Diffusive Lotka-Volterra competition models on periodically evolving domains

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Abstract

[1]The paper deals with the asymptotic dynamics of a diffusive Lotka-Volterra competition model with periodic coefficients and zero-flux boundary conditions acting on a spatially isotropic and temporally periodic evolving domain. We determine sufficient conditions for the global stability of the spatially homogeneous positive periodic solution. Moreover, the phenomenon of competitive exclusion is investigated.

Keywords. Lotka-Volterra diffusive model. Evolving domain. Periodicity. Global stability.

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1 Introduction

Reaction-diffusion equations on time-variable domains are intrinsically non-autonomous, even in the case of constant coefficients ([**?**]). Thus it is natural to deal with non-autonomous equations under the assumption the domain varies in time.

During the past decades, the reaction diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + u(a_1 - b_{11}u - b_{12}v) \\ \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + v(a_2 - b_{21}u - b_{22}v) \end{cases} \tag{1.1}$$

on some cylindrical time-space domain $\Omega \times [0, +\infty[$ has been widely studied. If the coefficients b_{ij} and d_i, $i, j = 1, 2$, are positive, system (**??**) is usually referred to in the literature as the Lotka-Volterra model for two competing species with

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diffusion. In (??) the unknowns $u(x,t), v(x,t)$ denote the population densities and the terms $d_1\Delta u$, $d_2\Delta v$ are due to the dispersion in the habitat Ω.

When a homogeneous boundary Neumann condition is imposed to the solutions, the question of coexistence or, more generally, the long-term behavior of positive solutions has been investigated in numerous articles; see, for example, [?, ?, ?, ?, ?, ?, ?] and the references quoted therein.

In this paper, system (??) is discussed under the assumption that a_i, d_i, b_{ij}, $i, j = 1, 2$, are continuous functions of t alone, periodic of period T, which means the environment is temporally periodic. In this case, a solution to the ODE system

$$\begin{cases} u' = u(a_1 - b_{11}u - b_{12}v) \\ \\ v' = v(a_2 - b_{21}u - b_{22}v) \end{cases} \tag{1.2}$$

may be viewed as a spatially homogeneous solution to the system of reaction-diffusion equations (??). Thus, it doesn't catch that, for a reaction-diffusion Lotka-Volterra competition model with spatially independent reaction terms, the asymptotic dynamics is deeply influenced by the asymptotic behavior of solutions to (??).

In all the aforementioned articles, the investigations on system (??) are tackled under the assumption the domain Ω is fixed. On the other hand, the habitat of species often changes due to many reasons; for example, the depth and area of rivers and lakes may change due to seasons effect. We say the habitat is an evolving domain when it exhibits spells of growth and spells of contraction, hence it is a particular time-varying domain Ω_t. Moreover, in this paper, we assume that Ω_t varies periodically in time.

A natural question arises about the reaction of the species to the changes in their habitat and in particular the effects of evolving domains on long term behavior of solutions. However, in most cases, it is difficult to carry out stability analysis, because of the presence of some extra terms, namely an advection and a dilution term, appearing in diffusive models on evolving domains.

To simplify the model equations we use Lagrangian transformations (see [?]), changing (??) into a system on a fixed domain, and we assume that the domain evolution is periodic and isotropic. Under these assumptions, the presence of an evolving domain has the mathematical effect of modifying diffusion rates and growth rates of the populations, leading to system (??) on the fixed domain Ω_0. The new model preserves the main features of (??), that is it remains a competitive system with diffusion and time-periodic coefficients.

To the best of our knowledge, few analytical results are available regarding the stability of solutions to reaction-diffusion systems on variable domains. Indeed, the long-time properties are often carried out through numerical simulations ([?, ?]).

In most articles the dynamic properties of competition diffusion systems on fixed domains are obtained by constructing upper and lower solutions ([?, ?]) or using families of contracting rectangles ([?]). In this paper, we use the Lyapunov functional method ([?]) to explore the conditions underlying global stability of system (??). We prove, in particular, that its global dynamics is determined only by the coefficients of kinetic system (**??**) and by the evolution rate of the domain, regardless of the diffusion rates. Our main results, Theorems **??** and **??**, provide new sufficient conditions for stable coexistence and competitive exclusion of species in model (**??**), even in the case of fixed domains.

2 Preliminaries

Let $\Omega_t \subset \mathbb{R}^n$ $(n \geq 1)$ be a simply connected bounded evolving domain for all $t \geq 0$ and let $\partial\Omega_t$ be its evolving boundary. Also, let $u_1(x(t), t)$, $u_2(x(t), t)$ be the density of two species at time t and position $x(t)$.

The diffusive Lotka-Volterra competition system on Ω_t can be obtained from the application of Reynolds transport theorem (see [**?**]). The evolution of Ω_t generates a flow velocity field $\mathbf{a}(x, t)$, which has the effect of introducing the extra terms $\mathbf{a} \cdot \nabla u_i$ and $u_i(\nabla \cdot \mathbf{a})$ $(i = 1, 2)$ in the classical system on a fixed domain. $\mathbf{a} \cdot \nabla u_i$ is called advection term, while $u_i(\nabla \cdot \mathbf{a})$ is called dilution term.

Hence, we study the reaction diffusion system on the cylinder $\Omega_t \times [0, +\infty[$

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \mathbf{a} \cdot \nabla u_1 + u_1(\nabla \cdot \mathbf{a}) = d_1 \Delta u_1 + u_1(a_1 - b_{11}u_1 - b_{12}u_2) \\[2ex] \dfrac{\partial u_2}{\partial t} + \mathbf{a} \cdot \nabla u_2 + u_2(\nabla \cdot \mathbf{a}) = d_2 \Delta u_2 + u_2(a_2 - b_{21}u_1 - b_{22}u_2). \end{cases} \tag{2.1}$$

We assume that the environment is temporally periodic and spatially homogeneous, so that all coefficients depend only on time t and are supposed to be continuous and T-periodic. Moreover for $i, j = 1, 2$, $i \neq j$, $d_i(t), b_{ii}(t) > 0$, $b_{ij}(t) \geq 0$ and $[a_i(t)] > 0$, where, if f is a continuous T-periodic function,

$$[f(t)] = \frac{1}{T} \int_0^T f(t)\, dt$$

denotes the integral average (or mean value) of f.

We impose self-organization on the system through the Neumann boundary conditions

$$\frac{\partial u_1}{\partial \mathbf{n}} = \frac{\partial u_2}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega_t, \quad t > 0, \tag{2.2}$$

and we take initial conditions

$$u_1(x, 0) = \varphi_1(x),\ u_2(x, 0) = \varphi_2(x), \quad x \in \overline{\Omega}_0, \tag{2.3}$$

with $\varphi_1(x), \varphi_2(x) \in C^2(\Omega_0) \cap C(\overline{\Omega}_0)$, $\varphi_1(x), \varphi_2(x) \geq 0$ for every $x \in \overline{\Omega}_0$.

In particular, a solution of (**??**) is said to be positive if the initial data (**??**) satisfy $\varphi_1(x), \varphi_2(x) > 0$ for all $x \in \Omega_0$.

As a standard assumption in the derivation of reaction-diffusion equations on evolving domains (see [**?**, **?**]) we formulate the following hyphotesis:

(H_1) The flow velocity $\mathbf{a}(x, t)$ coincides with the domain velocity, i.e.,

$$\mathbf{a}(x, t) = (x_1'(t), \ldots, x_n'(t)). \tag{2.4}$$

For analytic convenience we introduce a transformation mapping model (**??**) into a system on a fixed domain. In order to do that, we assume a special class of domain evolution which satisfies the further assumption

(H_2) (Isotropic and periodic domain evolution). There exists a C^1, T-periodic function $\rho(t)$ subject to

$$\rho(0) = 1,\ \rho(t) > 0 \text{ for } t > 0,$$

such that, for every $x(t) \in \Omega_t$,

$$(x_1(t), \dots, x_n(t)) = \rho(t)(y_1, \dots, y_n), \quad y = (y_1, \dots, y_n) \in \Omega_0. \tag{2.5}$$

Then, (u_1, u_2) is mapped into the new function (v_1, v_2) defined by

$$\begin{aligned} v_1(y_1, \dots y_n, t) &= u_1(x_1(t), \dots, x_n(t), t) \\ v_2(y_1, \dots y_n, t) &= u_2(x_1(t), \dots, x_n(t), t). \end{aligned} \tag{2.6}$$

Previous assumptions (H_1) and (H_2) lead to a resulting model on the fixed domain Ω_0

$$\begin{cases} \dfrac{\partial v_1}{\partial t} + n\dfrac{\rho'(t)}{\rho(t)}v_1 = \dfrac{d_1(t)}{\rho^2(t)}\Delta v_1 + v_1(a_1(t) - b_{11}(t)v_1 - b_{12}(t)v_2) \\ \dfrac{\partial v_2}{\partial t} + n\dfrac{\rho'(t)}{\rho(t)}v_2 = \dfrac{d_2(t)}{\rho^2(t)}\Delta v_2 + v_2(a_2(t) - b_{21}(t)v_1 - b_{22}(t)v_2), \\ \dfrac{\partial v_1}{\partial \mathbf{n}} = \dfrac{\partial v_2}{\partial \mathbf{n}} = 0 \text{ in } \partial\Omega_0 \times]0, +\infty[, \\ v_1(y, 0) = \varphi_1(y),\ u_2(y, 0) = \varphi_2(y), \quad y \in \overline{\Omega}_0, \end{cases} \tag{2.7}$$

where n is the spatial dimension (see [**?**, **?**] for details).

Before analyzing (**??**), we consider the diffusive logistic equation on the continuously evolving domain Ω_t, i.e.,

$$\frac{\partial u}{\partial t} + \mathbf{a} \cdot \nabla u + u(\nabla \cdot \mathbf{a}) = d(t)\Delta u + u(a(t) - b(t)u), \tag{2.8}$$

subject to Neumann boundary condition and a positive initial condition $\varphi(x) \in C^2(\Omega_0) \cap C(\overline{\Omega}_0)$. Under assumptions ($H_1$) and ($H_2$) and transformation (**??**), we get the diffusive logistic equation on the cylinder $\Omega_0 \times [0, +\infty[$

$$\frac{\partial v}{\partial t} + n\frac{\rho'(t)}{\rho(t)}v = \frac{d(t)}{\rho^2(t)}\Delta v + v(a(t) - b(t)v), \tag{2.9}$$

equipped with the zero-flux condition

$$\frac{\partial v}{\partial \mathbf{n}}(y, t) = 0, \qquad y \in \partial\Omega_0, t > 0, \tag{2.10}$$

and the initial condition

$$v(y,0) = \varphi(y), \qquad y \in \overline{\Omega}_0. \tag{2.11}$$

We assume that the coefficients $a(t), b(t), d(t)$ are continuous T-periodic functions, $d(t), b(t) > 0$ and $[a(t)] > 0$.

Setting

$$\gamma(t) = a(t) - n\frac{\rho'(t)}{\rho(t)}, \tag{2.12}$$

one may write differential equation (**??**) as

$$\frac{\partial v}{\partial t} = \frac{d(t)}{\rho^2(t)}\Delta v + v(\gamma(t) - b(t)v), \tag{2.13}$$

which has the form of a diffusive logistic equation on the fixed domain Ω_0, with diffusion coefficients $\frac{d(t)}{\rho^2(t)}$ and carrying capacity $\gamma(t)$, where $[\gamma(t)] = [a(t)] > 0$.

It is well known ([**?**]) that, under our assumptions, the corresponding kinetic logistic equation

$$v' = v(\gamma(t) - b(t)v)$$

admits a unique positive periodic solution $v^\circ(t)$. The following result shows that $v^\circ(t)$ is globally asymptotically stable as a solution to (**??**).

Theorem 2.1. *The periodic solution $v^\circ(t)$ is globally attractive for* (**??**)-(**??**)*, i.e., every positive solution $v(y,t)$ of the initial value problem converges to the spatially homogeneous function $v^\circ(t)$ as $t \to +\infty$, that is*

$$\lim_{t\to+\infty} |v(y,t) - v^\circ(t)| = 0 \text{ uniformly w.r.t. } y \in \Omega_0. \tag{2.14}$$

Proof. Under the substitution

$$w(y,t) = \frac{v(y,t)}{v^\circ(t)} - 1, \tag{2.15}$$

equation (**??**) or, equivalently, equation (**??**), turns into

$$\frac{\partial w}{\partial t} = \frac{d(t)}{\rho^2(t)}\Delta w - (w+1)r(t)w, \tag{2.16}$$

where $r(t) = b(t)v^\circ(t)$. Obviously $w(y,t) = 0$ is a solution to (**??**).

Let us consider the Lyapunov function

$$W(t) = \int_{\Omega_0} (w - \log(1+w))\, dy;$$

then, by using the zero-flux condition on the boundary and the Gauss formula, we get

$$\begin{aligned}
W'(t) &= \int_{\Omega_0} \left(\frac{\partial w}{\partial t} - \frac{\partial w}{\partial t}\frac{1}{1+w}\right) dy = \int_{\Omega_0} \frac{\partial w}{\partial t}\frac{w}{w+1}\, dy \\
&= \frac{d(t)}{\rho^2(t)}\int_{\Omega_0} \frac{w}{w+1}\Delta w\, dy - r(t)\int_{\Omega_0} w^2\, dy \\
&= -\frac{d(t)}{\rho^2(t)}\int_{\Omega_0} \frac{|\nabla w|^2}{(1+w)^2}\, dy - r(t)\int_{\Omega_0} w^2\, dy.
\end{aligned}$$

Therefore $W'(t) < 0$ for $w \neq 0$ and there exists $\lambda > 0$ such that

$$W'(t) \leq -\lambda \int_{\Omega_0} w^2 \, dy. \tag{2.17}$$

Integrating (**??**) from a fixed $\bar{t} > 0$ to t, we obtain

$$\lambda \int_{\bar{t}}^{t} ds \left(\int_{\Omega_0} w^2 \, dy \right) \leq W(\bar{t}) - W(t) < W(\bar{t}) < +\infty.$$

Thus,

$$\int_{\bar{t}}^{+\infty} ds \left(\int_{\Omega_0} w^2 \, dy \right) < +\infty$$

and, consequently ([**?**, Lemma 2.1]),

$$\lim_{t \to +\infty} \|w(\cdot, t)\|_{L^2(\Omega_0)} = 0. \tag{2.18}$$

Fix $p > \max\{n, 2\}$; then the Sobolev inequality yields, for $(y,t) \in \Omega_0 \times [0, +\infty[$ ([**?**]),

$$\begin{aligned} |w(y,t)|^p &\leq \int_{\Omega_0} |w|^p \, dy + \int_{\Omega_0} |\nabla w|^p \, dy \\ &\leq c_1 \int_{\Omega_0} |w|^2 \, dy + c_2 \int_{\Omega_0} |\nabla w|^2 \, dy. \end{aligned} \tag{2.19}$$

Moreover,

$$\lim_{t \to +\infty} \int_{\Omega_0} |\nabla w|^2 \, dy = 0;$$

in fact, multiplying (**??**) by w and integrating over Ω_0, we get that there exists $c > 0$ satisfying

$$\int_{\Omega_0} |\nabla w|^2 \, dy \leq c \left(-\frac{1}{2} \frac{d}{dt} \int_{\Omega_0} w^2 \, dy + \int_{\Omega_0} w^2 \, dy \right).$$

From this, (**??**) and (**??**), it follows that $\lim_{t \to +\infty} |w(y,t)| = 0$ uniformly w.r.t. $y \in \Omega_0$.

Going back to $v(y,t)$ by means of (**??**), we have

$$\lim_{t \to +\infty} |v(y,t) - v^\circ(t)| = 0 \text{ uniformly w.r.t. } y \in \Omega_0$$

and this completes the proof. □

3 Coexistence

In this section we study the stable coexistence of the species in model (**??**).

From now on, we rewrite the equations in (**??**) as

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = \dfrac{d_1(t)}{\rho^2(t)}\Delta v_1 + v_1(\gamma_1(t) - b_{11}(t)v_1 - b_{12}(t)v_2) \\ \\ \dfrac{\partial v_2}{\partial t} = \dfrac{d_2(t)}{\rho^2(t)}\Delta v_2 + v_2(\gamma_2(t) - b_{21}(t)v_1 - b_{22}(t)v_2), \end{cases} \tag{3.1}$$

where $\gamma_i(t) = a_i(t) - n\dfrac{\rho'(t)}{\rho(t)}$, $i = 1, 2$.

Therefore we have transformed system (**??**) into a reaction-diffusion Lotka-Volterra system on the fixed domain Ω_0 with diffusion coefficients and carrying capacities depending on $\rho(t)$. Note that $[\gamma_i(t)] = [a_i(t)] > 0$, for $i = 1, 2$.

If either $b_{12} = 0$ or $b_{21} = 0$, then each population density $v_i(y, t)$, $i = 1, 2$, satisfies a logistic diffusive equation, so that, under Neumann boundary conditions and positive initial conditions, Theorem **??** applies.

The Lotka-Volterra diffusive system (**??**) admits two semitrivial periodic solutions, given by $(U(t), 0)$ and $(0, V(t))$, where $U(t)$ is the positive periodic solution to

$$u' = u\left(\gamma_1(t) - b_{11}(t)u\right) \tag{3.2}$$

and $V(t)$ is the positive periodic solution to

$$v' = v\left(\gamma_2(t) - b_{22}(t)v\right). \tag{3.3}$$

Under the assumptions

$$[a_1(t)] > [b_{12}(t)V(t)] \quad \text{and} \quad [a_2(t)] > [b_{21}(t)U(t)], \tag{3.4}$$

we denote by $U^0(t)$ the positive periodic solution to

$$u' = u(\gamma_1(t) - b_{12}(t)V(t) - b_{11}(t)u)$$

and by $V^0(t)$ the positive periodic solution to

$$v' = v\left(\gamma_2(t) - b_{21}(t)U(t) - b_{22}(t)v\right).$$

The following theorem gives a persistence result for the kinetic Lotka-Volterra competitive system corresponding to (**??**).

Theorem 3.1. *Consider the ODE system*

$$\begin{cases} u' = u\left(\gamma_1(t) - b_{11}(t)u - b_{12}(t)v\right) \\ \\ v' = v\left(\gamma_2(t) - b_{21}(t)u - b_{22}(t)v\right). \end{cases} \tag{3.5}$$

If (**??**) *holds true, then* (**??**) *admits a positive periodic solution* $(u^*(t), v^*(t))$ *such that*

$$U^0(t) \leq u^*(t) \leq U(t), \quad V^0(t) \leq v^*(t) \leq V(t), \quad t > 0. \tag{3.6}$$

Moreover, for any positive solution $(u(t), v(t))$ *to* (**??**) *there exists* $\bar{t} > 0$ *such that, for any* $t \geq \bar{t}$,

$$U^0(t) \leq u(t) \leq U(t), \quad V^0(t) \leq v(t) \leq V(t). \tag{3.7}$$

Proof. The proof of (**??**) comes directly from [**?**] (see also [**?**]).

We pass now to prove (**??**) and, to this end, we fix a positive solution $(u(t), v(t))$ to (**??**). If $u(0) \leq U(0)$, then $u(t) \leq U(t)$ for every $t \geq 0$.

Suppose $u(0) > U(0)$ and let $x(t)$ be the solution to (**??**) with $x(0) = u(0)$. By means of the comparison theorem, we get that $u(t) < x(t)$ and $U(t) < x(t)$ for any $t > 0$. Moreover (see [**?**]),

$$\lim_{t \to +\infty} |x(t) - U(t)| = 0. \tag{3.8}$$

From this it follows that there exists $t_0 > 0$ satisfying

$$u(t_0) \leq U(t_0). \tag{3.9}$$

In fact, by contradiction, suppose that $u(t) > U(t)$ for every $t > 0$; then

$$x(t) \geq u(t) > U(t), \quad t > 0.$$

Taking the above relation and (**??**) into account,

$$\lim_{t \to +\infty} (u(t) - U(t)) = 0,$$

so that $\lim\limits_{t \to +\infty} v(t) = 0$. On the other hand, $v(t)$ is the solution to the logistic equation

$$v' = v\left(\gamma_2(t) - b_{21}(t)u - b_{22}(t)v\right)$$

with $v(0) > 0$, hence it is bounded from below by a positive constant. This leads to a contradiction, thus (**??**) holds true and, consequently, for every $t \geq t_0$,

$$u(t) \leq U(t). \tag{3.10}$$

In the same way one proves that there exists $t_1 \geq t_0$ such that, for any $t \geq t_1$,

$$v(t) \leq V(t). \tag{3.11}$$

By using (**??**), (**??**) and the comparison theorem, we get the remaining inequalities. □

The next theorem shows a similar result holds for the corresponding reaction-diffusion system.

Theorem 3.2. *Under assumption* (**??**)*, the temporally periodic rectangle*

$$\Sigma(t) := \left[U^0(t), U(t)\right] \times \left[V^0(t), V(t)\right], \quad t \geq 0, \tag{3.12}$$

is invariant and attractive for (**??**)*. More precisely,*

(a) If $(v_1(y,t), v_2(y,t))$ is a positive solution to (**??**) *with $(\varphi_1(y), \varphi_2(y)) \in \Sigma(0)$ for every $y \in \overline{\Omega}_0$, then $(v_1(y,t), v_2(y,t)) \in \Sigma(t)$ for every $t > 0$ and $y \in \overline{\Omega}_0$.*

(b) If $(v_1(y,t), v_2(y,t))$ is a positive solution to (**??**)*, there exists $\overline{t} > 0$ such that $(v_1(y,t), v_2(y,t)) \in \Sigma(t)$ for every $t > \overline{t}$ and $y \in \overline{\Omega}_0$.*

Proof. Fix a positive solution $(v_1(y,t), v_2(y,t))$ to (**??**). Since $U(t)$ is the positive periodic solution to (**??**), it also satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{d_1(t)}{\rho^2(t)} \Delta u + u\left(\gamma_1(t) - b_{11}(t)u\right)$$

subject to $\dfrac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega_0 \times]0, +\infty[$. Hence, if $\varphi_1(y) = v_1(y,0) \le U(0)$ for every $y \in \overline{\Omega}_0$, by using the comparison theorem for parabolic equations, we get $v_1(y,t) \le U(t)$ for every $t > 0$ and $y \in \overline{\Omega}_0$.

Assume now that $M = \max_{y\in\overline{\Omega}_0} v_1(y,0) > U(0)$ and set $N = \min_{y\in\overline{\Omega}_0} v_2(y,0)$. Let us denote by $(u(t), v(t))$ the solution to

$$\begin{cases} u' = u\left(\gamma_1(t) - b_{11}(t)u - b_{12}(t)v\right) \\ \\ v' = v\left(\gamma_2(t) - b_{21}(t)u - b_{22}(t)v\right) \\ \\ u(0) = M, v(0) = N. \end{cases}$$

$(u(t), v(t))$ is a spatially homogeneous solution to (**??**), so that the comparison theorem for diffusive competition systems (see [**?**, **?**]) ensures that $v_1(y,t) \le u(t)$ for every $t > 0$ and $y \in \overline{\Omega}_0$. On the other hand, taking Theorem **??** into account, there exists $\bar{t} > 0$ such that, for every $t > \bar{t}$, $u(t) \le U(t)$. Consequently, for $t \ge \bar{t}$ and $y \in \overline{\Omega}_0$,

$$v_1(y,t) \le u(t) \le U(t).$$

Reasoning in the same way, one can prove the remaining inequalities. □

Below, the main result of the section. It provides a sufficient condition for the asymptotic stable coexistence of species $v_1(y,t), v_2(y,t)$ in model (**??**).

Theorem 3.3. *Assume that condition* (**??**) *is satisfied and let* $(u^*(t), v^*(t))$ *be a positive periodic solution to* (**??**) *as in Theorem* **??**. *If, in addition, for any* $t \in [0,T]$,

$$(b_{12}(t)v^*(t) + b_{21}(t)u^*(t))^2 - 4b_{11}(t)u^*(t) \cdot b_{22}(t)v^*(t) < 0, \tag{3.13}$$

then $(u^*(t), v^*(t))$ *is a globally attractive solution to system* (**??**).

Proof. Let $(v_1(y,t), v_2(y,t))$ be a positive solution to (**??**) and consider the Lyapunov function

$$W(t) = \int_{\Omega_0} \left[\frac{v_1}{u^*(t)} - 1 - \log\left(\frac{v_1}{u^*(t)}\right) + \frac{v_2}{v^*(t)} - 1 - \log\left(\frac{v_2}{v^*(t)}\right) \right] dy.$$

We have that $W(t) > 0$ if $(v_1, v_2) \neq (u^*, v^*)$. By easy calculations we get the following expression for the derivative of $W(t)$

$$
\begin{aligned}
W'(t) = & \int_{\Omega_0} \left[\frac{\partial}{\partial t}\left(\frac{v_1}{u^*(t)}\right)\left(\frac{v_1 - u^*(t)}{v_1}\right) + \frac{\partial}{\partial t}\left(\frac{v_2}{v^*(t)}\right)\left(\frac{v_2 - v^*(t)}{v_2}\right)\right] dy \\
= & \int_{\Omega_0} \frac{d_1(t)}{\rho^2(t)} \frac{\Delta v_1}{u^*(t)} \left(1 - \frac{u^*(t)}{v_1}\right) dy + \int_{\Omega_0} \frac{d_2(t)}{\rho^2(t)} \frac{\Delta v_2}{v^*(t)} \left(1 - \frac{v^*(t)}{v_2}\right) dy \qquad (3.14) \\
- & \int_{\Omega_0} A(y,t)\, dy,
\end{aligned}
$$

where

$$
\begin{aligned}
A(y,t) &= b_{11}(t) u^*(t) \left(\frac{v_1(y,t)}{u^*(t)} - 1\right)^2 \\
&+ (b_{12}(t) v^*(t) + b_{21}(t) u^*(t)) \left(\frac{v_1(y,t)}{u^*(t)} - 1\right)\left(\frac{v_2(y,t)}{v^*(t)} - 1\right) \qquad (3.15) \\
&+ b_{22}(t) v^*(t) \left(\frac{v_2(y,t)}{v^*(t)} - 1\right)^2 .
\end{aligned}
$$

We notice that, integrating by parts and taking the Neumann boundary conditions into account, one yields

$$
\int_{\Omega_0} \frac{d_1(t)}{\rho^2(t)} \frac{\Delta v_1}{u^*(t)} \left(1 - \frac{u^*(t)}{v_1}\right) dy = -\frac{d_1(t)}{\rho^2(t)} \int_{\Omega_0} \frac{|\nabla v_1|^2}{v_1^2} dy \qquad (3.16)
$$

and

$$
\int_{\Omega_0} \frac{d_2(t)}{\rho^2(t)} \frac{\Delta v_2}{v^*(t)} \left(1 - \frac{v^*(t)}{v_2}\right) dy = -\frac{d_2(t)}{\rho^2(t)} \int_{\Omega_0} \frac{|\nabla v_2|^2}{v_2^2} dy. \qquad (3.17)
$$

For any fixed $t > 0$, let us now consider the quadratic form

$$
B(t; x_1, x_2) = -b_{11}(t) u^*(t) x_1^2 - 2 \frac{(b_{12}(t) v^*(t) + b_{21}(t) u^*(t))}{2} x_1 x_2 - b_{22}(t) v^*(t) x_2^2.
$$

Under assumption (**??**), $B(t; x_1, x_2)$ is negative definite. Hence, denoting by $\Lambda(t)$ the largest eigenvalue of the symmetric matrix

$$
\begin{pmatrix}
-b_{11}(t) u^*(t) & -\dfrac{(b_{12}(t) v^*(t) + b_{21}(t) u^*(t)}{2} \\
-\dfrac{(b_{12}(t) v^*(t) + b_{21}(t) u^*(t)}{2} & -b_{22}(t) v^*(t),
\end{pmatrix},
$$

we deduce

$$
B(t; x_1, x_2) \leq -\Lambda(t)(x_1^2 + x_2^2).
$$

Setting

$$
\lambda = \min_{t \in [0,T]} \Lambda(t),
$$

previous inequality implies

$$
A(t,y) \leq -\lambda \left[\left(\frac{v_1(y,t)}{u^*(t)} - 1\right)^2 + \left(\frac{v_2(y,t)}{v^*(t)} - 1\right)^2 \right]. \qquad (3.18)
$$

Taking (**??**), (**??**)-(**??**) into account, we get that

$$W'(t) \leq -\lambda \int_{\Omega_0} \left[\left(\frac{v_1}{u^*(t)} - 1 \right)^2 + \left(\frac{v_2}{v^*(t)} - 1 \right)^2 \right] dy. \tag{3.19}$$

From this, reasoning as in the final part of the proof of Theorem **??**, we obtain

$$\lim_{t\to+\infty} \frac{v_1(y,t)}{u^*(t)} - 1 = 0 = \lim_{t\to+\infty} \frac{v_2(y,t)}{v^*(t)} - 1$$

uniformly w.r.t. $y \in \Omega_0$. We conclude that

$$\lim_{t\to+\infty} |v_1(y,t) - u^*(t)| = 0 = \lim_{t\to+\infty} |v_2(y,t) - v^*(t)|$$

uniformly w.r.t. $y \in \Omega_0$, ending the proof. □

In the case $\rho(t) = 1$, Ω_t is a fixed domain Ω and (**??**) is the classical reaction-diffusion system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 + u_1(a_1 - b_{11}u_1 - b_{12}u_2) \\ \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 + u_2(a_2 - b_{21}u_1 - b_{22}u_2), \end{cases} \tag{3.20}$$

endowed with the boundary conditions

$$\frac{\partial u_1}{\partial \mathbf{n}} = \frac{\partial u_2}{\partial \mathbf{n}} = 0 \quad (x,t) \in \partial\Omega \times]0, +\infty[, \tag{3.21}$$

which models the two-species competition diffusion phenomena with no-flux on the boundary in population dynamics.

Theorem **??**, when applied to (**??**)-(**??**), yields the following result.

Theorem 3.4. *Suppose that inequalities* (**??**) *hold true. Then* (**??**)-(**??**) *admits a positive* T*-periodic solution* $(u^\circ(t), v^\circ(t))$. *If, in addition,*

$$(b_{12}(t)v^\circ(t) + b_{21}(t)u^\circ(t))^2 < 4b_{11}(t)u^\circ(t) \cdot b_{22}(t)v^\circ(t), \tag{3.22}$$

for any positive solution (u_1, u_2) *of* (**??**)-(**??**) *with positive initial value,*

$$u_1(x,t) - u^\circ(t) \to 0, \quad u_2(x,t) - v^\circ(t) \to 0$$

as $t \to +\infty$, *uniformly w.r.t.* $x \in \Omega$.

Note that $(u^\circ(t), v^\circ(t))$ is exactly $(u^*(t), v^*(t))$ for $\rho(t) = 1$.

Model (**??**)-(**??**) has been studied in [**?**]. The more general case where the coefficients depend on (x,t) is considered in [**?**], while in [**?**] the authors investigate the almost periodic case.

The conditions for the global asymptotic stability introduced in [**?**, **?**] require the positivity of the coefficients $a_i(t)$, $i = 1, 2$, hypothesis which is not necessary in Theorem **??**. The common characteristic in all those conditions and (**??**) is a *sort of dominance property* of the coefficients $b_{11}(t)$ and $b_{22}(t)$ over $b_{12}(t)$ and $b_{21}(t)$.

Remark 3.1. *It is well known (see [?]) that, when the coefficients in (??) are positive constants, the inequalities*

$$a_1 > b_{12}\left(\frac{a_2}{b_{22}}\right) \qquad a_2 > b_{21}\left(\frac{a_1}{b_{11}}\right) \tag{3.23}$$

ensure the global asymptotic stability of the unique equilibrium point (u°, v°) *given by*

$$u^\circ = \frac{a_1 b_{22} - a_2 b_{12}}{b_{11} b_{22} - b_{12} b_{21}} \quad v^\circ = \frac{a_2 b_{11} - a_1 b_{22}}{b_{11} b_{22} - b_{12} b_{21}}.$$

We point out that condition (??) is equivalent to (??), as in the autonomous case $U(t) = \frac{a_1}{b_{11}}$, $V(t) = \frac{a_2}{b_{22}}$. *Therefore the results in Theorem ?? are in accordance with the ones available in the autonomous case.*

4 Competitive exclusion

Our aim is investigating the phenomenon of the competitive exclusion for (??), which entails that one of the two populations in the model, for large t, tends to occupy the whole environment, driving the other species to extinction.

We begin with some preliminaries about the kinetic Lotka-Volterra system (??) (see [?]).

Theorem 4.1. *Under the assumptions*

$$-b_{22}(t)[a_1(t)] + b_{12}(t)[a_2(t)] < 0 \text{ and } -b_{21}(t)[a_1(t)] + b_{11}(t)[a_2(t)] < 0, \tag{4.1}$$

the semitrivial solution $(U(t), 0)$ *is a globally attractive solution to system (??).*

The previous result also holds true in presence of diffusion, as the theorem below shows.

Theorem 4.2. *Under assumption (??)* $(U(t), 0)$ *is globally attractive as a solution to problem (??).*

Proof. Let $(v_1(y,t), v_2(y,t))$ be a positive solution to (??); we want to prove that

$$\lim_{t\to+\infty} |v_1(y,t) - U(t)| = 0 = \lim_{t\to+\infty} v_2(y,t)$$

uniformly w.r.t. $y \in \Omega_0$.

First of all, we notice that, by means of the comparison theorem, $0 < v_2(y,t) \le V(t)$, $y \in \Omega_0$; further, as shown in [?, ?], assumption (??) implies that

$$[a_1(t)] > [b_{12}(t)V(t)] \quad \text{and} \quad [a_2(t)] < [b_{21}(t)U(t)].$$

In particular, since $[a_1(t)] > [b_{12}(t)V(t)]$, from Theorem ?? we get that there exists $t_0 > 0$ such that

$$U^0(t) \le v_1(y,t) \le U(t) \tag{4.2}$$

for every $t > t_0$ and $y \in \Omega_0$.

Theorem **??** ensures that the ODE system (**??**) admits the global stable non negative solution $(U(t),0)$. Hence, according to the results of [**?**], any positive solution to (**??**) is asymptotically periodic and spatially homogeneous. It follows that

$$\nabla v_1 \to 0 \quad \text{and } \nabla v_2 \to 0 \tag{4.3}$$

as $t \to +\infty$, uniformly with respect to the spatial variables.

It remains to prove that (v_1, v_2) converges to $(U(t),0)$ as $t \to +\infty$.

Since (**??**) holds, we are allowed to choose $\alpha > 0$, $\beta > 2$ such that

$$\frac{[a_2(t)]}{[a_1(t)]} < \frac{\alpha}{\beta} < \min_{t\in[0,T]} \left\{ \frac{b_{21}(t)}{b_{11}(t)}, \frac{b_{22}(t)}{b_{12}(t)} \right\}. \tag{4.4}$$

Consider the function

$$Z(t) = \int_{\Omega_0} v_1^{-\alpha} v_2^{\beta}\, dy, \tag{4.5}$$

which is well defined thanks to (**??**). A direct calculation gives

$$\begin{aligned} Z'(t) =& \int_{\Omega_0} \left(-\alpha v_1^{-\alpha-1} v_2^{\beta} \frac{\partial v_1}{\partial t} + \beta v_1^{-\alpha} v_2^{\beta-1} \frac{\partial v_2}{\partial t} \right) dy \\ =& -\alpha \frac{d_1(t)}{\rho^2(t)} \int_{\Omega_0} v_1^{-\alpha-1} v_2^{\beta} \Delta v_1\, dy \\ &- \alpha \int_{\Omega_0} \left(v_1^{-\alpha} v_2^{\beta} \left(\gamma_1(t) - b_{11}(t)v_1 - b_{12}(t)v_2\right)\right) dy \\ &+ \beta \frac{d_2(t)}{\rho^2(t)} \int_{\Omega_0} v_1^{-\alpha} v_2^{\beta-1} \Delta v_2\, dy \\ &+ \beta \int_{\Omega_0} \left(v_1^{-\alpha} v_2^{\beta} \left(\gamma_2(t) - b_{21}(t)v_1 - b_{22}(t)v_2\right)\right) dy \\ =& I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned} \tag{4.6}$$

Integrating by parts and using the zero-flux boundary conditions,

$$\begin{aligned} I_1(t) &= \alpha \frac{d_1(t)}{\rho^2(t)} \int_{\Omega_0} \nabla(v_1^{-\alpha-1} v_2^{\beta}) \cdot \nabla v_1 dy \\ &= -\alpha(\alpha+1) \frac{d_1(t)}{\rho^2(t)} \int_{\Omega_0} v_1^{-\alpha-2} v_2^{\beta} |\nabla v_1|^2 dy \\ &+ \alpha\beta \frac{d_1(t)}{\rho^2(t)} \int_{\Omega_0} v_1^{-\alpha-1} v_2^{\beta-1} \nabla v_1 \cdot \nabla v_2\, dy. \end{aligned}$$

In the same way one can show that

$$\begin{aligned} I_3(t) &= -\beta(\beta-1) \frac{d_2(t)}{\rho^2(t)} \int_{\Omega_0} v_1^{-\alpha} v_2^{\beta-2} |\nabla v_2|^2\, dy \\ &+ \alpha\beta \frac{d_2(t)}{\rho^2(t)} \int_{\Omega_0} v_1^{-\alpha-1} v_2^{\beta-1} \nabla v_1 \cdot \nabla v_2\, dy. \end{aligned}$$

Concerning $I_2(t) + I_4(t)$, we find that

$$\begin{aligned} I_2(t) + I_4(t) &= (-\alpha\gamma_1(t) + \beta\gamma_2(t))\, Z(t) \\ &+ (\alpha b_{11}(t) - \beta b_{21}(t)) \int_{\Omega_0} v_1^{-\alpha+1} v_2^{\beta} dy \\ &+ (\alpha b_{12}(t) - \beta b_{22}(t)) \int_{\Omega_0} v_1^{-\alpha} v_2^{\beta+1} dy. \end{aligned} \tag{4.7}$$

Using (**??**), (**??**) and (**??**), we can write

$$Z'(t) \leq (-\alpha\gamma_1(t) + \beta\gamma_2(t))\, Z(t) + I_1(t) + I_3(t). \tag{4.8}$$

We point out that, by means of (**??**),

$$\lim_{t\to+\infty} I_1(t) = \lim_{t\to+\infty} I_3(t) = 0. \tag{4.9}$$

Moreover, condition (**??**) ensures that

$$[-\alpha\gamma_1(t) + \beta\gamma_2(t)] = [-\alpha a_1(t) + \beta a_2(t)] < 0,$$

which entails that the zero solution to

$$u'(t) = (-\alpha\gamma_1(t) + \beta\gamma_2(t))\, u(t)$$

is uniformly asymptotically stable. This, together with (**??**), (**??**) and Theorem 5.5.7 of [**?**], implies that

$$\lim_{t\to+\infty} Z(t) = 0.$$

Accordingly, taking the cited results in [**?**] and (**??**) into account, we deduce

$$\lim_{t\to+\infty} v_1^{-\alpha}(y,t) v_2^{\beta}(y,t) = 0$$

uniformly w.r.t. $y \in \Omega_0$. Hence

$$\lim_{t\to+\infty} v_2(y,t) = 0 \tag{4.10}$$

uniformly w.r.t. $y \in \Omega_0$.

Note that v_1 is a positive solution to the reaction-diffusion equation

$$\frac{\partial v}{\partial t} = \frac{d_1(t)}{\rho^2(t)} \Delta v + v((\gamma_1(t) - b_{12}(t) v_2)) - b_{11}(t) v). \tag{4.11}$$

The coefficient

$$\alpha(y,t) = \gamma_1(t) - b_{12}(t) v_2(y,t)$$

is asymptotic to $\gamma_1(t)$ by (**??**), that is

$$\lim_{t\to+\infty} (\alpha(y,t) - \gamma_1(t)) = 0$$

uniformly w.r.t. $y \in \Omega_0$. Adapting the argument of Theorem 2.4 in [?] to diffusive logistic equation (??), we obtain

$$\lim_{t\to+\infty} |v_1(y,t) - U(t)| = 0 \tag{4.12}$$

uniformly w.r.t. $y \in \Omega_0$. Below, the main steps of the proof.

Fix $\varepsilon > 0$ such that $0 < \varepsilon < [\gamma_1(t)]$. Then, by means of (??), there exists $t_\varepsilon > 0$ such that, for all $t > t_\varepsilon$ and all $y \in \Omega_0$,

$$\gamma_1(t) - \varepsilon < \gamma_1(t) - b_{12}(t)v_2(y,t) < \gamma_1(t) + \varepsilon.$$

Let us denote by $v_{\pm\varepsilon}(y,t)$ the solution to the diffusive logistic equation

$$\frac{\partial v}{\partial t} = \frac{d_1(t)}{\rho^2(t)}\Delta v + v((\gamma_1(t) \pm \varepsilon) - b_{11}(t)v),$$

equipped with zero-flux boundary condition and the initial condition

$$v(y,t_\varepsilon) = v_1(y,t_\varepsilon), \qquad y \in \overline{\Omega}_0,$$

and by $U_{\pm\varepsilon}(t)$ the periodic solution to the corresponding kinetic logistic equation.

We have that, for every $t > t_\varepsilon$ and $y \in \Omega_0$,

$$v_{-\varepsilon}(y,t) < v_1(y,t) < v_{+\varepsilon}(y,t).$$

Theorem **??** yields

$$\lim_{t\to+\infty} |v_{\pm\varepsilon}(y,t) - U_{\pm\varepsilon}(t)| = 0$$

uniformly w.r.t. $y \in \Omega_0$. Since

$$\lim_{\varepsilon\to 0} U_{\pm\varepsilon}(t) = U(t),$$

we get (??). □

In the same way, the following result provides a sufficient condition for the asymptotic extinction of species $v_1(y,t)$.

Corollary 4.1. *If the inequalities*

$$-b_{11}(t)[a_2(t)] + b_{21}(t)[a_1(t)] < 0 \textit{ and } -b_{12}(t)[a_2(t)] + b_{22}(t)[a_1(t)] < 0$$

are satisfied, then the semitrivial solution $(0, V(t))$ is globally attractive as a solution to problem (??).

Remark 4.1. *In the case of a fixed domain Ω ($\rho(t) = 1$) and positive spatially homogeneous coefficients, Ahmad and Lazer ([?]) show that any positive solution (u_1, u_2) of (??)-(??) satisfies*

$$\lim_{t\to+\infty}(u_1(x,t) - U(t)) = 0 \textit{ and } \lim_{t\to+\infty} u_2(x,t) = 0$$

uniformly w.r.t. $x \in \overline{\Omega}$ *if the following inequalities are verified:*

$$(a_1)_L > (b_{12})_M \frac{(a_2)_M}{(b_{22})_L}, \quad (a_2)_M < (b_{21})_L \frac{(a_1)_L}{(b_{11})_M}. \tag{4.13}$$

In (**??**)

$$(a_1)_L = \min_{t\in[0,T]} a_1(t),$$

$$(a_2)_M = \max_{t\in[0,T]} a_2(t),$$

and so on for the other coefficients.

Our assumption (**??**) *is weaker than* (**??**)*, thus, even for a fixed domain, Theorem* **??** *improves the corresponding result in [*__?__*].*

5 Conclusions

In this paper, we have studied the asymptotic dynamics of a diffusive Lotka-Volterra competition model with T-periodic coefficients on a T-periodically and isotropically evolving domain. Under our assumptions, the presence of an evolving domain has the effect of modifying the diffusion rates and also the growth rates of the populations, without changing their mean value. In particular, we have determined sufficient conditions for the stable coexistence and the competitive exclusion, which seem, even in the case of a fixed domain, to be weaker than the ones already present in the literature.

The no-flux boundary conditions, imposed above, mean that no individual can move in or out through the boundary of the habitat. Such boundary conditions lead to the following relevant property: any positive solution of our model is asymptotically spatially homogeneous, that is the population tends to spread out in the habitat in a uniform way.

The results of the paper show the influence of the domain evolution on the coexistence and competitive exclusion of the species. In particular,
(i) The conditions for the asymptotic coexistence provided in Theorem **??** depend on $(u^*(t), v^*(t))$ and, consequently on the evolution rate $\rho(t)$ and on the dimension n of the habitat.
(ii) On the contrary the phenomenon of extinction of only one species occurs under the same conditions for a fixed domain, but the limiting values $U(t)$ or $V(t)$ depend on $\rho(t)$ and n since the growth rates $\gamma_i(t) = a_i(t) - n\frac{\rho(t)}{\rho'(t)}$, $i = 1, 2$.

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