



Partially concentrating standing waves for weakly coupled Schrödinger systems

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Received: 13 November 2023 / Revised: 15 February 2024 / Accepted: 28 February 2024 /

Published online: 29 March 2024

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Abstract

We study the existence of standing waves for the following weakly coupled system of two Schrödinger equations

$$\begin{cases} i\hbar\partial_t\psi_1 = -\frac{\hbar^2}{2m_1}\Delta\psi_1 + V_1(x)\psi_1 - \mu_1|\psi_1|^2\psi_1 - \beta|\psi_2|^2\psi_1 \\ i\hbar\partial_t\psi_2 = -\frac{\hbar^2}{2m_2}\Delta\psi_2 + V_2(x)\psi_2 - \mu_2|\psi_2|^2\psi_2 - \beta|\psi_1|^2\psi_2, \end{cases}$$

where V_1 and V_2 are radial potentials bounded from below. We address the case $m_1 \sim \hbar^2 \rightarrow 0$, m_2 constant, and prove the existence of a standing wave solution with both nontrivial components satisfying a prescribed asymptotic profile. In particular, the second component of such solution exhibits a concentrating behavior, while the first one keeps a quantum nature.

Mathematics Subject Classification 35B25 · 35J47 · 35Q55

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1 Introduction

The mathematical analysis of singularly perturbed semilinear elliptic equations and systems has been the object of a wide range of studies in the last decades. Among the many motivations, a big role is provided by models in Quantum Mechanics, and in particular by the semiclassical analysis of Schrödinger-type equations. In this context, one postulates that the classical Newtonian Mechanics should be recovered from the Quantum one by letting the Planck constant \hbar vanish. Accordingly, the wave function of the quantum particle should concentrate and collapse to one or more Dirac's deltas, which position should describe the sharp location of classical particles. When different quantum waves interact, e.g. in the case of weakly coupled NLS systems, the commonly investigated setting is the one in which all the waves concentrate in point particles. From the analytical point of view, this study lets different challenges arise. On the one hand, one may ask what is the limit concentrating profile at specific energy levels, for instance for ground states; this is typically done exploiting variational methods and blow-up analysis. On the other hand, solutions concentrating with prescribed shape and position can be constructed, mainly using the Lyapunov–Schmidt reduction approach.

A largely studied model is the case of a binary mixture of Bose–Einstein condensates, usually described by the Gross–Pitaevskii system, namely a systems of two weakly coupled nonlinear Schrödinger equations

$$\begin{cases} i\hbar\partial_t\psi_1 = -\frac{\hbar^2}{2m_1}\Delta\psi_1 + V_1(x)\psi_1 - \mu_1|\psi_1|^2\psi_1 - \beta|\psi_2|^2\psi_1 & \text{in } \mathbb{R}^N \\ i\hbar\partial_t\psi_2 = -\frac{\hbar^2}{2m_2}\Delta\psi_2 + V_2(x)\psi_2 - \mu_2|\psi_2|^2\psi_2 - \beta|\psi_1|^2\psi_2 & \text{in } \mathbb{R}^N, \end{cases}$$

for $N = 1, 2, 3$. Here, ψ_1 and ψ_2 are the order parameters of the two components of the mixture, m_1 and m_2 the corresponding masses, and \tilde{V} and \tilde{W} the external potentials, bounded from below. In general, the (trapping) potentials may be different, opening interesting possibilities concerning the geometrical configurations of the condensates. Finally, the interaction parameters μ_1, μ_2, β depend on the scattering lengths associated to the different states. These interaction parameters can be fine-tuned across a wide range of values, profiting from the presence of a Feshbach resonance. For more details on this model we refer to the book by Pitaevskii and Stringari [21], in particular Chaps. 5 and 21.

Looking for standing waves $(\psi_1(x, t), \psi_2(x, t)) = (e^{iE_1t/\hbar}u(x), e^{iE_2t/\hbar}v(x))$ of frequencies E_i we have that (u, v) solves

$$\begin{cases} -\frac{\hbar^2}{2m_1}\Delta u + V(x)u = \mu_1u^3 + \beta v^2u & \text{in } \mathbb{R}^N, \\ -\frac{\hbar^2}{2m_2}\Delta v + W(x)v = \mu_2v^3 + \beta u^2v & \text{in } \mathbb{R}^N, \end{cases} \quad (1)$$

with $V(x) = E_1 + \tilde{V}(x)$ and $W(x) = E_2 + \tilde{W}(x)$, for E_i such that $\inf_{\mathbb{R}^N} V > 0$ and $\inf_{\mathbb{R}^N} W > 0$. The usually studied case is the one in which μ_i, β are both very large, or \hbar is very small with respect to the other parameters, so that one is lead to consider the singularly perturbed elliptic system

$$\begin{cases} -\varepsilon^2 \Delta u + V_1(x)u = \mu_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V_2(x)v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N. \end{cases} \tag{2}$$

While the autonomous case, namely the case $V_i \equiv \lambda_i$ with λ_i positive constants, has been widely studied in the last two decades, (see the recent paper [25] for an exhaustive list of references), a few results concerning the non-autonomous situation are known. The first result seems due to Lin and Wei [11] who studied the case of a binary mixture in a singularly perturbed regime and in presence of trapping potentials. They prove the existence of ground state solutions, derive their asymptotic behaviors as $\varepsilon \rightarrow 0$ and show that each component has one maximum point (possibly the same), called *spike*, which is trapped at the minimum points of the potentials V_i . The existence of a concentrating ground state solutions was also established by Montefusco, Pellacci and Squassina [15], Pomponio [22], Ikoma and Tanaka [7] and Byeon [4]. All the previous papers are concerned with system (2) where both the equations are affected by the presence of the small parameter ε , so that every wave (given by the components of the vector solution) concentrates as ε approaches zero.

Here, we focus on another type of regime, as it may happen that only some of the waves act in a semiclassical way, while the others persist in a quantum behavior. To the best of our knowledge, this kind of analysis is not present in the PDEs literature yet, and this paper is a first contribution in this direction.

More precisely, in our study we consider $2m_1 = \hbar^2$ and $m_2 = \frac{1}{2}$ in (1), so that (u, v) solves the following elliptic weakly coupled system

$$\begin{cases} -\Delta u + V(x)u = \mu_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + W(x)v = \mu_2 v^3 + \beta vu^2 & \text{in } \mathbb{R}^N, \end{cases} \tag{3}$$

where $\varepsilon^2 := \hbar^2$. According to the previous discussion, along this paper we deal with the system above in the singularly perturbed regime $\varepsilon \rightarrow 0$. Moreover, our study will deal with the case of $\mu_i > 0$ and $\beta < 0$, corresponding to positive intraspecies and to a negative interspecies scattering length, describing a repulsive interaction between the condensates.

Our analysis will include the class of potentials satisfying the following assumptions.

(V1) $V(x) = V(|x|)$ is a radially symmetric function satisfying

$$V \in C^0(\mathbb{R}^N) \text{ and } \inf_{\mathbb{R}^N} V(x) > 0. \tag{4}$$

Moreover, V is such that there exists $\Upsilon \in C^3(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ unique positive radial solution to

$$\begin{cases} -\Delta \Upsilon + V(x)\Upsilon = \mu_1 \Upsilon^3 \\ \Upsilon(x) \rightarrow 0 \text{ as } |x| \rightarrow +\infty, \end{cases} \tag{5}$$

which is non-degenerate in the space $H_e^1(\mathbb{R}^N)$ of functions even with respect to each variables, i.e.

$$H_e^1(\mathbb{R}^N) := \left\{ u \in H^1(\mathbb{R}^N) : u(x_1, \dots, x_i, \dots, x_N) = u(x_1, \dots, -x_i, \dots, x_N), i = 1, \dots, N \right\} \tag{6}$$

that is the only solutions to

$$\begin{cases} -\Delta z + V(x)z = 3\mu_1 \Upsilon^2 z \text{ in } \mathbb{R}^N \\ z \in H_e^1(\mathbb{R}^N) \end{cases}$$

is the trivial one.

(V₂) The potential V is such that for every $f \in L^m(\mathbb{R}^N)$, $2 \leq m < +\infty$, there exists a unique solution $u \in W^{2,m}(\mathbb{R}^N)$ of the equation

$$-\Delta u + V(x)u = f,$$

and

$$\|u\|_{W^{2,m}(\mathbb{R}^N)} \lesssim \|f\|_{L^m(\mathbb{R}^N)}.$$

(W) $W(x)$ is even with respect to all the variables,

$$W \in C^3(\mathbb{R}^N) \text{ and } \inf_{\mathbb{R}^N} W(x) > 0. \tag{7}$$

Our main result is stated as follows.

Theorem 1.1 *Let $N = 2, 3$, and suppose that (V₁), (V₂) and (W) hold. Assume $\beta < 0$. Let*

$$\omega(x) := W(x) - \beta^2 \Upsilon(x), \tag{8}$$

set $\omega_0 := \omega(0) > 0$ and assume that

$$\frac{\partial^2 \omega}{\partial x_1^2}(0) < 0. \tag{9}$$

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a solution $(u_\varepsilon, v_\varepsilon)$ of system (3) even with respect to each variable and having the following asymptotic profile as $\varepsilon \rightarrow 0$

$$u_\varepsilon(x) \sim \Upsilon(x), \quad v_\varepsilon(x) \sim \sqrt{\frac{\omega_0}{\mu_2}} \left[U\left(\omega_0 \frac{x - P_\varepsilon}{\varepsilon}\right) + U\left(\omega_0 \frac{x + P_\varepsilon}{\varepsilon}\right) \right], \tag{10}$$

where Υ solves (5), U is the positive radial solution of

$$-\Delta U + \omega_0 U = \mu_2 U^3 \text{ in } \mathbb{R}^N \tag{11}$$

and the peaks P_ε and $-P_\varepsilon$ collapse to the origin as

$$\pm P_\varepsilon = \rho_\varepsilon(\pm 1, 0, \dots, 0), \text{ with } \lim_{\varepsilon \rightarrow 0^+} \frac{\rho_\varepsilon}{\varepsilon \ln(1/\varepsilon)} = \frac{1}{\sqrt{\omega_0}}. \tag{12}$$

Theorem 1.1 states the existence of a solution whose first component looks like a genuine solution to (5), in particular it does not concentrate, and whose second component concentrates at two opposite points which collapse to the origin as ε goes to 0. As a consequence of the coupling in the equations, the first component of (3) plays the role of an additional potential in the singularly perturbed second equation, so that the concentration will be triggered by the *modified potential* $W - \beta\Upsilon^2$. Because of the assumption $\beta < 0$, we will obtain a solution in the repulsive regime, and, when the origin is a maximum point of W our solution exists for every β negative; while when $\partial_{11}^2 W(0) > 0$ we obtain a solution for $\beta < \beta_0 < 0$ (see (14)).

Let us make some comments.

Remark 1.2 We point out that, in case Υ satisfies

$$\Delta\Upsilon(0) \neq 0, \tag{13}$$

then assumption (9) is satisfied as long as

$$\beta < -\frac{\frac{\partial^2 W}{\partial x_1^2}(0)}{2\Upsilon(0)|\frac{\partial^2 \Upsilon}{\partial x_1^2}(0)|}. \tag{14}$$

On the other hand, assumption (13) is verified in case V is radially non-decreasing near 0 and Υ is radial and has a (local) strict maximum at 0, as one can verify applying Hopf’s Lemma. Indeed, first we observe that, in such a case, for any i the function $\partial_i \Upsilon$ solves

$$\Delta \partial_i \Upsilon - V \partial_i \Upsilon = -3\Upsilon^2 \partial_i \Upsilon + \Upsilon \partial_i V \geq 0 \text{ in } \Omega_i := \{|x| \leq r_0, x_i \geq 0\},$$

for r_0 small, because $\partial_i \Upsilon \leq 0$ and $\partial_i V \geq 0$ in Ω_i . Moreover

$$0 = \partial_i \Upsilon(0) = \max_{\Omega_i} \partial_i \Upsilon.$$

By Hopf’s Lemma we deduce $\partial_{ii} \Upsilon(0) < 0$ and (13) follows.

Remark 1.3 It is useful to recall the classical results concerning the case of constant potential, i.e.

$$-\Delta U + \lambda U = \mu U^3 \text{ in } \mathbb{R}^N, U \in H^1(\mathbb{R}^N). \tag{15}$$

It is well known that (15) has an unique positive solution which is radially symmetric and also that the set of solution of the corresponding linearized equation

$$-\Delta z + \lambda z = 3\mu U^2 z \text{ in } \mathbb{R}^N, \quad z \in H^1(\mathbb{R}^N)$$

is spanned by the N partial derivatives $\frac{\partial U}{\partial x_i}$ which are odd in each variable. In addition, U is radially decreasing and it satisfies the following exponential decay (see [2, 3, 9])

$$\lim_{|x| \rightarrow \infty} U(x) e^{\sqrt{\lambda}|x|} |x|^{\frac{N-1}{2}} = C_0 > 0, \quad \lim_{|x| \rightarrow \infty} \frac{U'(x)}{U(x)} = -1. \quad (16)$$

Remark 1.4 We observe that a class of potentials V which satisfy hypotheses (\mathbf{V}_1) and (\mathbf{V}_2) includes both the constant potentials and, at least in dimension $N = 3$, the trapping ones, like $V(x) = \lambda + |x|^m$ for some $\lambda > 0$ and $m > 0$. Indeed, (\mathbf{V}_1) follows from Remarks 1.2, 1.3 and [5, Corollary 1.5 and Appendix A]. On the other hand, (\mathbf{V}_2) follows by standard elliptic theory in the case of constant potentials (see e.g. [24, Thm. 3, p. 135]), while for trapping ones it is a consequence of [16, 17, 23] (as we mentioned, [23] deals only with dimensions $N \geq 3$; we believe that a version of such results should hold also in lower dimension, but this is far beyond the aim of this paper).

Remark 1.5 Our result relies on the symmetry of the potentials V and W which allows to build symmetric solutions with symmetric peaks P_ε and $-P_\varepsilon$ collapsing to the origin. We strongly believe that a similar construction could be carried out in a more general setting in the spirit of Kang and Wei [8], when the radial solution of the first equation (5) is non-degenerate in the whole space $H^1(\mathbb{R}^N)$ (i.e. V is a trapping potential as in [5, 23]). In that case, it should be possible to build a solutions whose first component resembles the radial solution Υ of (5) and the second component has two different peaks collapsing to a maximum point of the modified potential $W - \beta\Upsilon^2$.

Remark 1.6 We will prove Theorem 1.1 using a classical Lyapunov–Schmidt reduction. This will allow us to build each component with a prescribed profile: the first component will look as one bump solution for ε sufficiently small, while the second will develop two spikes collapsing at the origin and will be exponentially small far from them. In performing this classical procedure, we faced some new difficulties. First of all, in view of the square growth of the coupling term, we need to correct the ansatz of both the components to detect the suitably reduced problem. Moreover, the use of the regularity theory will be crucial in order to make suitable expansion of all the terms involved in the construction.

Our existence result does not cover the case $N = 1$, as in this case the size of the error term does not produce the suitable smallness of the reminders terms in the ansatz despite of the presence of the correction term. We think that this point could be managed introducing further correction terms, again in both the equations, which at the prices of heavy technicality should allow to construct a remaining term sufficiently small.

Remark 1.7 Our result deals with the case of a binary mixture and it is natural to ask if our construction can be extended to the case of a larger number of equations, i.e.

$$\begin{cases} -\Delta u + V(x)u = \mu_1 u^3 + u \sum_{j=1}^k \beta_j v_j^2 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v_j + W_j(x)v_j = \mu_j v_j^3 + v_j \left(\beta_j u^2 + \sum_{i \neq j}^k \beta_i v_i^2 \right) & \text{in } \mathbb{R}^N, \quad j = 2, \dots, k. \end{cases}$$

In particular, we wonder if it is possible to build a solution whose components v_j concentrate at different pairs of points $(P_j^\varepsilon, -P_j^\varepsilon)$ collapsing to the origin as ε goes to zero.

Remark 1.8 The unperturbed version of system (2) (let us say $\varepsilon = 1$) was firstly studied by Peng and Wang [19] who (in presence of radial potentials) constructed an unbounded sequence of non-radial solutions exhibiting an arbitrarily large number of peaks. Their result has been successively extended to the case of more than two equations by Pistoia and Vaira [20] and very recently by Li, Wei and Wu [10].

We wonder if it is possible, by combining the ideas used in the above papers, to produce a solution to system (3) with $\varepsilon = 1$ whose first component looks like the solution to (5) and second component concentrates at an arbitrary large number of points approaching infinity as ε goes to zero.

Remark 1.9 Let us finally observe that the existence of solutions to the system (2) is closely related to the study of the normalized solutions for nonlinear Schrödinger systems. We refer the reader to the recent papers by Lu [14], Liu and Yang [13], Guo and Xie [6] and Liu and Tian [12]. In particular, it would be interesting to produce normalized solutions using as a parameter their L^2 -norms, in the spirit of the results obtained by Pellacci, Pistoia, Vaira and Verzini [18].

The paper is organized as follows. In the next section we set the problem, by introducing the the main blocks of our construction and by reformulating problem (3) as a system of two equations, one set in an infinite dimensional set, the other, called the reduced problem, in a finite dimensional one. In Sect. 3 we solve the the infinite dimensional equation. Finally, in Sect. 4 we study the reduced problem and we complete the proof of Theorem 1.1.

2 Setting of the problem

Let us introduce the Banach spaces

$$\begin{aligned} H_V^2 &:= \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < +\infty \right\}, \\ H_{W_\varepsilon}^2 &:= \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} W(\varepsilon x)u^2 < +\infty \right\}. \end{aligned} \tag{17}$$

equipped with the norms

$$\|u\|_V := \left(\int_{\mathbb{R}^N} \sum_{|\alpha|=2} |D^\alpha u|^2 + \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V(x)u^2 \right)^{\frac{1}{2}}$$

and

$$\|u\|_\varepsilon := \left(\int_{\mathbb{R}^N} \sum_{|\alpha|=2} |D^\alpha u|^2 + \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} W(\varepsilon)(x)u^2 \right)^{\frac{1}{2}}$$

Henceforth, we omit the subscript ε in u, v and we agree that $a \lesssim b$ means $|a| \leq c|b|$ for some constant c which does not depend on a and b .

Performing a change of variable in the second equation, we are lead to seek a solution (u, v) of

$$\begin{cases} -\Delta u + V(x)u = \mu_1 u^3 + \beta uv^2 \left(\frac{x}{\varepsilon}\right) & \text{in } \mathbb{R}^N, \\ -\Delta v + W(\varepsilon x)v = \mu_2 v^3 + \beta u^2(\varepsilon x)v & \text{in } \mathbb{R}^N. \end{cases} \tag{18}$$

in the space

$$X = \{(u, v) \in H_V^2 \times H_{W_\varepsilon}^2 \mid u, v \text{ are even functions (see (6))}\}.$$

In the next subsection, we introduce the main building blocks in the construction of our solution.

2.1 The ansatz and the correction terms

In view of assumptions (V_1) and (V_2) , we can consider Υ the solution of

$$-\Delta \Upsilon + V(x)\Upsilon = \mu_1 \Upsilon^3 \text{ in } \mathbb{R}^N \tag{19}$$

and U be the solution of

$$-\Delta U + \omega_0 U = \mu_2 U^3 \text{ in } \mathbb{R}^N, \tag{20}$$

where, since $\beta < 0$

$$\omega_0 := W(0) - \beta \Upsilon^2(0) > 0.$$

We look for a solution (u, v) of (18) of the form

$$u(x) = \underbrace{\Upsilon(x) + \beta \Phi_\varepsilon(x)}_{=: \Xi_\varepsilon(x)} + \varphi(x), \quad v(x) = \underbrace{U_\varepsilon(x) + \beta \Psi_\varepsilon(x)}_{=: \Theta_\varepsilon(x)} + \psi(x). \tag{21}$$

where

$$U_\varepsilon(x) := U\left(x - \frac{P_\varepsilon}{\varepsilon}\right) + U\left(x + \frac{P_\varepsilon}{\varepsilon}\right) = U_{-P_\varepsilon}(x) + U_{P_\varepsilon}(x)$$

and the concentration points satisfy (see (12))

$$\begin{aligned} P_\varepsilon &= \rho_\varepsilon P_0 = \rho_\varepsilon(1, 0, \dots, 0), \quad \rho_\varepsilon \\ &= d\varepsilon \ln(1/\varepsilon), \quad d \in \left(\frac{1}{\sqrt{\omega_0}} - \delta, \frac{1}{\sqrt{\omega_0}} + \delta\right) \text{ for some } \delta > 0. \end{aligned} \tag{22}$$

The function Φ_ε and Ψ_ε , are suitable correction terms, whose existence and properties are established in Lemma 2.1, 2.3. The remainder terms φ and ψ belong to the space

$$K^\perp := \left\{ (\varphi, \psi) \in X \mid \int_{\mathbb{R}^N} \psi(x) Z_\varepsilon(x) dx = 0 \right\},$$

where $K := \text{span}\{(0, Z_\varepsilon)\} \subset H^2$ and

$$Z_\varepsilon(x) := \partial_1 U\left(x + \frac{P_\varepsilon}{\varepsilon}\right) - \partial_1 U\left(x - \frac{P_\varepsilon}{\varepsilon}\right), \tag{23}$$

solves the linear equation

$$-\Delta Z_\varepsilon + \omega_0 Z_\varepsilon = 3\mu_2 \left(U_{P_\varepsilon}^2 \partial_{x_1} U_{P_\varepsilon} - U_{-P_\varepsilon}^2 \partial_{x_1} U_{-P_\varepsilon} \right). \tag{24}$$

Moreover, it is worthwhile to point out that all the functions Φ_ε , U_ε and Z_ε are even functions.

In the following we introduce the two correction terms we need in our construction of the solution. Let us start from the term in the first component.

Lemma 2.1 *There exists a unique even $\Phi_\varepsilon \in H_V^2(\mathbb{R}^N)$ solution of the equation*

$$-\Delta \Phi_\varepsilon + \left(V(x) - 3\mu_1 \Upsilon^2(x) \right) \Phi_\varepsilon = \Upsilon(x) U_\varepsilon^2 \left(\frac{x}{\varepsilon} \right). \tag{25}$$

Moreover, Φ_ε satisfies $\|\Phi_\varepsilon\|_{W^{2,m}(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{m}}$ and $\|\Phi_\varepsilon\|_{C^{1,1-\frac{N}{m}}(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{m}}$ for any $m \geq 2$.

Proof By exploiting assumptions $(V_1) - (V_2)$ we deduce that for any even function $f \in L^2(\mathbb{R}^N)$ the problem

$$-\Delta \Phi + \left(V(x) - 3\mu_1 \Upsilon^2(x) \right) \Phi = f \text{ in } \mathbb{R}^N$$

has a unique even solution Φ such that $\|\Phi\|_V \leq c\|f\|_{L^2(\mathbb{R}^N)}$, for some constant c which does not depend on f . Now we point that the function $f(x) = \Upsilon(x) U_\varepsilon^2 \left(\frac{x}{\varepsilon} \right)$ is

an even function with

$$\left\| \Upsilon U_\varepsilon^2 \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{2}}.$$

Indeed, by scaling $x = \varepsilon y$ we immediately deduce

$$\int_{\mathbb{R}^N} \Upsilon^2(x) U_\varepsilon^4 \left(\frac{x}{\varepsilon} \right) dx \lesssim \int_{\mathbb{R}^N} \Upsilon^2(x) \left(U^4 \left(\frac{x - P_\varepsilon}{\varepsilon} \right) + U^4 \left(\frac{x + P_\varepsilon}{\varepsilon} \right) \right) dx \lesssim \varepsilon^N. \tag{26}$$

As a direct consequence, we infer

$$\|\Phi_\varepsilon\|_{H^2(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{2}}.$$

This implies that $\Phi_\varepsilon \in C^{0, \frac{1}{2}}(\mathbb{R}^N)$ and $\|\Phi_\varepsilon\|_{C^{0, \frac{1}{2}}(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{2}}$.

Now, we write (25) as

$$-\Delta \Phi + V(x)\Phi = \underbrace{-3\mu_1 \Upsilon^2(x)\Phi(x)}_{=f_1(x)} + \underbrace{\Upsilon(x)U_\varepsilon \left(\frac{x}{\varepsilon} \right)}_{=f_2(x)} \text{ in } \mathbb{R}^N$$

and we observe that, reasoning as in (26), we deduce that

$$\|f_1\|_{L^m(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{2}} \text{ and } \|f_2\|_{L^m(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{m}}, \text{ for any } m \geq 2;$$

then, assumptions (V₂) implies,

$$\|\Phi_\varepsilon\|_{W^{2,m}(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{m}},$$

so that, choosing $m > N$, Sobolev embedding $W^{2,m}(\mathbb{R}^N) \hookrightarrow C^{1, 1-\frac{N}{m}}(\mathbb{R}^N)$ yields the claim. □

Remark 2.2 It is useful to remark that by Lemma (2.1) since $\nabla \Phi_\varepsilon(0) = 0$ we deduce

$$|\Phi_\varepsilon(y) - \Phi_\varepsilon(0)| \lesssim |y|^{2-\frac{N}{m}} \|\Phi_\varepsilon\|_{C^{1, 1-\frac{N}{m}}(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{m}} |y|^{2-\frac{N}{m}} \tag{27}$$

Moreover, since $\nabla \Upsilon(0) = 0$ we also have

$$|\Upsilon(y) - \Upsilon(0)| \lesssim |y|^2 \|\Upsilon\|_{C^2(\mathbb{R}^N)} \lesssim |y|^2. \tag{28}$$

Lemma 2.3 *There exists a unique $\Psi_\varepsilon \in H^2(\mathbb{R}^N)$, solution of the equation*

$$-\Delta \Psi_\varepsilon + \left(\omega_0 - 3\mu_2 \left(U_{P_\varepsilon}^2(x) + U_{-P_\varepsilon}^2(x) \right) \right) \Psi_\varepsilon = 2\beta \Phi_\varepsilon(0) \Upsilon(0) U_\varepsilon(x). \tag{29}$$

Moreover, $\|\Psi_\varepsilon\|_{H^2(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{2}}$ and there exist $B, \gamma, R_0 > 0$ such that

$$|\Psi_\varepsilon(x)| \lesssim \varepsilon^{N/2} \left(e^{-\gamma|x+\frac{P_\varepsilon}{\varepsilon}|} + e^{-\gamma|x-\frac{P_\varepsilon}{\varepsilon}|} \right), \quad \forall |x| \geq R_0.$$

Proof As U is radial, there exists a unique Ψ radial solution to

$$-\Delta \Psi + \left(\omega_0 - 3\mu_2 U^2(x) \right) \Psi = U(x), \tag{30}$$

then the function Ψ_ε defined as

$$\Psi_\varepsilon(x) = 2\beta \Phi_\varepsilon(0) \Upsilon(0) \left[\Psi \left(x + \frac{P_\varepsilon}{\varepsilon} \right) + \Psi \left(x - \frac{P_\varepsilon}{\varepsilon} \right) \right]$$

solves (29). The regularity properties of Ψ are consequence of the regularity properties of U , while the bound from above of the $H^2(\mathbb{R}^N)$ norm follows from the upper bound on the $L^\infty(\mathbb{R}^N)$ norm of Φ_ε .

The exponential decay of Ψ_ε will follow from the analogous decay of Ψ . In order to prove this property, let us first show that there exists $R > 1$ such that $\Psi(r) \geq 0$ for every $r > R$.

By contradiction there exists a sequence $r_n \rightarrow +\infty$ of minimum point at a negative level for Ψ . Then

$$\begin{aligned} 0 > -\Psi_{rr}(r_n) - \frac{N-1}{r} \Psi_r(r_n) &= -\omega_0 \Psi(r_n) + 3\mu_2 U^2(r_n) \Psi(r_n) + U(r_n) \\ &\geq -\Psi(r_n) \left(\omega_0 - 3\mu_2 U^2(r_n) \right) \geq 0 \end{aligned}$$

as soon as r_n is sufficiently large so that the parenthesis is positive.

Let us now fix $A > 0$ such that $3\mu_2 U \Psi + 1 \leq A$ and let $v(r) = B e^{-\gamma r}$ with $\gamma^2 < \omega_0$. Then $w = v - \Psi$ solves

$$\begin{aligned} -\Delta w + \omega_0 w &= v \left(\omega_0 - \gamma^2 + \frac{N-1}{r} \gamma \right) - U(3\mu_2 \Psi U + 1) \\ &\geq v \left(\omega_0 - \gamma^2 + \frac{N-1}{r} \gamma \right) - AU \\ &\geq B e^{-\gamma r} \left(\omega_0 - \gamma^2 + \frac{N-1}{r} \gamma \right) - A e^{-\sqrt{\omega_0} r} \\ &= e^{-\gamma r} \left[B \left(\omega_0 - \gamma^2 + \frac{N-1}{r} \gamma \right) - A e^{-(\sqrt{\omega_0} - \gamma)r} \right] > 0 \end{aligned}$$

as $\gamma^2 < \omega_0$ for $r > R_1$ sufficiently large. In addition, we can choose B such that $B e^{-\gamma r} \geq \max_{|x|=R_1} \Psi$, then the maximum principle yields $0 \leq \Psi \leq v$ for $|x| > R_1$. \square

Remark 2.4 Let us point out that assumptions (\mathbf{V}_1) and (\mathbf{V}_2) are satisfied by constant potentials as well as by potentials of the type $V(x) = |x|^m$ with $m > 0$ as shown in [5, 23].

Remark 2.5 As a consequence of Sobolev embedding, the bounds from above stated in Lemma 2.1 hold for Ψ_ε as well.

2.2 Rewriting the problem

Let us introduce the orthogonal projections

$$\tilde{\Pi} : L^2(\mathbb{R}^N) \mapsto \tilde{K}, \quad \text{and} \quad \tilde{\Pi}^\perp : L^2(\mathbb{R}^N) \mapsto \tilde{K}^\perp$$

where

$$\begin{aligned} \tilde{K} &:= \text{span}\{(0, Z_\varepsilon)\} \subset L^2, \\ \tilde{K}^\perp &:= \left\{ (f, g) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} g Z_\varepsilon dx = 0 \right\}. \end{aligned}$$

Plugging the ansatz $u = \Xi_\varepsilon + \varphi$ and $v = \Theta_\varepsilon + \psi$ (see (21)) into (18), one obtains the following equivalent system

$$\begin{cases} \tilde{\Pi} \{ \mathcal{L}(\varphi, \psi) - \mathcal{E} - \mathcal{N}(\varphi, \psi) \} = 0 \\ \tilde{\Pi}^\perp \{ \mathcal{L}(\varphi, \psi) - \mathcal{E} - \mathcal{N}(\varphi, \psi) \} = 0 \end{cases} \tag{31}$$

Here the linear operator $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ is defined by

$$\begin{aligned} \mathcal{L}_1(\varphi, \psi) &:= -\Delta\varphi + V(x)\varphi - \left(3\mu_1 \Xi_\varepsilon^2 + \beta \Theta_\varepsilon^2 \left(\frac{x}{\varepsilon} \right) \right) \varphi - 2\beta \Theta_\varepsilon \left(\frac{x}{\varepsilon} \right) \Xi_\varepsilon \psi \left(\frac{x}{\varepsilon} \right) \\ \mathcal{L}_2(\varphi, \psi) &:= -\Delta\psi + W(\varepsilon x)\psi - \left(3\mu_2 \Theta_\varepsilon^2 + \beta \Xi_\varepsilon^2(\varepsilon x) \right) \psi - 2\beta \Theta_\varepsilon \Xi_\varepsilon(\varepsilon x) \varphi(\varepsilon x). \end{aligned} \tag{32}$$

The error term $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$ is defined by

$$\begin{aligned} \mathcal{E}_1 &:= 3\mu_1 \Upsilon(x) \beta^2 \Phi_\varepsilon^2 + \mu_1 \beta^3 \Phi_\varepsilon^3 + \beta^3 \Upsilon(x) \Psi_\varepsilon \left(\frac{x}{\varepsilon} \right) + 2\beta^2 \Upsilon(x) U_\varepsilon \left(\frac{x}{\varepsilon} \right) \Psi_\varepsilon \left(\frac{x}{\varepsilon} \right) \\ &\quad + \beta^2 \Phi_\varepsilon(x) U_\varepsilon^2 \left(\frac{x}{\varepsilon} \right) + \beta^4 \Phi_\varepsilon \Psi_\varepsilon \left(\frac{x}{\varepsilon} \right) + 2\beta^2 \Phi_\varepsilon U_\varepsilon \left(\frac{x}{\varepsilon} \right) \Psi_\varepsilon \left(\frac{x}{\varepsilon} \right) \\ \mathcal{E}_2 &:= (\omega_0 - \omega(\varepsilon x)) \Theta_\varepsilon + 2\beta^2 U_\varepsilon (\Upsilon(\varepsilon x) \Phi_\varepsilon(\varepsilon x) - \Phi_\varepsilon(0) \Upsilon(0)) \\ &\quad + \mu_2 (U_\varepsilon^3 - U_{P_\varepsilon}^3 - U_{-P_\varepsilon}^3) + 3\beta \mu_2 (U_\varepsilon^2 - U_{P_\varepsilon}^2 - U_{-P_\varepsilon}^2) \Psi_\varepsilon \\ &\quad + \beta^3 \Phi_\varepsilon^2(\varepsilon x) \Theta_\varepsilon + 2\beta^3 \Psi_\varepsilon \Upsilon(\varepsilon x) \Phi_\varepsilon(\varepsilon x) + 3\mu_2 \beta^2 U_\varepsilon \Psi_\varepsilon^2 + \mu_2 \beta^3 \Psi_\varepsilon^3 \end{aligned} \tag{33}$$

where

$$\omega(x) := W(x) - \beta \Upsilon^2(x) \quad \text{and} \quad \omega(0) = \omega_0.$$

Moreover, by applying Lemma 2.3 we obtain

$$\int_{\mathbb{R}^N} \Theta_{\varepsilon_n}^2 \left(\frac{x}{\varepsilon_n} \right) |\varphi_n \chi| \lesssim \|\varphi_n\|_2 \int_{\mathbb{R}^N} \left[\Theta_{\varepsilon_n}^4 \left(\frac{x}{\varepsilon_n} \right) \right]^{\frac{1}{2}} \lesssim \varepsilon_n^{\frac{N}{2}} \|\Theta_{\varepsilon_n}^2\|_{L^2(\mathbb{R}^N)}.$$

Arguing analogously on the other terms on the right hand side, it follows that φ solves

$$-\Delta\varphi + V(x)\varphi = 3\mu_1\Upsilon^2\varphi \text{ in } \mathbb{R}^N.$$

Since φ is an even function, by the assumption (\mathbf{V}_1) we get $\varphi \equiv 0$.

In order to show that $\varphi_n \rightarrow 0$ strongly in $H_V^2(\mathbb{R}^N)$, it is enough to exploit Lemma 2.1 and 2.3 to verify that the $L^2(\mathbb{R}^N)$ -norm of the right hand side (R.H.S. for short) of the first equation goes to zero, and then apply hypothesis (\mathbf{V}_1) . Indeed, as $\|(\varphi_n, \psi_n)\|_{H_V^2(\mathbb{R}^N) \times H_W^2(\mathbb{R}^N)} = 1$, Sobolev embeddings yield $\|\varphi_n\|_{L^2(\mathbb{R}^N)}, \|\psi_n\|_{L^2(\mathbb{R}^N)} \leq 1$ and $\|\varphi_n\|_{L^\infty(\mathbb{R}^N)} \leq 1$. Then

$$\begin{aligned} \|R.H.S.\|_{L^2(\mathbb{R}^N)} &\lesssim \|\Upsilon^2\varphi_n\|_{L^2(\mathbb{R}^N)} + \left\| \left[\Phi_{\varepsilon_n}(\beta\Phi_{\varepsilon_n} + 2\Upsilon) + \beta\Theta_{\varepsilon_n}^2 \left(\frac{x}{\varepsilon_n} \right) \right] \varphi_n \right\|_{L^2(\mathbb{R}^N)} \\ &\quad + \left\| (\Upsilon + \beta\Phi_{\varepsilon_n})\Theta_{\varepsilon_n} \left(\frac{x}{\varepsilon_n} \right) \psi_n \left(\frac{x}{\varepsilon_n} \right) \right\|_{L^2(\mathbb{R}^N)} + \|f_n\|_{L^2(\mathbb{R}^N)} \\ &\lesssim \|\Upsilon^2\varphi_n\|_{L^2(\mathbb{R}^N)} + \|\Phi_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^N)}\|\varphi_n\|_{L^2(\mathbb{R}^N)} + \varepsilon^{\frac{N}{2}}\|\varphi_n\|_{L^\infty(\mathbb{R}^N)} \\ &\quad + \varepsilon^{\frac{N}{2}}\|\psi_n\|_{L^2(\mathbb{R}^N)} + \|f_n\|_{L^2(\mathbb{R}^N)} \\ &= o(1), \end{aligned}$$

where we have also taken into account that Υ decays exponentially and $\varphi \rightarrow 0$ strongly in $L^2_{loc}(\mathbb{R}^N)$, so that we also have

$$\int_{\mathbb{R}^N} \Upsilon^4 \varphi_n^2 = o(1).$$

Step 2. We now study the second equation in (35) and we prove that $t_n \rightarrow 0$. We test with Z_{ε_n} and we remind that Z_{ε_n} solves

$$-\Delta Z_{\varepsilon_n} + \omega_0 Z_{\varepsilon_n} = 3\mu_2 \left(U_{P_{\varepsilon_n}}^2 \partial_{x_1} U_{P_{\varepsilon_n}} - U_{-P_{\varepsilon_n}}^2 \partial_{x_1} U_{-P_{\varepsilon_n}} \right).$$

Therefore we get

$$\begin{aligned} t_n \int_{\mathbb{R}^N} Z_{\varepsilon_n}^2 &= \int_{\mathbb{R}^N} \left(W(\varepsilon_n x) - \beta\Upsilon^2(\varepsilon_n x) - \omega_0 \right) Z_{\varepsilon_n} \psi_n \\ &\quad - 3\mu_2 \int_{\mathbb{R}^N} \left[U_{\varepsilon_n}^2 Z_{\varepsilon_n} - \left(U_{-P_{\varepsilon_n}}^2 \partial_{x_1} U_{-P_{\varepsilon_n}} - U_{P_{\varepsilon_n}}^2 \partial_{x_1} U_{P_{\varepsilon_n}} \right) \right] \psi_n \\ &\quad - 2\beta \int_{\mathbb{R}^N} \Theta_{\varepsilon_n} \Xi(\varepsilon_n x) \varphi_n(\varepsilon_n x) Z_{\varepsilon_n} - 2\beta^2 \int_{\mathbb{R}^N} \Upsilon(\varepsilon_n x) \Phi_{\varepsilon_n}(\varepsilon_n x) \psi_n Z_{\varepsilon_n} \end{aligned}$$

$$\begin{aligned}
 & -\beta^3 \int_{\mathbb{R}^N} \Phi^2(\varepsilon_n x) \psi_n Z_{\varepsilon_n} - 3\mu_2 \beta \int_{\mathbb{R}^N} \Psi_{\varepsilon_n} (2U_{\varepsilon_n} + \beta \Psi_{\varepsilon_n}) \psi_n Z_n \\
 \lesssim & \left\| \left(W(\varepsilon_n x) - \beta \Upsilon^2(\varepsilon_n x) - \omega_0 \right) Z_{\varepsilon_n} \right\|_{L^2(\mathbb{R}^N)} \|\psi_n\|_{L^2(\mathbb{R}^N)} \\
 & + \left\| U_{\varepsilon_n}^2 Z_{\varepsilon_n} - \left(U_{-P_{\varepsilon_n}}^2 \partial_{x_1} U_{-P_{\varepsilon_n}} - U_{P_{\varepsilon_n}}^2 \partial_{x_1} U_{P_{\varepsilon_n}} \right) \right\|_{L^2(\mathbb{R}^N)} \|\psi_n\|_{L^2(\mathbb{R}^N)} \\
 & + \|\varphi_n\|_{L^\infty(\mathbb{R}^N)} + \|\Phi_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^N)} + \|\Psi_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^N)} \\
 = & o(1).
 \end{aligned} \tag{36}$$

Indeed, we use the exponential decay of U and of its derivatives. Since $\omega_0 = W(0) - \beta \Upsilon^2(0)$ we get

$$|W(y) - \beta \Upsilon^2(y) - \omega_0| \lesssim |y| \quad \text{if } |y| \leq \sigma$$

for some $\sigma > 0$ and so

$$\left\| \left(W(\varepsilon_n x) - \beta \Upsilon^2(\varepsilon_n x) - \omega_0 \right) Z_{\varepsilon_n} \right\|_{L^2(\mathbb{R}^N)} = o(1).$$

Moreover a direct computation and Lemma A.2 shows that

$$\left\| U_{\varepsilon_n}^2 Z_{\varepsilon_n} - \left(U_{-P_{\varepsilon_n}}^2 \partial_{x_1} U_{-P_{\varepsilon_n}} - U_{P_{\varepsilon_n}}^2 \partial_{x_1} U_{P_{\varepsilon_n}} \right) \right\|_{L^2(\mathbb{R}^N)} = o(1).$$

It is possible to show that all the other integral terms on the left hand side tend to zero by applying Lemma 2.1 and 2.3, Step 1. and Sobolev embeddings.

Finally, since it is immediate to check that $\|Z_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)} = C + o(1)$ for some $C > 0$, we deduce that $t_n = o(1)$.

Step 3. Let us now introduce the sequences

$$\tilde{\psi}_{-P_n}(x) := \psi_n \left(x - \frac{P_{\varepsilon_n}}{\varepsilon_n} \right), \quad \tilde{\psi}_{+P_n}(x) := \psi_n \left(x + \frac{P_{\varepsilon_n}}{\varepsilon_n} \right).$$

We will show that (up to subsequences) $\tilde{\psi}_{\pm P_n} \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$.

Both these sequences are bounded in $H^2_{W_{\varepsilon_n}}(\mathbb{R}^N)$, so that, up to subsequences, $\tilde{\psi}_{\pm P_n} \rightharpoonup \tilde{\psi}_{\pm}$ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$. Let us first show that $\tilde{\psi}_+ \equiv 0$, then an analogous argument will yield that $\tilde{\psi}_+ \equiv 0$.

In the following we will use the notation $\tilde{\psi}_{+P_n}(x) = \tilde{\psi}_{P_n}(x)$. Recalling (35), the function $\tilde{\psi}_{P_n}$ satisfies the equation

$$\begin{aligned} -\Delta \tilde{\psi}_{P_n} + W(\varepsilon_n x + P_n) \tilde{\psi}_{P_n} &= 3\mu_2 \left(U(x) + U\left(x + 2\frac{P_n}{\varepsilon_n}\right) + \beta \Psi_{\varepsilon_n}\left(x + \frac{P_n}{\varepsilon_n}\right) \right)^2 \tilde{\psi}_{P_n} \\ &\quad + \beta \Upsilon^2(\varepsilon_n x + P_n) \tilde{\psi}_{P_n} \\ &\quad + \beta \Phi_{\varepsilon_n}(\varepsilon_n x + P_n) (\beta \Phi_{\varepsilon_n}(\varepsilon_n x + P_n) + 2\Upsilon(\varepsilon_n x + P_n)) \tilde{\psi}_{P_n} \\ &\quad + 2\beta \Theta_{\varepsilon_n}\left(x + \frac{P_n}{\varepsilon_n}\right) \Xi_{\varepsilon_n}\left(x + \frac{P_n}{\varepsilon_n}\right) \varphi_n(\varepsilon_n x + P_n) \\ &\quad + g_n\left(x + \frac{P_n}{\varepsilon_n}\right) + t_n Z_{\varepsilon_n}\left(x + \frac{P_n}{\varepsilon_n}\right). \end{aligned}$$

Arguing as in the first step, applying Lemma 2.1 and Lemma 2.3, and taking into account that $\|\varphi_n\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$, we obtain that the limit function $\tilde{\psi}_+$ solves the limit problem

$$-\Delta \tilde{\psi}_+ + \omega_0 \tilde{\psi}_+ = 3\mu_2 U^2 \tilde{\psi}_+ \text{ in } \mathbb{R}^N. \tag{37}$$

On the other hand, the function $\tilde{\psi}_+$ inherits the symmetry properties of the function $\tilde{\psi}_{\pm P_n}$, namely it is even in the last two variables and it satisfies the orthogonality condition

$$\int_{\mathbb{R}^N} \tilde{\psi}_+(y) \partial_1 U(y) dy = 0,$$

as

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \psi_n(x) Z_{\varepsilon_n}(x) dx = \int_{\mathbb{R}^N} \tilde{\psi}_{+P_n}(y) \left(\partial_1 U(y) - \partial_1 U\left(y - 2\frac{P_n}{\varepsilon_n}\right) \right) dy \\ &= \int_{\mathbb{R}^N} \tilde{\psi}_+(y) \partial_1 U(y) dy + o(1). \end{aligned}$$

This, together with (37), yields $\tilde{\psi}_+ \equiv 0$.

Step 4. Let us prove that a contradiction arises. First, let us prove that $\|\psi_n\|_{L^2(\mathbb{R}^N)} = o(1)$. By testing the second equation with ψ_n , and recalling that $\beta < 0$ in view of (14), we deduce that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \psi_n|^2 + W(\varepsilon_n x) \psi_n^2 &= \int_{\mathbb{R}^N} 3\mu_2 U_{\varepsilon_n}^2 \psi_n^2 + \beta \int_{\mathbb{R}^N} \Upsilon^2(\varepsilon_n x) \psi_n^2 \\ &\quad + \int_{\mathbb{R}^N} \beta \Phi_{\varepsilon_n}(\varepsilon_n x) (\beta \Phi_{\varepsilon_n}(\varepsilon_n x) + 2\Upsilon(\varepsilon_n x)) \psi_n^2 \\ &\quad + \int_{\mathbb{R}^N} 2\beta U_{\varepsilon_n} \Xi_{\varepsilon_n}(\varepsilon_n x) \varphi_n(\varepsilon_n x) \psi_n + \int_{\mathbb{R}^N} g_n \psi_n \\ &\lesssim \int_{\mathbb{R}^N} 3\mu_2 U_{\varepsilon_n}^2 \psi_n^2 + o(1), \end{aligned}$$

where we have repeatedly applied Lemma 2.1, 2.3, that $\|\phi_n\|_{L^\infty(\mathbb{R}^N)} = o(1)$ and that $g_n \rightarrow 0$ strongly in $L^2(\mathbb{R}^N)$. Concerning the last term, we have that

$$\begin{aligned} \int_{\mathbb{R}^N} U_{\varepsilon_n}^2 \psi_n^2 &= \int_{\mathbb{R}^N} U_{P_{\varepsilon_n}}^2 \psi_n^2 + \int_{\mathbb{R}^N} U_{-P_{\varepsilon_n}}^2 \psi_n^2 + 2 \int_{\mathbb{R}^N} U_{P_{\varepsilon_n}} U_{-P_{\varepsilon_n}} \psi_n^2 \\ &\lesssim \int_{\mathbb{R}^N} U^2 \tilde{\psi}_{+P_n}^2 + \int_{\mathbb{R}^N} U^2 \tilde{\psi}_{-P_n}^2 + \|\psi_n\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} U_{P_{\varepsilon_n}} U_{-P_{\varepsilon_n}} = o(1) \end{aligned}$$

because $\tilde{\psi}_{\pm P_n} \rightarrow 0$ strongly in $L^2_{loc}(\mathbb{R}^N)$ (as shown in the previous step) and U decays exponentially. This implies that $\psi_n \rightarrow 0$ strongly in $H^1(\mathbb{R}^N)$, thanks to (7).

Finally, let us prove that a contradiction arises by showing that also $\psi_n \rightarrow 0$ strongly in $H^2_{W_\varepsilon}(\mathbb{R}^N)$. In order to show this, it is enough to use hypothesis (V₂) and to check that the $L^2(\mathbb{R}^N)$ -norm of the right hand side of the second equation in (35) goes to zero. Indeed, by Lemma 2.1, taking into account that $\|\psi_n\|_{L^2(\mathbb{R}^N)} \rightarrow 0$ and $\|\phi_n\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$

$$\begin{aligned} \|R.H.S.\|_{L^2(\mathbb{R}^N)} &\lesssim \|3\mu_2 U_{\varepsilon_n}^2 \psi_n\|_{L^2(\mathbb{R}^N)} + \left\| (\Upsilon(\varepsilon_n x) + \beta \Phi_{\varepsilon_n}(\varepsilon_n x))^2 \psi_n \right\|_{L^2(\mathbb{R}^N)} \\ &\quad + \|U_{\varepsilon_n} (\Upsilon(\varepsilon_n x) + \beta \Phi_{\varepsilon_n}(\varepsilon_n x)) \phi_n(\varepsilon_n x)\|_{L^2(\mathbb{R}^N)} + \|g_n\|_{L^2(\mathbb{R}^N)} \\ &\quad + |t_n| \|Z_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)} \\ &\lesssim \|\psi_n\|_{L^2(\mathbb{R}^N)} + \|\phi_n\|_{L^\infty(\mathbb{R}^N)} + \|g_n\|_{L^2(\mathbb{R}^N)} + |t_n| \\ &= o(1) \end{aligned}$$

□

Remark 3.2 Let us observe that the hypothesis $\beta < 0$ is needed only in the proof of Step 4. Moreover, it is not needed in the case of a constant potential $W(x) \equiv W(0) = W_0$. Indeed, in this case if $\beta \geq 0$

$$\beta \Upsilon^2(x) \leq \beta \Upsilon^2(0) < W(0), \quad \text{as } \omega_0 > 0.$$

so that the sequence $(W_0 - \beta \Upsilon^2(\varepsilon_n x)) \psi_n^2 \geq 0$ for every x and the final contradiction can be obtained applying Fatou Lemma. However, even if at this point we can manage $\beta \geq 0$ (in the case of W constant), the study of the finite dimensional problem will require $\beta < 0$ as shown in hypothesis (14).

3.1 The size of the error term

In this subsection we compute the $L^2(\mathbb{R}^N)$ of \mathcal{E} which will determine the norm of the remainder term (φ, ψ) .

Proposition 3.3 *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ it results*

$$\|\mathcal{E}_1\|_{L^2(\mathbb{R}^N)} = \mathcal{O}(\varepsilon^N) \quad \text{and} \quad \|\mathcal{E}_2\|_{L^2(\mathbb{R}^N)} = \mathcal{O}(\varepsilon^2 |\ln \varepsilon|^2)$$

so that,

$$\|\mathcal{E}\|_{L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)} = \mathcal{O}(\varepsilon^2 |\ln \varepsilon|^2)$$

Proof Let us start studying \mathcal{E}_1 given in (33). We have

$$\begin{aligned} \|\mathcal{E}_1\|_{L^2(\mathbb{R}^N)} &\lesssim \|\Upsilon \Phi_\varepsilon^2\|_{L^2(\mathbb{R}^N)} + \|\Phi_\varepsilon^3\|_{L^2(\mathbb{R}^N)} + \left\| \Phi_\varepsilon U_\varepsilon^2 \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} + \left\| \Upsilon \Psi_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} \\ &\quad + \left\| \Upsilon U_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \Psi_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} + \left\| \Phi_\varepsilon \Psi_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} \\ &\quad + \left\| \Phi_\varepsilon U_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \Psi_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Lemma 2.1 and Sobolev embedding imply that (see Remark 2.2) $\|\Phi_\varepsilon\|_\infty \lesssim \varepsilon^{\frac{N}{2}}$. Then we deduce

$$\|\Upsilon \Phi_\varepsilon^2\|_{L^2(\mathbb{R}^N)} \lesssim \|\Phi_\varepsilon\|_\infty^2 \lesssim \varepsilon^N, \quad \|\Phi_\varepsilon^3\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{3N}{2}}$$

and

$$\left\| \Phi_\varepsilon U_\varepsilon^2 \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^N} \Phi_\varepsilon^2 U_\varepsilon^4 \left(\frac{x}{\varepsilon} \right) dx \right)^{\frac{1}{2}} \lesssim \varepsilon^{\frac{N}{2}} \left(\int_{\mathbb{R}^N} U_\varepsilon^4 \left(\frac{x}{\varepsilon} \right) dx \right)^{\frac{1}{2}} \lesssim \varepsilon^N.$$

Moreover by applying Lemma 2.3 we obtain

$$\left\| \Upsilon \Psi_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{2}} \|\Psi_\varepsilon\|_{H^2(\mathbb{R}^N)} \lesssim \varepsilon^N.$$

As far as concern the last three terms, similar computations show that

$$\left\| \Upsilon U_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \Psi_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} + \left\| \Phi_\varepsilon \Psi_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} + \left\| \Phi_\varepsilon U_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \Psi_\varepsilon \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^N.$$

Let us now study the $L^2(\mathbb{R}^N)$ norm of the terms in (34). In view of (7) and (12), recalling that $x_0 = 0$ is a critical point of ω by symmetry,

$$\begin{aligned} \|(\omega_0 - \omega(\varepsilon x))U_\varepsilon\|_{L^2(\mathbb{R}^N)} &\lesssim \left(\int_{\mathbb{R}^N} \varepsilon^4 |x|^4 U^2 \left(x - \frac{P_\varepsilon}{\varepsilon} \right) dx \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{\mathbb{R}^N} \varepsilon^4 \left| x + \frac{P_\varepsilon}{\varepsilon} \right|^4 U^2(x) dx \right)^{\frac{1}{2}} \tag{38} \\ &\lesssim \rho_\varepsilon^2 \lesssim \varepsilon^2 |\ln \varepsilon|^2. \end{aligned}$$

On the other hand Lemma 2.3 allows us to deduce that

$$\begin{aligned} & \|(\omega_0 - \omega(\varepsilon x))\Psi_\varepsilon\|_{L^2(\mathbb{R}^N)} \\ & \lesssim \int_{\mathbb{R}^N} \left(\varepsilon^4 \left| x + \frac{P_\varepsilon}{\varepsilon} \right|^4 \Psi^2(x) \right)^{\frac{1}{2}} \lesssim \rho_\varepsilon^2 \|\Psi\|_{L^2(\mathbb{R}^N)} \lesssim \rho_\varepsilon^2 \lesssim \varepsilon^2 |\ln \varepsilon|^2. \end{aligned}$$

By (27) one has

$$|\Phi_\varepsilon(\varepsilon x) - \Phi_\varepsilon(0)| \leq \varepsilon^{\frac{N}{m}} |\varepsilon x|^{2 - \frac{N}{m}} = \varepsilon^2 |x|^{1 - \frac{N}{m}}$$

so that, we obtain

$$\begin{aligned} \|U_\varepsilon(\Upsilon(\varepsilon x)\Phi_\varepsilon(\varepsilon x) - \Phi_\varepsilon(0)\Upsilon(0))\|_{L^2(\mathbb{R}^N)} & \lesssim \|U_\varepsilon \Upsilon(\varepsilon x)(\Phi_\varepsilon(\varepsilon x) - \Phi_\varepsilon(0))\|_{L^2(\mathbb{R}^N)} \\ & \quad + \|U_\varepsilon(\Upsilon(\varepsilon x) - \Upsilon(0))\Phi_\varepsilon(0)\|_{L^2(\mathbb{R}^N)} \\ & \lesssim \varepsilon^2 \left(\int_{\mathbb{R}^N} \left(U(x) \left| x + \frac{P_\varepsilon}{\varepsilon} \right|^{2 - \frac{N}{m}} \right)^2 dx \right)^{\frac{1}{2}} \\ & \quad + \varepsilon^2 \|\Phi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \left(U(x) \left| x + \frac{P_\varepsilon}{\varepsilon} \right|^2 \right)^2 dx \right)^{\frac{1}{2}} \\ & \lesssim \varepsilon^2 \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{2 - \frac{N}{m}} + \varepsilon^{2 + \frac{N}{2}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^2 \\ & \lesssim \varepsilon^2 |\ln \varepsilon|^{2 - \frac{N}{m}} + \varepsilon^{2 + \frac{N}{2}} |\ln \varepsilon|^2 \lesssim \varepsilon^2 |\ln \varepsilon|^{2 - \frac{N}{m}} \end{aligned}$$

By Lemma A.2 - (i) one also has

$$\begin{aligned} \|U_\varepsilon^3 - U_{P_\varepsilon}^3 - U_{-P_\varepsilon}^3\|_{L^2(\mathbb{R}^N)} & \lesssim \|U_{P_\varepsilon}^2 U_{-P_\varepsilon}\|_{L^2(\mathbb{R}^N)} + \|U_{-P_\varepsilon}^2 U_{P_\varepsilon}\|_{L^2(\mathbb{R}^N)} \\ & \lesssim \left(\int_{\mathbb{R}^N} U^2 \left(x - 2 \frac{P_\varepsilon}{\varepsilon} \right) U^4(x) dx \right)^{\frac{1}{2}} \lesssim e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \\ & \lesssim \varepsilon \rho_\varepsilon \lesssim \varepsilon^2 |\ln \varepsilon|. \end{aligned}$$

Moreover, conclusion (ii) of Lemma A.2 and (12) yield

$$\begin{aligned} \|(U_\varepsilon^2 - U_{P_\varepsilon}^2 - U_{-P_\varepsilon}^2)\Psi_\varepsilon\|_{L^2(\mathbb{R}^N)} & \lesssim \|\Psi_\varepsilon\|_\infty \left(\int_{\mathbb{R}^N} U^2 \left(x - \frac{2P_\varepsilon}{\varepsilon} \right) U^2(x) dx \right)^{\frac{1}{2}} \\ & \lesssim \varepsilon^{\frac{N}{2}} \begin{cases} e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{1}{4}} & \text{if } N = 2 \\ e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-1} \left| \ln \frac{\rho_\varepsilon}{\varepsilon} \right|^{\frac{1}{2}} & \text{if } N = 3 \end{cases} \\ & = o(\varepsilon^2 |\ln \varepsilon|^2) \end{aligned}$$

as, by using (12) it follows that $\rho_\varepsilon \sim \frac{1}{\sqrt{\omega_0}} \varepsilon |\ln \varepsilon|$ and hence $\varepsilon^{\frac{N}{2}} e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} = o(\varepsilon^2 |\ln \varepsilon|^2)$.

Let us finally study the last terms in (34)

$$\|\Phi_\varepsilon^2(\varepsilon x)\Theta_\varepsilon\|_{L^2(\mathbb{R}^N)} \lesssim \|\Phi_\varepsilon\|_\infty^2 \lesssim \varepsilon^N; \quad \|\Psi_\varepsilon\Upsilon(\varepsilon x)\Phi_\varepsilon(\varepsilon x)\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^N$$

and

$$\|U_\varepsilon\Psi_\varepsilon^2\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^N; \quad \|\Psi_\varepsilon^3\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{3N}{2}},$$

concluding the proof. □

3.2 Solving the second equation in (31)

Lemma 3.1 and Proposition 3.3 yields the following result

Proposition 3.4 *There exists $\varepsilon > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a unique solution $(\varphi, \psi) \in K^\perp$ of the equation*

$$\tilde{\Pi}^\perp (\mathcal{L}(\varphi, \psi) - \mathcal{E} - \mathcal{N}(\varphi, \psi)) = 0.$$

Furthermore,

$$\|(\varphi, \psi)\|_X = \mathcal{O}\left(\varepsilon^2 |\ln \varepsilon|^2\right). \tag{39}$$

Proof We will obtain the result by applying the contraction principle to the continuous map

$$T : C_\varepsilon := \left\{ (\varphi, \psi) \in K^\perp, : \|(\varphi, \psi)\|_X \leq A\tau_\varepsilon \right\} \mapsto C_\varepsilon$$

$$T(\varphi, \psi) := \tilde{\mathcal{L}} \left[\tilde{\Pi}^\perp (\mathcal{E} + \mathcal{N}(\varphi, \psi)) \right]$$

where

$$\tau_\varepsilon = \varepsilon^2 |\ln \varepsilon|^2, \quad \tilde{\mathcal{L}} := \left(\tilde{\Pi}^\perp \circ \mathcal{L} \right)^{-1},$$

is well defined thanks to Lemma 3.1, and A is a suitable positive constant to be chosen. In order to find A , it is sufficient to prove that

$$\|\mathcal{N}(\varphi, \psi)\|_{L^2(\mathbb{R}^N)} = o(\tau_\varepsilon), \quad \text{forall } (\varphi, \psi) \in C_\varepsilon. \tag{40}$$

Let us start studying \mathcal{N}_1 , taking into account that $\|\Xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq C$, it results

$$\|\mathcal{N}_1(\varphi, \psi)\|_{L^2(\mathbb{R}^N)}^2 \lesssim \int_{\mathbb{R}^N} \varphi^6 + \Xi_\varepsilon^2 \varphi^4 + \psi^2 \left(\frac{x}{\varepsilon}\right) \varphi^2 \Theta_\varepsilon^2 \left(\frac{x}{\varepsilon}\right) + \psi^4 \left(\frac{x}{\varepsilon}\right) \varphi^2 + \Xi_\varepsilon^2 \psi^4 \left(\frac{x}{\varepsilon}\right)$$

$$\begin{aligned}
 &\lesssim \|\varphi\|_V^6 + \|\varphi\|_V^4 + \|\varphi\|_4^2 \left(\int_{\mathbb{R}^N} \psi^4 \left(\frac{x}{\varepsilon} \right) \Theta_\varepsilon^4 \left(\frac{x}{\varepsilon} \right) \right)^{\frac{1}{2}} \\
 &\quad + \left(\int_{\mathbb{R}^N} \psi^6 \left(\frac{x}{\varepsilon} \right) \right)^{\frac{2}{3}} \|\varphi\|_6^2 \\
 &\quad + \int_{\mathbb{R}^N} \psi^4 \left(\frac{x}{\varepsilon} \right) \\
 &\lesssim \|\varphi\|_V^6 + \|\varphi\|_V^4 + \varepsilon^{\frac{N}{2}} \|\varphi\|_V^2 \|\psi\|_\varepsilon^2 + \varepsilon^{\frac{2N}{3}} \|\varphi\|_V^2 \|\psi\|_\varepsilon^4 + \varepsilon^N \|\psi\|_\varepsilon^4 \\
 &\lesssim \tau_\varepsilon^6 + \tau_\varepsilon^4 + \varepsilon^{\frac{N}{2}} \tau_\varepsilon^4 + \varepsilon^{\frac{2N}{3}} \tau_\varepsilon^6 + \varepsilon^N \tau_\varepsilon^4 = o(\tau_\varepsilon^2). \tag{41}
 \end{aligned}$$

In addition,

$$\begin{aligned}
 \|\mathcal{N}_2(\varphi, \psi)\|_{L^2(\mathbb{R}^N)}^2 &\lesssim \int_{\mathbb{R}^N} \psi^6 + \Theta_\varepsilon^2 \psi^4 + \psi^2 \varphi^2(\varepsilon x) \Xi_\varepsilon^2(\varepsilon x) + \psi^2 \varphi^4(\varepsilon x) + \Theta_\varepsilon^2 \varphi^4(\varepsilon x) \\
 &\lesssim \|\psi\|_\varepsilon^6 + \|\psi\|_\varepsilon^4 + \|\psi\|_4^2 \left(\int_{\mathbb{R}^N} \varphi^4(\varepsilon x) \right)^{\frac{1}{2}} + \|\psi\|_6^2 \left(\int_{\mathbb{R}^N} \varphi^6(\varepsilon x) \right)^{\frac{2}{3}} \\
 &\quad + \left(\int_{\mathbb{R}^N} \varphi^6(\varepsilon x) \right)^{\frac{2}{3}} \\
 &\lesssim \|\psi\|_\varepsilon^6 + \|\psi\|_\varepsilon^4 + \varepsilon^{-\frac{N}{2}} \|\psi\|_\varepsilon^2 \|\varphi\|_V^2 \\
 &\quad + \varepsilon^{-\frac{2N}{3}} \|\psi\|_\varepsilon^2 \|\varphi\|_V^4 + \varepsilon^{-\frac{2N}{3}} \|\varphi\|_V^4 = o(\tau_\varepsilon^2).
 \end{aligned}$$

this, together with (41), implies (40). Then, the claim follows by the contraction mapping theorem. □

Remark 3.5 Note that, by the Sobolev embeddings $H^2(\mathbb{R}^3) \hookrightarrow C^{0, \frac{1}{2}}(\mathbb{R}^3)$, $H^2(\mathbb{R}^2) \hookrightarrow C^{0, \alpha}(\mathbb{R}^2)$ for any $\alpha \in (0, 1)$, we deduce that

- $\|\varphi\|_{C^{0, \frac{1}{2}}(\mathbb{R}^3)} + \|\psi\|_{C^{0, \frac{1}{2}}(\mathbb{R}^3)} \lesssim \varepsilon^2 |\ln \varepsilon|^2$,
- $\|\varphi\|_{C^{0, \alpha}(\mathbb{R}^2)} + \|\psi\|_{C^{0, \alpha}(\mathbb{R}^2)} \lesssim \varepsilon^2 |\ln \varepsilon|^2$, for every $\alpha \in (0, 1)$,

so that

- $|\varphi(\varepsilon x) - \varphi(0)| \leq \varepsilon^2 |\ln \varepsilon|^2 (\varepsilon|x|)^{1/2} = \varepsilon^{\frac{5}{2}} |\ln \varepsilon|^2 |x|^{1/2}$ for $N = 3$.
- $|\varphi(\varepsilon x) - \varphi(0)| \leq \varepsilon^2 |\ln \varepsilon|^2 (\varepsilon|x|)^\alpha = \varepsilon^{2+\alpha} |\ln \varepsilon|^2 |x|^\alpha$ for $N = 2$, for every $\alpha \in (0, 1)$,

and analogous estimates hold for ψ .

4 Solving the reduced problem

In this section, we are going to study the first equation in (31). Let (φ, ψ) the solution of the second equation in (31), then

$$\widetilde{\Pi} \{ \mathcal{L}(\varphi, \psi) - \mathcal{E} - \mathcal{N}(\varphi, \psi) \} = c_0 Z_\varepsilon$$

where

$$c_0 := \frac{(\mathcal{L}_2(\varphi, \psi) - \mathcal{E}_2 - \mathcal{N}_2(\varphi, \psi), Z_\varepsilon)_{L^2(\mathbb{R}^N)}}{\|Z_\varepsilon\|^2}. \tag{42}$$

Our goal will be to prove that $c_0 = 0$. From now on, we fix (φ, ψ) given in Proposition 3.4.

Lemma 4.1 *It results*

$$|(\mathcal{N}_2(\varphi, \psi) - \mathcal{L}_2(\varphi, \psi), Z_\varepsilon)_{L^2(\mathbb{R}^N)}| \leq o(\varepsilon^2 |\ln \varepsilon|)$$

Proof Arguing as in the proof of Proposition 3.4 it is easy to obtain that

$$\int_{\mathbb{R}^N} \mathcal{N}_2(\varphi, \psi) Z_\varepsilon \, dx = \mathcal{O}(\|(\varphi, \psi)\|^2) = o(\varepsilon^2 |\ln \varepsilon|).$$

Now by using (24) and (32) we get that

$$\begin{aligned} \int_{\mathbb{R}^N} \mathcal{L}_2(\varphi, \psi) Z_\varepsilon \, dx &= \int_{\mathbb{R}^N} \nabla \psi \nabla Z_\varepsilon + W(\varepsilon x) \psi Z_\varepsilon - (3\mu_2 \Theta_\varepsilon^2 + \beta \Xi^2(\varepsilon x)) \psi Z_\varepsilon \\ &\quad - 2\beta \int_{\mathbb{R}^N} \Theta_\varepsilon \Xi(\varepsilon x) \varphi(\varepsilon x) Z_\varepsilon \\ &= \int_{\mathbb{R}^N} (W(\varepsilon x) - \omega_0) \psi Z_\varepsilon + 3\mu_2 (U_{P_\varepsilon}^2 \partial_{x_1} U_{P_\varepsilon} - U_{-P_\varepsilon}^2 \partial_{x_1} U_{-P_\varepsilon}) \psi \\ &\quad - \int_{\mathbb{R}^N} (3\mu_2 \Theta_\varepsilon^2 + \beta \Xi^2(\varepsilon x)) \psi Z_\varepsilon - 2\beta \int_{\mathbb{R}^N} \Theta_\varepsilon \Xi(\varepsilon x) \varphi(\varepsilon x) Z_\varepsilon \\ &= \int_{\mathbb{R}^N} (\omega(\varepsilon x) - \omega(0)) Z_\varepsilon \psi \, dx + 3\mu_2 \int_{\mathbb{R}^N} U_{P_\varepsilon}^2 \partial_1 U_{-P_\varepsilon} \psi \, dx \\ &\quad - 3\mu_2 \int_{\mathbb{R}^N} U_{-P_\varepsilon}^2 \partial_1 U_{P_\varepsilon} \psi - 6\mu_2 \int_{\mathbb{R}^N} U_{P_\varepsilon} U_{-P_\varepsilon} Z_\varepsilon \psi \, dx \\ &\quad - 3\mu_2 \beta^2 \int_{\mathbb{R}^N} \Psi_\varepsilon^2 Z_\varepsilon \psi \, dx - 6\mu_2 \beta \int_{\mathbb{R}^N} U_\varepsilon \Psi_\varepsilon Z_\varepsilon \psi \, dx \\ &\quad - \beta^3 \int_{\mathbb{R}^N} \Phi_\varepsilon^2(\varepsilon x) Z_\varepsilon \psi \, dx - 2\beta^2 \int_{\mathbb{R}^N} \Upsilon(\varepsilon x) \Phi_\varepsilon(\varepsilon x) Z_\varepsilon \psi \, dx \\ &\quad - 2\beta^2 \int_{\mathbb{R}^N} \Psi_\varepsilon \Upsilon(\varepsilon x) \varphi(\varepsilon x) Z_\varepsilon \, dx - 2\beta^3 \int_{\mathbb{R}^N} \Psi_\varepsilon \Phi_\varepsilon(\varepsilon x) \varphi(\varepsilon x) Z_\varepsilon \, dx \\ &\quad - 2\beta \int_{\mathbb{R}^N} U_\varepsilon \Upsilon(\varepsilon x) \varphi(\varepsilon x) Z_\varepsilon \, dx - 2\beta^2 \int_{\mathbb{R}^N} U_\varepsilon \Phi_\varepsilon(\varepsilon x) \varphi(\varepsilon x) Z_\varepsilon \, dx. \end{aligned} \tag{43}$$

$$\tag{44}$$

Let us study the right hand side. First of all, arguing as in (38) and taking into account (39), we have

$$\begin{aligned} \int_{\mathbb{R}^N} (\omega(\varepsilon x) - \omega(0)) Z_\varepsilon \psi \, dx &\lesssim \left(\int_{\mathbb{R}^N} |\omega(\varepsilon x) - \omega(0)|^2 Z_\varepsilon^2 \, dx \right)^{\frac{1}{2}} \|\psi\|_{L^2(\mathbb{R}^N)} \\ &\lesssim \rho_\varepsilon^2 \|\psi\|_{L^2(\mathbb{R}^N)} = o(\varepsilon^2 |\ln \varepsilon|). \\ \int_{\mathbb{R}^N} U_{P_\varepsilon}^2 \partial_1 U_{-P_\varepsilon} \psi \, dx &\lesssim \left(\int_{\mathbb{R}^N} U_{P_\varepsilon}^4 U_{-P_\varepsilon}^2 \, dx \right)^{\frac{1}{2}} \|\psi\|_{L^2(\mathbb{R}^N)} \\ &\lesssim e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \|\psi\|_{L^2(\mathbb{R}^N)} = o(\varepsilon^2 |\ln \varepsilon|), \end{aligned}$$

where we have applied Lemma A.2. Similarly

$$\begin{aligned} \int_{\mathbb{R}^N} U_{-P_\varepsilon}^2 \partial_1 U_{P_\varepsilon} \psi &= o(\varepsilon^2 |\ln \varepsilon|), \\ \int_{\mathbb{R}^N} U_{P_\varepsilon} U_{-P_\varepsilon} Z_\varepsilon \psi \, dx &\lesssim \left(\int_{\mathbb{R}^N} U_{P_\varepsilon}^4 U_{-P_\varepsilon}^2 \, dx \right)^{\frac{1}{2}} \|\psi\|_{L^2(\mathbb{R}^N)} = o(\varepsilon^2 |\ln \varepsilon|). \end{aligned}$$

In addition, (39) and Lemma 2.1, 2.3 yield

$$\begin{aligned} \int_{\mathbb{R}^N} \Psi_\varepsilon^2 Z_\varepsilon \psi \, dx &\lesssim \|\Psi_\varepsilon\|_{L^4(\mathbb{R}^N)}^2 \|\psi\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^N \|\psi\|_{L^2(\mathbb{R}^N)} = o(\varepsilon^2 |\ln \varepsilon|). \\ \int_{\mathbb{R}^N} U_\varepsilon \Psi_\varepsilon Z_\varepsilon \psi \, dx &\lesssim \|\Psi_\varepsilon\|_{L^2(\mathbb{R}^N)} \|\psi\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{2}} \|\psi\|_{L^2(\mathbb{R}^N)} \\ &\lesssim \varepsilon^{\frac{N}{2}} \varepsilon^2 |\ln \varepsilon|^2 = o(\varepsilon^2 |\ln \varepsilon|) \\ \int_{\mathbb{R}^N} \Phi_\varepsilon^2(\varepsilon x) Z_\varepsilon \psi \, dx &\lesssim \|\Phi_\varepsilon\|_\infty^2 \|\psi\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^N \|\psi\|_{L^2(\mathbb{R}^N)} = o(\varepsilon^2 |\ln \varepsilon|) \\ \int_{\mathbb{R}^N} \Upsilon(\varepsilon x) \Phi_\varepsilon(\varepsilon x) Z_\varepsilon \psi \, dx &\lesssim \|\Phi_\varepsilon\|_\infty \|\psi\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^{\frac{N}{2}} \varepsilon^2 |\ln \varepsilon|^2 = o(\varepsilon^2 |\ln \varepsilon|) \end{aligned}$$

and the terms in (43) can be handled analogously. Let us focus on the terms in (44). Recalling that

$$\int_{\mathbb{R}^N} U_{\pm P_\varepsilon} \partial_1 U_{\pm P_\varepsilon} \, dx = 0, \tag{45}$$

we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} U_\varepsilon \Upsilon(\varepsilon x) \varphi(\varepsilon x) Z_\varepsilon \, dx &= \int_{\mathbb{R}^N} \Upsilon(\varepsilon x) (\varphi(\varepsilon x) - \varphi(0)) U_\varepsilon Z_\varepsilon \, dx \\ &\quad + \int_{\mathbb{R}^N} \varphi(0) (\Upsilon(\varepsilon x) - \Upsilon(0)) U_\varepsilon Z_\varepsilon \, dx \\ &\quad + \Upsilon(0) \varphi(0) \int_{\mathbb{R}^N} U_\varepsilon Z_\varepsilon \, dx \\ &= \int_{\mathbb{R}^N} \Upsilon(\varepsilon x) (\varphi(\varepsilon x) - \varphi(0)) U_\varepsilon Z_\varepsilon \, dx \end{aligned}$$

$$\begin{aligned}
 &+ \int_{\mathbb{R}^N} \varphi(0)(\Upsilon(\varepsilon x) - \Upsilon(0))U_\varepsilon Z_\varepsilon \, dx \\
 &- \Upsilon(0)\varphi(0) \int_{\mathbb{R}^N} (U_{P_\varepsilon} \partial_1 U_{-P_\varepsilon} - U_{-P_\varepsilon} \partial_1 U_{P_\varepsilon}) \, dx.
 \end{aligned}
 \tag{46}$$

Taking into account Remark 3.5 (choosing $\alpha = \frac{1}{2}$ for $N = 2$), we infer

$$\begin{aligned}
 \int_{\mathbb{R}^N} \Upsilon(\varepsilon x)(\varphi(\varepsilon x) - \varphi(0))U_\varepsilon Z_\varepsilon \, dx &\lesssim \varepsilon^{\frac{5}{2}} |\ln \varepsilon|^2 \int_{\mathbb{R}^N} |x|^{\frac{1}{2}} U_{P_\varepsilon}^2 \, dx \\
 &\lesssim \varepsilon^{\frac{5}{2}} |\ln \varepsilon|^2 \int_{\mathbb{R}^N} \left| x + \frac{P_\varepsilon}{\varepsilon} \right|^{\frac{1}{2}} U^2 \, dx \\
 &\lesssim \varepsilon^2 \rho_\varepsilon^{\frac{1}{2}} |\ln \varepsilon|^2 = o(\varepsilon^2 |\ln \varepsilon|).
 \end{aligned}$$

Moreover, (12) and (39) yield

$$\int_{\mathbb{R}^N} \varphi(0)(\Upsilon(\varepsilon x) - \Upsilon(0))U_\varepsilon Z_\varepsilon \, dx \lesssim \varepsilon^2 \|\varphi\|_\infty \int_{\mathbb{R}^N} |x|^2 U_{P_\varepsilon}^2 \, dx \lesssim \rho_\varepsilon^2 \|\varphi\|_V = o(\varepsilon^2 |\ln \varepsilon|).$$

The last two terms in (46) can be studied similarly, by applying Lemma A.2. It results

$$\begin{aligned}
 \Upsilon(0)\varphi(0) \int_{\mathbb{R}^N} U_{P_\varepsilon} \partial_1 U_{-P_\varepsilon} \, dx &\lesssim \|\varphi\|_\infty \int_{\mathbb{R}^N} U_{P_\varepsilon} U_{-P_\varepsilon} \, dx \lesssim \|\varphi\|_V e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \\
 &= o(\varepsilon^2 |\ln \varepsilon|).
 \end{aligned}$$

The last term in (44) can be easier studied as

$$\left| \int_{\mathbb{R}^N} U_\varepsilon \Phi_\varepsilon(\varepsilon x)\varphi(\varepsilon x)Z_\varepsilon \, dx \right| \lesssim \|\Phi_\varepsilon\|_\infty \|\varphi\|_\infty \lesssim \varepsilon^{N/2} \|\varphi\|_V = o(\varepsilon^2 |\ln \varepsilon|)$$

concluding the proof. □

We are now in position to study the relevant term in (42).

Lemma 4.2 *It results*

$$\int_{\mathbb{R}^N} \mathcal{E}_2 Z_\varepsilon \, dx = \left[-\partial_{11}\omega(0)\mathfrak{b}\varepsilon\rho_\varepsilon - 2\mu_2 c e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \right] (1 + o(1))$$

for some positive constants \mathfrak{b} and c .

Proof Recalling the definition of \mathcal{E}_2 given in (34) we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \mathcal{E}_2 Z_\varepsilon dx &= \int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) \Theta_\varepsilon Z_\varepsilon dx + 2\beta^2 \int_{\mathbb{R}^N} U_\varepsilon (\Upsilon(\varepsilon x) \Phi_\varepsilon(\varepsilon x) \\ &\quad - \Phi_\varepsilon(0) \Upsilon(0)) Z_\varepsilon dx \\ &\quad + \mu_2 \int_{\mathbb{R}^N} (U_\varepsilon^3 - U_{P_\varepsilon}^3 - U_{-P_\varepsilon}^3) Z_\varepsilon dx \\ &\quad + 3\beta\mu_2 \int_{\mathbb{R}^N} (U_\varepsilon^2 - U_{P_\varepsilon}^2 - U_{-P_\varepsilon}^2) \Psi_\varepsilon Z_\varepsilon dx \tag{47} \\ &\quad + \beta^3 \int_{\mathbb{R}^N} \Phi_\varepsilon^2(\varepsilon x) \Theta_\varepsilon Z_\varepsilon dx + 2\beta^3 \int_{\mathbb{R}^N} \Psi_\varepsilon \Upsilon(\varepsilon x) \Phi_\varepsilon(\varepsilon x) Z_\varepsilon dx \\ &\quad + 3\mu_2\beta^2 \int_{\mathbb{R}^N} U_\varepsilon \Psi_\varepsilon^2 Z_\varepsilon dx + \mu_2\beta^3 \int_{\mathbb{R}^N} \Psi_\varepsilon^3 Z_\varepsilon dx. \end{aligned}$$

Direct computations yield

$$\begin{aligned} \int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) \Theta_\varepsilon Z_\varepsilon dx &= \int_{\mathbb{R}^N} (\omega(0) - \omega(\varepsilon x)) U_\varepsilon(x) Z_\varepsilon dx \\ &\quad + \beta \int_{\mathbb{R}^N} (\omega(0) - \omega(\varepsilon x)) \Psi_\varepsilon Z_\varepsilon dx \\ &= \int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) U_{P_\varepsilon} \partial_1 U_{P_\varepsilon} dx \\ &\quad - \int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) U_{P_\varepsilon} \partial_1 U_{-P_\varepsilon} dx \tag{48} \\ &\quad + \int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) U_{-P_\varepsilon} \partial_1 U_{P_\varepsilon} dx \\ &\quad - \int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) U_{-P_\varepsilon} \partial_1 U_{-P_\varepsilon} dx \\ &\quad + \beta \int_{\mathbb{R}^N} (\omega(0) - \omega(\varepsilon x)) \Psi_\varepsilon Z_\varepsilon dx. \end{aligned}$$

Let us start studying the first term on the right hand side. Taking into account (45), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) U_{P_\varepsilon} \partial_1 U_{P_\varepsilon} dx &= -\frac{1}{2} \int_{\mathbb{R}^N} \langle D^2\omega(0)\varepsilon x, \varepsilon x \rangle U \left(x + \frac{P_\varepsilon}{\varepsilon}\right) \partial_1 U \left(x + \frac{P_\varepsilon}{\varepsilon}\right) dx \\ &\quad + \mathcal{O} \left(\int_{\mathbb{R}^N} \varepsilon^3 \left|x + \frac{P_\varepsilon}{\varepsilon}\right|^3 U^2(y) dx \right) \\ &= -\frac{1}{2} \int_{\mathbb{R}^N} \langle D^2\omega(0)(\varepsilon y - P_\varepsilon), \varepsilon y - P_\varepsilon \rangle U(y) \partial_1 U(y) dy \\ &\quad + o(\varepsilon\rho_\varepsilon) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \langle D^2\omega(0)\varepsilon y, P_\varepsilon \rangle U(y) \partial_1 U(y) dy \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{2} \int_{\mathbb{R}^N} \langle D^2 \omega(0) P_\varepsilon, \varepsilon y \rangle U(y) \partial_1 U(y) dy + o(\varepsilon \rho_\varepsilon) \\
 &= \varepsilon \rho_\varepsilon \underbrace{\int_{\mathbb{R}^N} \langle D^2 \omega(0) P_0, y \rangle U(y) \partial_1 U(y) dy}_{:=A} + o(\varepsilon \rho_\varepsilon) \\
 &= \partial_{11}^2 \omega(0) \mathfrak{b} \varepsilon \rho_\varepsilon + o(\varepsilon \rho_\varepsilon).
 \end{aligned}$$

Note that, from (12) and from the fact that U is radially decreasing we deduce that

$$A := \underbrace{\partial_{11}^2 \omega(0) \int_{\mathbb{R}^N} y_1^2 U(y) \frac{U'(y)}{|y|} dy}_{<0} + \sum_{i=2}^N \partial_{1i}^2 \omega(0) \underbrace{\int_{\mathbb{R}^N} y_1 y_i U(y) \frac{U'(y)}{|y|} dy}_{=0}$$

so that

$$A := -\partial_{11}^2 \omega(0) \mathfrak{b} \text{ and } \mathfrak{b} := - \int_{\mathbb{R}^N} y_1^2 U(y) \frac{U'(y)}{|y|} dy > 0 \tag{49}$$

since $\partial_{11}^2 \omega(0) < 0$, thanks to (9). In a similar way we get that

$$- \int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) U_{-P_\varepsilon} \partial_1 U_{-P_\varepsilon} dx = A \varepsilon \rho_\varepsilon + o(\varepsilon \rho_\varepsilon).$$

We claim that the other terms on the right hand side of (48) are of higher order with respect to $\varepsilon \rho_\varepsilon$. Indeed, Lemma A.3 and (12) yield

$$\begin{aligned}
 &\int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) U_{-P_\varepsilon} \partial_1 U_{P_\varepsilon} dx = -\frac{1}{2} \int_{\mathbb{R}^N} \langle D^2 \omega(0) \varepsilon x, \varepsilon x \rangle \\
 &U\left(x - \frac{P_\varepsilon}{\varepsilon}\right) \partial_1 U\left(x + \frac{P_\varepsilon}{\varepsilon}\right) dx \\
 &+ \mathcal{O}\left(\int_{\mathbb{R}^N} \varepsilon^3 |x|^3 U\left(x - \frac{P_\varepsilon}{\varepsilon}\right) U\left(x + \frac{P_\varepsilon}{\varepsilon}\right) dx\right) \\
 &= -\frac{1}{2} \int_{\mathbb{R}^N} \langle D^2 \omega(0) P_\varepsilon, P_\varepsilon \rangle U\left(y - 2\frac{P_\varepsilon}{\varepsilon}\right) \partial_1 U(y) dy + o(\varepsilon \rho_\varepsilon) \\
 &= -\frac{1}{2} \rho_\varepsilon^2 \langle D^2 \omega(0) P_0, P_0 \rangle \int_{\mathbb{R}^N} U\left(y - 2\frac{P_\varepsilon}{\varepsilon}\right) \partial_1 U(y) dy + o(\varepsilon \rho_\varepsilon) \\
 &= \frac{1}{2} \mathfrak{c} \rho_\varepsilon^2 \langle D^2 \omega(0) P_0, P_0 \rangle e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon}\right)^{-(N-1) + \frac{N+1}{2}} + o(\varepsilon \rho_\varepsilon) \\
 &= \frac{1}{2} \mathfrak{c} \rho_\varepsilon^2 \langle D^2 \omega(0) P_0, P_0 \rangle e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon}\right)^{-\frac{N-1}{2}} \frac{\rho_\varepsilon}{\varepsilon} + o(\varepsilon \rho_\varepsilon) \\
 &= o(\varepsilon \rho_\varepsilon).
 \end{aligned}$$

Similarly

$$\int_{\mathbb{R}^N} (\omega_0 - \omega(\varepsilon x)) U_{P_\varepsilon} \partial_1 U_{-P_\varepsilon} dx = o(\varepsilon \rho_\varepsilon).$$

The last term in (48) can be managed, by applying Lemma 2.3. Indeed, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} (\omega(0) - \omega(\varepsilon x)) \Psi_\varepsilon Z_\varepsilon dx \right| \\ & \leq c\varepsilon^2 \|\Psi_\varepsilon\|_{L^2(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |x|^4 U_{P_\varepsilon}^2 dx \right)^{\frac{1}{2}} \leq \varepsilon^{\frac{N}{2}} \rho_\varepsilon^2 = o(\varepsilon \rho_\varepsilon). \end{aligned}$$

Let us now study the second term on the right hand side of (47). It holds

$$\begin{aligned} \int_{\mathbb{R}^N} U_\varepsilon (\Upsilon(\varepsilon x) \Phi_\varepsilon(\varepsilon x) - \Phi_\varepsilon(0) \Upsilon(0)) Z_\varepsilon dx &= \int_{\mathbb{R}^N} U_\varepsilon \Upsilon(\varepsilon x) (\Phi_\varepsilon(\varepsilon x) - \Phi_\varepsilon(0)) Z_\varepsilon dx \\ &+ \int_{\mathbb{R}^N} U_\varepsilon (\Upsilon(\varepsilon x) - \Upsilon_\varepsilon(0)) \Phi_\varepsilon(0) Z_\varepsilon dx. \end{aligned}$$

In view of Remark 2.2, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^N} U_\varepsilon \Upsilon(\varepsilon x) (\Phi_\varepsilon(\varepsilon x) - \Phi_\varepsilon(0)) Z_\varepsilon dx \right| &\lesssim \int_{\mathbb{R}^N} |\Phi_\varepsilon(\varepsilon x) - \Phi_\varepsilon(0)| U_{P_\varepsilon}^2 dx \\ &\lesssim \varepsilon^2 \int_{\mathbb{R}^N} |x|^{2-\frac{N}{m}} U_{P_\varepsilon}^2 dx \\ &\lesssim \varepsilon^2 \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{2-\frac{N}{m}} = o(\varepsilon \rho_\varepsilon) \text{ if } m > N \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} U_\varepsilon (\Upsilon(\varepsilon x) - \Upsilon_\varepsilon(0)) \Phi_\varepsilon(0) Z_\varepsilon dx \right| \\ & \leq C\varepsilon^2 \|\Phi_\varepsilon\|_\infty \int_{\mathbb{R}^N} |x|^2 U_{P_\varepsilon}^2 dx \leq \varepsilon^{\frac{N}{2}} \rho_\varepsilon^2 = o(\varepsilon \rho_\varepsilon). \end{aligned}$$

Let us study the cubic and square term in U_{P_ε} in (47). We have

$$\begin{aligned} \mu_2 \int_{\mathbb{R}^N} (U_\varepsilon^3 - U_{P_\varepsilon}^3 - U_{-P_\varepsilon}^3) Z_\varepsilon dx &= 3\mu_2 \int_{\mathbb{R}^N} U_{P_\varepsilon}^2 U_{-P_\varepsilon} Z_\varepsilon dx \\ &+ 3\mu_2 \int_{\mathbb{R}^N} U_{P_\varepsilon} U_{-P_\varepsilon}^2 Z_\varepsilon dx \\ &= 3\mu_2 \int_{\mathbb{R}^N} U_{P_\varepsilon}^2 U_{-P_\varepsilon} \partial_1 U_{P_\varepsilon} dx \\ &- 3\mu_2 \int_{\mathbb{R}^N} U_{P_\varepsilon}^2 U_{-P_\varepsilon} \partial_1 U_{-P_\varepsilon} dx \end{aligned}$$

$$\begin{aligned}
 &+3\mu_2 \int_{\mathbb{R}^N} U_{P_\varepsilon} U_{-P_\varepsilon}^2 \partial_1 U_{P_\varepsilon} dx \\
 &-3\mu_2 \int_{\mathbb{R}^N} U_{P_\varepsilon} U_{-P_\varepsilon}^2 \partial_1 U_{-P_\varepsilon} dx \\
 = &3\mu_2 \int_{\mathbb{R}^N} U \left(y - 2\frac{P_\varepsilon}{\varepsilon} \right) U^2(y) \partial_1 U(y) dy \\
 &-3\mu_2 \int_{\mathbb{R}^N} U^2 \left(y + 2\frac{P_\varepsilon}{\varepsilon} \right) U(y) \partial_1 U(y) dy \\
 &+3\mu_2 \int_{\mathbb{R}^N} U^2 \left(y - 2\frac{P_\varepsilon}{\varepsilon} \right) U(y) \partial_1 U(y) dy \\
 &-3\mu_2 \int_{\mathbb{R}^N} U \left(y + 2\frac{P_\varepsilon}{\varepsilon} \right) U^2(y) \partial_1 U(y) dy.
 \end{aligned}$$

By exploiting Lemma A.3-(i) with $s = 1$ and $t = 3$ we deduce

$$\begin{aligned}
 3\mu_2 \int_{\mathbb{R}^N} U \left(y - 2\frac{P_\varepsilon}{\varepsilon} \right) U^2(y) \partial_1 U(y) dy &= \mu_2 \int_{\mathbb{R}^N} U \left(y - 2\frac{P_\varepsilon}{\varepsilon} \right) \partial_1 U^3(y) dy \\
 &= -\mu_2 c e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} (1 + o(1))
 \end{aligned}$$

and similarly

$$-3\mu_2 \int_{\mathbb{R}^N} U \left(y + 2\frac{P_\varepsilon}{\varepsilon} \right) U^2(y) \partial_1 U(y) dy = -\mu_2 c e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} (1 + o(1)).$$

Applying again Lemma A.3-(ii) with $s = t = 2$ we infer

$$\begin{aligned}
 3\mu_2 \int_{\mathbb{R}^N} U^2 \left(y + 2\frac{P_\varepsilon}{\varepsilon} \right) U(y) \partial_1 U(y) dy &= \frac{3}{2} \mu_2 \int_{\mathbb{R}^N} U^2 \left(y + 2\frac{P_\varepsilon}{\varepsilon} \right) \partial_1 U^2(y) dy \\
 &= \begin{cases} \bar{c} e^{-4\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{1}{2}} (1 + o(1)) & \text{if } N = 2 \\ \bar{c} e^{-4\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-2} \ln \frac{\rho_\varepsilon}{\varepsilon} (1 + o(1)) & \text{if } N = 3 \end{cases} \\
 &= o \left(e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \right)
 \end{aligned}$$

and similarly

$$3\mu_2 \int_{\mathbb{R}^N} U^2 \left(y - 2\frac{P_\varepsilon}{\varepsilon} \right) U(y) \partial_1 U(y) dy = o \left(e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \right).$$

Hence

$$\mu_2 \int_{\mathbb{R}^N} (U_\varepsilon^3 - U_{P_\varepsilon}^3 - U_{-P_\varepsilon}^3) Z_\varepsilon dx = -2\mu_2 c e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} (1 + o(1)).$$

Again, by using Lemma 2.3 and Lemma A.2, it follows

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (U_\varepsilon^2 - U_{P_\varepsilon}^2 - U_{-P_\varepsilon}^2) \Psi_\varepsilon Z_\varepsilon dx \right| &\leq C \left[\int_{\mathbb{R}^N} U_{P_\varepsilon}^2 U_{-P_\varepsilon} |\Psi_\varepsilon| + \int_{\mathbb{R}^N} U_{-P_\varepsilon}^2 U_{P_\varepsilon} |\Psi_\varepsilon| \right] \\ &\leq C \|\Psi_\varepsilon\|_{L^2(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} U^2 \left(y - 2 \frac{P_\varepsilon}{\varepsilon} \right) U^4(y) dy \right)^{\frac{1}{2}} \\ &\leq C \varepsilon^{\frac{N}{2}} e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \\ &= o \left(e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \right). \end{aligned}$$

By Lemma 2.1 and Lemma 2.3 we also obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \Phi_\varepsilon^2(\varepsilon x) \Theta_\varepsilon Z_\varepsilon dx \right| &\leq C \int_{\mathbb{R}^N} |\Phi_\varepsilon(\varepsilon x)|^2 U_{P_\varepsilon}^2 dx + C \int_{\mathbb{R}^N} |\Phi_\varepsilon(\varepsilon x)|^2 |\Psi_\varepsilon| dx \lesssim \varepsilon^N \\ &= o(\varepsilon \rho_\varepsilon) \\ \left| \int_{\mathbb{R}^N} \Psi_\varepsilon \Upsilon(\varepsilon x) \Phi_\varepsilon(\varepsilon x) Z_\varepsilon dx \right| &\leq C \|\Phi_\varepsilon\|_\infty \|\Psi_\varepsilon\|_{L^2(\mathbb{R}^N)} \lesssim \varepsilon^N \\ &= o(\varepsilon \rho_\varepsilon), \end{aligned}$$

and the same upper bound holds for the last two terms in (47), concluding the proof. \square

Proof of Theorem 1.1: completed First notice that (u, v) is a solution of (3) iff (u, v) is a solution of (18). Then, we look for a solution of (18) of the form given in (21). So that, we are lead to look for $(\varphi, \psi) \in X$ satisfying the system (31). Proposition 3.4 yields the existence of (φ, ψ) satisfying the second equation in (31). The first equation is solved as well, if we show that $c_0 = 0$ where c_0 is given in (42). Then, Lemma 4.1 and 4.2 imply that we have to find $d = d_\varepsilon$ in (22) such that

$$\left[-\partial_{11}\omega(0) b_\varepsilon \rho_\varepsilon - 2\mu_2 c e^{-2\sqrt{\omega_0} \frac{\rho_\varepsilon}{\varepsilon}} \left(\frac{\rho_\varepsilon}{\varepsilon} \right)^{-\frac{N-1}{2}} \right] (1 + o(1)) = 0,$$

which is equivalent (taking into account that $\rho_\varepsilon = d_\varepsilon \varepsilon \ln \frac{1}{\varepsilon}$) to

$$(1 - \sqrt{\omega_0} d) \ln \varepsilon + o(\ln \varepsilon) = 0$$

and this is solvable by d_ε such that $d_\varepsilon \rightarrow \frac{1}{\sqrt{\omega_0}}$ as $\varepsilon \rightarrow 0$. That concludes the proof. \square

Acknowledgements The authors warmly thank the anonymous referee for her/his precious comments, and in particular for having pointed out a gap in the proof of Lemma 2.1 in a previous version of this manuscript.

Funding Open access funding provided by Università degli Studi della Campania Luigi Vanvitelli within the CRUI-CARE Agreement. Work partially supported by the MUR-PRIN-20227HX33Z ‘‘Pattern formation in nonlinear phenomena’’, the project HORIZON EUROPE SEEDS – S51 STEPS: ‘‘STEerability and controllability of PDES in Agricultural and Physical models’’, the project ‘‘Start’’ within the program of the University ‘‘Luigi Vanvitelli’’ reserved to young researchers, Piano strategico 2021–2023, the MUR grant

Dipartimento di Eccellenza 2023-2027, the Portuguese government through FCT/Portugal under the project PTDC/MAT-PUR/1788/2020, the INdAM-GNAMPA group.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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Appendix A. Technical Lemma

Lemma A.2 (Lemma 3.7 in [1]) *Let $u, v : \mathbb{R}^N \rightarrow \mathbb{R}$ be two positive continuous radial functions such that*

$$u(x) \sim |x|^a e^{-b|x|}, \quad v(x) \sim |x|^{a'} e^{-b'|x|}$$

as $|x| \rightarrow \infty$, where $a, a' \in \mathbb{R}$, and $b, b' > 0$. Let $\xi \in \mathbb{R}^N$ such that $|\xi| \rightarrow \infty$. We denote $u_\xi(x) = u(x - \xi)$. Then the following asymptotic estimates hold:

(i) *If $b < b'$,*

$$\int_{\mathbb{R}^N} u_\xi v \sim e^{-b|\xi|} |\xi|^a.$$

A similar expression holds if $b > b'$, by replacing a and b with a' and b' .

(ii) *If $b = b'$, suppose that $a \geq a'$. Then:*

$$\int_{\mathbb{R}^N} u_\xi v \sim \begin{cases} e^{-b|\xi|} |\xi|^{a+a'+\frac{N+1}{2}} & \text{if } a' > -\frac{N+1}{2}, \\ e^{-b|\xi|} |\xi|^a \log |\xi| & \text{if } a' = -\frac{N+1}{2}, \\ e^{-b|\xi|} |\xi|^a & \text{if } a' < -\frac{N+1}{2}. \end{cases}$$

Lemma A.3 ((Lemma A.2 in [20]) *Let $U_{\lambda,\mu} = \sqrt{\frac{\lambda}{\mu_1}} U(\sqrt{\lambda}x)$ where U is the solution of (15). Let $s, t \geq 1$ and consider the following integral*

$$\Theta_{s,t}(\zeta) := \int_{\mathbb{R}^N} U_{\lambda,\mu}^s(x + \zeta) \partial_{x_1} U_{\lambda,\mu}^t(x) dx, \quad \zeta \in \mathbb{R}^N.$$

(i) If $s < t$ then

$$\Theta_{s,t}(\zeta) \sim c s \frac{\zeta_1}{|\zeta|} e^{-s\sqrt{\lambda}|\zeta|} |\zeta|^{-s\frac{N-1}{2}} \quad \text{as } |\zeta| \rightarrow +\infty;$$

(ii) If $s = t$ then

$$\Theta_{s,t}(\zeta) \sim \begin{cases} c s \frac{\zeta_1}{|\zeta|} e^{-s\sqrt{\lambda}|\zeta|} |\zeta|^{-s(N-1)+\frac{N+1}{2}} & \text{if } s < \frac{N+1}{N-1}; \\ c s \frac{\zeta_1}{|\zeta|} e^{-s\sqrt{\lambda}|\zeta|} |\zeta|^{-s\frac{(N-1)}{2}} \ln |\zeta| & \text{if } s = \frac{N+1}{N-1}; \\ c s \frac{\zeta_1}{|\zeta|} e^{-s\sqrt{\lambda}|\zeta|} |\zeta|^{-s\frac{(N-1)}{2}} & \text{if } s > \frac{N+1}{N-1}. \end{cases}$$

as $|\zeta| \rightarrow +\infty$.

Here c denotes a positive constant. In case $N = 1$ the first option in (ii) holds, namely

$$\Theta_{s,s}(\zeta) \sim c s \frac{\zeta_1}{|\zeta|} e^{-s|\zeta|} |\zeta| \quad \text{as } |\zeta| \rightarrow +\infty.$$

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