]. Taking into account of Theorem 1, it remains to estimate only the nonlinear contribution to the solution for (21). Let $f(u) = |u|^p$. We first estimate:

$$\left\| \int_0^t (S(t, s, \cdot) - G(t, s, \cdot)) * f(u)(s, \cdot) ds \right\|_{L^2},$$

where we denote by $S(t, s, x)$ the solution operator to (53), that is,

$$u(t, \cdot) = S(t, s, \cdot) * f(s, \cdot),$$

and we define

$$G(t, s, x) = (G_2(t-s, \cdot) * G_1(\Lambda(t) - \Lambda(s), \cdot))(x).$$

Let us assume $\ell \geq 1/2$. Thanks to Lemma 6 with $\eta = 1$, using (64) and that $p > \bar{p}$, we may estimate

$$\begin{aligned} & \left\| \int_0^t (S(t, s, \cdot) - G(t, s, \cdot)) * f(u)(s, \cdot) ds \right\|_{L^2} \\ & \lesssim \int_0^t o(1+t-s)^{-\frac{n}{8}} \|u(s, \cdot)\|_{L^p \cap L^{2p}}^p ds \\ & \lesssim \int_0^t o(1+t-s)^{-\frac{n}{8}} (1+s)^{-\frac{n}{4}(p-1)} ds = o(1+t)^{-\frac{n}{8}}. \end{aligned}$$

We proceed similarly for $\ell \in (0, 1/2)$, but now, for a sufficiently small δ , we obtain:

$$\begin{aligned}
& \left\| \int_0^t (S(t, s, \cdot) - G(t, s, \cdot)) * f(u)(s, \cdot) ds \right\|_{L^2} \\
& \lesssim \int_0^t (1 + \Lambda(t) - \Lambda(s))^{-\frac{n}{4} - \delta} \|u(s, \cdot)\|_{L^p \cap L^{2p}}^p ds \\
& \lesssim (1 + t)^{\frac{n}{4} + \delta} \int_0^t (1 + t - s)^{-\frac{n}{4} - \delta} (1 + s)^{-\frac{n(1-\ell)}{2}(p-1)} ds \\
& \lesssim (1 + t)^{-(\frac{n}{4} + \delta)(1-\ell)} = o(1 + t)^{-\frac{n}{4}(1-\ell)},
\end{aligned}$$

where we assumed that δ was sufficiently small to verify

$$\frac{n(1-\ell)}{2}(p-1) > 1 + \delta.$$

Now it remains to estimate

$$\left\| \int_0^t G(t, s, \cdot) * f(u(s, \cdot)) ds - G(t, \cdot) \int_0^\infty \int_{\mathbb{R}^n} f(u(s, y)) dy ds \right\|_{L^2}.$$

For $s \geq t/2$ we do not use cancelations. First, let $\ell \geq 1/2$. Then

$$\begin{aligned}
\left\| \int_{t/2}^t G(t, s, \cdot) * f(u(s, \cdot)) ds \right\|_{L^2} & \lesssim (1 + t)^{-\frac{n}{4}(p-1)} \int_{t/2}^t (1 + t - s)^{-\frac{n}{8}} ds \\
& \approx (1 + t)^{-\frac{n}{8} + 1 - \frac{n}{4}(p-1)} = o(t^{-\frac{n}{8}}), \\
\|G(t, \cdot)\|_{L^2} \int_{t/2}^\infty \int_{\mathbb{R}^n} f(u(s, y)) dy ds & \lesssim (1 + t)^{-\frac{n}{8}} \int_{t/2}^\infty (1 + s)^{-\frac{n}{4}(p-1)} ds \\
& \lesssim (1 + t)^{-\frac{n}{8} + 1 - \frac{n}{4}(p-1)} = o(t^{-\frac{n}{8}}).
\end{aligned}$$

We proceed similarly for $\ell \in (0, 1/2)$. In this case, we have

$$\begin{aligned}
\left\| \int_{t/2}^t G(t, s, \cdot) * f(u(s, \cdot)) ds \right\|_{L^2} & \lesssim (1 + \Lambda(t))^{-\frac{n}{2}(p-1)} \int_{t/2}^t (1 + \Lambda(t) - \Lambda(s))^{-\frac{n}{4}} ds \\
& \lesssim (1 + t)^{-\frac{n(1-\ell)}{2}(p-1)} (1 + t)^{\frac{n}{4}\ell} \int_{t/2}^t (1 + t - s)^{-\frac{n}{4}} ds \\
& \approx (1 + t)^{-\frac{n(1-\ell)}{4} + 1 - \frac{n(1-\ell)}{2}(p-1)} \log(1 + t) = o(t^{-\frac{n(1-\ell)}{4}}), \\
\|G(t, \cdot)\|_{L^2} \int_{t/2}^\infty \int_{\mathbb{R}^n} f(u(s, y)) dy ds & \lesssim (1 + \Lambda(t))^{-\frac{n}{4}} \int_{t/2}^\infty (1 + s)^{-\frac{n(1-\ell)}{2}(p-1)} ds \\
& \lesssim (1 + t)^{-\frac{n(1-\ell)}{4} + 1 - \frac{n(1-\ell)}{2}(p-1)} = o(t^{-\frac{n(1-\ell)}{4}}).
\end{aligned}$$

Now we have to estimate

$$\begin{aligned} & \left\| \int_0^{t/2} G(t, s, \cdot) * f(u(s, \cdot)) ds - G(t, \cdot) \int_0^{t/2} \int_{\mathbb{R}^n} f(u(s, y)) dy ds \right\|_{L^2} \\ &= \left\| \int_0^{t/2} \int_{\mathbb{R}^n} (G(t, s, \cdot - y) - G(t, \cdot)) f(u(s, y)) dy ds \right\|_{L^2}. \end{aligned}$$

First, let $\ell \geq 1/2$. For any small $\delta \in (0, 1/2)$, let us define

$$\begin{aligned} \Omega_\delta(t) &= [0, \delta t] \times \left\{ y \in \mathbb{R}^n : |y| \leq \delta t^{\frac{1}{4}} \right\}, \\ \Omega_\delta(t)^- &= ([0, t/2] \times \mathbb{R}^n) \setminus \Omega_\delta(t), \\ \Omega_\delta(t)^c &= ([0, \infty) \times \mathbb{R}^n) \setminus \Omega_\delta(t). \end{aligned}$$

Now we may estimate

$$\int_{\Omega_\delta(t)^-} \|G(t, s, \cdot - y) - G(t, \cdot)\|_{L^2} |u(s, y)|^p dy ds \lesssim t^{-\frac{n}{8}} \int_{\Omega_\delta(t)^-} |u(s, y)|^p dy ds,$$

and, for any fixed $\delta > 0$, the latter integral tends to 0, as $t \rightarrow \infty$. Indeed $u \in L^p([0, \infty) \times \mathbb{R}^n)$, since

$$\|u\|_{L^p([0, \infty) \times \mathbb{R}^n)}^p \lesssim \int_0^\infty (1 + \tau)^{-\frac{n}{4}(p-1)} d\tau \leq C.$$

Therefore, since $\Omega_\delta(t)$ monotonously tends to $[0, \infty) \times \mathbb{R}^n$ as $t \rightarrow \infty$, we get that

$$\int_{\Omega_\delta(t)^-} |u(s, y)|^p dy ds \leq \int_{\Omega_\delta(t)^c} |u(s, y)|^p dy ds \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Now, we cannot use the self-similar profile, as in [24], to deal with $\Omega_\delta(t)$, since our $G(t, s, \cdot)$ is the convolution of two different diffusive-type kernels. However, both G_2 and G_1 are self-similar, that is

$$G_2(t, x) = \rho^{-\frac{n}{4}} G_2(\rho^{-1}t, \rho^{-\frac{1}{4}}x), \quad G_1(t, x) = \rho^{-\frac{n}{2}} G_1(\rho^{-1}t, \rho^{-\frac{1}{2}}x),$$

for any $\rho > 0$. Therefore, setting

$$\begin{aligned} \tilde{G}(t, s, t^{-\frac{1}{4}}(x - y)) &\doteq G_2(1 - s/t, \cdot) * G_1(t^{-\frac{1}{2}}(\Lambda(t) - \Lambda(s)), \cdot)(t^{-\frac{1}{4}}(x - y)), \\ G^\sharp(t, t^{-\frac{1}{4}}x) &\doteq G_2(1, \cdot) * G_1(t^{-\frac{1}{2}}\Lambda(t), \cdot)(t^{-\frac{1}{4}}x), \end{aligned}$$

we obtain

$$\begin{aligned} G(t, s, x - y) &= G_2(t - s, \cdot) * G_1(\Lambda(t) - \Lambda(s), \cdot)(x - y) \\ &= t^{-\frac{n}{4}} \tilde{G}(t, s, t^{-\frac{1}{4}}(x - y)) \\ G(t, x) &= G_2(t, \cdot) * G_1(\Lambda(t), \cdot)(x) = t^{-\frac{n}{4}} G^\sharp(t, t^{-\frac{1}{4}}x). \end{aligned}$$

Now, as a consequence of $s \leq \delta t$ and $|y| \leq t^{\frac{1}{4}} \delta$ in $\Omega_\delta(t)$, we have that

$$s/t \leq \delta, \quad |t^{-\frac{1}{4}} y| \leq \delta, \quad t^{-\frac{1}{2}} \Lambda(s) \leq t^{-\frac{1}{2}} \Lambda(\delta t) \lesssim \delta,$$

where in the last inequality we used the assumption $\ell \geq 1/2$. As a consequence of the continuity, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \sup_{\Omega_\delta(t)} \|G(t, s, \cdot - y) - G(t, \cdot)\|_{L^2} &= t^{-\frac{n}{4}} \sup_{\Omega_\delta(t)} \left\| \tilde{G}(t, s, t^{-\frac{1}{4}}(\cdot - y)) - G^\sharp(t, t^{-\frac{1}{4}} \cdot) \right\|_{L^2} \\ &= t^{-\frac{n}{8}} \sup_{s \leq \delta t, |y| \leq \delta} \left\| \tilde{G}(t, s, \cdot - y) - G^\sharp(t, \cdot) \right\|_{L^2} \leq \epsilon t^{-\frac{n}{8}}, \end{aligned}$$

so that

$$\left\| \int_{\Omega_\delta(t)} (G(t, s, \cdot - y) - G(t, \cdot)) f(u(s, y)) dy ds \right\|_{L^2} \leq \epsilon t^{-\frac{n}{8}} \int_{\Omega_\delta(t)} f(u(s, y)) dy ds \lesssim \epsilon t^{-\frac{n}{8}}.$$

Taking the limit as $\epsilon \rightarrow 0$, we conclude the proof. We proceed similarly for $\ell \in (0, 1/2)$, but now using

$$\begin{aligned} G(t, s, x - y) &= G_2(t - s, \cdot) *_{(x)} G_1(\Lambda(t) - \Lambda(s), \cdot)(x - y) \\ &= G_2((t - s)/\Lambda(t), \cdot) * G_1(1 - \Lambda(s)/\Lambda(t), \cdot)(\Lambda(t)^{-\frac{1}{2}}(x - y)) \\ &=: \tilde{G}(t, s, \Lambda(t)^{-\frac{1}{2}}(x - y)), \\ G(t, x) &= G_2(t/\Lambda(t), \cdot) * G_1(1, \cdot)(\Lambda(t)^{-\frac{1}{2}}x) =: G^\sharp(t, \Lambda(t)^{-\frac{1}{2}}x), \end{aligned}$$

and we conclude the proof. $\square$

7. The case of a null moment condition

The result in Theorem 4 may appear confusing, since we didn't exclude the chance that the moment of the initial data is zero, that is,

$$M_0 = \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) dx = 0.$$

The above condition implies that $\hat{u}_0 + \hat{u}_1$ vanishes at $\xi = 0$. As a consequence, the solution v to problems (49)-(50)-(51) has the property

$$\int_{\mathbb{R}^n} v(t, x) dx = \hat{v}(t, 0) = 0,$$

for any $t \geq 0$. As a matter of fact, we possibly have that the solution v to problems (49)-(50)-(51) is identically zero, in the special case $u_0(x) + u_1(x) = 0$, for any $x \in \mathbb{R}^n$. This suggests that the properties

for the solution u to problems (1) and (21) may be proved independently of the diffusion phenomenon, when the null moment condition $M_0 = 0$ holds.

Indeed, if $M_0 = 0$ and $u_0, u_1 \in L^1$, then we may replace the statement of Lemma 3 (with $\eta = 1$ and $k = 0$) by the following.

Lemma 7. *Let $n \geq 1$ and $\lambda = \lambda(t)$ be as in (7). Assume $(u_0, u_1) \in \mathcal{A} = (H^2 \cap L^1) \times (L^2 \cap L^1)$, be such that*

$$M_0 = \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) dx = 0.$$

Let $u \in \mathcal{C}([0, \infty), H^2) \cap \mathcal{C}^1([0, \infty), L^2)$ be the solution to (1). Then we have the following estimate

$$\|u(t, \cdot)\|_{L^2} = o\left(t^{-\max\{\frac{n}{8}, \frac{n}{4}(1-\ell)\}}\right) \|(u_0, u_1)\|_{\mathcal{A}}, \quad t \rightarrow \infty. \quad (69)$$

Proof of Lemma 7. In view of Lemma 3, it is sufficient to prove that

$$\|v(t, \cdot)\|_{L^2} = o\left(t^{-\max\{\frac{n}{8}, \frac{n}{4}(1-\ell)\}}\right) \|v_0\|_{L^1 \cap L^2}, \quad t \rightarrow \infty, \quad (70)$$

where v is the solution to (51). However, due to

$$v(t, x) = G(t, \cdot) * v_0$$

where $G(t, x)$ is as in (6), we may apply Lemma 4 with

$$\int_{\mathbb{R}^n} v(0, x) dx = 0,$$

as we did in the proof of Theorem 1, and obtain (69). $\square$

By the same reasoning, estimates (9) or, respectively, (11), or (13), remain valid also for the solution to the nonlinear problem (21), when $M_0 = 0$, that is,

$$M = \int_0^\infty \int_{\mathbb{R}^n} |u(t, x)|^p dx dt,$$

in (24). Moreover, for nontrivial data, the above quantity is nonzero. That is, we may determinate the asymptotic profile of the solution to (21) using $G(t, x)$ in (6), even when the moment condition for the initial data in (3) is not verified.

8. Proof of Theorem 6

To prove Theorem 6, we shall now deal with a nonlinearity $\partial_x(|u_x|^{p-1}u_x)$, instead of $|u|^p$.

Proof of Theorem 6. We follow the proof of Theorem 3, but now the Duhamel's operator is

$$Fu(t, x) \doteq \int_0^t K(t, s, x) *_{(x)} \partial_x (|u_x|^{p-1} u_x) ds, \quad (71)$$

where $K(t, s, x)$ is the fundamental solution to (53). Again we shall prove (60) and (61), where

$$X(T) = \mathcal{C}([0, T], H^2) \cap \mathcal{C}^1([0, T], L^2),$$

equipped with the norm in (58), that is,

$$\|u\|_{X(T)} = \max_{t \in [0, T]} \sum_{j+k=0}^1 (1+t)^{j+(\frac{1}{4}+k) \max\{1-\ell, \frac{1}{2}\}} \|\partial_t^j \partial_x^{2k} u(t, \cdot)\|_{L^2}. \quad (72)$$

For any $u \in X(T)$, we may estimate

$$\|u(s, \cdot)\|_{L^2} \lesssim \|u\|_{X(T)} (1+s)^{-\frac{1}{4} \max\{(1-\ell), \frac{1}{2}\}},$$

$$\|u_{xx}(s, \cdot)\|_{L^2} \lesssim \|u\|_{X(T)} (1+s)^{-\frac{5}{4} \max\{(1-\ell), \frac{1}{2}\}},$$

for any $s \in [0, T]$. By Gagliardo-Nirenberg inequality, we also obtain

$$\|u_x(s, \cdot)\|_{L^q} \lesssim \|u\|_{X(T)} (1+s)^{-(1-\frac{1}{2q}) \max\{(1-\ell), \frac{1}{2}\}}, \quad \text{for any } q \in [2, \infty].$$

As a consequence,

$$\begin{aligned} \|\partial_x (|u_x(s, \cdot)|^{p-1} u_x(s, \cdot))\|_{L^1} &\lesssim \|u_{xx}(s, \cdot)\|_{L^2} \|u_x(s, \cdot)\|_{L^{2(p-1)}}^{p-1} \\ &\lesssim \|u\|_{X(T)}^p (1+s)^{-p \max\{(1-\ell), \frac{1}{2}\}}, \\ \|\partial_x (|u_x(s, \cdot)|^{p-1} u_x(s, \cdot))\|_{L^2} &\lesssim \|u_{xx}(s, \cdot)\|_{L^2} \|u_x(s, \cdot)\|_{L^\infty}^{p-1} \\ &\lesssim \|u\|_{X(T)}^p (1+s)^{-(p+\frac{1}{4}) \max\{(1-\ell), \frac{1}{2}\}}. \end{aligned}$$

We first assume $\ell \geq 1/2$. We employ Lemma 5 with $\eta = 1$ for $s \in [0, t/2]$ and Lemma 5 with $\eta = 2$ for $s \in [t/2, t]$, so that, for $j+k=0, 1$, we may estimate

$$\begin{aligned} \|\partial_t^j \partial_x^{2k} Fu(t, \cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{1}{8}-j-\frac{k}{2}} \|\partial_x (|u_x(s, \cdot)|^{p-1} u_x(s, \cdot))\|_{L^1 \cap L^2} ds \\ &\quad + \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} \|\partial_x (|u_x(s, \cdot)|^{p-1} u_x(s, \cdot))\|_{L^2} ds \\ &\lesssim \|u\|_{X(T)}^p \int_0^{t/2} (1+t-s)^{-\frac{1}{8}-j-\frac{k}{2}} (1+s)^{-\frac{p}{2}} ds \end{aligned}$$

$$\begin{aligned}
& + \|u\|_{X(T)}^p \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} (1+s)^{-\frac{p}{2}-\frac{1}{8}} ds \\
& \approx \|u\|_{X(T)}^p (1+t)^{-\frac{1}{8}-j-\frac{k}{2}} \int_0^{t/2} (1+s)^{-\frac{p}{2}} ds \\
& + \|u\|_{X(T)}^p (1+t)^{-\frac{1}{8}-\frac{p}{2}} \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} ds,
\end{aligned}$$

thanks to the property $1+t-s \approx 1+t$ for $s \in [0, t/2]$ and $1+s \approx 1+t$ for $s \in [t/2, t]$. Due to the fact that

$$j + \frac{k}{2} \leq 1 < \frac{p}{2},$$

for $j+k=0, 1$, thanks to the assumption of $p > 2$, the two integrals may be estimated by

$$\begin{aligned}
& \|u\|_{X(T)}^p (1+t)^{-\frac{1}{8}-j-\frac{k}{2}} \int_0^{t/2} (1+s)^{-\frac{p}{2}} ds = C \|u\|_{X(T)}^p (1+t)^{-\frac{1}{8}-j-\frac{k}{2}}, \\
& \|u\|_{X(T)}^p (1+t)^{-\frac{1}{8}-\frac{p}{2}} \int_{t/2}^t (1+t-s)^{-j-\frac{k}{2}} ds \lesssim \|u\|_{X(T)}^p (1+t)^{-\frac{1}{8}-j-\frac{k}{2}}.
\end{aligned}$$

This concludes the proof for $\ell \geq 1/2$. Now let $\ell \in (0, 1/2)$. We employ again Lemma 5 with $\eta = 1$ for $s \in [0, t/2]$ and Lemma 5 with $\eta = 2$ for $s \in [t/2, t]$. Moreover:

- for $s \in [t/2, t]$, we use (56), that is, $\Lambda(t) - \Lambda(s) \gtrsim (t-s)(1+t)^{-\ell}$, and $1+s \approx 1+t$;
- for $s \in [0, t/2]$, we use $\Lambda(t) - \Lambda(s) \approx \Lambda(t)$, and $1+t-s \approx 1+t$.

Therefore, we get

$$\begin{aligned}
& \|\partial_t^j \partial_x^{2k} Fu(t, \cdot)\|_{L^2} \lesssim (1+t)^{-j-(1-\ell)(\frac{1}{4}+k)} \int_0^{t/2} \|\partial_x(|u_x(s, \cdot)|^{p-1} u_x(s, \cdot))\|_{L^1 \cap L^2} ds \\
& + (1+t)^{k\ell} \int_{t/2}^t (1+t-s)^{-j-k} \|\partial_x(|u_x(s, \cdot)|^{p-1} u_x(s, \cdot))\|_{L^2} ds \\
& \lesssim \|u\|_{X(T)}^p (1+t)^{-j-(1-\ell)(\frac{1}{4}+k)} \int_0^{t/2} (1+s)^{-(1-\ell)p} ds \\
& + \|u\|_{X(T)}^p (1+t)^{k\ell-(1-\ell)(\frac{1}{4}+p)} \int_{t/2}^t (1+t-s)^{-j-k} ds.
\end{aligned}$$

Due to the fact that

$$j+k \leq 1 < (1-\ell)p,$$

for $j+k=0, 1$, thanks to the assumption of $p \geq 2 > (1-\ell)^{-1}$, the both two integrals may be estimated by

$$\|u\|_{X(T)}^p (1+t)^{-j-(1-\ell)(\frac{1}{4}+k)}.$$

In order to prove (61), we need more computation than in Theorem 3. We first write

$$\begin{aligned}\partial_x(|u_x|^{p-1}u_x) - \partial_x(|v_x|^{p-1}v_x) &= p|u_x|^{p-1}u_{xx} - p|v_x|^{p-1}v_{xx} \\ &= p|u_x|^{p-1}(u_{xx} - v_{xx}) + pv_{xx}(|u_x|^{p-1} - |v_x|^{p-1}),\end{aligned}$$

then we use the inequality

$$||u_x|^{p-1} - |v_x|^{p-1}| \lesssim |u_x - v_x| (|u_x|^{p-2} + |v_x|^{p-2}),$$

valid for $p \geq 2$, and the Hölder inequality to get

$$\begin{aligned}\|\partial_x(|u_x(s, \cdot)|^{p-1}u_x(s, \cdot)) - \partial_x(|v_x(s, \cdot)|^{p-1}v_x(s, \cdot))\|_{L^1} \\ \lesssim \|u_{xx}(s, \cdot) - v_{xx}(s, \cdot)\|_{L^2} \|u_x(s, \cdot)\|_{L^{2(p-1)}}^{p-1} \\ + \|v_{xx}(s, \cdot)\|_{L^2} \|u_x(s, \cdot) - v_x(s, \cdot)\|_{L^{2(p-1)}} (\|u_x(s, \cdot)\|_{L^{2(p-1)}}^{p-2} + \|v_x(s, \cdot)\|_{L^{2(p-1)}}^{p-2}),\end{aligned}$$

and similarly for the L^2 norm. We then prove (61), as we did in the proof of Theorem 5.

To prove the asymptotic profile, we shall estimate the nonlinear contribution to the solution for (28). We follow the proof of Theorem 4, but now $f = f(u_x) = \partial_x(|u_x|^{p-1}u_x)$. Let $\ell \geq 1/2$. Using $p > 2$, we easily obtain

$$\begin{aligned}\left\| \int_0^t (S(t, s, \cdot) - G(t, s, \cdot)) * f(u_x)(s, \cdot) ds \right\|_{L^2} \\ \lesssim \int_0^t o(1+t-s)^{-\frac{1}{8}} (1+s)^{-\frac{p}{2}} ds = o(1+t)^{-\frac{1}{8}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\| \int_{t/2}^t G(t, s, \cdot) * f(u_x(s, \cdot)) ds \right\|_{L^2} &\lesssim (1+t)^{-\frac{p}{2}} \int_{t/2}^t (1+t-s)^{-\frac{1}{8}} ds = o(t^{-\frac{1}{8}}), \\ \|G(t, \cdot)\|_{L^2} \int_{t/2}^\infty \int_{\mathbb{R}} f(u_x(s, y)) dy ds &\lesssim (1+t)^{-\frac{1}{8}} \int_{t/2}^\infty (1+s)^{-\frac{p}{2}} ds = o(t^{-\frac{1}{8}}).\end{aligned}$$

We notice that $\partial_x(|u_x|^{p-1}u_x) \in L^1([0, \infty) \times \mathbb{R})$, due to

$$\|\partial_x(|u_x|^{p-1}u_x)\|_{L^1([0, \infty) \times \mathbb{R})} \lesssim \int_0^\infty (1+\tau)^{-\frac{p}{2}} d\tau \leq C,$$

and this implies that the integral in the right-hand side in

$$\begin{aligned}\int_{\Omega_\delta(t)^-} \|G(t, s, \cdot - y) - G(t, \cdot)\|_{L^2} |\partial_x(|u_x(s, y)|^{p-1}u_x(s, y))| dy ds \\ \lesssim t^{-\frac{1}{8}} \int_{\Omega_\delta(t)^-} |\partial_x(|u_x(s, y)|^{p-1}u_x(s, y))| dy ds,\end{aligned}$$

tends to 0, as $t \rightarrow \infty$. The rest of the proof is as in the proof of Theorem 4. $\square$

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