TWISTOR GEOMETRY OF THE FLAG MANIFOLD

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ABSTRACT. A study is made of algebraic curves and surfaces in the flag manifold $\mathbb{F} = SU(3)/T^2$, and their configuration relative to the twistor projection π from $\mathbb{F}$ to the complex projective plane $\mathbb{P}^2$, defined with the help of an anti-holomorphic involution j. This is motivated by analogous studies of algebraic surfaces of low degree in the twistor space $\mathbb{P}^3$ of S^4. Deformations of twistor fibres project to real surfaces in $\mathbb{P}^2$, whose metric geometry is investigated. Attention is then focussed on toric Del Pezzo surfaces that are the simplest type of surfaces in $\mathbb{F}$ of bidegree $(1,1)$. These surfaces define orthogonal complex structures on specified dense open subsets of $\mathbb{P}^2$ relative to its Fubini-Study metric. The discriminant loci of various surfaces of bidegree $(1,1)$ are determined, and bounds given on the number of twistor fibres that are contained in more general algebraic surfaces in $\mathbb{F}$.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to study the complex 3-dimensional flag manifold $\mathbb{F}$ and some of the associated geometrical structures arising from its description as the homogeneous space $SU(3)/T^2$. If we fix an invariant complex structure on $\mathbb{F}$ then there are three natural projections from $\mathbb{F}$ to the complex projective plane $\mathbb{P}^2$, one of which (call it π) is neither holomorphic nor anti-holomorphic. The resulting three fibrations play an implicit role in the classification of harmonic maps of surfaces into the complex projective planes $\mathbb{P}^2$ [18, 19], though in this paper, we shall be more concerned with real branched coverings of $\mathbb{P}^2$ defined by the choice of an algebraic surface in $\mathbb{F}$.

Let p be a point of $\mathbb{P}^2$ and ℓ a line in $\mathbb{P}^2$. The pair (p, ℓ) defines a point of $\mathbb{F}$ if $p \in \ell$. We can regard a line ℓ in $\mathbb{P}^2$ as a point in the dual complex projective plane $\mathbb{P}^{2\vee}$, so that $\mathbb{F}$ is naturally an algebraic subvariety of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$. This is the standpoint that we adopt in the early sections of this paper, in which our notation exploits to a maximum the underlying elementary linear algebra. For example, the relation $p \in \ell$ is equivalent to the vanishing of the pairing $p\ell$, and the line through two distinct points p, q can be represented by a cross product $p \times q$. Section 2 uses the resulting

double fibration to compute Hodge numbers defined by line bundles $\mathcal{O}(a, b)$. This also enables us to associate a bidegree to both curves and surfaces in $\mathbb{F}$.

Section 3 focusses attention on the most basic families of curves and surfaces in the flag variety $\mathbb{F}$. A family $\mathcal{V}$ of curves $L_{q,m}$ of bidegree $(1, 1)$ is parametrized by the complement of $\mathbb{F}$ in $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$, and realizes each element of $\mathcal{V}$ as the intersection ${}_qH \cap H_m$ of two Hirzebruch surfaces of type 1. Each of these surfaces can, merely by their description as a subvariety of $\mathbb{F}$, be viewed simultaneously as a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ and as the blowup of $\mathbb{P}^2$ at one point. An arbitrary smooth curve of bidegree $(1, 1)$ has the form $L_{q,m}$ for some (q, m) with $qm \neq 0$. We study intersections between members of this family and various Hirzebruch surfaces.

Section 4 deals with the classification of surfaces S in $\mathbb{F}$ of bidegree $(1, 1)$, each of which corresponds to a complex 3×3 matrix A up to the addition of a scalar multiple of the identity and rescaling. There is an analogy with the simultaneous diagonalization of quadratic forms, but in general A will not be diagonalizable, which leads to singular and reducible examples. Indeed, the classification of all such surfaces provides a geometrical illustration of Jordan canonical form in the simplest of cases. We then proceed to study the equivalence of such surfaces under unitary transformations.

We begin to examine the twistor picture in section 5. The Hermitian structure allows one to associate to p a line $p^* \in \mathbb{P}^{2\vee}$, and to ℓ a point $\ell^* \in \mathbb{P}^2$. The anti-linear involution $j \colon (p, \ell) \mapsto (\ell^*, p^*)$ of $\mathbb{F}$ has no fixed points. We can then define π by mapping (p, ℓ) to the point $p^* \times \ell \in \mathbb{P}^2$. Then π commutes with j, and exhibits $\mathbb{F}$ as the twistor space of the complex projective plane $\mathbb{P}^2$ with its standard (self-dual) Fubini-Study metric. The distinction between $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$ is now less important, and we obtain a triple π_1, π, π_2^* of fibrations of $\mathbb{F} \to \mathbb{P}^2$. They are permuted by means of outer automorphisms (of which j is one) arising from the Weyl group of $SU(3)$, but are distinguished by our choice of complex structure on $\mathbb{F}$.

Local sections s of π parametrize almost complex structures J on open sets of $\mathbb{P}^2$, and a fundamental property of a twistor space asserts that the image of s is holomorphic if and only if J is complex, i.e., its Nijenhuis tensor vanishes. The involution j maps J to $-J$ and, in the twistor context, j-invariant objects are called 'real'. In the analogous situation of the Penrose fibration $\mathbb{P}^3 \to S^4$, there has been extensive study of algebraic surfaces in $\mathbb{P}^3$ and their associated orthogonal complex structures in domains of S^4 [35, 2, 3, 4, 5, 15, 6, 7, 21, 8].

Section 6 explains the relevance of the basic geometry of curves and surfaces to the twistor theory. The fibres of the twistor fibration π form a real subfamily of $\mathcal{V}$, whereas a generic smooth curve $L_{q,m}$ projects to a surface of revolution, whose first fundamental form (induced from the Fubini-Study metric) we identify. As q approaches the line m, the image predictably acquires a dumbbell shape, reflecting the degeneration of $L_{q,m}$ to two lines. The underlying $U(1)$ symmetry enables us to visualize this in Figure 7, and other relevant surfaces of revolution are constructed in Section 9 and diplayed in Figure 7.

If a complex 3×3 matrix A has three distinct eigenvalues, then the associated surface S in $\mathbb{F}$ is a smooth Del Pezzo surface of degree 6. Any such surface is invariant by the action of the maximal torus T^2 of diagonal matrices in $SU(3)$, and this allows us to use toric methods to describe its behaviour relative to the twistor projection. Such surfaces S have bidegree $(1, 1)$, and are the analogues of quadrics in $\mathbb{P}^3$. For example, S is j-invariant if and only if A is Hermitian, and in this case S contains a family of twistor fibres parametrized by a circle. It then becomes a natural problem to understand the configuration of such a surface relative to π, and to determine how many fibres of π it can contain.

The problem of determining the branch locus of certain (real or toric) surfaces of bidegree $(1, 1)$ relative to π is considered in Section 7. In Section 8, we learn that a $(1, 1)$ surface can contain zero, one, two, or (if real) infinitely many twistor fibres. This is first proved by Bézout-type methods, and then more explicitly. All these cases are realized by various examples. The final Section 9 describes the twistor fibres and branch loci of smooth but non-real $(1, 1)$ surfaces.

We conclude with some observations that will not be pursued in this paper, but which suggest alternative approaches to, and generalizations of, our work.

There are close analogues of our results with those of [35] on the Penrose fibration $\mathbb{P}^3 \to S^4$. This is to be expected since the twistor spaces $\mathbb{P}^3$ and $\mathbb{F}$ incorporate an open orbit of a complex Heisenberg group, and are birationally equivalent, a fact that extends to any two Wolf spaces of

the same dimension [13]. From the twistor viewpoint, surfaces of bidegree $(1,1)$ in $\mathbb{F}$ evidently correspond to quadrics in $\mathbb{P}^3$; this is the conclusion of the investigation in Section 9. We expect surfaces of bidegree $(2,1)$ or $(1,2)$ ('Del Pezzo double planes') to relate to cubic surfaces in $\mathbb{P}^3$. The latter can contain at most 5 twistor lines (and some do) [8], and according Corollary 8.3 the former can contain at most 6 twistor lines (but this may not be optimal). On the other hand, the flag variety is a richer environment in which to study surfaces, owing to its natural fibrations to planes. For example, surfaces of bidegree $(2,2)$ in $\mathbb{F}$ are K3 surfaces for which the non-commuting involutions arising from π_1 and π_2 give rise to non-trivial dynamics [37].

There is a also a link between $\mathbb{F}$ and $\mathbb{P}^3$ at the topological level. Reducing the structure group of $\mathbb{F}$ to $SO(3)$, one can define complex conjugation as an anti-holomorphic involution acting simultaneously on each factor of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$. Composing this involution with j gives rise to a holomorphic involution $\tilde{\sigma}$ of $\mathbb{F}$ that covers complex conjugation σ in $\mathbb{P}^2$. It is well known that the orbifold $\mathbb{P}^2/\langle\sigma\rangle$ is homeomorphic to S^4, now thought of as the sphere inside the irreducible 5-dimensional representation of $SO(3)$. The quotient $\mathbb{F}/\langle\sigma\rangle$ can be identified with the secant variety of a rational normal curve in $\mathbb{P}^4$, and corresponds to the twistor space of S^4 with its orbifold metric [26].

For a generalization of the approach of Section 5 and triple fibrations in the context of $Spin(7)$ and triality, we cite [32].

2. PRELIMINARIES

In this section, we set up notation that will allow us to work with the flag manifold $\mathbb{F}$.

Throughout the paper, we denote by $\mathbb{P}^n$ the complex projective space $\mathbb{P}(\mathbb{C}^{n+1})$, and by $\mathbb{P}^{n\vee}$ its dual $\mathbb{P}((\mathbb{C}^{n+1})^\vee)$. For the most part, n will equal 2. We shall always regard elements of $\mathbb{C}^3$ as row vectors, and elements of the dual space $(\mathbb{C}^3)^\vee$ as column vectors, with transpose also indicated by the superscript $^\vee$. We shall retain this distinction at the level of homogeneous coordinates, so that the row vector $p = (p_0, p_1, p_2)$ defines $[p_0 : p_1 : p_2]$ in $\mathbb{P}^2$, and the column vector $\ell = (\ell_0, \ell_1, \ell_2)^\vee$ defines $[\ell_0 : \ell_1 : \ell_2]^\vee$ in $\mathbb{P}^{2\vee}$. Assuming our vectors are non-zero, we shall abuse notation by writing $p \in \mathbb{P}^2$ and $\ell \in \mathbb{P}^{2\vee}$, so that the assertions $p \in \ell$ (geometry) and $p\ell = 0$ (algebra) can be used interchangeably.

We study now the bi-projective space $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$. Its Segre embedding into $\mathbb{P}^8$ is induced by the map $(p, \ell) \mapsto \ell p$, in homogeneous coordinates

$$\left([p_0 : p_1 : p_2], \ [\ell_0 : \ell_1 : \ell_2]^\vee \right) \longmapsto \begin{bmatrix} p_0\ell_0 & p_1\ell_0 & p_2\ell_0 \\ p_0\ell_1 & p_1\ell_1 & p_2\ell_1 \\ p_0\ell_2 & p_1\ell_2 & p_2\ell_2 \end{bmatrix}.$$

Its image (the Segre variety) is a fourfold of degree 6 in $\mathbb{P}^8$, see for example [24]. We have a diagram

$$\begin{array}{ccc} & \mathbb{P}^2 \times \mathbb{P}^{2\vee} & \\ \Pi_1 \swarrow & & \searrow \Pi_2 \\ \mathbb{P}^2 & & \mathbb{P}^{2\vee} \end{array}$$

in which Π_1 and Π_2 are the standard projections with $\Pi_1(p, \ell) = p$ and $\Pi_2(p, \ell) = \ell$. Their fibres are linear sections of the Segre variety of codimension 2.

What follows is some relevant algebra. Let $\mathcal{R} := \mathbb{C}[p_0, p_1, p_2]$ be the complex vector space of all homogeneous polynomials in the variables p_0, p_1, p_2. Analogously, set $\mathcal{R}^\vee := \mathbb{C}[\ell_0, \ell_1, \ell_2]$. The spaces $\mathcal{R}$ and $\mathcal{R}^\vee$ are graded in the usual way, so that $\mathcal{R}_a$ denotes the vector space of homogeneous polynomials of degree a in the variables p_0, p_1, p_2. In view of the Segre embedding, we consider $\mathcal{P} := \mathcal{R} \otimes_\mathbb{C} \mathcal{R}^\vee$. Then $\mathcal{P}$ is a polynomial ring in the variables p_i, ℓ_j, bigraded in the following way. Set $\mathcal{P}_{a,b} := \mathcal{R}_a \otimes_\mathbb{C} \mathcal{R}^\vee_b$ for $(a, b) \in \mathbb{N}^2$, so that

$$(1) \qquad \mathcal{P} = \bigoplus_{(a,b)\in\mathbb{N}^2} \mathcal{P}_{a,b}, \quad \dim \mathcal{P}_{a,b} = \binom{a+2}{2}\binom{b+2}{2}$$

Multiplication of polynomials induces a bilinear map $\mathcal{P}_{a,b} \times \mathcal{P}_{c,d} \to \mathcal{P}_{a+c,b+d}$.

For any $a, b \in \mathbb{Z}$, we set $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a, b) = \Pi_1^* \mathcal{O}_{\mathbb{P}^2}(a) \otimes \Pi_2^* \mathcal{O}_{\mathbb{P}^{2\vee}}(b)$. The Leray-Hirsch theorem (or the Künneth formula) implies that $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^{2\vee}) \cong \mathbb{Z}^2$ is freely generated as an abelian group by

$\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(1,0)$ and $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(0,1)$. Since the canonical line bundle $\omega_{\mathbb{P}^2}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-3)$, we have the expression

$$\omega_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}} \cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(-3,-3).$$

for the canonical bundle of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$.

Next, we compute Hodge numbers using the Künneth formula. This gives

$$(2) \qquad h^i(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a,b)) = \sum_{j=0}^{i} h^j(\mathcal{O}_{\mathbb{P}^2}(a)) \, h^{i-j}(\mathcal{O}_{\mathbb{P}^{2\vee}}(b))$$

for all $i \in \mathbb{N}$ and $(a,b) \in \mathbb{Z}^2$. From (2), Serre duality and the cohomology of line bundles on $\mathbb{P}^2$, we deduce

Lemma 2.1.   (i) $h^0(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a,b)) = \begin{cases} \binom{a+2}{2}\binom{b+2}{2} & \text{if} \quad (a,b) \in \mathbb{N}^2 \\ 0 & \text{if either } a < 0 \text{ or } b < 0; \end{cases}$

(ii) $h^1(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a,b)) = 0$ for all $(a,b) \in \mathbb{Z}^2$;

(iii) $h^2(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a,b)) = 0$ if either $a \geqslant 0$ and $b \geqslant -2$, or $a = -1$, or $a \leqslant -2$ and $b < 0$;

(iv) $h^3(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a,b)) = 0$ for all $(a,b) \in \mathbb{Z}^2$;

(v) $h^4(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a,b)) = 0$ if either $a \geqslant -2$, or $b \geqslant -2$.

These results will be refined in the next subsection.

2.1. **The flag manifold.** We next define the main object of study.

Definition 2.2. The *flag manifold* is the algebraic subvariety of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$ given by

$$\mathbb{F} := \big\{ (p,\ell) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} \mid p \in \ell \big\}.$$

Since the condition $p \in \ell$ is equivalent to $p\ell = 0$, we have the coordinate description

$$\mathbb{F} := \{([p_0 : p_1 : p_2], [\ell_0 : \ell_1 : \ell_2]^\vee) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} \mid p_0\ell_0 + p_1\ell_1 + p_2\ell_2 = 0\}.$$

In future, we shall favour the algebraic way of expressing incidence. We shall denote the restrictions of the standard projections to the flag manifold in lower case: $\pi_i := \Pi_i|_{\mathbb{F}}$ for $i = 1,2$. The two maps $\pi_1 : \mathbb{F} \to \mathbb{P}^2$ and $\pi_2 : \mathbb{F} \to \mathbb{P}^{2\vee}$ are locally trivial $\mathbb{P}^1$-bundles. In particular, $\mathbb{F} = \mathbb{P}(\Omega^1_{\mathbb{P}^2}(1))$ with $\pi_1 : \mathbb{P}(\Omega^1_{\mathbb{P}^2}(1)) \to \mathbb{P}^2$ the natural projection as a $\mathbb{P}^1$-bundle.

The fibres of π_1 and π_2 can easily be described explicitly. Let $q \in \mathbb{P}^2$ and $m \in \mathbb{P}^{2\vee}$. Then

$$\pi_1^{-1}(q) = \{(p,\ell) \in \mathbb{F} \mid q\ell = 0\},$$
$$\pi_2^{-1}(m) = \{(p,\ell) \in \mathbb{F} \mid pm = 0\}$$

are linear sections of codimension 3 and so smooth rational curves. Observe that $\pi_1^{-1}(q) \cap \pi_2^{-1}(m) \neq \varnothing$ if and only if $qm = 0$.

With notation from the previous subsection, we can regard $p\ell = p_0\ell_0 + p_1\ell_1 + p_2\ell_2$ as an element of $\mathcal{P}_{1,1}$. Set $\mathcal{S} := \mathcal{P}/(p\ell)$. Since $\mathcal{S}$ is the quotient of a bigraded polynomial ring by a principal ideal generated by a bi-homogeneous polynomial,

$$(3) \qquad \mathcal{S} = \bigoplus_{a,b} \mathcal{S}_{a,b}$$

is also bi-graded. Here, $\mathcal{S}_{a,b}$ is a complex vector space, and multiplication in the ring $\mathcal{S}$ induces a bilinear map $\mathcal{S}_{a,b} \times \mathcal{S}_{c,d} \to \mathcal{S}_{a+c,b+d}$.

Set

$$\mathcal{O}_{\mathbb{F}}(a,b) := \pi_1^* \mathcal{O}_{\mathbb{P}^2}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{2\vee}}(b).$$

The Leray-Hirsch theorem applied to the map $\pi_1 : \mathbb{P}(\Omega^1_{\mathbb{P}^{2\vee}}(1)) \to \mathbb{P}^{2\vee}$ implies that any line bundle on $\mathbb{F}$ is isomorphic to $\mathcal{O}_{\mathbb{F}}(a,b)$ for a unique $(a,b) \in \mathbb{Z}^2$. Since $\mathbb{F}$ is an effective divisor of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$, we can apply this statement to the line bundle that generates $\mathbb{F}$. Indeed, since the fibers of Π_i are linear sections of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$, we must have $\mathbb{F} \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(1,1)|$.

Since $\omega_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}} \cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(-3,-3)$, the adjunction formula gives the expression

$$(4) \qquad \omega_{\mathbb{F}} \cong \mathcal{O}_{\mathbb{F}}(-2,-2)$$

for the canonical bundle of the flag manifold.

From the exact sequence

(5) $$0 \to \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a-1, b-1) \to \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(a, b) \to \mathcal{O}_{\mathbb{F}}(a, b) \to 0$$

and Lemma 2.1 we get:

Lemma 2.3. (i) $h^0(\mathcal{O}_{\mathbb{F}}(a,b)) = \begin{cases} \binom{a+2}{2}\binom{b+2}{2} - \binom{a+1}{2}\binom{b+1}{2} & \text{if} \quad (a,b) \in \mathbb{N}^2 \\ 0 & \text{if either } a < 0 \text{ or } b < 0 \end{cases}$

  (ii) $h^1(\mathcal{O}_{\mathbb{F}}(a,b)) = 0$ if either $a \geqslant 0$ and $b \geqslant 0$, or $a \leqslant 0$ and $b \leqslant 0$;
  (iii) $h^2(\mathcal{O}_{\mathbb{F}}(a,b)) = 0$ if either $a \geqslant 0$ and $b \geqslant -2$, or $a = -1$, or $a \leqslant -2$ and $b < 0$.

Since $\mathcal{S}_{a,b} = H^0(\mathcal{O}_{\mathbb{F}}(a,b))$, we have that $\mathcal{S}_{a,b}$ has dimension $\binom{a+2}{2}\binom{b+2}{2} - \binom{a+1}{2}\binom{b+1}{2}$, for all $(a,b) \in \mathbb{N}^2$. In particular $\dim(\mathcal{S}_{1,1}) = 8$.

Basic results regarding the flag manifold from a related point of view can also be found in [31, §1.1].

2.2. **Automorphisms.**
This subsection briefly describes the projective and unitary automorphisms of the flag manifold $\mathbb{F}$. We shall only need the latter from subsection 4.3 onwards, so we start from the ambient bi-projective space.

The family of automorphisms of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$ is generated by pairs (B_1, B_2) in $SL(3, \mathbb{C}) \times SL(3, \mathbb{C})$, acting via matrix multiplication as

$$(B_1, B_2) \cdot (p, \ell) = (pB_1^{-1}, B_2\ell).$$

To obtain the family of automorphisms of $\mathbb{F}$ it is sufficient to consider those transformations that preserve the equation $p\ell = 0$. Applying (B_1, B_2), one gets

$$p\ell = (pB_1^{-1})(B_2\ell) = p(B_1^{-1}B_2)\ell,$$

so $B_1 = B_2$. We deduce that the automorphisms of $\mathbb{F}$ are generated by matrices $B \in SL(3, \mathbb{C})$ acting as

(6) $$B \cdot (p, \ell) = (pB^{-1}, B\ell).$$

Of course, $e^{2\pi i/3}I$ acts trivially on $\mathbb{F}$, so the projective group acting effectively is $PGL(3, \mathbb{C}) = SL(3, \mathbb{C})/\mathbb{Z}_3$.

The subgroup of projective unitary transformations is defined by imposing a reduction from $SL(3, \mathbb{C})$ to the special unitary group $SU(3)$. For the purposes of calculation, it will be more convenient to allow $B \in U(3)$, given that the centre of $U(3)$ will always act trivially. The realization of the flag manifold as the twistor space $\mathbb{F}$ of $\mathbb{P}^2$ will require us to restrict to this compact subgroup.

Having fixed an origin in $\mathbb{F}$, we can further reduce $SU(3)$ to its standard maximal torus T consisting of diagonal matrices. Because the resulting isotropy representation of $\mathbb{F} = SU(3)/T$ has three irreducible real components, there is (up to homothety) a 2-parameter family of $SU(3)$-invariant Riemannian metrics on $\mathbb{F}$. This family includes two Einstein metrics, a Kähler and a nearly-Kähler one. Both can be constructed as submersions over $\mathbb{P}^2$ with the Fubini-Study metric g_{FS} on $\mathbb{P}^2$, which will however be the only metric relevant to this paper.

The choice of the maximal torus T gives rise to the Weyl group $W = N(T)/T \cong S_3$, where $N(T)$ is the normalizer of T in $SU(3)$. An element of W can be represented by a matrix in $SU(3)$ permuting the coordinates. This projective action will be relevant in the classification of canonical forms for surfaces, see Remark 4.4. There is however a different representation of W that acts non-trivially on cohomology and is especially relevant to the twistor geometry of section 5, see Remark 5.3.

3. Some curves and surfaces in the flag manifold

We start with the concept of bidegree for a curve:

Definition 3.1. Let $C \subset \mathbb{F}$ be an algebraic curve. We define its bidegree $\mathrm{bdeg}(C) = (d_1, d_2)$ as follows: we say that $d_i = 0$ if $\pi_i(C)$ is a point; otherwise $d_i = a_i b_i$, where $a_i = \deg(\pi_i(C))$ and $b_i = \deg(\pi|_C)$.
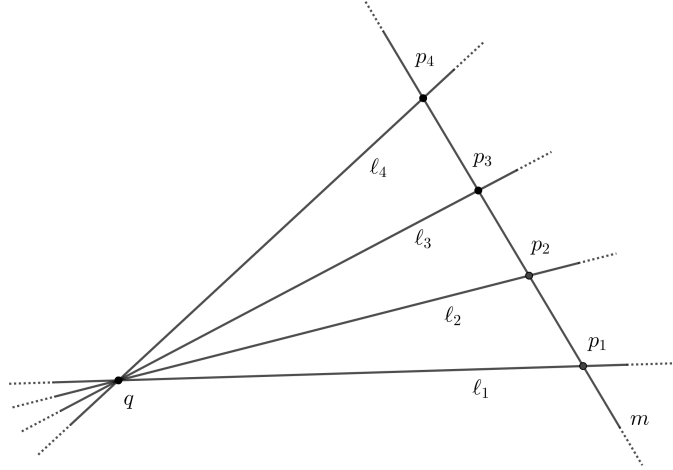
FIGURE 1. Suppose that $q \notin m$. Then $L_{q,m}$ consists of pairs (p, ℓ) such that $q \in \ell$ and $p = \ell \cap m$. In the figure, (p_i, ℓ_i) are four elements of $L_{q,m}$.

The fibres of π_1 and π_2 provide the most obvious examples. For any $p \in \mathbb{P}^2$ and $m \in \mathbb{P}^{2\vee}$, we have

$$\begin{aligned} \mathrm{bdeg}(\pi_1^{-1}(p)) &= (0,1), \\ \mathrm{bdeg}(\pi_2^{-1}(m)) &= (1,0). \end{aligned}$$

Moreover, we can identify

$$(7) \qquad N_{\pi_1^{-1}(p),\mathbb{F}} = \pi_1^*(N_{p,\mathbb{P}^2}) = \pi_1^*(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{F}} \oplus \mathcal{O}_{\mathbb{F}}.$$

as the normal bundle of a fibre of π_1 in $\mathbb{F}$.

We recall that an algebraic submanifold is said to be *integral* if it is reduced and irreducible. The symbol $\simeq$ will be used throughout this paper to indicate biholomorphism.

Remark 3.2. Let $C \subset \mathbb{F}$ be an integral projective curve with $\mathrm{bdeg}(C) = (d_1, d_2)$. Then, for $i \in \{1, 2\}$, we have the following:

- If $d_i = 1$, then C is rational. Indeed, $\pi_i(C) \simeq \mathbb{P}^1$ and $\deg(\pi_i(C)) = 1$.
- If $d_i = 0$, then $d_{3-i} = 1$. Indeed, if $\pi_i(C) = \{p\}$, then $C \subseteq \pi_i^{-1}(p) \simeq \mathbb{P}^1$ and hence, since C is integral, $C = \pi_i^{-1}(p)$. In particular, by (7), $N_{C,\mathbb{F}} = \mathcal{O}_{\mathbb{F}}^2$.

This concludes an analysis of the easy cases.

3.1. **Curves of bidegree (1,1).** We next define the fundamental family of curves of bidegree $(1, 1)$. See [9, p. 438, Example 3], [25, p. 147], [20, §4.3] and [31, Section 1.1].

Definition 3.3. Fix $(q, m) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ such that $(q, m) \in (\mathbb{P}^2 \times \mathbb{P}^{2\vee}) \setminus \mathbb{F}$, so $qm \neq 0$. The formula

$$L_{q,m} := \{(p, \ell) \in \mathbb{F} \mid p \in m, \ \ell \ni q\} = \{(p, \ell) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} \mid p\ell = 0, \ q\ell = 0, \ pm = 0\}$$

defines a family $\mathcal{V}$ of curves in $\mathbb{F}$.

Given $p, q \in \mathbb{P}^2$, the line passing through p and q is represented by the column vector

$$(p_1 q_2 - p_2 q_1 : p_2 q_0 - p_0 q_2 : p_0 q_1 - p_1 q_0)^\vee,$$

which we shall denote by $p \times q$. The cross product is the natural isomorphism

$$(8) \qquad \Lambda^2(\mathbb{C}^3) \xrightarrow{\cong} (\mathbb{C}^3)^\vee,$$

induced by an $SL(3, \mathbb{C})$ structure on $\mathbb{C}^3$. In the same way, the intersection $\ell \cap m$ of two lines is the point represented by the row vector

$$(\ell_1 m_2 - \ell_2 m_1, \ \ell_2 m_0 - \ell_0 m_2, \ \ell_0 m_1 - \ell_1 m_0),$$

which we denote by $\ell \times m$. We further abuse notation by writing $p \times q \in \mathbb{P}^{2\vee}$ and $\ell \times m \in \mathbb{P}^2$.

The cross product formulae are suggestive of computation, and we shall use them below in preference to the equivalent set theoretic statements.

Remark 3.4. It is easy to see that $\mathrm{bdeg}(L_{q,m}) = (1,1)$. Indeed, $L_{q,m} \cap \pi_1^{-1}(q') \neq \varnothing$ if and only if $q'm = 0$, in which case
$$L_{q,m} \cap \pi_1^{-1}(q') = \{(q', q \times q')\}.$$
Note that $q \times q'$ is defined because q does not lie on m. Similarly, $L_{q,m} \cap \pi_2^{-1}(m') \neq \varnothing$ if and only if $qm' = 0$, in which case
$$L_{q,m} \cap \pi_2^{-1}(m') = \{(m \times m', m')\}.$$
By Remark 3.2, we have $L_{q,m} \simeq \mathbb{P}^1$ for any $(q, m) \in (\mathbb{P}^2 \times \mathbb{P}^{2\vee}) \setminus \mathbb{F}$.

The geometry of such curves is described further in Lemma 3.13 below.

Our next result describes the possible intersections of two elements in $\mathcal{V}$.

Lemma 3.5. *Let* $(q, m), (q', m')$ *be distinct points of* $(\mathbb{P}^2 \times \mathbb{P}^{2\vee}) \setminus \mathbb{F}$.

   (i) *If* $q \neq q'$ *and* $m \neq m'$, *then* $L_{q,m} \cap L_{q',m'} \neq \varnothing$ *if and only if the point* $m \cap m'$ *lies on the line* qq', *in which case* $L_{q,m} \cap L_{q',m'} = \{(m \times m', \, q \times q')\}$.
   (ii) *If* $q = q'$ *and* $m \neq m'$, *then* $L_{q,m} \cap L_{q',m'} = \{(m \times m', q \times (m \times m'))\}$.
   (iii) *If* $q \neq q'$ *and* $m = m'$, *then* $L_{q,m} \cap L_{q',m'} = \{((q \times q') \times m, \, q \times q')\}$.

Proof. The results rely on an analysis of the following system, representing a point (p, ℓ) of intersection of the $(1,1)$-curves $L_{q,m}$ and $L_{q',m'}$:

(9)
$$\begin{cases} p_0 \ell_0 + p_1 \ell_1 + p_2 \ell_2 = 0 \\ q_0 \ell_0 + q_1 \ell_1 + q_2 \ell_2 = 0 \\ p_0 m_0 + p_1 m_1 + p_2 m_2 = 0 \\ q'_0 \ell_0 + q'_1 \ell_1 + q'_2 \ell_2 = 0 \\ p_0 m'_0 + p_1 m'_1 + p_2 m'_2 = 0 \end{cases}$$

Assume first that $q \neq q'$ and $m \neq m'$. By considering the third and fifth equations, we get that the first component p of the intersection must lie on m and m', i.e. $p = m \times m'$. Similarly, by from the second and fourth equation, the second component ℓ equals $q \times q'$. Given the first equation (characterising $\mathbb{F}$), this solution is admissible if and only $(m \times m')(q \times q') = 0$.

Assume next that $q = q'$ and $m \neq m'$. As before, we get that $p = m \times m'$. The latter is distinct from q (for otherwise q would lie in $m \cap m'$), hence we see that ℓ must be the line pq. If follows that $\ell = q \times (m \times m')$, and (ii) is established.

Case (iii) is completely analogous. $\qquad\square$

The following is an immediate consequence of the previous lemma:

Corollary 3.6. *Let* $(q, m), (q', m') \in (\mathbb{P}^2 \times \mathbb{P}^{2\vee}) \setminus \mathbb{F}$. *Then* $L_{q,m}$ *intersects* $L_{q',m'}$ *if and only if* $(m \times m')(q \times q') = 0$.

Example 3.7. Take $L_{q,m}, L_{q',m'} \in \mathcal{V}$ such that
$$q = [i:0:0], \quad m = [1:1:0], \quad \text{and} \quad q' = [1:1:0], \ m' = [i:0:0].$$
Then $L_{q,m} \cap L_{q',m'} = \varnothing$. For $q \times q' = [0:0:i]^\vee$ and $m \cap m' = [0:0:i]$.

We have seen that $(\mathbb{P}^2 \times \mathbb{P}^{2\vee}) \setminus \mathbb{F}$ parametrizes a complex 4-dimensional family $\mathcal{V}$ of rational curves in $\mathbb{F}$. If we extend the definition of $L_{q,m}$ to the case $q \in m$, we find that

(10)
$$L_{q,m} = \pi_1^{-1}(q) \cup \pi_2^{-1}(m)$$

is the union of the respective fibres meeting in $(q, m) \in \mathbb{F}$. Referring to Definition 3.3, these two fibres correspond to the respective possibilities that $p = q$ (which forces $q\ell = 0$) or $\ell = m$ (which forces $pm = 0$). The algebraic closure $\overline{\mathcal{V}}$ is therefore $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$, formed by adjoining these reducible curves. In other words, $\overline{\mathcal{V}}$ is the Hilbert scheme that parametrizes all closed subschemes of $\mathbb{F}$ having Hilbert polynomial $2t + 1$ with respect to the Segre embedding. See also [31, Lemma 1.5]).

3.2. **Surfaces of bidegree (1,0) and (0,1).** As for the case of curves, but perhaps more naturally, we can consider a notion of bidegree. Subsequently, we shall focus mainly on the cases of 'low' bidegree.

Let $d_1, d_2 \in \mathbb{N}$ be any pair of natural numbers. We set $\mathcal{O}_{\mathbb{F}}(d_1, d_2) = \pi_1^* \mathcal{O}_{\mathbb{P}^2}(d_1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{2\vee}}(d_2)$, and use $|\mathcal{O}_{\mathbb{F}}(d_1, d_2)|$ to denote the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{F}}(d_1, d_2)))$. From now on we denote $\mathcal{O}_{\mathbb{F}}$ by $\mathcal{O}$ if no confusion will arise.

Definition 3.8. Let $S \subset \mathbb{F}$ be an algebraic surface in $|\mathcal{O}(d_1, d_2)|$. Then we say that S has bidegree (d_1, d_2), and we write $\mathrm{bdeg}(S) = (d_1, d_2)$.

Fix $m \in \mathbb{P}^{2\vee}$ and $q \in \mathbb{P}^2$. The former represents a line in $\mathbb{P}^2$, which set-theoretically equals $\pi_1(\pi_2^{-1}(m))$. The latter defines a line in $\mathbb{P}^{2\vee}$ (corresponding to a pencil of lines in $\mathbb{P}^2$), which is of course $\pi_2(\pi^{-1}(q))$. These objects pull back to surfaces in $\mathbb{F}$:

Definition 3.9. Given $m \in \mathbb{P}^{2\vee}$ and $q \in \mathbb{P}^2$, set

$$H_m := \pi_1^{-1}(\pi_1(\pi_2^{-1}(m))) = \{(p, \ell) \in \mathbb{F} \mid p \in m\}$$
$$_qH := \pi_2^{-1}(\pi_2(\pi_1^{-1}(q))) = \{(p, \ell) \in \mathbb{F} \mid \ell \ni q\}.$$

In words, H_m is the set of pairs $(p, \ell) \in \mathbb{F}$ such that p moves on m, while $_qH$ is the set of pairs $(p, \ell) \in \mathbb{F}$ such that ℓ contains q. Using the cross product, we also have

$$(11) \qquad H_m := \{(p, \ell) \mid p\ell = 0, \ pm = 0\} = \{(\ell \times m, \ell) \mid \ell \in \mathbb{P}^{2\vee}\}$$
$$(12) \qquad _qH := \{(p, \ell) \mid p\ell = 0, \ q\ell = 0\} = \{(p, p \times q) \mid p \in \mathbb{P}^2\}.$$

If p, q are distinct points of $\mathbb{P}^2$, then

$$\pi_1^{-1}(p) \cap {}_qH = \{(p, p \times q)\}$$

consists of a single point. Similarly, if $l \neq m$ then

$$\pi_2^{-1}(\ell) \cap H_m = \{(\ell \times m, \ell)\}.$$

It follows that $\mathrm{bdeg}(_qH) = (0, 1)$ and $\mathrm{bdeg}(H_m) = (1, 0)$. We can also deduce this from (1), since $_qH$ is defined by the equation $q\ell = 0$ that is linear (of degree one) in the ℓ_j and H_m is defined by one linear in the p_i.

On the other hand, it is easy to see that any surface $S \in |\mathcal{O}(0, 1)|$ has the form $_qH$ for some $q \in \mathbb{P}^2$, and any surface $S \in |\mathcal{O}(1, 0)|$ has the form H_m for some $m \in \mathbb{P}^{2\vee}$. Hence the family of surfaces $S \in |\mathcal{O}(0, 1)|$ is parametrized by $\mathbb{P}^2$ and has complex dimension 2. Any two elements of the family are projectively equivalent because $SL(3, \mathbb{C})$ acts transitively on $\mathbb{P}^2$. The same considerations hold for elements in $|\mathcal{O}(1, 0)|$.

Since $\pi^{-1}(q) \subset {}_qH$ and $\pi^{-2}(m) \subset H_m$, we obtain the following result.

Corollary 3.10. *Both H_m and $_qH$ are Hirzebruch surfaces of type* 1. *Indeed, π_1 blows up $\mathbb{P}^2$ at q, and π_2 blows up $\mathbb{P}^{2\vee}$ at m.*

Any two Hirzebruch surfaces of 'like' type intersect in a fibre of π_1 or π_2. Indeed, for $m \neq m'$ and $q \neq q'$, we have

$$H_m \cap H_{m'} = \pi_1^{-1}(m \times m'), \qquad {}_qH \cap {}_{q'}H = \pi_2^{-1}(q \times q').$$

It follows that any two bidegree $(1, 0)$ or $(0, 1)$ surfaces always meet, while the intersection three generic $(1, 0)$ (respectively $(0, 1)$) surfaces is empty. Moreover, for any $q \in \mathbb{P}^2$ and $m \in \mathbb{P}^{2\vee}$ such that $q \notin m$, we have

$$_qH \cap H_m = L_{q,m}$$

while, if $q \in m$, we get

$$_qH \cap H_m = \pi_1^{-1}(q) \cap \pi_2^{-1}(m).$$

It follows that the triple intersection

$$H_m \cap H_{m'} \cap {}_qH = \pi_1^{-1}(m \times m') \cap {}_qH = \{(m \times m', q \times (m \times m'))\}$$

is a single point. In conclusion,

Proposition 3.11. *The intersection of Hirzebruch surfaes can be summarized by the products*

$$\mathcal{O}(1,0) \cdot \mathcal{O}(1,0) \cdot \mathcal{O}(1,0) = 0,$$
$$\mathcal{O}(1,0) \cdot \mathcal{O}(0,1) \cdot \mathcal{O}(1,0) = 1,$$
$$\mathcal{O}(0,1) \cdot \mathcal{O}(1,0) \cdot \mathcal{O}(0,1) = 1,$$
$$\mathcal{O}(0,1) \cdot \mathcal{O}(0,1) \cdot \mathcal{O}(0,1) = 0.$$

Remark 3.12. Thanks to the previous proposition, we are able to compute c_1^2 for a generic surface S of bidegree (a,b). Indeed, given that $\omega_{\mathbb{F}} = \mathcal{O}(-2,-2)$, we also have that $\omega_S = \mathcal{O}(a-2,b-2)$, and hence $c_1^2 = \mathcal{O}(a-2,b-2) \cdot \mathcal{O}(a-2,b-2) \cdot \mathcal{O}(a,b)$. Therefore

$$c_1^2 = [(a-2)\mathcal{O}(1,0) + (b-2)\mathcal{O}(0,1)] \cdot [(a-2)\mathcal{O}(1,0) + (b-2)\mathcal{O}(0,1)] \cdot [a\mathcal{O}(1,0) + b\mathcal{O}(0,1)]$$
$$= 3a^2b + 3ab^2 - 4a^2 - 4b^2 - 16ab + 12a + 12b.$$

We now show an important interplay between curves of bidegree $(1,1)$ and surfaces of bidegree $(0,1)$ and $(1,0)$.

Lemma 3.13. *Let $C \subset \mathbb{F}$ be a connected curve of bidegree $(1,1)$. Then $C = {}_qH \cap H_m$ for some $(q,m) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee}$. In particular, if C is smooth then $qm \neq 0$ and $C = L_{q,m}$.*

Proof. Since $\pi_1(C)$ is a line, then $C \subset H_m$ for some line $m \in \mathbb{P}^{2\vee}$. Similarly, $C \subset {}_qH$ for some point $q \in \mathbb{P}^2$. Therefore

$$C \subseteq {}_qH \cap H_m$$

and, since $\mathrm{bdeg}(L_{q,m}) = (1,1)$, we conclude that $C \simeq {}_qH \cap H_m$. $\square$

Note that the union $\pi_1^{-1}(q) \cup \pi_2^{-1}(m)$ with $q \notin m$ consists of two skew lines, in contrast to (10). It is therefore a reducible $(1,1)$-curve, given by two disjoint components of bidegree $(1,0)$ and $(0,1)$.

We now pass to work on the normal bundle. Recall [34] that any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$.

Lemma 3.14. *Let $C \subset \mathbb{F}$ be any smooth rational curve. Then C has bidegree $(1,1)$ if and only if its normal bundle N_C is isomorphic to the direct sum of two line bundles of degree 1.*

Proof. Let (d_1, d_2) be the bidegree of C and let $N_C \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$.

Since C is smooth, we have the exact sequence

$$(13) \qquad\qquad 0 \to T_C \to T_{\mathbb{F}|C} \to N_C \to 0,$$

where the first non-trivial map is the inclusion and the second is the projection on the quotient. Notice the since $\mathbb{F}$ is homogeneous, then $T_{\mathbb{F}}$ is globally generated, and the same holds for N_C. Hence $a_1, a_2 \geqslant 0$.

Since $\omega_{\mathbb{F}} \cong \mathcal{O}(-2,-2)$, we have $\det(T_{\mathbb{F}}) \cong \mathcal{O}_{\mathbb{F}}(2,2)$. Thus $\deg(T_{\mathbb{F}|C}) = 2d_1 + 2d_2$. Moreover, since C is rational, then $T_C \simeq \mathcal{O}_{\mathbb{P}^1}(2)$. Hence $\deg(T_C) = 2$ and $\deg(N_C) = a_1 + a_2$. Therefore, from the exact sequence in (13), we have

$$a_1 + a_2 = 2d_1 + 2d_2 - 2.$$

Now if $a_1 = a_2 = 1$ we have $d_1 + d_2 = 2$ and we conclude by Remark 3.2.

On the other, if $d_1 = d_2 = 1$ we have $a_1 + a_2 = 2$, with $a_1, a_2 \geqslant 0$. Moreover $a_1 \neq 0 \neq a_2$, by Remark 3.2. $\square$

This lemma is relevant to the deformation of twistor fibres, see forward to Remark 5.7.

3.3. **Surfaces of bidegree (0,d).** We add this subsection for completeness. Let C be a curve of degree d in $\mathbb{P}^2$ and consider the surface $S = \pi_1^{-1}(C)$. The surface S has bidegree $(0,d)$. Then the restriction

$$\pi_2|_S : S \longrightarrow \mathbb{P}^{2\vee}$$

is a cover of degree d of $\mathbb{P}^2$, branched over the dual curve $C^\vee$. Indeed, $\pi_2^{-1}(\ell) = \{(p,\ell) : p \in \ell \cap C\}$. The following lemma states that any $(0,d)$ surface in $\mathbb{F}$ arises in this way:

Lemma 3.15. *Given an integer $d > 0$, if S is an integral surface in $|\mathcal{O}(0,d)|$, then $S = \pi_1^{-1}(C)$, where $C \subset \mathbb{P}^2$ is a degree d integral curve.*

Proof. Since $S \in |\mathcal{O}(0, d)|$, we have $\dim \pi_1(S) \leqslant 1$, while $\pi_{2|S}$ is a $d : 1$ covering. Moreover, it is easy to see that $\pi_1(S)$ is a plane integral curve C of degree $\tilde{d}$. Clearly, for a general line $\mathcal{L}_m \subset \mathbb{P}^2$, $C \cap \mathcal{L}_m = \{p_1, \ldots, p_{\tilde{d}}\}$. Now

$$S \cap H_m = \pi_1^{-1}(C \cap \mathcal{L}_m) = \bigcup_{k=1}^{\tilde{d}} \pi_1^{-1}(p_k).$$

Hence, by Proposition 3.11, for a general $q \in \mathbb{P}^2$, we get

$$\tilde{d} = |(S \cap H_m) \cap {}_q H| = \mathcal{O}_F(0, d) \cdot \mathcal{O}_F(0, 1) \cdot \mathcal{O}_F(1, 0) = d,$$

where the last equality follows from Remark 3.11. $\qquad \square$

We omit the proof of the following result.

Proposition 3.16. *Fix $d > 0$ and let S_1, S_2 be integral surfaces of bidegree $(0, d)$. Let $C_i = \pi_2(S_i)$ for $i = 1, 2$. Then S_1 and S_2 are isomorphic projective varieties if and only if C_1 and C_2 are isomorphic projective varieties.*

4. Classification of surfaces of bidegree $(1, 1)$

This section provides a description of all $(1, 1)$-surfaces contained in the flag manifold, and a classification of them up to projective equivalence. We then proceed to classify the *smooth* surfaces up to unitary equivalence, in preparation for the twistor geometry that is introduced in the next section. Recall that, by Remark 2.3, the set of the $(1, 1)$-surfaces has complex dimension 7.

Given a 3×3 complex non-scalar matrix A, we define

(14) $$S_A = \{(p, \ell) \in \mathbb{F} \mid pA\ell = 0\}.$$

This is a surface of bidegree $(1, 1)$, since its intersection with a generic fibre of each projection is one point. Indeed, being the zero locus of an element of $\mathcal{S}_{1,1}$ (cf. (3)), it belongs to $|\mathcal{O}(1, 1)|$. Conversely, any surface of bidegree $(1, 1)$ in $\mathbb{F}$ will be defined by an element of $\mathcal{P}_{1,1}$, equivalently by a suitable matrix A. The surfaces of bidegree $(1, 1)$ are therefore parametrized by the matrices of the Kronecker pencil (see e.g. [28, §10.3]) of the form $sA + tI$, where A is a complex 3×3 non-scalar matrix and $s, t \in \mathbb{C}$, with $s \neq 0$.

For each class, we can (i) choose a representative with a zero eigenvalue, and (ii) if there is a non-zero eigenvalue we can assume it equals 1. This can be achieved with suitable choices s and t. By listing the resulting Jordan canonical forms, we deduce

Lemma 4.1. *Any $(1, 1)$-surface is projectively equivalent to S_A, where A is one of the following matrices:*

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

We proceed to give geometric descriptions of some of the resulting classes of surfaces.

4.1. **Smooth (1,1)-surfaces.** This class is a natural one to distinguish. Up to projective equivalence, it can only arise from matrices of type A_1.

Proposition 4.2. *Let S be a smooth $(1, 1)$-surface in $\mathbb{F}$. Then S is a Del Pezzo surface of degree 6, and is unique up to biholomorphism.*

Proof. By the adjunction formula we have $\omega_S \cong \mathcal{O}(-1, -1)_{|S}$. Hence S is a Del Pezzo surface of degree 6 (by Remark 3.12). From the classification in [17, §8.4.2] we get the uniqueness up to biholomorphism. $\qquad \square$

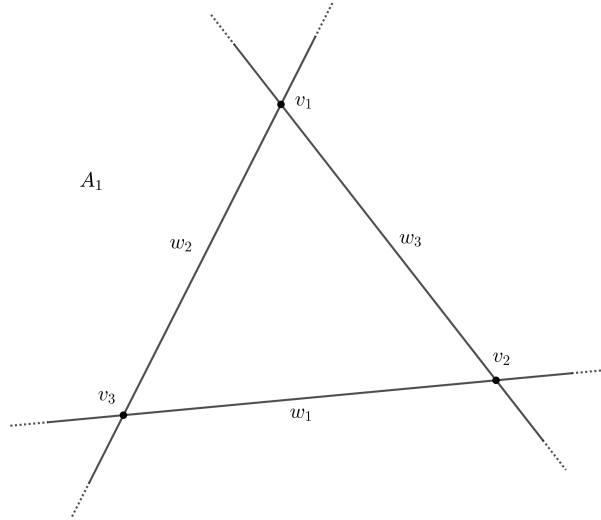We can describe these Del Pezzo surfaces more explicitly:

FIGURE 2. A smooth $(1,1)$-surfaces of type A_1 can be seen as the blowup at three points in general position in either $\mathbb{P}^2$ or $\mathbb{P}^{2\vee}$.

Proposition 4.3. *If A is a 3×3 complex matrix A that admits three distinct eigenvalues, then the associated surface S_A is smooth. It can be realized as the blowup (i) of $\mathbb{P}^2$ via π_1 at three points corresponding to the left eigenvectors of A, or (ii) of $\mathbb{P}^{2\vee}$ via π_2 at three points corresponding to right eigenvectors of A.*

By left (respectively, right) eigenvector, we mean a row (respectively, column) vector satisfying the obvious equation. Of course, the left eigenvectors are transposes of the (more usual) right eigenvectors of $A^\vee$. But the left-right formalism is more in keeping with our approach.

Proof. For (i), let p_1, p_2, p_3 be points in $\mathbb{P}^2$ corresponding to three linearly independent left eigenvectors. We prove that the restriction of π_1 to S_A is a blowup of $\mathbb{P}^2$ in such points. This is indeed a degree 6 Del Pezzo surface.

Given $q \in \mathbb{P}^2$, we have

$$\pi_1^{-1}(q) \cap S_A = \big\{(q,m) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} \mid qm = 0, \ qAm = 0\big\}.$$

The two conditions $qm = 0$ and $qAm = 0$ are dependent if and only if q is a left eigenvector of A. In this case, we have $\pi_1^{-1}(q) \subset S_A$. Since the set $\pi_1^{-1}(q) \cap S_A$ is a point for generic q, we conclude that $(\pi_1)_{|S_A}$ is the blowup of $\mathbb{P}^2$ at p_1, p_2, p_3.

Case (ii) is similar. $\square$

Remark 4.4. If the three left eigenvectors of A correspond to points p_1, p_2, p_3 in $\mathbb{P}^2$, then the right eigenvectors of A correspond to the lines $p_2 \times p_3$, $p_3 \times p_1$, $p_1 \times p_2$ in $\mathbb{P}^{2\vee}$; see Figure 2). A triple of eigenvalues $(\alpha_1, \alpha_2, \alpha_3)$ is projectively equivalent to $(0, 1, \lambda)$, where $\lambda = (\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)$. Let

$$(15) \qquad \Lambda = \Big\{ \lambda, \ \frac{1}{\lambda}, \ 1 - \lambda, \ \frac{1}{1-\lambda}, \ \frac{\lambda}{\lambda-1}, \ \frac{\lambda-1}{\lambda} \Big\}.$$

By permuting the eigenvalues, we see that the sets $\{0, 1, \lambda'\}$ with $\lambda' \in \Lambda$ are all projectively equivalent. We may regard λ as a real cross ratio.

4.2. **Singular (1,1)-surfaces.** We now describe the set of singular surfaces of bidegree $(1,1)$. We shall prove the next proposition directly using linear algebra, and then reconcile its statements with Lemma (4.1).

Proposition 4.5. *The family of singular $(1,1)$-surfaces contained in $\mathbb{F}$ is parametrized by an irreducible variety of dimension 6 and each irreducible singular $(1,1)$-surface has exactly one singular point. The family of reducible $(1,1)$-surfaces in $\mathbb{F}$ has dimension 4 and each reducible $(1,1)$-surface is of the form $_qH \cup H_m$, for some $(q,m) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee}$.*

Proof. Let S_A be a $(1,1)$-surface defined by a matrix A. The Jacobian of the map $\mathbb{C}^3 \times (\mathbb{C}^3)^\vee \to \mathbb{C}^2$ defined by

$$(16) \qquad ((p_0, p_1, p_2), (\ell_0, \ell_1, \ell_2)) \longmapsto \begin{pmatrix} pA\ell \\ p\ell \end{pmatrix}$$

can be represented by the matrix

$$\begin{pmatrix} A\ell & Ap \\ \ell & p \end{pmatrix}.$$

It has rank less than 2 at the point $(p, \ell) \in \mathbb{F}$ if and only if p is a left eigenvector and ℓ is a right eigenvector of A, with the same eigenvalue. Imposing these conditions, given that $p\ell = 0$, we automatically get $(p, \ell) \in S_A$.

Thus S_A is singular if and only if the system

$$(17) \qquad \begin{cases} pA = \lambda p \\ A\ell = \lambda \ell \\ p\ell = 0 \end{cases}$$

admits a non-trivial solution $(p, \ell) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee}$. We want to prove that this happens if and only if λ has algebraic multiplicity greater than one. This would imply that the family of singular $(1,1)$-surfaces has codimension 1 in the 7-dimensional family of $(1,1)$-surfaces, and (with reference to Lemma 4.1) is irreducible.

Suppose that p is a left eigenvector of A with eigenvalue λ of algebraic multiplicity 1, and that μ is an eigenvalue of A distinct from λ. If $(A - \mu I)^2 \ell = 0$ then

$$0 = p(A - \mu I)^2 \ell = (\lambda - \mu)^2 p\ell$$

so $p\ell = 0$. It follows that the 2-dimensional annihilator of the row vector p is spanned by right eigenvectors or generalized eigenvectors with eigenvalues distinct from λ. Hence, there is no non-zero column vector ℓ that solves (17).

Suppose that λ is an eigenvalue of A of algebraic multiplicity 2. Denote by V_λ and W_λ the respective left and right eigenspaces for λ. We have the following two cases.

(1) Suppose that $V_\lambda = \{p\}$ has dimension 1, and set $W_\lambda = \{\ell\}$. Then there exists a generalized eigenvector $\tilde{p}$, satisfying $\tilde{p}(A - \lambda I) = p$, and

$$(18) \qquad p\ell = \tilde{p}(A - \lambda I)\ell = 0.$$

It follows that (p, ℓ) is a solution of (17). The argument above shows that there are no more solutions (projectively speaking).

(2) Suppose that V_λ has dimension 2. In this case, $\mathbb{P}(V_\lambda)$ represents an element $m \in \mathbb{P}^{2\vee}$, and $\mathbb{P}(W_\lambda)$ represents an element $q \in \mathbb{P}^2$. There two subcases.

   (i) Firstly, assume that there is an eigenvalue μ distinct from λ, and that q is a left eigenvector for μ. Since $A(p \times q)$ is a multiple of $(pA) \times (qA)$ by (8), it follows that $p \times q \in W_\lambda$, and $(p, p \times q)$ solves (17) for any $p \in V_\lambda$. These solutions give rise to the curve

$$L_{q,m} = \{(p, \ell) : p \in m, \ \ell \ni q\}$$

   of singular points in S_A. It follows from (11) that S_A is the reducible surface $_qH \cup H_m$, where $qm \neq 0$.

   (ii) Secondly, assume that A has a unique eigenvalue λ (which we could take to be 0). Then the solutions to (17) are given by

$$\{(q, \ell) : \ell \in W_\lambda\} \cup \{(p, m) : p \in V_\lambda\} = \pi_1^{-1}(q) \cup \pi_2^{-1}(m).$$

   We again have $S_A = {}_qH \cup H_m$, but in this subcase $mq = 0$.

Since we have seen that each reducible $(1,1)$-surface is of the form $_qH \cup H_m$ for $(q, m) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee}$, we conclude that the set of reducible $(1,1)$-surfaces has complex dimension 4. $\qquad \square$

We end this section by summarizing the behaviour of the five canonical cases:
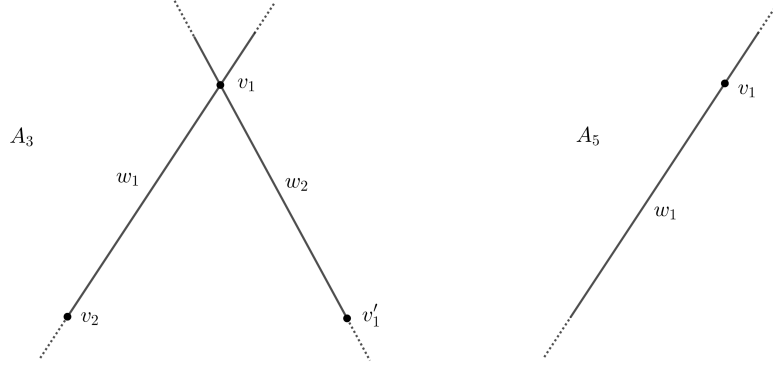
- S_{A_1} is smooth.

FIGURE 3. Non-smooth non-reducible $(1,1)$-surfaces S_A have only one singular point. The disposition of the eigenvectors of A and $A^\vee$ is described here.
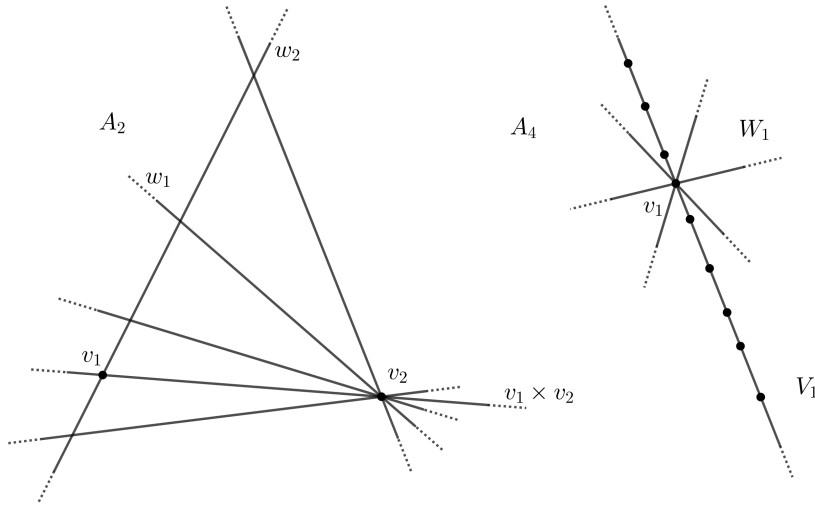

FIGURE 4. Non-smooth reducible $(1,1)$-surfaces are singular along an element of $\overline{\mathcal{V}}$: either a curve $L_{q,m}$ for the case A_2 or the union of two intersecting fibres $\pi_1^{-1}(q) \cup \pi_2^{-1}(m)$ for A_4.

- $S_{A_2} = {}_{[0:0:1]}H \cup H_{[0:0:1]}$ is reducible and singular on

$$L_{[0:0:1],[0:0:1]} = \{([p_0 : p_1 : 0], [\ell_0 : \ell_1 : 0]^\vee) \mid p_0\ell_0 + p_1\ell_1 = 0\}.$$

- S_{A_3} has only one singular point $([1:0:0], [0:1:0])$.
- $S_{A_4} = {}_{[0:1:0]}H \cup H_{[1:0:0]}$ is reducible and singular on

$$\pi_1^{-1}[1:0:0] \cup \pi_2^{-1}[0:1:0] = \{([p_0 : 0 : p_2], [0 : \ell_1 : \ell_2] \mid p_2\ell_2 = 0\}.$$

- S_{A_5} has a unique singular point $([1:0:0], [0:0:1])$.

4.3. **Unitary equivalence.** The realization of the flag manifold of the twistor space $\mathbb{F}$ of $\mathbb{P}^2$ will require us to restrict to *unitary automorphisms* of $\mathbb{F}$, as defined in subsection 2.2.

Given a $(1,1)$-surface S_A in $\mathbb{F}$, we know that A is projectively equivalent to ones of the matrices A_i of Lemma 4.1. Hence there is a non-singular matrix C such that

$$C^{-1}AC = A_i,$$

By 'QR-factorization', C can itself be decomposed as $C = QR$ where $Q \in U(3)$ and R is upper triangular. Hence,

$$Q^{-1}AQ = RA_iR^{-1}$$

is upper triangular. In other words, up to unitary equivalence, we can still assume (at least) that A is upper triangular.

Let us begin with the smooth case. Recall that S_A is smooth if and only if A has three distinct eigenvalues. After adding a scalar multiple and re-scaling, we can assume that these are $0, 1, \lambda$ with $\lambda \in \mathbb{C} \setminus \{0, 1\}$, so that

$$(19) \qquad A = \begin{pmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{pmatrix}.$$

However, this representation is not unique. Let

$$X = \begin{pmatrix} e^{i\vartheta_1 + i\vartheta_2} & 0 & 0 \\ 0 & e^{i\vartheta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be a diagonal unitary matrix. Then

$$(20) \qquad XAX^{-1} = \begin{pmatrix} 0 & ae^{i\vartheta_1} & be^{i\vartheta_1 + i\vartheta_2} \\ 0 & 1 & ce^{i\vartheta_2} \\ 0 & 0 & \lambda \end{pmatrix}$$

parametrizes the orbit of A under the standard maximal torus T^2.

Proposition 4.6. *Let A, A' be the matrices defined by respectively* (19) *and*

$$A' = \begin{pmatrix} 0 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & \lambda' \end{pmatrix}.$$

Then A and A' are unitarily equivalent if and only if $\lambda = \lambda'$ and A' has the form (20).

Proof. Let us first assume that the matrices A and A' are unitarily equivalent, so that there exists $X = (x_{ij})$ in $U(3)$ such that $XA = A'X$. This equation translates into the system

$$\begin{cases} 0 = x_{21}a' + x_{31}b' \\ 0 = x_{21} + x_{31}c' \\ 0 = x_{31}\lambda' \\ x_{11}a + x_{12} = x_{22}a' + x_{32}b' \\ x_{21}a + x_{22} = x_{22} + x_{32}c' \\ x_{31}a + x_{32} = x_{32}\lambda' \\ x_{11}b + x_{12}c + x_{13}\lambda = x_{23}a' + x_{33}b' \\ x_{21}b + x_{22}c + x_{23}\lambda = x_{23} + x_{33}c' \\ x_{31}b + x_{32}c + x_{33}\lambda = x_{33}\lambda', \end{cases}$$

and immediately gives $x_{31} = x_{21} = x_{32} = 0$. But as X is unitary and now upper triangular, it must be diagonal. Thus, $x_{13} = x_{12} = x_{23} = 0$ and $|x_{11}| = |x_{22}| = |x_{22}| = 1$. Substituting above, we get

$$\begin{cases} x_{11}a = x_{22}a' \\ x_{11}b = x_{33}b' \\ x_{22}c = x_{33}c' \\ x_{33}\lambda = x_{33}\lambda', \end{cases}$$

from which we get

(21) $$abc\lambda = \lambda'; \quad |a'| = |a|, \ |b'| = |b|, \ |c'| = |c|; \quad ab'c = a'bc'.$$

This implies that $a' = e^{i\vartheta_1}a, c' = e^{i\vartheta_2}c$ and $b' = e^{i(\vartheta_1+\vartheta_2)}b$, and leads to case (i). Conversely, we have already seen that any matrix (20) is unitarily equivalent to A. $\square$

Remark 4.7. Recall that the surface of bidegree $(1,1)$ S_A defined by a matrix A of the form (19) is invariant if we modify A by adding multiples of the identity and rescaling. But in order to return to the canonical form we would need to apply an element of the Weyl group of $SU(3)$, which acts by permuting the standard coordinates of $\mathbb{C}^3$. Or, we could reverse the order and start by transforming A into one of

$$\begin{pmatrix} 0 & b & a \\ 0 & \lambda & 0 \\ 0 & c & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 & 0 \\ c & 1 & 0 \\ b & a & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & c \\ a & 0 & b \\ 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 & 0 \\ b & 0 & a \\ c & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & c & 0 \\ 0 & \lambda & 0 \\ a & b & 0 \end{pmatrix}.$$

The first three matrices are the result of applying a 2-cycle, and the last two of applying a 3-cycle. Then in special cases, if one or more of a, b, c vanish in A, the new matrices can be put in to canonical forms as follows:

$$\begin{pmatrix} 0 & \frac{b}{\lambda} & \frac{a}{\lambda} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\lambda}{\lambda-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -c \\ 0 & 1 & -b \\ 0 & 0 & 1-\lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{a}{\lambda} \\ 0 & 0 & \frac{\lambda-1}{\lambda} \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{c}{\lambda-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{1-\lambda} \end{pmatrix}.$$

Then if we denote by $S(\lambda, a, b, c)$ the surface defined by a matrix A of the form (19), we deduce that $S(\lambda, 0, 0, 0)$ is unitarily equivalent to $S(\lambda', 0, 0, 0)$ for any $\lambda' \in \Lambda$ defined in Remark 4.4. Moreover, we have the following relations of unitarily equivalence:

$$S(\lambda, a, b, 0) \sim S\left(\frac{1}{\lambda}, \frac{b}{\lambda}, \frac{a}{\lambda}, 0\right) \qquad S(\lambda, 0, b, c) \sim S(1-\lambda, 0, -c, -b),$$

$$S(\lambda, a, 0, 0) \sim S\left(\frac{\lambda-1}{\lambda}, 0, 0, -\frac{a}{\lambda}\right), \qquad S(\lambda, 0, 0, c) \sim S\left(\frac{1}{1-\lambda}, \frac{c}{1-\lambda}, 0, 0\right).$$

Notice that, even in the general case, when $abc \neq 0$, it is possible to find different canonical forms. For example consider the following example: Let A of the form (19), with $a \in \mathbb{R}$ and take

$$X = \begin{pmatrix} -\frac{a}{\sqrt{a^2+1}} & -\frac{1}{\sqrt{a^2+1}} & 0 \\ \frac{1}{\sqrt{a^2+1}} & -\frac{a}{\sqrt{a^2+1}} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is easy to check that X is unitary, and to compute that

$$A' = XAX^T = \begin{pmatrix} 1 & a & \frac{(ba+c)}{\sqrt{a^2+1}} \\ 0 & 0 & \frac{(ac-b)}{\sqrt{a^2+1}} \\ 0 & 0 & \lambda \end{pmatrix},$$

hence, for $a \in \mathbb{R}$, we obtain that

$$S(\lambda, a, b, c) \sim S\left(1-\lambda, -a, -\frac{(ba+c)}{\sqrt{a^2+1}}, -\frac{(ac-b)}{\sqrt{a^2+1}}\right).$$

Corollary 4.8. *Let S_A be a smooth $(1,1)$-surface where A is as in Formula (19), with $[a : b : c] \in \mathbb{P}^2$. Then its stabilizer is trivial.*

Proof. The proof is a direct consequence of the computations given to prove Theorem 4.6 applied to A and $A' = A$. $\square$

5. Twistor theory

We now study the flag manifold as the twistor space of $\mathbb{P}^2$, in the context of 4-dimensional Riemannian geometry. Following [9, Example 3, p. 438], [20, §4.3, p. 500], [25, Remark p.147] and [30, Ex p. 0], we construct the twistor projection explicitly.

Consider the Hermitian product

$$\langle p, q \rangle = pq^* = p_0 \overline{q}_0 + p_1 \overline{q}_1 + p_2 \overline{q}_2,$$

where $p = (p_0, p_1, p_2)$ and $q = (q_0, q_1, q_2)$ are row vectors, and $q^* = \overline{q}^\vee$. This pairing induces the anti-linear bijection

$$(22) \qquad \mathbb{C}^3 \to (\mathbb{C}^3)^\vee, \quad q \mapsto q^*.$$

We shall extend the asterisk notation to projective classes by writing

$$p = [p_0 : p_1 : p_2] \in \mathbb{P}^2 \quad \Rightarrow \quad p^* = [\overline{p}_0 : \overline{p}_1 : \overline{p}_2]^\vee \in \mathbb{P}^{2\vee},$$

and p^* can be thought of as the line at infinity ℓ_p^∞. We can similarly convert a line into a point:

$$\ell = [\ell_0 : \ell_1 : \ell_2]^\vee \in \mathbb{P}^{2\vee} \quad \Rightarrow \quad \ell^* = [\overline{\ell}_0 : \overline{\ell}_1 : \overline{\ell}_2] \in \mathbb{P}^2.$$

The twistor projection $\pi : \mathbb{F} \to \mathbb{P}^2$ is defined by

$$(23) \qquad \pi(p, \ell) = p^* \times \ell.$$

Recall that this cross product is the point of intersection of the two lines, i.e. $p^* \cap \ell = \{p^* \times \ell\}$. The equation

$$\pi([p_0 : p_1 : p_2], [\ell_0 : \ell_1 : \ell_2]^\vee) = [\overline{p_1}\ell_2 - \overline{p_2}\ell_1 : \overline{p_2}\ell_0 - \overline{p_0}\ell_2 : \overline{p_0}\ell_1 - \overline{p_1}\ell_0]$$

gives a coordinate representation of π.

The introduction of the bijection (22) allows us to regard the projections π_1, π_2, π on the same footing. If we set $\pi_2^*(p, \ell) = \ell^*$, then π_1, π, π_2^* are all maps $\mathbb{F} \to \mathbb{P}^2$. A point $(p, \ell) \in \mathbb{F}$ defines a unitary splitting

$$(24) \qquad \mathbb{C}^3 = \langle p \rangle \oplus \langle p^* \times \ell \rangle \oplus \langle \ell^* \rangle,$$

and our projections now correspond to the three components. Note that π_2^* is now anti-holomorphic, while the twistor projection π is neither holomorphic nor anti-holomorphic.

$$
\begin{array}{ccc}
 & \mathbb{P}^2 & \\
 & \pi \uparrow & \\
 & \mathbb{F} & \\
\pi_1 \swarrow & & \searrow \pi_2^* \\
\mathbb{P}^2 & & \mathbb{P}^2
\end{array}
$$

5.1. **Almost complex structures.** Fix a point $x = (p, \ell)$ in $\mathbb{F}$, and set $q = \pi(x) = p^* \times \ell$. Referring to (24), we have an identification

$$(25) \qquad (T_x\mathbb{F}, \mathbb{J}) \cong (\langle p^* \rangle \otimes \langle \ell^* \rangle) \oplus (\langle p^* \rangle \otimes \langle q \rangle) \oplus (\langle q^* \rangle \otimes \langle \ell^* \rangle).$$

of the holomorphic tangent space to $\mathbb{F}$ for the complex structure $\mathbb{J}$ that arises from $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$, which we have been considering. These choices ensure that the projections π_1, π, π_2 behave appropriately with respect to $\mathbb{J}$, in particular that π_1 is holomorphic. It is well known that $\pi_1 : \mathbb{F} \to \mathbb{P}^2$ can be identified with the projectivized tangent bundle $\mathbb{P}(T\mathbb{P}^2)$. The same is true of the non-holomorphic fibration $\pi : \mathbb{F} \to \mathbb{P}^2$ except that in this case the realization of $\mathbb{J}$ is more complicated; this is the heart of Penrose's twistor space concept [9, 33].

The analogue of (25) for the standard complex structure J on the projective plane is

$$(26) \qquad (T_q\mathbb{P}^2, J) \cong (\langle q^* \rangle \otimes \langle \ell^* \rangle) \oplus (\langle q^* \rangle \otimes \langle p \rangle) = L \oplus L^\top.$$

Geometrically, L is tangent to the line ℓ and $L^\top$ is tangent to the line p^* (both lines pass through q). We can now see that (25) is 'constructed' by combining a vertical component (the first summand)

with a horizontal component whose almost complex structure J' is a twisted version of (26) defined by setting

$$J' = \left\{ \begin{array}{ll} J & \text{on } L \\ -J & \text{on } L^\top, \end{array} \right.$$

In this way, each fibre $\pi^{-1}(q) \simeq U(2)/T^2$ parametrizes almost complex structures on $T_q\mathbb{P}^2$ that are orthogonal relative to the Fubini-Study metric:

$$(27) \qquad\qquad g_{\text{FS}}(J'X, J'Y) = g_{\text{FS}}(X, Y).$$

In the context of this paper, we record

Definition 5.1. An *orthogonal complex structure* is a complex structure defined on an open subset of $\mathbb{P}^2$ satisfying (27) at each point.

5.2. **Symmetries.** One can relate the fibers of π_1, π, π_2^* by exploiting the obvious 3-fold symmetry inherent in the unitary description of $\mathbb{F}$.

Definition 5.2. Let j_1, j, j_2 be the diffeomorphims of $\mathbb{F}$ defined by

$$\begin{array}{rcl} j_1(p, \ell) & = & (p, p \times \ell^*) \\ j(p, \ell) & = & (\ell^*, p^*) \\ j_2(p, \ell) & = & (p^* \times \ell, \ell). \end{array}$$

Each of these transformations are involutions. For example,

$$(j_1)^2(p, \ell) = (p, p \times (p \times \ell^*)^*) = (p, p \times (p^* \times \ell)),$$

but the projective class $p \times (p^* \times \ell)$ equals ℓ by the well-known vector identity, given that the scalar product $p\ell$ vanishes. Each involution j_1, j, j_2 permutes the projections complementary to π_1, π, π_2^* respectively. In particular,

$$\pi = \pi_1 \circ j_2 = \pi_2^* \circ j_1.$$

The involutions therefore generate the symmetry group S_3 that permutes the projections π_1, π, π_2. Notice that (only) j is anti-holomorphic.

Remark 5.3. The group S_3 we have just defined is a representation of the Weyl group W of $SU(3)$, mentioned in subsection 2.2. To understand this, regard a point of $\mathbb{F}$ as a right coset gT, where $T = T^2$ is the isotropy subgroup of $SU(3)$ fixing a point (the 'origin') of $\mathbb{F}$. If $w \in N(T)$ then we can define an action

$$w \cdot (gT) = (gT)w^{-1} = (gw)T.$$

Suppose that $w \notin T$ (so that w is not the identity in W). Then w does not act on $\mathbb{F}$ as a holomorphic isometry (that would be the action $gT \mapsto wgT$), and fixes no point $gT \in \mathbb{F}$.

The six complex structures induced on $\mathbb{F}$ by S_3 are precisely the $SU(3)$ invariant complex structures considered in [14]. Each can be defined in terms of decomposition analogous to (25), which was used to specify the standard one $\mathbb{J}$. These six structures are supplemented by two non-integrable almost complex structures $\pm J'$, which render the three fibrations π_1, π, π_2 equivalent. Moreover, even permutations in S_3 act J'-holomorphically on $\mathbb{F}$, and it is this fact that allows one to generate harmonic maps of Riemann surfaces into $\mathbb{P}^2$ from holomorphic ones [18].

Any element of the groupup S_3 generated by Definition 5.2 commutes with the action of group of unitary automorphisms discussed in subsection 2.2. For example,

$$B \cdot j(p, \ell) = (\ell^* B^{-1}, Bp^*) = ((B\ell)^*, (pB^{-1})^*) = j(B \cdot (p, \ell)),$$

for $B \in SU(3)$. Moreover, these equations yield

Lemma 5.4. *A projective automorphism of $\mathbb{F}$ induced by $B \in SL(3, \mathbb{C})$ is unitary if and only if $B \circ j = j \circ B$.*

A unitary automorphism of $\mathbb{F}$ commutes with each of the three projections π_1, π, π_2, and is completely determined by its action on the base $\mathbb{P}^2$ of any one of these projections. The relevance of an anti-holomorphic involution in the definition of twistor spaces is discussed in [9, 20, 25, 29, 30].

5.3. **Twistor fibres.** This subsection analyses more carefully the fibres of π.

Definition 5.5. Given $q \in \mathbb{P}^2$ we call $\pi^{-1}(q)$ a *twistor fibre* or *twistor line*.

Given $q \in \mathbb{P}^2$, we have
$$\pi^{-1}(q) = \{(p, \ell) \in \mathbb{F} \mid p^* \times \ell = q\} = \{(p, \ell) \in \mathbb{F} \mid q\ell = 0, \ pq^* = 0\}.$$
From the general theory in the citations above, we know that the fibers of π are rational j-invariant curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Lemma 3.13 and 3.14 tell us each curve $L_{q,m}$ has the same properties once we impose j-invariance. Indeed, Definition 3.3 implies that
$$\pi^{-1}(q) = L_{q,q^*}.$$
In particular, any twistor line also has bidegree $(1,1)$.

The family of twistor fibres is obtained as the set of fixed points of the anti-holomorphic involution j acting on the space of parameters of the closure of $(1,1)$-curves. Whence,

Lemma 5.6. *The set of twistor fibers is a Zariski dense subset of $\mathcal{V}$.*

Notice the analogy between the previous lemma and [2, Lemma 3.2].

Remark 5.7. Since $j(L_{q,m}) = L_{m^*,q^*}$, the curve $L_{q,m}$ is j-invariant if and only if it is a twistor fibre. The latter define a real (i.e. j-invariant) slice of $\mathcal{V}$, see Definition 3.3. Conversely, $\overline{\mathcal{V}} \cong \mathbb{P}^2 \times \mathbb{P}^2$ can be thought of as the compactified complexified space of twistor fibres, analogous to the role that the Klein quadric $\mathbb{Gr}_2(\mathbb{C}^4)$ plays in Penrose's twistor theory for $\mathbb{P}^3$.

Given the vector identity
$$(m \times q^*)(q \times m^*) = -|m \times q^*|^2$$
(before we projectivize), Lemma 3.5 implies that $L_{q,m} \cap j(L_{q,m}) = L_{q,m} \cap L_{m^*,q^*}$ is empty if and only if q, m^* are distinct points of $\mathbb{P}^2$, i.e. $L_{q,m}$ is not a twistor fibre. On the other hand, provided $pq^* \neq 0$,
$$\pi_1^{-1}(p) \cap j(\pi_1^{-1}(q)) = \pi_1^{-1}(p) \cap \pi_2^{-1}(q^*) = \varnothing,$$
so in this case j generates a skew pair of fibres.

Thanks to Remark 3.4, we are able to compute the twistor projections $\pi(\pi_i^{-1}(q))$ of curves of bidegree $(0,1)$ $(i = 1)$ and $(1,0)$ $(i = 2)$. For any $q \in \mathbb{P}^2$, we have
$$\pi(\pi_1^{-1}(q)) = \{z \in \mathbb{P}^2 \mid \pi^{-1}(z) \cap \pi_1^{-1}(q) \neq \varnothing\},$$
but $\pi^{-1}(z) = L_{z,z^*}$, hence $\pi^{-1}(z) \cap \pi_1^{-1}(q) \neq \varnothing$ if and only if $qz^* = 0$. Collecting everything:
$$\pi(\pi_1^{-1}(q)) = \{z \in \mathbb{P}^2 \mid zq^* = 0\},$$
Analogously,
$$\pi(\pi_2^{-1}(m)) = \{z \in \mathbb{P}^2 \mid zm^* = 0\}.$$
Each image is a projective line. The same conclusion can be reached by means of the identity $\pi = \pi_i \circ j_i$ for $i = 1, 2$.

6. CURVES AND SURFACES REVISITED

We begin this section by revisiting the description of surfaces of bidegree $(1,0)$ and $(0,1)$. Any two surfaces of type $(0,1)$ (respectively, $(1,0)$) are equivalent under unitary transformation. Using the projections π_1, π_2, this is a consequence of the fact $SU(3)$ acts transitively on both $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$.

Lemma 6.1. $_qH$ *is invariant by j_2, and H_m is invariant by j_1.*

Proof. This follows immediately from Definitions 3.9 and 5.2. The condition that a point of $\mathbb{F}$ lies in $_qH$ depends only on its image under π_2^*. But this is unchanged by j_2. Similarly for the triple H_m, π_1, j_1. $\square$

Recall Corollary 3.10: π_1 realizes $_qH$ as the blow-up of $\mathbb{P}^2$ at q. The same is true of π, since
$$\pi^{-1}(p) = (\pi_1 \circ j_2)^{-1}(p) = j_2(\pi_1^{-1}(p)),$$
which is a single point unluess $p = q$. A similar argument applies to H_m:

Corollary 6.2. *The twistor projection π realizes $_qH$ as the blow-up of $\mathbb{P}^2$ at q, and H_m as the blow up of $\mathbb{P}^2$ at m^*.*

6.1. **Orthogonal complex structures.** We are now in a position to use the theory of subsection 5.1 to deduce an application of Corollary 6.2. Recall the concept (Definition 5.1) of an orthogonal complex structure (that we shall abbreviate OCS) relative to the Fubini-Study metric.

Proposition 6.3. *The complement of a point q in $\mathbb{P}^2$ admits an OCS J_q, inducing the opposite orientation to the standard complex structure J. Moreover, the action of J_q on $T_p\mathbb{P}^2$ is defined by reversing the sign of J on the complex line L tangent to the projective line containing p and q.*

Proof. The complex structure J_q is that of $_qH \setminus \pi^{-1}(q)$, which π bijects onto $\mathbb{P}^2 \setminus \{p\}$. The fact that this complex structure satisfies (27) follows from the remarks above. The second statement is a consequence of the fact that the complex surface $_qH$ is a union of fibres $\pi_2^{-1}(\ell)$ each of which projects to a line ℓ passing through q. □

Remark 6.4. Note that J_q does not extend to $\mathbb{P}^2$, for one thing π has no global sections over $\mathbb{P}^2$ even topologically. Thus, $\mathbb{P}^2 \setminus \{q\}$ is a maximal domain of definition for J_q.

The results of this subsection point to the analogy between the family of $(1,0)$ (respectively, $(0,1)$) surfaces in the twistor space $\mathbb{F}$ and projective planes in the twistor space $\mathbb{P}^3$. Two members of each family intersect in a projective line, and each member contains exactly one twistor fibre, cf. [35, 36].

In the next subsection, we shall effectively illustrate the appearance of some genus zero J_q-holomorphic curves in the Fubini-Study domain $\mathbb{P}^2 \setminus \{q\}$.

6.2. **Spheres and dumbbells.** We shall now consider the effect of the twistor fibration on the simplest holomorphic objects of bidegree $(1,1)$ that we have defined, namely the curves

$$L_{q,m} = \{(p,\ell) \mid p\ell = 0, \ q\ell = 0, \ pm = 0\}.$$

They form a family $\overline{\mathcal{V}}$ that includes two special cases:

- If $q \in m$ then π maps $L_{q,m} = \pi_1^{-1}(q) \cup \pi_2^{-1}(m)$ to the union $q^* \cup m$ of the two lines in $\mathbb{P}^2$ intersecting in the point $q^* \times m$. For example, if $q = [1:0:0]$ and $m = [0:1:0]^\vee$ then

$$q^* \cup m = \{[z_0 : z_1 : z_2] : z_0 = 0 \text{ or } z_1 = 0\}.$$

- If $q^* = m$ so that then our curve $L_{q,m} = \pi^{-1}(q)$ is a twistor fibre, and projects to the single point $q \in \mathbb{P}^2$.

In the generic case, $L_{q,m}$ is biholomorphic to $\mathbb{P}^1$, and π embeds it in $\mathbb{P}^2$. For if $(p,\ell) \in L_{q,m}$ and $\pi(p,\ell) = r$ then $\ell = r \times q$ and $p = \ell \times m$. The image of $L_{q,m}$ will therefore be homeomorphic to a 2-sphere. Our next results make this precise.

Lemma 6.5. *Let $L_{q,m} \in \mathcal{V}$, with $q = [q_0 : q_1 : q_2]$, $m = [m_0 : m_1 : m_2]$ and $qm \neq 0$. Then $\pi(L_{q,m}) = \{z \in \mathbb{P}^2 \mid z\,\Phi z^* = 0\}$, where*

$$(28) \qquad \Phi = \begin{pmatrix} m_1q_1 + m_2q_2 & -m_0q_1 & -m_0q_2 \\ -m_1q_0 & m_0q_0 + m_2q_2 & -m_1q_2 \\ -m_2q_0 & -m_2q_1 & m_0q_0 + m_1q_1 \end{pmatrix}.$$

Proof. The set $\pi(L_{q,m})$ coincides with the set of points $z \in \mathbb{P}^2$ such that the fiber $\pi^{-1}(z)$ meets $L_{q,m}$. But $\pi^{-1}(z) = L_{z,z^*}$, so

$$\pi(L_{q,m}) = \{z \in \mathbb{P}^2 \mid L_{z,z^*} \cap L_{q,m} \neq \varnothing\}.$$

In view of Lemma 3.5 and Corollary 3.6, we deduce that $L_{q,m}$ and L_{z,z^*} have a non-empty intersection if and only if (choosing vector representatives) the expression

$$(z^* \times m)(z \times q) = (qm)(zz^*) - (zm)(qz^*)$$

vanishes. The right-hand side equals $z\Phi z^*$ where Φ is the 3×3 matrix $(qm)I - mq$, where $qm = q_0m_0 + q_1m_1 + q_2m_2$. whose entries are as stated. □

Proposition 6.6. *Let $qm \neq 0$ and $q^* \neq m$. Then π maps $L_{q,m}$ onto a round 2-sphere in real affine coordinates.*

Proof. The matrix $\Phi = (qm)I - mq$ is singular, we see immediately that $q\Phi = 0$ and $\Phi m = 0$. In order to analyse the equation $z\Phi z^* = 0$ we can first apply a unitary transformation so as to assume that

$$q = [1:0:0], \qquad m = [1:2\rho:0],$$

with ρ a positive real number (which will turn out to be the radius of our sphere).

$$\Phi = \begin{pmatrix} 0 & 0 & 0 \\ -2\rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we obtain

$$|z_1|^2 + |z_2|^2 - 2\rho\bar{z}_0 z_1 = 0. \tag{29}$$

There are no solutions with $z_0 = 0$, so we set $z_0 = 1$ and work with the affine coordinates

$$(z_1, z_2) = (x_1 + iy_1, \ x_2 + iy_2),$$

and their real and imaginary components on the right-hand side. Then

$$\begin{cases} x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2\rho x_1 = 0 \\ y_1 = 0. \end{cases}$$

These equations obviously define a 2-sphere in the Euclidean space $\mathbb{R}^3$ defined by $y_1 = 0$. For the remark below, it will be convenient to set $x = x_1 - \rho$, so that the 2-sphere becomes

$$x^2 + x_2^2 + y_2^2 = \rho^2$$

and with centre $(x, x_2, y_2) = (0, 0, 0)$. $\qquad\qquad\square$

Expanding slightly the proof verifies the special cases discussed above. In the limit $m_1 \to 0$ (so $q^* = m$) the 2-sphere degenerates to a single point. If, on the other hand, we had allowed $m_0 = 0$, we would have been led to the union of the projective lines $z_0 = 0$ and $z_1 = 0$, again as expected. These observations are illustrated by Figure 5, whose justification we work towards.

The 2-sphere $\pi(L_{q,m})$ is not round relative to the Fubini-Study metric because it is not an orbit of $SU(3)$. However, it is a union of circles

$$\{(x, x_2, y_2) : x^2 + x_2^2 + y_2^2 = \rho^2\}$$

that are orbits of $U(1) \subset SU(3)$. It is therefore isometric to a surface of revolution with the induced Fubini-Study metric. In order to identify the shape of this, we shall use the methods of [1, §8]. Set

$$z_1 = x = \rho \sin v, \qquad z_2 = x_2 + iy_2 = \rho e^{iu} \cos v,$$

so the u and v represent longitude and latitude respectively on the 2-sphere. In our inhomogeneous coordinates (z_1, z_2), the restriction of the Fubini-Study metric to $\pi(L_{q,m})$ takes the form

$$\frac{|dz_1|^2 + |dz_2|^2}{1 + \rho^2} - \frac{|\bar{z}_1 dz_1 + \bar{z}_2 dz_2|^2}{(1 + \rho^2)^2}.$$

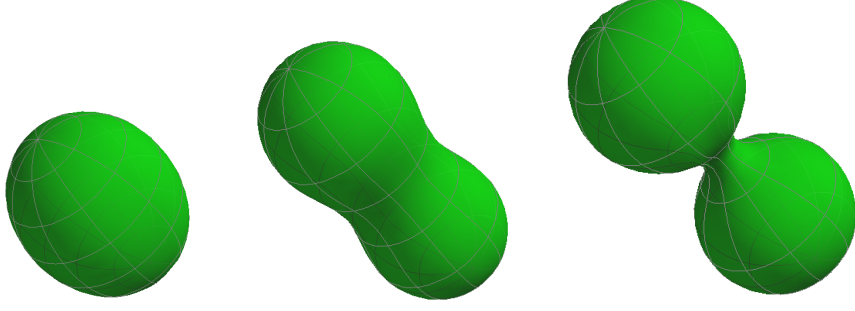A computation shows that this reduces to the first fundamental form

$$\frac{\rho^2 \cos^2 v (1 + \rho^2 \sin^2 v)}{(1 + \rho^2)^2} du^2 + \frac{\rho^2}{1 + \rho^2} dv^2.$$

We seek a profile curve $(f(v), g(v))$ which generates a surface of revolution with the same first fundamental form, which requires setting

$$f(v) = \frac{\rho \cos v \sqrt{1 + \rho^2 \sin^2 v}}{1 + \rho^2}$$

$$f'(v)^2 + g'(v)^2 = \frac{\rho^2}{1 + \rho^2}.$$

In order to plot Figure 5, we determined $g(v)$ by numerical integration.

FIGURE 5. Twistor images of the curve $L_{[1:0:0],[1:2\rho:0]}$ with $\rho = \frac{1}{2}, 2, 8$

6.3. **Surfaces of bidegree (1,d).** This interesting class of rational surfaces S incorporates increasingly wild examples as d becomes large. On the one hand, S is obtained by blowing up k points in $\mathbb{P}^2$, on the other it defines a holomorphic d-fold branched cover over $\mathbb{P}^{2\vee}$. Using Remark 3.12, we deduce that

$$9 - k = c_1^2 = -d^2 - d + 8.$$

Such a surface will hit a generic twistor line in $d + 1$ points, a number that can be regarded as its 'twistor degree', by comparison to the case of $\mathbb{P}^3 \to S^4$.

The case $(1, 2)$ is mentioned in the Introduction and corresponds to a Del Pezzo surface P of degree 2 that double covers $\mathbb{P}^2$ branched over a quartic. Here, P is the zero set of a polynomial lying in the summand $\mathcal{P}_{1,2}$ in the notation of (1).

Example 6.7. Let us consider the surface in the flag defined by

$$p_0(2\ell_0\ell_1 + \ell_2^2) + p_1(2\ell_0\ell_2 + \ell_1^2) + p_2(2\ell_1\ell_2 + \ell_0^2) = 0.$$

This can be seen as the blow up of $\mathbb{P}^{2\vee}$ in 7 points in the following way. Such points can be obtained by imposing that, for a fixed p the previous equation is linearly dependent of $p\ell = 0$, i.e.

$$\begin{pmatrix} 2\ell_0\ell_1 + \ell_2^2 & \ell_0 \\ 2\ell_0\ell_2 + \ell_1^2 & \ell_1 \\ 2\ell_1\ell_2 + \ell_0^2 & \ell_2 \end{pmatrix}$$

has rank one, that is if and only if

$$\begin{cases} \ell_2^3 - \ell_0^3 = 0 \\ \ell_0\ell_1^2 + \ell_1\ell_2^2 - 2\ell_0^2\ell_2 = 0 \\ \ell_0^2\ell_1 + \ell_1^2\ell_2 - 2\ell_0\ell_2^2 = 0 \end{cases}$$

Solving this system we get the following seven points. There is a trivial solution for $\ell_0 = \ell_1 = \ell_2$. The other 6 solutions arise in couples by imposing, $\ell_2 = \eta^k \ell_0$ for $k = 0, 1, 2$ and $\eta = e^{2\pi i/3}$. Therefore the seven points are

$$[0 : 1 : 0], \quad [1 : 1 : 1], \quad [1 : -2 : 1], \quad [1 : \overline{\eta} : \eta], \quad [1 : -2\overline{\eta} : \eta], \quad [1 : \eta : \overline{\eta}], \quad [1 : -2\eta : \overline{\eta}].$$

The surface can be seen as a double cover of $\mathbb{P}^2$ branched over a quartic. We identify the latter by studying the system

$$\begin{cases} p_0(2\ell_0\ell_1 + \ell_2^2) + p_1(2\ell_0\ell_2 + \ell_1^2) + p_2(2\ell_1\ell_2 + \ell_0^2) = 0 \\ p_0 l_0 + p_1 l_1 + p_2 l_2 = 0 \end{cases}$$

If $p_1 \neq 0$, multiplying by p_1 the first equation, we get

$$p_1 p_2 \ell_0^2 + 2p_1^2 \ell_0\ell_2 + p_0 p_1 \ell_2^2 - (p_0\ell_0 + p_2\ell_2)^2 = 0,$$

that is

$$(p_1 p_2 - p_0^2)\ell_0^2 + 2(p_1^2 - p_0 p_2)\ell_0\ell_2 + (p_0 p_1 - p_2^2)\ell_2^2 = 0,$$

and the discriminant is:

$$\tfrac{1}{4}\Delta = (p_1^2 - p_0 p_2)^2 - (p_1 p_2 - p_0^2)(p_0 p_1 - p_2^2) = p_1^4 - 3p_0 p_1^2 p_2 + p_1 p_2^3 + p_0^3 p_1 - p_0^2 p_2^2$$

If $p_1 = 0$ it is easy to check that we still obtain points in the previous quartic.

Example 6.8. Let us consider the surface in the flag defined by

$$p_0 \ell_1^2 + p_1 \ell_2^2 + p_2 \ell_0^2 = 0.$$

This can be seen as the blow up of $\mathbb{P}^{2\vee}$ in 7 points in the following way. Such points can be obtained by imposing that, for a fixed p the previous equation is linearly dependent of $p\ell = 0$, i.e.

$$\begin{pmatrix} \ell_1^2 & \ell_0 \\ \ell_2^2 & \ell_1 \\ \ell_0^2 & \ell_2 \end{pmatrix}$$

has rank one, that is if and only if

$$\begin{cases} \ell_1^3 - \ell_0 \ell_2^2 = 0 \\ \ell_0^3 - \ell_2 \ell_1^2 = 0 \\ \ell_2^3 - \ell_1 \ell_0^2 = 0 \end{cases}$$

Solving this system we get the following seven points for $k = 0, \dots, 6$

$$[\zeta^{3k} : \zeta^k : 1],$$

where ζ is any seventh root of one.

The surface can be seen as a double cover of $\mathbb{P}^2$ branched over a quartic. We identify the latter by studying the system

$$\begin{cases} p_0 \ell_1^2 + p_1 \ell_2^2 + p_2 \ell_0^2 = 0 \\ p_0 l_0 + p_1 l_1 + p_2 l_2 = 0 \end{cases}$$

If $p_0 \neq 0$, multiplying by p_0^2 the first equation, we get

$$p_0^3 \ell_1^2 + p_0^2 p_1 \ell_2^2 + p_0^2 p_2 \ell_0^2 = 0$$

that is

$$(p_0^3 + p_1^2 p_2)\ell_1^2 + 2p_1 p_2^2 \ell_1 \ell_2 + (p_0^2 p_1 + p_2^3)\ell_2^2 = 0,$$

and the discriminant is:

$$\tfrac{1}{4}\Delta = -p_0^2 (p_0^3 p_1 + p_1^3 p_2 + p_2^3 p_0).$$

If $p_0 = 0$ it is easy to check that we still obtain points in the quartic defined by $p_0^3 p_1 + p_1^3 p_2 + p_2^3 p_0 = 0$.

7. TWISTOR PROJECTIONS OF SMOOTH TORIC SURFACES

Having studied the twistor images of curves and (at the start of section 6) surfaces of degree one, we turn attention to the next simplest case. Our general goal will be to determine the subsets of $\mathbb{P}^2$ defined by

Definition 7.1. Let S be any element of $|\mathcal{O}(a,b)|$. Its *twistor discriminant locus* $\mathcal{D}(S)$ is the union

$$\mathcal{D} = \mathcal{D}_0 \cup \mathcal{R},$$

where $\mathcal{D}_0 = \{q \in \mathbb{P}^2 \mid \pi^{-1}(q) \subset S\}$ and $\mathcal{R}$ is the branch locus of $\pi_{|S} : S \to \mathbb{P}^2$ consisting of points $p \in \mathbb{P}^2$ for which $|\pi^{-1}(p)| < a + b$.

We shall determine the discriminant locus for smooth surfaces in $\mathbb{F}$ of bidegree $(1,1)$. Any such surface is defined by the canonical form

$$(30) \qquad\qquad A = A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad \lambda \in \{0, 1\},$$

from Lemma 4.1. In order to make sense of twistorial proprties, and since we are ultimately intereested in the metric geometry of $\mathcal{D}$, we work up to unitary equivalence.

Recall that $S_A, S_{A'}$ are unitarily equivalent if and only if there exists X in $SU(3)$ (equivalently, in $U(3)$) such that

$$A' = X^* A X = X^{-1} A X.$$

Since (30) commutes with elements of the standard maximal torus T^2 of $SU(3)$, our constructions will be T^2-invariant. This is not surprising, since it is well known that the Del Pezzo surfaces arising in Proposition 4.2 are in fact toric. We first treat the case in which S_A is real (meaning that $j(S) = S$), whose conclusion will motivate the non-real case.

7.1. **Real (1,1)-surfaces.** Here we focus on the j-invariant case, for the moment without assuming smoothness. It follows from the definitions (14) and (5.2) that S_A is j-invariant if and only if $A = A^*$, i.e. A is Hermitian. Such a matrix has real eigenvalues, is diagonalizable over $SU(3)$, and S_A can be represented by a real diagonal matrix. Proposition 4.5 implies that S_A is either smooth and of type A_1 in Lemma 4.1, or reducible and of type A_2.

The reducible surfaces correspond to matrices A with two distinct eigenvalues; any one has the form ${}_qH \cup H_{q^*}$, is singular on the twistor line L_{q,q^*}, and contains no other twistor line. It follows that any two are unitarily equivalent, so little more needs to be written.

In the smooth case, we can impose (30) with $\lambda \in (1, 2]$. This restriction arises from Remark 4.4, and ensures that the unitary equivalence class of S_A is uniquely determined by a point of the interval $(1, 2]$.

Theorem 7.2. *Let S be a smooth j-invariant surface of bidegree $(1, 1)$. The twistor projection π restricted to S is a degree 2 cover of $\mathbb{P}^2$ without ramification, i.e. $\mathcal{R} = \varnothing$. Moreover, S contains infinitely many twistor fibers and $\mathcal{D}_0$ is parametrized by a circle that is the orbit of a maximal torus in $SU(3)$.*

Proof. Fix $q \in \mathbb{P}^2$. Recall that its twistor fibre is
$$\pi^{-1}(q) = L_{q,q^*} = \{(p, \ell) \mid p\ell = 0,\ q\ell = 0,\ pq^* = 0\} = {}_qH \cap H_{q^*}.$$
Now
$$S_A \cap {}_qH = \{(p, \ell) \mid p\ell = 0,\ pA\ell = 0,\ q\ell = 0\}$$
is non-empty if and only if

(31)
$$\det(p \mid pA \mid q) = 0.$$

This equation can be written as $p\,Cp^\vee = 0$, where
$$C = \begin{pmatrix} 0 & q_2 & -\lambda q_1 \\ q_2 & 0 & (\lambda - 1)q_0 \\ -\lambda q_1 & (\lambda - 1)q_0 & 0 \end{pmatrix},$$
which defines a conic $\mathcal{C}$. Applying the involution j shows that
$$\pi(S_A \cap {}_qH) = \mathcal{C} = \pi_2^*(S_A \cap H_{q^*}).$$
The remaining equation

(32)
$$pq^* = 0$$

asserts that p lies on the line q^*, so there will be two such points on C for generic q. If C contains the twistor fibre $\pi^{-1}(q)$ then C must be reducible and
$$0 = \det C = 2\lambda(\lambda - 1)q_0 q_1 q_2,$$
so that one of q_0, q_1, q_2 vanishes. We obtain the following three equations for $\mathcal{C}$:
$$\begin{cases} p_0(q_2 p_1 - q_1 \lambda p_2) = 0 & \text{if } q_0 = 0 \\ p_1(q_2 p_0 + q_0(\lambda - 1)p_2) = 0 & \text{if } q_1 = 0 \\ p_2(-q_1 \lambda p_0 + q_0(\lambda - 1)p_1) = 0 & \text{if } q_2 = 0 \end{cases}$$
Combining these with (32),

(33)
$$\begin{cases} q_0 = 0 & \text{and} & |q_1|^2 \lambda + |q_2|^2 = 0 \\ q_1 = 0 & \text{and} & |q_2|^2 - |q_0|^2(\lambda - 1) = 0 \\ q_2 = 0 & \text{and} & |q_1|^2 \lambda + |q_0|^2(\lambda - 1) = 0. \end{cases}$$

However, our assumption that $\lambda > 1$ rules out all but the second possibility, so $q_1 = 0$ and $|q_2/q_0| = \sqrt{\lambda - 1}$. This is indeed a circular orbit of T with a 1-dimensional stabilizer.

We shall now determine the branch locus $\mathcal{R}$. If $q \in \mathcal{R}$ then clearly the line (32) is tangent to C. If $q_0 q_1 q_2 = 0$ then C is reducible and tangency would imply $q \in \mathcal{D}_0$. We can therefore assume that $q_0 q_1 q_2 \neq 0$. In general, tangency implies that

$$(34) \qquad\qquad 0 = q^* \times (Cp^\vee) = (q^* \times C)\, p^\vee,$$

in which the cross product $q^* \times C$ is a 3×3 matrix computed from C column by column. It follows that

$$\left(\begin{array}{ccc} \lambda|q_1|^2 + |q_2|^2 & -(\lambda-1)q_0\bar{q}_1 & (\lambda-1)q_0\bar{q}_2 \\ -\lambda q_1\bar{q}_0 & (\lambda-1)|q_0|^2 - |q_2|^2 & \lambda q_1\bar{q}_2 \\ -q_2\bar{q}_0 & q_2\bar{q}_1 & -(\lambda-1)|q_0|^2 - \lambda|q_1|^2 \\ \hline \bar{q}_0 & \bar{q}_1 & \bar{q}_2 \end{array} \right) p^\vee = 0.$$

The 4×3 matrix incorporates $q^* \times C$ and (32), and must therefore have rank at most 2. Note that $q^* \cdot (q^* \times C) = 0$, so $\det(q^* \times C) = 0$. Since $q_0 q_1 q_2 \neq 0$, the remaining 3×3 minors each tell us that

$$(|q_0|^2(\lambda-1) + |q_1|^2\lambda - |q_2|^2)^2 + 4|q_0|^2|q_2|^2(\lambda-1) = 0.$$

Since $\lambda > 0$ and $q_0 q_2 \neq 0$, we conclude that $\mathcal{R} = \varnothing$. $\qquad\square$

Remark 7.3. Theorem 7.2 gives a clear expectation of how the discriminant locus behaves for a smooth surface S_A with A diagonalizable, and therefore toric. Consider the moment mapping

$$\mu\colon \qquad \begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \Delta \subset \mathbb{R}^3 \\ [q_0 : q_1 : q_2] & \mapsto & (|q_0|^2, |q_1|^2, |q_2|^2)/\|q\|^2 \end{array}$$

corresponding to the action of T on $\mathbb{P}^2$, equipped with its Fubini-Study symplectic 2-form. Its image is a 2-simplex Δ, and the union of lines $q_0 q_1 q_2 = 0$ is mapped onto the boundary $\partial\Delta$.

If A is Hermitian, its discriminant circle is mapped to a point of $\partial\Delta$. This is a midpoint of an edge of $\partial\Delta$ if $\lambda \in \{-1, \frac{1}{2}, 2\}$, in which case the circle is maximal. Such circles played a key role in the configuration of equidistant points in $\mathbb{P}^2$ [27]. The prohibited values $\lambda \in \{0, 1, \infty\}$ correspond to the vertices of Δ for which S_A is reducible and the discriminant locus becomes a single point. We shall see that if A is diagonalizable, but not Hermitian, its discrimant locus is a smooth 2-torus that maps to an interior point of Δ.

Corollary 7.4. *Let S be as in the hypothesis of Theorem 7.2. Then $\mathbb{P}^2 \setminus \mathcal{D}_0$ is the maximal domain of an orthogonal complex structure J_S.*

Proof. First, we observe that $S_A \setminus \pi^{-1}(\mathcal{D}_0)$ has two connected components. This follows directly from the generalized Jordan Curve Theorem [11, Theorem VI.8.8], given that S_A is the blow-up of $\mathbb{P}^2$ at three points. The components are interchanged by j. Either one projects bijectively onto $\mathbb{P}^2 \setminus \mathcal{D}_0$ and induces the OCS J_S. $\qquad\square$

7.2. Toric non-real (1,1)-surfaces. In this subsection, we shall describe the twistor discriminant locus for a surface S_A with A in the diagonal form (30) with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This condition on λ ensures that S_A is smooth but not j-invariant.

Theorem 7.5. *Let S_A be the surface defined by (30) with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the twistor projection π restricted to S is a degree 2 cover of $\mathbb{P}^2$ whose branch locus $\mathcal{R} \subset \mathbb{P}^2$ is the zero set of*

$$R(q) = (|q_0|^2(\lambda-1) + |q_1|^2\lambda + |q_2|^2)^2 - 4|q_0|^2|q_2|^2(\lambda-1) = 0\}.$$

Moreover, S contains no twistor fibres, i.e. $\mathcal{D}_0 = \varnothing$.

Proof. Inevitably, we follow the same first steps of the proof of Theorem 7.2.

The surface S_A contains the fiber $\pi^{-1}(q)$ if and only if $q_0 = 0$ and $|q_1|^2\lambda + |q_2|^2 = 0$ or $q_1 = 0$ and $|q_2|^2 - |q_0|^2(\lambda-1) = 0$ or $q_2 = 0$ and $|q_1|^2\lambda + |q_0|^2(\lambda-1) = 0$. But in all three cases, as $\lambda_1 \neq 0$, we have no solution $q \in \mathbb{P}^2$.

We now want to study the branch locus. As in the proof of Theorem 7.2, if $q_0 q_1 q_2 = 0$ there is no branch locus. Hence for the remainder, assume $q_0 q_1 q_2 \neq 0$. Again, following the proof of Theorem 7.2, we obtain that the branch locus of the map $\pi_{|S_A}$ is given $R(q) = 0$ where $R(q)$ is defined as in the statement of the theorem. $\qquad\square$

In the following corollary we give a geometric description of the set $\mathcal{R}$.

Corollary 7.6. *Let S_A be a $(1,1)$-surface as in Theorem 7.5. Let $\alpha = \alpha_0 + i\alpha_1$ be a square root of $\lambda - 1$ and $\beta = \beta_0 + i\beta_1$ a square root of λ, such that $\alpha_1\beta_0 > 0$ and $\frac{\alpha_0\beta_0 + \alpha_1\beta_1}{\alpha_1} > 0$. Then the branch locus of $\pi_{|S_A}$ is*

$$\mathcal{R} = \{q \in \mathbb{P}^2 \mid |q_0|\alpha - i|q_1|\beta - |q_2| = 0\}.$$

In particular, the twistor discriminant locus of S_A is a smooth 2-dimensional real torus.

Proof. First of all notice that, since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, as $\alpha^2 = \lambda - 1$ and $\beta^2 = \lambda$, then $\alpha_0\alpha_1 \neq 0$ and $\beta_0\beta_1 \neq 0$. Moreover up to changes of signs of α and β we can always reduce to the assumptions of the statement. Given that, the proof is smoothly obtained thanks to the following computation:

$$\begin{aligned}
R(q) &= (|q_0|^2(\lambda-1) + |q_1|^2\lambda + |q_2|^2)^2 - 4|q_0|^2|q_2|^2(\lambda-1) \\
&= (|q_0|^2(\lambda-1) + |q_1|^2\lambda + |q_2|^2 - 2|q_0||q_2|\alpha)(|q_0|^2(\lambda-1) + |q_1|^2\lambda + |q_2|^2 + 2|q_0||q_2|\alpha) \\
&= ((|q_0|\alpha - |q_2|)^2 + |q_1|^2\lambda)((|q_0|\alpha + |q_2|)^2 + |q_1|^2\lambda) \\
&= (|q_0|\alpha - i|q_1|\beta - |q_2|)(|q_0|\alpha + i|q_1|\beta - |q_2|)(|q_0|\alpha - i|q_1|\beta + |q_2|)(|q_0|\alpha + i|q_1|\beta + |q_2|).
\end{aligned}$$

We claim that only the first term of this factorization vanishes. Indeed, setting

$$R_{--}(q) = |q_0|\alpha - i|q_1|\beta - |q_2|, \qquad R_{+-}(q) = |q_0|\alpha + i|q_1|\beta - |q_2|,$$
$$R_{-+}(q) = |q_0|\alpha - i|q_1|\beta + |q_2|, \qquad R_{++}(q) = |q_0|\alpha + i|q_1|\beta + |q_2|,$$

And splitting into real and imaginary part, we get that $R_{**}(q) = 0$ ($* = +, -$) if and only if

$$\begin{cases} |q_0|\alpha_0 \mp i|q_1|\beta_1 \pm |q_2| = 0 \\ |q_0|\alpha_1 \pm i|q_1|\beta_0 = 0 \end{cases}$$

By looking at the second equation we get that $R_{-*}(q) = 0$ admits solutions and $R_{+*}(q) = 0$ doesn't, since $\alpha_1\beta_0 > 0$. Moreover, we have that $|q_0| = |q_1|\frac{\beta_0}{\alpha_1}$ and hence, the first equation can be written as

$$|q_1|\frac{\alpha_0\beta_0 + \alpha_1\beta_1}{\alpha_1} \pm |q_2| = 0.$$

Therefore, $R_{--}(q) = 0$ admits solutions and $R_{-+}(q) = 0$ doesn't, since $\frac{\alpha_0\beta_0 + \alpha_1\beta_1}{\alpha_1} > 0$. $\qquad\square$

Remark 7.7. In the previous corollary, we saw by direct computations that the set $\mathcal{R}$ is a torus. This fact can also be seen because $\mathcal{R}$ is invariant under the torus action $(\vartheta_1, \vartheta_2) \cdot [q_0 : q_1 : q_2] = [q_0 : e^{i\vartheta_1}q_1, e^{i\vartheta_2}q_2]$, for any $(\vartheta_1, \vartheta_2) \in S^1 \times S^1$.

Theorems 7.2 and 7.5 can be considered as *toy models* to study the most general case. We will see now that the general analysis of the twistor discriminant locus of a smooth $(1,1)$ surface is highly non-trivial. In particular we decided to split the analysis of the twistor discriminant locus into two parts: first we study the set $\mathcal{D}_0$ of twistor fibers contained in a smooth $(1,1)$ surface, then we give Cartesian equations for the set $\mathcal{R}$.

8. Counting twistor lines in a surface

In this section, we give a first upper bound on the number of twistor lines contained in a surface S which is not j-invariant, based on a Bezoút type method. For higher bidegrees, the bounds are not likely to be optimal. In particular, we shall prove in Corollary 8.4 that a non j-invariant $(1,1)$ surface contains at most two twistor lines, and in this case the bound is actually sharp.

First of all, notice that if $S \in |\mathcal{O}(a,b)|$, then $j(S) \in |\mathcal{O}(b,a)|$.

The next result gives a first bound on the number of $(1,1)$-curves contained in a non j-invariant surface S.

Proposition 8.1. *Fix $a, b \geqslant 0$ and an integral $S \in |\mathcal{O}(a,b)|$. If $j(S) \neq S$, then $S \cap j(S)$ contains at most $a^2 + ab + b^2$ integral curves of bidegree $(1,1)$.*

Proof. Since S is integral and $j(S) \neq S$, then $S \cap j(S)$ is a finite union of (possibly non-reduced) curves, that is

$$S \cap j(S) = \sum m_i C_i$$

with m_i positive integers, and C_i an irreducible curve with bidegree (E_i, F_i) for any i. Recall that $E_i = \deg(\pi_1(C_i)) \deg(\pi_{1|C_i})$ and $F_i = \deg(\pi_2(C_i)) \deg(\pi_{2|C_i})$. Then

$$\sum m_i \deg(\pi_{1|C_i}) = \mathcal{O}(a, b) \cdot \mathcal{O}(b, a) \cdot \mathcal{O}(1, 0)$$

as well as

$$\sum m_i \deg(\pi_{2|C_i}) = \mathcal{O}(a, b) \cdot \mathcal{O}(b, a) \cdot \mathcal{O}(0, 1).$$

But we have.

$$\mathcal{O}(a, b) \cdot \mathcal{O}(b, a) \cdot \mathcal{O}(1, 0) = (a\mathcal{O}(1, 0) + b\mathcal{O}(0, 1)) \cdot (b\mathcal{O}(1, 0) + a\mathcal{O}(0, 1)) \cdot \mathcal{O}(1, 0)$$
$$= (ab\mathcal{O}(1, 0) \cdot \mathcal{O}(1, 0) + a^2\mathcal{O}(1, 0)\mathcal{O}(0, 1) + b^2\mathcal{O}(1, 0)\mathcal{O}(0, 1) + ab\mathcal{O}(0, 1)\mathcal{O}(0, 1)) \cdot \mathcal{O}(1, 0),$$

and

$$\mathcal{O}(a, b) \cdot \mathcal{O}(b, a) \cdot \mathcal{O}(0, 1) = (a\mathcal{O}(1, 0) + b\mathcal{O}(0, 1)) \cdot (b\mathcal{O}(1, 0) + a\mathcal{O}(0, 1)) \cdot \mathcal{O}(0, 1)$$
$$= (ab\mathcal{O}(1, 0) \cdot \mathcal{O}(1, 0) + a^2\mathcal{O}(1, 0)\mathcal{O}(0, 1) + b^2\mathcal{O}(1, 0)\mathcal{O}(0, 1) + ab\mathcal{O}(0, 1)\mathcal{O}(0, 1)) \cdot \mathcal{O}(0, 1).$$

Therefore, by Proposition 3.11, we have

$$\mathcal{O}(a, b) \cdot \mathcal{O}(b, a) \cdot \mathcal{O}(1, 0) = \mathcal{O}(a, b) \cdot \mathcal{O}(b, a) \cdot \mathcal{O}(0, 1) = a^2 + ab + b^2.$$

Hence S contains at most $a^2 + ab + b^2$ integral curves of bidegree $(1, 1)$ (just setting $E_i = F_i = 1$). $\square$

By means of Lemma 2.3, we are able to prove the following proposition that will be used to refine the previous bound.

Proposition 8.2. *Let a, b, c, d be positive integers, $S \in |\mathcal{O}_{\mathbb{F}}(a, b)|$ and $D \in |\mathcal{O}_S(b, d)|$. Then $H^0(\mathcal{O}_S) = H^0(\mathcal{O}_D) = \mathbb{C}$, i.e. every holomorphic function on D is constant. Thus S and D are connected.*

Proof. Consider the exact sequence

$$(35) \qquad\qquad\qquad 0 \to \mathcal{O}_{\mathbb{F}}(-a, -b) \to \mathcal{O}_{\mathbb{F}} \to \mathcal{O}_S \to 0$$

Since, by Lemma 2.3, $H^0(\mathcal{O}_{\mathbb{F}}) = \mathbb{C}$ and $H^1(\mathcal{O}_{\mathbb{F}}(-a, -b)) = 0$, we get $H^0(\mathcal{O}_S) = \mathbb{C}$. Tensoring (35) by $\mathcal{O}_{\mathbb{F}}(-c, -d)$ and using Lemma 2.3 we also get that $H^1(\mathcal{O}_S(-c, -d)) = 0$.

Hence by the exact sequence

$$(36) \qquad\qquad\qquad 0 \to \mathcal{O}_S(-c, -d) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$

we also get $H^0(\mathcal{O}_D) = \mathbb{C}$. $\square$

Recall that a twistor line is a j-invariant integral curve of bidegree $(1, 1)$. A surface of bidegree $(1, 0)$ or $(0, 1)$ contains exactly one twistor line. As a consequence of the previous two results, we get the following bounds on the number of twistor lines contained in a surface which is not j-invariant.

Corollary 8.3. *Fix $a, b > 0$ and an integral $S \in |\mathcal{O}(a, b)|$. If $S \neq j(S)$, then S contains at most $a^2 + ab + b^2 - 1$ twistor lines.*

Proof. Notice that if L is a twistor line contained in S, then $L \in S \cap j(S)$. Hence by Proposition 8.1, we know that the number of twistor lines contained in S is at most $a^2 + ab + b^2$. We prove now that S does not contain exactly $a^2 + ab + b^2$ twistor lines. Assume that this is the case. Then we would have that $S \cap j(S)$ is the union of $a^2 + ab + b^2 \geqslant 2$ pairwise disjoint curves. In particular $S \cap j(S)$ is not connected. But by Proposition 8.2, noting that $D = S \cap j(S) \in |\mathcal{O}_S(b, a)|$, we have that $S \cap j(S)$ is connected. This gives a contradiction and concludes the proof. $\square$

Corollary 8.4. *Let $S \in |\mathcal{O}(1, 1)|$. If $S \neq j(S)$, then S contains at most 2 twistor lines.*

8.1. **Configurations of lines.** The bound from Corollary 8.3 demonstrates an interesting phenomenon. As a non j-invariant $S \in |\mathcal{O}(1,1)|$ contains at most 2 twistor lines, then, by Proposition 8.1 the intersection $S \cap j(S)$ contains at most just one integral $(1,1)$ curve C non j-invariant, but this is impossible, otherwise $S \cap j(S)$ would contains $j(C)$ as well contradicting the bound (see Remark 5.7). This means that in this case, as $S \cap j(S)$ is connected, there should be another j-invariant set of curves of possible bidegrees $(1,0)$ or $(0,1)$ that connect the two twistor fibers. The next examples illustrate exactly this situation.

Example 8.5. Consider the $(1,1)$ surface $S_A \in \mathbb{F}$ defined by the matrix

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then A define a smooth non j-invariant $(1,1)$-surface. The Cartesian equations of S and $j(S)$ are the following

$$S = \{(p,\ell) \in \mathbb{F} \mid (3\ell_0 + \ell_1)p_1 + \ell_2 p_2 = 0\}, \quad j(S) = \{(p,\ell) \in \mathbb{F} \mid (3p_0 + p_1)\ell_1 + p_2 \ell_2 = 0\}.$$

Therefore the intersection $S \cap j(S)$ in the flag $\mathbb{F}$ is given by the following system

(37)
$$\begin{cases} \ell_0 p_0 + \ell_1 p_1 + \ell_2 p_2 = 0 \\ (3\ell_0 + \ell_1)p_1 + 2\ell_2 p_2 = 0 \\ (3p_0 + p_1)\ell_1 + 2p_2 \ell_2 = 0 \end{cases}$$

By interpreting the previous intersection as a linear system with l unknowns and p as parameters, we get that it admits a solution in $\mathbb{P}^\vee$ if and only if the determinant of the associated matrix vanishes, i.e. when the following equation holds

(38)
$$-3p_2(p_0 - p_1)(2p_0 - p_1) = 0.$$

By changing the roles of l and p, we get that it admits a solution $p \in \mathbb{P}$ if and only if

(39)
$$3\ell_2(\ell_0 - \ell_1)(2\ell_0 - \ell_1) = 0.$$

By analyzing the factors of Formula (38) we obtain that

- if $p_2 = 0$, then $\ell_1 = 0$ and $\ell_0 = 0$; hence, in this case System (37) has the following set of solutions $(p,\ell) = ([* : * : 0], [0 : 0 : 1])$ that is $\pi_2^{-1}([0,0,1])$;
- if $p_0 - p_1 = 0$, then, by analyzing the second and the third equation in System (37), we get that $\ell_0 - \ell_1 = 0$ and that $2\ell_0 p_0 + \ell_2 p_2 = 0$, but these are the equation of the twistor fibre $L_{[1:-1:0]}$;
- if $2p_0 - p_1 = 0$, in complete analogy with the previous case, we get that $2\ell_0 - \ell_1 = 0$ ¡and that $5\ell_0 p_0 + \ell_2 p_2 = 0$ and these are the equation of the twistor fibre $L_{[2:-1:0]}$.

By performing the same analysis of the factor of Formula (39), we obtain that

- if $\ell_2 = 0$, then $p_1 = 0$ and $p_0 = 0$; hence, in this case System (37) has the following set of solutions $(p,\ell) = ([0 : 0 : 1], [* : * : 0])$ that is $\pi_1^{-1}([0,0,1])$;
- if $\ell_0 - \ell_1 = 0$, as before we get the twistor fibre $L_{[1:-1:0]}$;
- if $2\ell_0 - \ell_1 = 0$, as before we get the twistor fibre $L_{[2:-1:0]}$.

Notice that $j(\pi_2^{-1}([0,0,1])) = \pi_1^{-1}([0,0,1])$, so the set of solution of System (37) is indeed j-invariant. Of course, $L_{[1:-1:0]}$ and $L_{[1:-2:0]}$ are disjoint but these two lines should intersect somewhere with $\pi_2^{-1}([0,0,1]) \cup \pi_1^{-1}([0,0,1])$. We now compute these mutual intersections. First, recall from Remark 5.7 that $\pi_2^{-1}([0,0,1]) \cap \pi_1^{-1}([0,0,1]) = \varnothing$. By direct inspection we get the following intersections

$$\begin{cases} A = L_{[1:-1:0]} \cap \pi_2^{-1}([0,0,1]) = ([1 : 1 : 0], [0 : 0 : 1]) \\ B = L_{[1:-1:0]} \cap \pi_1^{-1}([0,0,1]) = ([0 : 0 : 1], [1 : 1 : 0]) \\ C = L_{[2:-1:0]} \cap \pi_2^{-1}([0,0,1]) = ([1 : 2 : 0], [0 : 0 : 1]) \\ D = L_{[2:-1:0]} \cap \pi_1^{-1}([0,0,1]) = ([0 : 0 : 1], [1 : 2 : 0]) \end{cases}$$

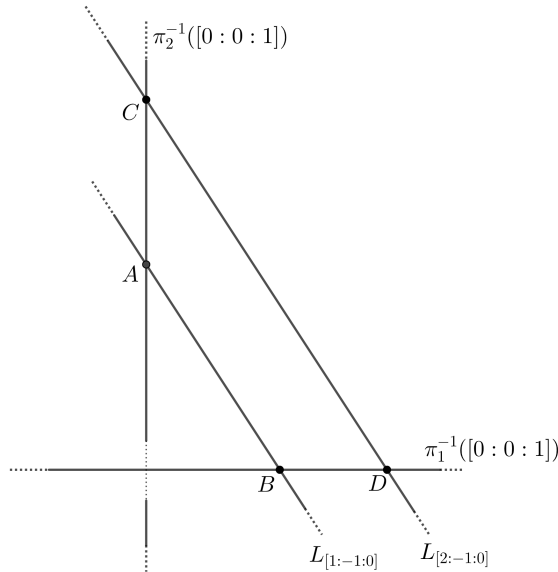We now give other two examples of smooth $(1,1)$-surfaces containing just one or none twistor fibres.

FIGURE 6. Intersection of S_A and $j(S_A)$. Notice how $S_A \cap j(S_A)$ is a connected set of two $(1,1)$-curves and two canonical (skew) fibres.

Example 8.6. Consider now the $(1,1)$ surface $S_A \in \mathbb{F}$ defined by the matrix

$$A = \begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

With analogous computations as in the previous example we can see that the intersection $S \cap j(S)$ is given by the union of the twistor line $L_{[1:\sqrt{2}:0]}$ and the two fibers $\pi_1^{-1}([0,0,1])$ and $\pi_2^{-1}([0,0,1])$.

Example 8.7. Consider finally the $(1,1)$ surface $S_A \in \mathbb{F}$ defined by the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In this case we get that the intersection $S \cap j(S)$ is given by the union of two lines $L_1 = L_{[1:1+i:0],[1:1+i:0]}$ and $L_2 = L_{[1:1+i:0],[1:1+i:0]}$ and the two fibers $\pi_1^{-1}([0,0,1])$ and $\pi_2^{-1}([0,0,1])$. Notice that $j(L_1) = L_2$ and there are no twistor lines contained in S.

We conclude this section with the following easy consequence of Proposition 8.1.

Corollary 8.8. *Let $S \in |\mathcal{O}(a,b)|$. If S contains infinitely many twistor lines, then S is j-invariant and hence $a = b$.*

9. TWISTOR PROJECTIONS OF SOME NON-REAL SURFACES

In section 8, we gave upper bounds on the number of twistor fibers contained in a non j-invariant integral surface and showed with an example, that such bound is attained in the case of $(1,1)$-surfaces. In this section we give an explicit proof that any non j invariant smooth $(1,1)$ surface contains at most 2 twistor lines.

We will need the following variations on the theme of solving quadratic equations:

Lemma 9.1. *Let $f = f_0 + if_1 \in \mathbb{C}$. If $f_0 + f_1^2 \leqslant \frac{1}{4}$ then the equation*

$$|z|^2 + z + f = 0$$

has one or two solutions

$$z = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4f_0 - 4f_1^2} - if_1.$$

If $f_0 + f_1^2 > \frac{1}{4}$ then it has no solution.

If we set $z = x + iy$ then obviously $y = -f_1$ and the lemma follows immediately from the usual quadratic formula applied to $x = \operatorname{Re} z$.

Corollary 9.2. *Let e be a non-zero complex number, and consider the equation*

$$(40) \qquad |z|^2 + ez + f = 0.$$

Set $\Delta = |e|^4 - 4f_0|e|^2 - 4f_1^2$. Then (40) admits a solution if and only if $\Delta \geqslant 0$, in which case

$$z = -\left(\tfrac{1}{2}|e|^2 \pm \tfrac{1}{2}\sqrt{\Delta} + if_1 \right) e^{-1}.$$

Proof. Set $Z = ez/|e|^2$ and $F = f/|e|^2$. Dividing (40) by $|e|^2$, we obtain

$$|Z|^2 + Z + F = 0.$$

Lemma 9.1 gives

$$Z = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4F_0 - 4F_1^2} - iF_1,$$

provided the square root is real. The statement follows. $\qquad\square$

9.1. Twistor lines again.

Corollary 9.2 enables us to state the main result of this section in which we establish real algebraic conditions on the coefficients of the matrix A for which the surface S_A contains 2, 1 or no twistor fibers.

Theorem 9.3. *Let $S = S_A$ be a smooth surface of bidegree $(1,1)$ such that $S \neq j(S)$ and the matrix A has the form*

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{pmatrix},$$

where $\lambda = \lambda_0 + i\lambda_1 \in \mathbb{C} \setminus \{0\}$ and $[a : b : c] \in \mathbb{P}^2$. If one of the following mutually exclusive conditions is satisfied, then S contains exactly one twistor fibre:

- *(i) $b \neq 0 \neq c$ and $|b|^2(1 - \lambda) - |c|^2\lambda - a\bar{b}c = 0$;*
- *(ii) $a \neq 0 \neq c$ and $|a|^2(1 - \lambda) + |c|^2 + a\bar{b}c = 0$;*
- *(iii) $a \neq 0 \neq b$ and $|a|^2\lambda + |b|^2 - a\bar{b}c = 0$;*
- *(iv) $b = 0 = c$ and $|a|^4 - 4|a|^2(|\lambda|^2 - \lambda_0) - 4\lambda_1^2 = 0$;*
- *(v) $a = 0 = c$ and $|b|^4 - 4(1 - \lambda_0)|b|^2 - 4\lambda_1^2 = 0$;*
- *(vi) $a = 0 = b$ and $|c|^4 - 4\lambda_0|c|^2 - 4\lambda_1^2 = 0$.*

If one of the following mutually exclusive conditions is satisfied, then S contains exactly two twistor fibres:

- *(vii) $b = 0 = c$ and $|a|^4 - 4|a|^2(|\lambda|^2 - \lambda_0) - 4\lambda_1^2 > 0$;*
- *(viii) $a = 0 = c$ and $|b|^4 - 4(1 - \lambda_0)|b|^2 - 4\lambda_1^2 > 0$;*
- *(ix) $a = 0 = b$ and $|c|^4 - 4\lambda_0|c|^2 - 4\lambda_1^2 > 0$.*

Proof. Let S_A be the surface defined by the matrix

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{pmatrix}$$

with $a, b, c, \lambda \in \mathbb{C}$, and $\lambda = \lambda_0 + i\lambda_1 \neq 0, 1$.

As in the proof of Theorem 7.2 we impose

$$(41) \qquad \det(p \mid pA \mid q) = 0,$$

and we get the conic C

$$(q_2)p_0 p_1 + (-\lambda q_1 + c q_2)p_0 p_2 + (-a q_2)p_1^2 + (-c q_0 + b q_1)p_2^2 + ((\lambda - 1)q_0 + a q_1 - b q_2)p_1 p_2 = 0.$$

Intersecting it with the line $p\bar{q} = 0$, we get generically two intersection points which give the two pre-images of q. The conic C is singular when the matrix

$$H = \begin{pmatrix} 0 & q_2 & -\lambda q_1 + c q_2 \\ q_2 & -2 a q_2 & (\lambda - 1) q_0 + a q_1 - b q_2 \\ -\lambda q_1 + c q_2 & (\lambda - 1) q_0 + a q_1 - b q_2 & 2(-c q_0 + b q_1) \end{pmatrix}$$

has vanishing determinant, i.e.

$$2 q_2 ((1 - \lambda) q_1 + c q_2)(\lambda q_0 - a \lambda q_1 + (ac - b) q_2) = 0.$$

This equation describes the union of three lines

$$\begin{aligned} r_1 : &\quad q_2 = 0, \\ r_2 : &\quad (1 - \lambda) q_1 + c q_2 = 0, \\ r_3 : &\quad \lambda q_0 - a \lambda q_1 + (ac - b) q_2 = 0. \end{aligned}$$

We analyze now the three cases in which $\det H$ vanishes as those points q form the set where $\pi^{-1}(q) \subset S_A$. Recall that the only missing condition is $p\bar{q} = 0$.

If $q \in r_1$, i.e. $q_2 = 0$, then the conic is reducible as

$$p_2((-\lambda q_1) p_0 + ((\lambda - 1) q_0 + a q_1) p_1 + (-c q_0 + b q_1) p_2) = 0,$$

and again as in the proof of Theorem 7.2, the surface contains the line L_q only if

$$\text{rank} \begin{pmatrix} \bar{q}_0 & \bar{q}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \quad \text{or} \quad \text{rank} \begin{pmatrix} \bar{q}_0 & \bar{q}_1 & 0 \\ -\lambda q_1 & (\lambda - 1) q_0 + a q_1 & -c q_0 + b q_1 \end{pmatrix} = 1.$$

Clearly the first matrix has always rank equal to 2, so we are left to study the second one. As the first column is always different from zero, this has rank equal to one if and only if

$$(42) \qquad \begin{cases} (\lambda - 1)|q_0|^2 + a \bar{q}_0 q_1 + \lambda |q_1|^2 = 0 \\ \bar{q}_0(-c q_0 + b q_1) = 0. \end{cases}$$

It is clear that, as $\lambda \neq 0, 1$, then $q_0 = 0$ if and only if $q_1 = 0$, but as we are in the case in which $q_2 = 0$, this option is not possible. Hence we have $b q_1 = c q_0$. We have that $b = 0$ if and only if $c = 0$. Hence, if $b \neq 0 \neq c$ we obtain a unique solution whenever b and c are such that

$$(43) \qquad -(1 - \lambda)|b|^2 + \lambda|c|^2 + a\bar{b}c = 0.$$

Assume now that $b = c = 0$, then we are left to deal with the equation $(\lambda - 1)|q_0|^2 + a \bar{q}_0 q_1 + \lambda|q_1|^2 = 0$. By setting $z = q_1/q_0$, the last equation is equivalent to

$$(44) \qquad |z|^2 + \frac{a}{\lambda} z + \frac{\lambda - 1}{\lambda} = 0.$$

Now, if $a \neq 0$, it is sufficient to apply Corollary 9.2: in this case we have

$$\Delta = \frac{1}{|\lambda|^4} \left(|a|^4 - 4|a|^2(|\lambda|^2 - \lambda_0) - 4\lambda_1^2 \right)$$

If $\Delta > 0$, then we have two solutions, while if $\Delta = 0$, then we have a unique solution.

If $q \in r_2$ we obtain

$$(45) \qquad |a|^2(1 - \lambda) + |c|^2 + a\bar{b}c = 0,$$

while, if $q \in r_3$ we get

$$(46) \qquad |a|^2 \lambda + |b|^2 - a\bar{b}c = 0.$$

Now, the analysis of these two conditions is completely analogous to the first case and we find cases $(2), (5), (8)$ or $(3), (6), (9)$, respectively.

Finally, as the conditions $a = 0 = b$, $a = 0 = c$ and $b = 0 = c$ are mutually exclusive, we only need to check that the twistor line obtained in the case $a \neq 0$, $b \neq 0$ and $c \neq 0$ is at most one. This means that no two of the conditions given in Formulas (43), (45) and (46) can coexist, i.e., the following systems have no solution $[a : b : c] \in \mathbb{P}^2$:

$$\begin{cases} (1 - \lambda)|b|^2 - \lambda|c|^2 - a\bar{b}c = 0 \\ -|a|^2(1 - \lambda) - |c|^2 - a\bar{b}c = 0, \end{cases} \quad \begin{cases} (1 - \lambda)|b|^2 - \lambda|c|^2 - a\bar{b}c = 0 \\ |a|^2 \lambda + |b|^2 - a\bar{b}c = 0, \end{cases} \quad \begin{cases} -|a|^2(1 - \lambda) - |c|^2 - a\bar{b}c = 0 \\ |a|^2 \lambda + |b|^2 - a\bar{b}c = 0. \end{cases}$$

But these three systems are equivalent to

$$\begin{cases} (1-\lambda)|b|^2 - \lambda|c|^2 - a\bar{b}c = 0 \\ (|a|^2 + |b|^2 + |c|^2)(1-\lambda) = 0, \end{cases} \quad \begin{cases} (1-\lambda)|b|^2 - \lambda|c|^2 - a\bar{b}c = 0 \\ \lambda(|a|^2 + |b|^2 + |c|^2) = 0, \end{cases} \quad \begin{cases} -|a|^2(1-\lambda) - |c|^2 - a\bar{b}c = 0 \\ |a|^2 + |b|^2 + |c|^2 = 0, \end{cases}$$

respectively, and hence, none of these admit a solution $[a : b : c] \in \mathbb{P}^2$. $\qquad\square$

Remark 9.4. We point out that Example 8.5 provides and example of surface containing two twistor lines (case (7) in the previous Proposition), Example 8.6 provides and example of surface containing one twistor line (case (4)) and Example 8.7 gives an example of surface without twistor lines.

9.2. **Ramification again.** Theorem 9.3 analysed the locus $\mathcal{D}_0 = \{z \in \mathbb{P}^2 \mid \pi^{-1}(z) \subset S\}$, (cf. Definition 7.1). We know that it consists of zero, one, or two points, depending on conditions on a, b, c, λ. Now we want to analyse the branch locus of a smooth non j-invariant $(1,1)$-surface.

Theorem 9.5. *Let S_A be a smooth non j-invariant surface of bidegree $(1,1)$, such that A is given by (19). Then the twistor projection π restricted to S is a degree 2 cover of $\mathbb{P}^2$ whose branch locus $\mathcal{R} \subset \mathbb{P}^2$ is the zero set of*

$$\begin{aligned} R(q) \;=\; & \left\{|q_0|^2(\lambda-1) + |q_1|^2\lambda + |q_2|^2 + a\bar{q}_0 q_1 + b\bar{q}_0 q_2 + c\bar{q}_1 q_2\right\}^2 \\ & -4\left\{|q_0|^2|q_2|^2(\lambda-1) + a\left[c\bar{q}_0 q_2(|q_0|^2 + |q_1|^2(\lambda+1) + |q_2|^2) - \bar{q}_0 q_1|q_2|^2(\lambda-1)\right]\right. \\ & \left. -b\bar{q}_0 q_1|q_1|^2 + c\bar{q}_1 q_2(|q_0|^2\lambda + |q_1|^2\lambda + |q_2|^2)\right\}. \end{aligned}$$

Proof. Following the proof of Theorem 7.2, we have that the branch locus is given by the set of points $q \in \mathbb{P}^2$ such that

$$(47) \qquad \mathrm{rank} \begin{pmatrix} \bar{q}_0 & q_2 p_1 + (-q_1\lambda + cq_2)p_2 \\ \bar{q}_1 & q_2 p_0 - 2aq_2 p_1 + (q_0(\lambda-1) + aq_1 - bq_2)p_2 \\ \bar{q}_2 & (-q_1\lambda + cq_2)p_0 + (q_0(\lambda-1) + aq_1 - bq_2)p_1 + 2(-cq_0 + bq_1)p_2 \end{pmatrix} = 1$$

which, together with $\bar{q} \cdot p = 0$ gives the following linear system in $[p_1 : p_2 : p_3] \in \mathbb{P}^2$

$$(48) \qquad \begin{cases} (\bar{q}_0 q_2)p_0 + (-2a\bar{q}_0 q_2 - \bar{q}_1 q_2)p_1 + (|q_0|^2(\lambda-1) + a\bar{q}_0 q_1 - b\bar{q}_0 q_2 + |q_1|^2\lambda - c\bar{q}_1 q_2)p_2 = 0 \\ (-\bar{q}_0 q_1\lambda + \bar{q}_0 q_2 c)p_0 + (|q_0|^2(\lambda-1) + a\bar{q}_0 q_1 - b\bar{q}_0 q_2 - |q_2|^2)p_1 \\ \qquad + (-2c|q_0|^2 + 2b\bar{q}_0 q_1 + \bar{q}_2 q_1\lambda - c|q_2|^2)p_2 = 0 \\ (-|q_1|^2\lambda + \bar{q}_1 q_2 c - |q_2|^2)p_0 + (\bar{q}_1 q_0(\lambda-1) + a|q_1|^2 - b\bar{q}_1 q_2 + 2a|q_2|^2)p_1 \\ \qquad + (-2c\bar{q}_1 q_0 + 2b|q_1|^2 - q_0\bar{q}_2(\lambda-1) - a\bar{q}_2 q_1 + b|q_2|^2)p_2 = 0 \\ \bar{q}_0 p_0 + \bar{q}_1 p_1 + \bar{q}_2 p_2 = 0. \end{cases}$$

Such system admits a non-zero solution if and only if the associated matrix

$$M = \begin{pmatrix} \bar{q}_0 q_2 & -2a\bar{q}_0 q_2 - \bar{q}_1 q_2 & \begin{matrix} |q_0|^2(\lambda-1) + a\bar{q}_0 q_1 - b\bar{q}_0 q_2 \\ +|q_1|^2\lambda - c\bar{q}_1 q_2 \end{matrix} \\ -\bar{q}_0 q_1\lambda + \bar{q}_0 q_2 c & \begin{matrix} |q_0|^2(\lambda-1) + a\bar{q}_0 q_1 \\ -b\bar{q}_0 q_2 - |q_2|^2 \end{matrix} & \begin{matrix} -2c|q_0|^2 + 2b\bar{q}_0 q_1 \\ +\bar{q}_2 q_1\lambda - c|q_2|^2 \end{matrix} \\ -|q_1|^2\lambda + \bar{q}_1 q_2 c - |q_2|^2 & \begin{matrix} \bar{q}_1 q_0(\lambda-1) + a|q_1|^2 \\ -b\bar{q}_1 q_2 + 2a|q_2|^2 \end{matrix} & \begin{matrix} -2c\bar{q}_1 q_0 + 2b|q_1|^2 - q_0\bar{q}_2(\lambda-1) \\ -a\bar{q}_2 q_1 + b|q_2|^2 \end{matrix} \\ \bar{q}_0 & \bar{q}_1 & \bar{q}_2 \end{pmatrix}$$

has rank strictly less than three. This happens if and only if the determinants of the 3×3 submatrices all vanish.

If we denote by M_i the determinant of the submatrix obtained from M by removing the ith row, then we have

$$M_1 = -\bar{q}_2 R, \quad M_2 = -\bar{q}_1 R, \quad M_3 = -\bar{q}_0 R, \quad M_4 = 0,$$
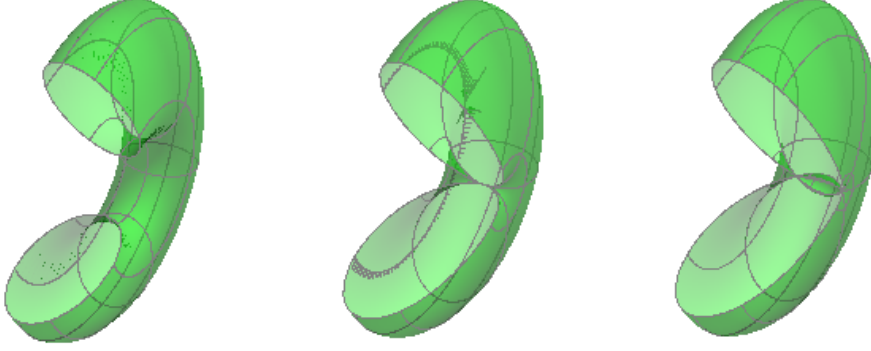
where $R = R(q)$ is as stated. $\qquad\square$

FIGURE 7. Part of the branch loci of $S_{2;a,0,0}$ for $a = 1$, $a = 2\sqrt{2}$, $a = 6$

Remark 9.6. Observe that if $a = b = c = 0$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then $R(q) = 0$ corresponds to the equations obtained in Theorems 7.2 and 7.5.

We shall conclude by investigating a case in which (to simplify matters) $b = c = 0$, and both a and λ are real. In this case S_A is invariant by the action of $U(1)$ on p_2 and ℓ_2 in (p, ℓ). This residual symmetry enables us to exhibit the branch locus as an explicit surface of revolution in $\mathbb{R}^3$.

Corollary 9.7. *Suppose that A is given by (19) with $a \in \mathbb{R}$, $a \neq 0$, $b = c = 0$, and $\lambda = 2$. Then the branch locus of $\pi \colon S_A \to \mathbb{P}^2$ is homeomorphic to a torus with 0, 1 or 2 singular points.*

Proof. We are assuming that $b = c = 0$ and $\lambda = 2$. Set $q_1 = x + iy$ and $q_2 = u + iv$. If $q_0 = 0$ then
$$R(q) = \left(2(x^2 + y^2)^2 + u^2 + v^2\right)^2,$$
and the zero set is empty in $\mathbb{P}^2$. So we can set $q_0 = 1$ and use x, y, u, v as real inhomogeneous coordinates. We set $r = |q_1|$ and $s = |q_2|$. It follows that
$$\operatorname{Im} R(q) = 2ay\left(1 + ax + 2r^2 + 3s^2\right),$$
and
$$\operatorname{Re} R(q) = (1 + 2r^2 + s^2 + ax)^2 - a^2y^2 - 4s^2 + 4axs^2.$$
To study $R(q) = 0$, we first assume $y \neq 0$. The imaginary part being zero implies that $ax < 0$ and
$$\operatorname{Re} R(q) = (-2s^2)^2 - a^2y^2 - 4s^2 + 4axs^2 = 4s^4 - 4(1 - ax)s^2 - a^2y^2.$$
Now, this vanishes if and only if
$$2s^2 = (1 - ax) + \sqrt{(1 - ax)^2 + a^2y^2}.$$
Going back to the imaginary part, we obtain that
$$\operatorname{Im} R(q) = 1 + ax + 2r^2 + \tfrac{3}{2}\left((1 - ax) + \sqrt{(1 - ax)^2 + a^2y^2}\right) = \tfrac{5}{2} - \tfrac{1}{2}ax + 2r^2 + \tfrac{3}{2}\sqrt{(1 - ax)^2 + a^2y^2},$$
which is strictly positive and hence there are no solutions.

Setting $y = 0$, gives $\operatorname{Im} R(q) = 0$ and we are reduced to study
$$(49) \qquad \operatorname{Re} R(q) = s^4 + (4x^2 + 6ax - 2)s^2 + (1 + ax + 2x^2)^2 = 0.$$
This defines a doubled limaçon-shaped curve in the (x, s) plane. The resulting surface of revolution is formed by rotating this profile around the x-axis. If $s = 0$ then $2x = -a \pm \sqrt{a^2 - 8}$. It follows that the surface is smooth if $0 < |a| < 2\sqrt{2}$, has one singular point if $|a| = 2\sqrt{2}$, and two if $|a| > 2\sqrt{2}$. The torus is not circular because we do not have the full T^2 symmetry. $\qquad \square$

Remark 9.8. As it stands, Corollary 9.7 is consistent with Theorem 9.3, and with the examples contained in Section 8.1. If, on the other hand, $a = 0$, we are back in the situation of a real $(1, 1)$ surface, see Subsection 7.1. The choice $\lambda = 2$ places us in the second line of (33) with $0 = q_1 = x + iy$. This is completely consistent with (49), which reduces to
$$s^4 + 2(2x^2 - 1)s^2 + (2x^2 + 1)^2 = 0,$$

which has a unique solution $x = 0$ and $s = 1$. After rotation, this is a circle in the $q_2 = u + iv$ plane.

Cut-away versions of the associated surfaces are shown in Figure 7. The three values of a correspond to a smooth torus, a horn torus and a spindle torus respectively [22, pp. 305–306]. These images provide a vivid analogy with the branch loci of quadrics in $\mathbb{P}^3$ [35]. One might conjecture that the branch loci of more general $(1,1)$ surfaces in $\mathbb{F}$ with finitely many twistor fibres fall topologically into the three categories illustrated by Figure 7, but verification is beyond the scope of the present paper.

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